A Galois lattice for qualitative spatial reasoning and representation

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Abstract. This paper presents an original approach to qualitative spatial representation and reasoning with topological relations based on the use of a Galois lattice of topological relations. This approach has been developed for qualitative spatial reasoning in the domain of agricultural landscape analysis. The paper describes first the general framework of topological relations, and then the design of a Galois lattice of topological relations. The elements of representation and reasoning based on this Galois lattice are discussed and examples are given, together with a brief description of the implementation of the Galois lattice within an object-based representation system.

Keywords: Galois lattice, Galois connection, lattice-based classification, topological relations, hierarchical knowledge representation, classification-based reasoning.

1 Introduction

This paper shows how a Galois lattice –also called a concept lattice structure in the following– can be used in qualitative spatial reasoning, by linking *qualitative models* with *quantitative data*. More precisely, in our framework, we design and use a concept lattice structure of topological relations for qualitative spatial reasoning: a concept lattice structure emphasizes the links between qualitative models, i.e. topological relations, and quantitative data, i.e. vector or raster data. Moreover, a concept lattice structure provides a natural basis for implementing qualitative models of topological relations, and for qualitative spatial reasoning as well.

This research work has been carried out in the context of the design of a knowledge-based system for agricultural landscape analysis and called LOLA. The main objective of this system is to recognize *landscape models* on land-use maps extracted from satellite images. Landscape models are abstract models describing agricultural spatial structures as sets of spatial entities and qualitative spatial relations between these entities [9]. They are used to classify *zones* extracted from the maps. A zone is a collection of raster regions, i.e. connected sets of pixels with the same label denoting the land-use category, e.g. crops, meadows, forest, buildings, etc.

From an implementation point of view, an object-based knowledge representation system, or OKR system, equipped with a classification process, has been used [15]. In this framework, the exploitation of land-use maps for landscape analysis may be considered as an instance classification problem, where landscape models correspond to classes, while zones correspond to instances that have to be classified according to landscape model classes. The classification process for landscape analysis is mainly based on spatial relations, and especially on topological relations, since they are (i) the most characteristic elements of the landscape models, (ii) sufficient for the current landscape analysis application, and (iii) they can be represented and manipulated on the basis of a well defined theoretical framework.

Following these needs, we have designed a hierarchical representation of topological relations based on a concept lattice structure, relying on the Galois lattice theory [1, 5, 19, 7]. In a concept lattice structure, a concept may be defined by an *extension*, i.e. the set of individuals being instances of the concept, and by an *intension*, i.e. the set of properties shared by all individuals. In our framework, the extension of concepts corresponds to topological relations between regions existing in an image, and the intension of concepts corresponds to properties computed on that image (*computational operations*). From a reasoning point of view, a concept lattice structure can be viewed as a hierarchical conceptual clustering of individuals, and as a representation of all implications between the attributes of the concepts [19].

The present work brings several new and original aspects in the management of topological relations, regarding representation, reasoning and implementation. It is one of the unique works on spatial reasoning relying on the Galois lattice theory, that is usually associated with formal concept analysis [19, 7], or with knowledge discovery [8, 18]. Classification-based reasoning is then used for inferring topological relations and spatial structures, involving both classification of entities and relations. Furthermore, our work combines a qualitative as well as a quantitative approach to landscape analysis on land-use maps, because the chosen implementation framework, namely OKR systems, provides both reasoning and computation facilities. Moreover, our system provides results that agronomists can easily read and understand. The results of this research work can be naturally reused in the design of geographical information systems.

The paper is organized as follows. The following section introduces the context of our work, i.e. topological relations and computational operations on an image. Then, the design of a concept lattice structure of topological relations is detailed, with the characteristics of the lattice elements. The reasoning capabilities resulting from the hierarchical organization of the topological relations are then described and discussed. Finally, a number of discussion elements and research perspectives are presented and conclude this paper.

2 Topological relations

2.1 Mereology and mereotopology

Mereology is a theory of the *part-whole* relation. It was introduced by Lesniewski at the beginning of 20th century as an alternative to the set theory [12]. It is an axiomatic base for topology and geometry of regions rather than of points. Mereology defines the relation $P(x,y)$, for *"x is a part of y"*, which is a partial order since it is reflexive, anti-symmetric and transitive.

The part-whole relation can be formalized using the "connection" relation [2]: two regions x and y are said to be connected –denoted by $C(x,y)$– if they share a point. The $C(x,y)$ relation is defined by the following axioms:

$$
\begin{aligned}
\forall x \in \mathcal{D} &\ :\ C(x,x) \\
\forall x,y \in \mathcal{D} &\ :\ C(x,y) \rightarrow C(y,x) \\
\forall x,y \in \mathcal{D} &\ :\ (\forall z \in \mathcal{D} : C(z,x) \leftrightarrow C(z,y)) \rightarrow x = y
\end{aligned}
$$

Relying on the connection relation, five relations can be defined accordingly: $DC(x,y)$ *"x is disconnected from y"*; $P(x,y)$ *"x is a part of y"*; $PP(x,y)$ *"x is a proper part of y"*; $O(x,y)$ *"x overlaps y"*; $DR(x,y)$ *"x is discrete from y"*. The theory based on these relations is interesting because it makes a difference between a point and a region, thanks to the relations $C(x,y)$ –

x and y share a point– and $O(x, y)$ –x and y share a region. This differentiation allows the definition of three additional relations: $EC(x, y)$ *“x is externally connected with y”*, $TP(x, y)$ *“x is a tangential part of y”*, $NTP(x, y)$ *“x is a non tangential part of y”* (see Table 1). Finally this differentiation is used to introduce topological notions –*interior*, *boundary*– into the purely set-based notions of the mereology, and is therefore called *mereotopology*.

$$
\begin{array}{|lll|}
DC(x, y) & \equiv_{def} & \neg C(x, y) \\
P(x, y) & \equiv_{def} & \forall z \ : \ C(z, x) \rightarrow C(z, y) \\
PP(x, y) & \equiv_{def} & P(x, y) \wedge \neg P(y, x) \\
O(x, y) & \equiv_{def} & \exists z \ : \ P(z, x) \wedge P(z, y) \\
DR(x, y) & \equiv_{def} & \neg O(x, y) \\
EC(x, y) & \equiv_{def} & C(x, y) \wedge \neg O(x, y) \\
TP(x, y) & \equiv_{def} & P(x, y) \wedge \exists z \ : \ EC(z, x) \wedge EC(z, y) \\
NTP(x, y) & \equiv_{def} & P(x, y) \wedge \neg \exists z \ : \ EC(z, x) \wedge EC(z, y)
\end{array}
$$

Table 1: Mereotopological relations [2]

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The most studied topological relations are those of the so-called **RCC-8** theory [17, 3] that is also based on the connection relation. This theory defines a set of eight base relations, namely the $\mathcal{B}$ set, which are exhaustive and mutually exclusive. Their names and iconic representations are given in Table 2 and their definitions in Table 3. The regions considered are potentially infinite in number and any degree of connection between them is allowed. These regions are closed, non empty, not necessarily internally connected, and they may overlap, i.e. a pixel may belong to several regions.

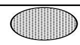
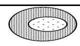
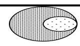
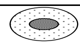
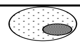
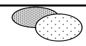
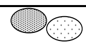
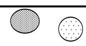
relation	notation	icons
“x is identical with y”	$EQ(x, y)$	
“x is a non tangential proper part of y”	$NTPP(x, y)$	
“x is a tangential proper part of y”	$TPP(x, y)$	
“x non tangentially contains as a proper part y”	$NTPP^{-1}(x, y)$	
“x tangentially contains as a proper part y”	$TPP^{-1}(x, y)$	
“x partially overlaps y”	$PO(x, y)$	
“x is externally connected with y”	$EC(x, y)$	
“x is disconnected from y”	$DC(x, y)$	

Table 2: Names and icons associated to the eight base relations of the **RCC-8** theory.

Following the work presented in [6] where different computational operations have been used, we propose a method for checking topological relations on regions based on the *set-difference* and the *intersection* of interior sets and boundaries [10, 11]). The following four *computational operations* are taken into account: the intersection of the interior sets, $x^\circ \cap y^\circ$; the intersection of the boundary sets, $\partial x \cap \partial y$; the two differences of the interior sets, $x^\circ - y^\circ$ and $y^\circ - x^\circ$. From these four operations eight conditions have been derived:

$$\begin{array}{lll}
EQ(x,y) & \equiv_{def} & P(x,y) \wedge P(y,x) \\
NTPP(x,y) & \equiv_{def} & PP(x,y) \wedge \neg\exists z \;:\; EC(z,x) \wedge EC(z,y) \\
TPP(x,y) & \equiv_{def} & PP(x,y) \wedge \exists z \;:\; EC(z,x) \wedge EC(z,y) \\
NTPP^{-1}(x,y) & \equiv_{def} & NTPP(y,x) \\
TPP^{-1}(x,y) & \equiv_{def} & TPP(y,x) \\
PO(x,y) & \equiv_{def} & O(x,y) \wedge \neg P(x,y) \wedge \neg P(y,x) \\
EC(x,y) & \equiv_{def} & C(x,y) \wedge \neg O(x,y) \\
DC(x,y) & \equiv_{def} & \neg C(x,y)
\end{array}$$

Table 3: The base relations of **RCC-8** [17]. Their definitions rely on the mereotopology.

- $(x^\circ - y^\circ = \emptyset)$: *"x is a part of y"*, denoted by $P(x,y)$[1]
- $(x^\circ - y^\circ \neq \emptyset)$: *"x is not a part of y"*, denoted by $Dx(x,y)$
- $(y^\circ - x^\circ = \emptyset)$: *"x contains y"* (*"y is a part of x"*), denoted by $P^{-1}(x,y)$
- $(y^\circ - x^\circ \neq \emptyset)$: *"x does not contain y"* (*"y is not a part of x"*), denoted by $Dy(x,y)$
- $(x^\circ \cap y^\circ = \emptyset)$: *"x is discrete from y"*, denoted by $DR(x,y)$
- $(x^\circ \cap y^\circ \neq \emptyset)$: *"x overlaps y"*, denoted by $O(x,y)$
- $(\partial x \cap \partial y = \emptyset)$: *"x does not share a boundary with y"*, denoted by $NA(x,y)$
- $(\partial x \cap \partial y \neq \emptyset)$: *"x shares a boundary with y"*, denoted by $A(x,y)$

This set of conditions is called CM*-8*. It is important to notice that some conditions imply others because of the properties of the image regions. For instance, each of the conditions $x^\circ - y^\circ = \emptyset$ or $y^\circ - x^\circ = \emptyset$ implies the condition $x^\circ \cap y^\circ \neq \emptyset$ (the regions are non empty); the condition $\partial x \cap \partial y \neq \emptyset$ is implied by the conjunction of the same conditions (the regions are non empty and closed). Finally these four operations allow to check **RCC-8** relations on the images. The correspondence between each relation and the conditions are described on Table 4: each relation is equivalent to a subset of conditions of CM*-8*.

x,y	$x^\circ - y^\circ$	$y^\circ - x^\circ$	$x^\circ \cap y^\circ$	$\partial x \cap \partial y$
$EQ(x,y)$	$\emptyset$	$\emptyset$	$\neq \emptyset$	$\neq \emptyset$
$NTPP(x,y)$	$\emptyset$	$\neq \emptyset$	$\neq \emptyset$	$\emptyset$
$TPP(x,y)$	$\emptyset$	$\neq \emptyset$	$\neq \emptyset$	$\neq \emptyset$
$NTPP^{-1}(x,y)$	$\neq \emptyset$	$\emptyset$	$\neq \emptyset$	$\emptyset$
$TPP^{-1}(x,y)$	$\neq \emptyset$	$\emptyset$	$\neq \emptyset$	$\neq \emptyset$
$PO(x,y)$	$\neq \emptyset$	$\neq \emptyset$	$\neq \emptyset$	$\neq \emptyset$
$EC(x,y)$	$\neq \emptyset$	$\neq \emptyset$	$\emptyset$	$\neq \emptyset$
$DC(x,y)$	$\neq \emptyset$	$\neq \emptyset$	$\emptyset$	$\emptyset$

Table 4: The correspondence between our computational operations for checking relations on the image and the **RCC-8** relations [13].

[1]The regions are closed and thus $x^\circ - y^\circ = \emptyset \leftrightarrow x - y = \emptyset$.

3 A Galois lattice of topological relations

In this section, we introduce a Galois lattice of topological relations, based on the set $\mathcal{B}$ and on the set CM-*8* of conditions. In the following, we denote by $\mathcal{D}$ a set of spatial regions that are non empty, closed, and non necessarily internally connected. The lattice has to respect the following requirements: it should contain the $\mathcal{B}$ and CM-*8* sets; it should be ordered by the implication relation between the eight relations of $\mathcal{B}$ and the eight conditions of CM-*8*. The lattice is built thanks to a Galois connection $\{f, g\}$ between $\mathcal{B}$ and CM-*8* as explained below.

3.1 A Galois lattice based on $\mathcal{B} \times CM$-*8*

Definition 1 *The function f maps a relation r of $\mathcal{B}$ to a subset of CM-8 where all conditions hold for a pair (x, y) whenever the relation r holds for (x, y):*

$$f: \begin{array}{ccl} 2^{\mathcal{B}} & \mapsto & 2^{CM\text{-}8} \\ \{r\} & \mapsto & f(\{r\}) = \{c \in CM\text{-}8 \mid \forall(x,y) \in \mathcal{D}^2 : r(x,y) \rightarrow c(x,y)\} \end{array}$$

Furthermore, f maps every subset R of $\mathcal{B}$ to the subset C of CM-*8* whose conditions are implied by *all* the relations of R: $f(R) = \cap_{r \in R} f(\{r\})$.

Definition 2 *The function g maps a condition c of CM-8 to the subset of $\mathcal{B}$ where all relations imply the condition c:*

$$g: \begin{array}{ccl} 2^{CM\text{-}8} & \mapsto & 2^{\mathcal{B}} = RCC\text{-}8 \\ \{c\} & \mapsto & g(\{c\}) = \{r \in \mathcal{B} \mid \forall(x,y) \in \mathcal{D}^2 : r(x,y) \rightarrow c(x,y)\} \end{array}$$

Furthermore g maps every subset C of CM-*8* to a subset R of $\mathcal{B}$ whose relations imply *all* the conditions of C: $g(C) = \cap_{c \in C} g(\{c\})$. The Galois connection $\{f, g\}$ is expressed in Table 5.

	P	Dx	P^{-1}	Dy	DR	O	NA	A
EQ	1	0	1	0	0	1	0	1
$NTPP$	1	0	0	1	0	1	1	0
TPP	1	0	0	1	0	1	0	1
$NTPP^{-1}$	0	1	1	0	0	1	1	0
TPP^{-1}	0	1	1	0	0	1	0	1
PO	0	1	0	1	0	1	0	1
EC	0	1	0	1	1	0	0	1
DC	0	1	0	1	1	0	1	0

Table 5: The table represents elements (r, c), with $r \in \mathcal{B}$ (line) and $c \in CM$-*8* (column). An element $(r, c) = 1$ if $\forall(x, y) \in \mathcal{D}^2, r(x, y) \rightarrow c(x, y)$. Otherwise $(r, c) = 0$.

The two functions f and g are used to define a closure operator h of RCC-*8* and a closure operator h' of $2^{CM\text{-}8}$:

$$h: \begin{array}{ccc} RCC\text{-}8 & \mapsto & RCC\text{-}8 \\ h(R) & = & g \circ f(R) \end{array} \qquad h': \begin{array}{ccc} 2^{CM\text{-}8} & \mapsto & 2^{CM\text{-}8} \\ h'(C) & = & f \circ g(C) \end{array}$$

The Galois lattice $\mathcal{T}_{\mathcal{G}}$ is built on the basis of the two closure operators h and h'. Its elements are pairs $(C, R) \in 2^{CM\text{-}8} \times RCC\text{-}8$, where $h(R) = R$ and $C = f(R) = h'(C)$. Such a pair is also a formal concept derived from the implication relation between $\mathcal{B}$ and CM-*8*, C being the intension and R the extension of this concept.

Definition 3 *The Galois lattice $\mathcal{T}_\mathcal{G}$ based on the implication between the relations of $\mathcal{B}$ and the conditions of CM-8 is the structure $\langle \mathcal{E}_\mathcal{G}, \sqsubseteq, \frown, \smile, (CM\text{-}8, \emptyset), (\emptyset, \mathcal{B})\rangle$ where:*

- *$\mathcal{E}_\mathcal{G}$ is the set of all pairs (C, R) where R is a subset of $\mathcal{B}$ closed for h, C is a subset of CM-8 closed for h', $f(R) = C$ and $g(C) = R$.*
- *The ordering $\sqsubseteq$ between two elements (C_1, R_1) and (C_2, R_2) is defined as follows:*

$$((C_1, R_1) \sqsubseteq (C_2, R_2)) \leftrightarrow \begin{cases} C_2 \subseteq C_1 \\ R_1 \subseteq R_2 \end{cases}$$

  where $\subseteq$ is the set-inclusion.
- *The greatest lower bound (denoted by $\frown$) of two elements is defined as follows:*

$$(C_1, R_1) \frown (C_2, R_2) = (h'(C_1 \cup C_2), R_1 \cap R_2)$$

- *The least upper bound (denoted by $\smile$) of two elements is defined as follows:*

$$(C_1, R_1) \smile (C_2, R_2) = (C_1 \cap C_2, h(R_1 \cup R_2))$$

  where $\cup$ and $\cap$ are the set-union and the set-intersection.
- *The bottom element is $(CM\text{-}8, \emptyset)$,*
- *The top element is $(\emptyset, \mathcal{B})$.*

3.2 The lattice ordering and the implication between relations

We define the π_r function that maps an element (C, R) of $\mathcal{T}_\mathcal{G}$ onto the disjunction of the relations of R (denoted $\pi_r(C, R)$):

$$\forall (x, y) \in \mathcal{D}^2, \;\; \pi_r(C, R)(x, y) =_{def} \bigvee_{r \in R} r(x, y)$$

Similarly we introduce the π_c function that maps an element (C, R) of $\mathcal{T}_\mathcal{G}$ onto the conjunction of the elements of C:

$$\forall (x, y) \in \mathcal{D}^2, \;\; \pi_c(C, R)(x, y) =_{def} \bigwedge_{c \in C} c(x, y)$$

It is possible to show that the two formulas resulting from the application of π_c and π_r are equivalent [10]. This equivalence is used to name the elements of $\mathcal{T}_\mathcal{G}$, either as a disjunction of relations or as a conjunction of conditions[2]. On the figure 1, elements are denoted by a name or an icon: a name represents a conjunction of conditions, e.g. `OetA` denotes the element $(\{O, A\}, R)$, where `et` stands for "and", or a disjunction of relations, e.g. `PP` denotes the element $(C, \{TPP, NTPP\})$, since $PP = TPP \vee NTTP$. The icons at the bottom of

[2]In the following A denotes the condition and `A` denotes the lattice element where the condition is represented; respectively EC and `EC` to denote the topological relation and the element of the lattice.

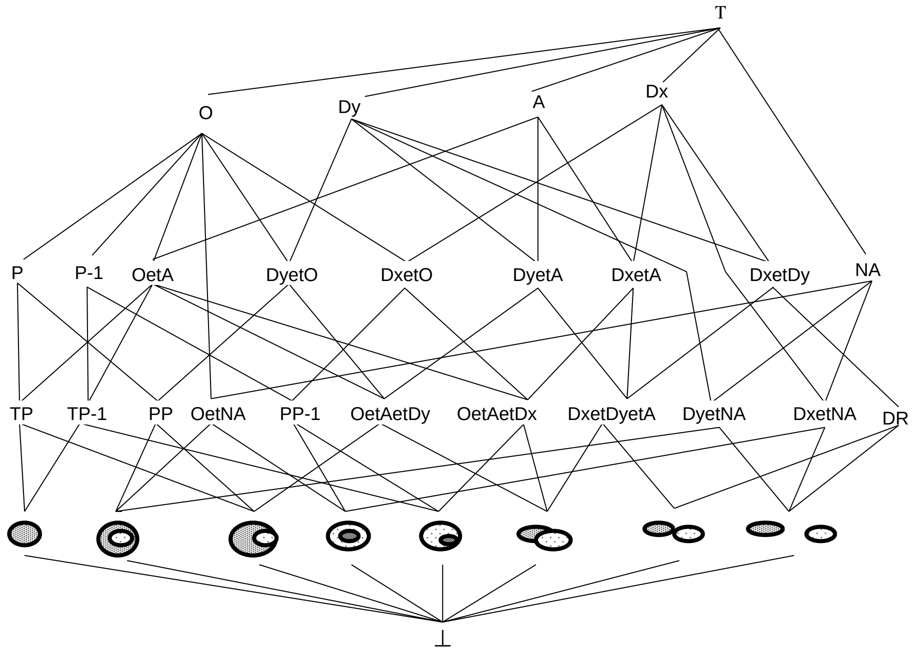


Figure 1: The Galois lattice built on the basis of Table 5: the pairs (C, R) are denoted by the relation, or by the condition, to which they are associated by the relations π_r, or π_c.

the lattice represent the elements associated to the eight base relations of **RCC-8**, e.g. EQ = $(\{O, A, P, P^{-1}\}, \{EQ\})$.

The π_c and π_r functions are also used to make explicit the link between the lattice ordering and the implication relation on the topological relations. Actually the following property is proved for all pairs of elements of $\mathcal{T}_\mathcal{G}$ (see [10]):

$$(C_1, R_1) \sqsubseteq (C_2, R_2) \leftrightarrow \begin{cases} \forall (x,y) \in \mathcal{D}^2, \bigvee_{r \in R_1} r(x,y) \rightarrow \bigvee_{r \in R_2} r(x,y) \\ \forall (x,y) \in \mathcal{D}^2, \bigwedge_{c \in C_1} c(x,y) \rightarrow \bigwedge_{c \in C_2} c(x,y) \end{cases}$$

Thus, if an element $\mathtt{E}_1 = (C_1, R_1)$ (e.g. EQ) is smaller than another element $\mathtt{E}_2 = (C_2, R_2)$ (e.g. TP), the disjunction of the relations of R_1 implies the disjunction of the relations of R_2 and the conjunction of the conditions of C_1 implies the conjunction of the conditions of C_2:

$$\begin{array}{lll} \mathtt{EQ} \sqsubseteq \mathtt{TP} \leftrightarrow & \forall (x,y) & EQ(x,y) \rightarrow (EQ \vee TPP)(x,y) \\ & \text{and} & (A \wedge P \wedge P^{-1} \wedge O)(x,y) \rightarrow (A \wedge P \wedge O)(x,y) \end{array}$$

3.3 The characterization of the *glb* ($\frown$) and the *lub* ($\smile$) in $\mathcal{T}_\mathcal{G}$

The operator $\frown$ in the lattice is equivalent to the operator $\wedge$ (conjunction) on the relations.

Property 1 *An element* E *of* $\mathcal{T}_\mathcal{G}$ *is the glb of two elements* $\mathtt{E}_1$ *and* $\mathtt{E}_2$ *if and only if* $\pi_r(\mathtt{E})$ *is equivalent to the conjunction of the two relations* $\pi_r(\mathtt{E}_1)$ *and* $\pi_r(\mathtt{E}_2)$*:*

$$(\mathtt{E} = \mathtt{E}_1 \frown \mathtt{E}_2) \leftrightarrow (\forall (x,y) \in \mathcal{D}^2,\ \pi_r(\mathtt{E})(x,y) \leftrightarrow \pi_r(\mathtt{E}_1)(x,y) \wedge \pi_r(\mathtt{E}_2)(x,y))$$

Furthermore, each element $\mathtt{E}$ *of* $\mathcal{T}_{\mathcal{G}}$ *is characterized by the equivalence between the disjunction* $\pi_r(\mathtt{E})$ *and the conjunction* $\pi_c(\mathtt{E})$*.*

$$(\mathtt{E} = \mathtt{E}_1 \frown \mathtt{E}_2) \leftrightarrow (\forall (x, y) \in \mathcal{D}^2,\ \pi_c(\mathtt{E})(x, y) \leftrightarrow \pi_c(\mathtt{E}_1)(x, y) \wedge \pi_c(\mathtt{E}_2)(x, y))$$

Property 2 *For all pairs of lattice elements* $(\mathtt{E}_1, \mathtt{E}_2)$*, the lub* $\mathtt{E}' = \mathtt{E}_1 \smile \mathtt{E}_2$ *is such that the disjunction of* $\pi_r(\mathtt{E}_1)$ *and* $\pi_r(\mathtt{E}_2)$ *implies the relation* $\pi_r(\mathtt{E}')$*:*

$$(\mathtt{E}' = \mathtt{E}_1 \smile \mathtt{E}_2) \rightarrow \left(\forall (x, y) \in \mathcal{D}^2,\ \pi_r(\mathtt{E}')(\mathtt{x}, \mathtt{y}) \leftarrow \pi_{\mathtt{r}}(\mathtt{E}_1)(\mathtt{x}, \mathtt{y}) \vee \pi_{\mathtt{r}}(\mathtt{E}_2)(\mathtt{x}, \mathtt{y})\right)$$

This property is easy to prove since the relation set R' of the element $\mathtt{E}' = \mathtt{E}_1 \smile \mathtt{E}_2$ is the closure of the union of the two relation sets R_1, R_2 of the elements $\mathtt{E}_1$ and $\mathtt{E}_2$ (cf. Definition 3). The reciprocal property of Property 2 is not true in the $\mathcal{T}_{\mathcal{G}}$ lattice because an element of $\mathcal{T}_{\mathcal{G}}$ can be the *lub* of several elements: $\mathtt{E}'$ can be both the *lub* of the pair $\mathtt{E}_1$, $\mathtt{E}_2$ and the *lub* of the pair $\mathtt{E}_3$, $\mathtt{E}_4$. Then:

$$\begin{array}{r}\mathtt{E}' = \mathtt{E}_1 \smile \mathtt{E}_2 : \pi_r(\mathtt{E}_1) \vee \pi_{\mathtt{r}}(\mathtt{E}_2) \rightarrow \pi_{\mathtt{r}}(\mathtt{E}') \\ \mathtt{E}' = \mathtt{E}_3 \smile \mathtt{E}_4 : \pi_r(\mathtt{E}_2) \vee \pi_{\mathtt{r}}(\mathtt{E}_4) \rightarrow \pi_{\mathtt{r}}(\mathtt{E}') \\ \text{and}\ \ \pi_r(\mathtt{E}') \rightarrow \pi_{\mathtt{r}}(\mathtt{E}_1) \vee \pi_{\mathtt{r}}(\mathtt{E}_2) \vee \pi_{\mathtt{r}}(\mathtt{E}_3) \vee \ldots\end{array}$$

For instance, the *lub* of the elements $\mathtt{TP}$ and $\mathtt{TP}^{-1}$ of $\mathcal{T}_{\mathcal{G}}$ is the element `OetA` which is also the *lub* of the elements `OetAetDx` and `OetAetDy` (see Figure 1). This property has to be linked to a property of closed sets: the *lub* of two closed sets is generally not a closed set, whereas the *glb* is a closed set [1, 5].

3.4 Reasoning with the Galois lattice $\mathcal{T}_{\mathcal{G}}$

According to its properties, the lattice $\mathcal{T}_{\mathcal{G}}$ can be used for spatial reasoning and especially for linking quantitative and qualitative reasoning. Actually, each relation of the lattice is equivalent to a set of quantitative conditions. Furthermore, the $\mathcal{T}_{\mathcal{G}}$ lattice is ordered and closed under conjunction and converse. Its main drawback is that it is not closed under disjunction and composition.

Conjunction. According to Property 1 the conjunction of two relations of the lattice is also a relation of the lattice, actually the *glb* of the two elements. For example, the conjunction of the relations $P = \{EQ, TPP, NTPP\}$ and $P^{-1} = \{EQ, TPP^{-1}, NTPP^{-1}\}$ is the relation $\{EQ\}$ and $\mathtt{P} \frown \mathtt{P}^{-1} = \mathtt{EQ}$. In other words, the intersection of two sets of relations of the lattice is a set of relations of the lattice, see the definition of the *glb*:

$$(C_1,\ R_1) \frown (C_2,\ R_2) = (h'(C_1 \cup C_2),\ R_1 \cap R_2)$$

Converse. Each relation of $\mathcal{T}_{\mathcal{G}}$ has a converse in $\mathcal{T}_{\mathcal{G}}$. This property is due to the fact that both the RCC-*8* relations (e.g. $TPP^{-1}(x, y) = TPP(y, x)$) and the CM-*8* conditions have a converse (e.g. $A(x, y) = A(y, x)$, $Dx(x, y) = Dy(y, x)$). Thus, the converse of an element $\mathtt{E} = (C,\ R)$ is the element $\mathtt{E}^{-1} = (h'(C^{-1}),\ R^{-1})$ where C^{-1} is the set of the converses of the conditions of C and R^{-1} is the set of the converses of the relations of R.

Disjunction and composition. According to Property 2, the disjunction of two relations of $\mathcal{T}_{\mathcal{G}}$ is not necessarily a relation of $\mathcal{T}_{\mathcal{G}}$. For example, the disjunction of the relations $TP = \{EQ, TPP\}$ and $TP^{-1} = \{EQ, TPP^{-1}\}$ is the relation $\{EQ, TPP, TPP^{-1}\}$, whereas the *lub* of the elements $\mathtt{TP}$ and $\mathtt{TP}^{-1}$ is the element `OetA` that represents the relation $\{PO, EQ, TPP, TPP^{-1}\}$.

In other words, the union of two sets of relations of the lattice is not necessarily a set of relations of the lattice, see the definition of the *lub*:

$$(C_1,\, R_1) \smile (C_2,\, R_2) = (C_1 \cap C_2,\, h(R_1 \cup R_2))$$

Consequently, the composition of two relations is *a priori* not necessarily a relation of $\mathcal{T}_{\mathcal{G}}$. Thus, the $\mathcal{T}_{\mathcal{G}}$ lattice is not closed under composition.

Actually, the $\mathcal{T}_{\mathcal{G}}$ lattice is not complemented and the *lub* of two elements is not equivalent to the disjunction of the corresponding relations (see Section 3.3). For example, if the relation $(TP \vee TP^{-1})(x, y)$ holds, it is concluded in $\mathcal{T}_{\mathcal{G}}$ that $(TPP \vee TPP^{-1} \vee EQ \vee PO)(x, y)$ holds (`OetA` is the *lub* of `TP` and $\texttt{TP}^{-1}$). By contrast, it is concluded in the boolean lattice $\mathcal{T}_{\mathcal{P}}$ –the lattice $\mathcal{T}_{\mathcal{P}}$ is the power set of $\mathcal{B}$, i.e. $\mathbf{2}^{\mathcal{B}}$, that is ordered by the inclusion and that includes the whole **RCC-8** set– that $(TPP \vee TPP^{-1} \vee EQ)(x, y)$ holds. This last conclusion cannot be derived in $\mathcal{T}_{\mathcal{G}}$ since the corresponding element does not exist.

4 Lattice-based classification in practice

Lattices have been used for automatic theorem proving and relations inferring in the **RCC-8** framework [16]. Moreover, they are well adapted for classification-based reasoning. The Galois lattices in particular are well adapted for the classification and computation of topological relations. Their main drawback is that reasoning is not complete [11]. This drawback is in balance with the low number of elements in the Galois lattices, making them practically more manageable than the boolean lattice $\mathcal{T}_{\mathcal{P}}$.

4.1 Inferences on the relations

The lattice $\mathcal{T}_{\mathcal{G}}$ has been implemented within a frame-based representation system [13]. The elements of the lattices have been represented within relation classes organized according to the lattice ordering. In the following, the same notation is used for an element and the class representing this element, i.e. `E` stands for the element `E` and for the class representing `E`.

Figure 2 shows the generic class `SPATIAL-RELATION` that represents the top element of the lattice, and three other classes. The class `O` is a subclass of the generic class `SPATIAL-RELATION`; the class `DR` is a subclass of the class `DxetDy` and the class `EC` is a subclass of the classes `DR` and `DxetDyetA`.

For example, let us suppose that the EQ relation, expressed by the class `EQ`, has to be checked for two regions in the image; this class has a set of superclasses whose `condition` attribute value is reduced to a unique condition. This set is $\mathcal{S} = \{\texttt{P}, \texttt{P}^{-1}, \texttt{A}, \texttt{O}\}$. The associated conditions are: P, P^{-1}, A and O. The common descendants of the elements of $\mathcal{S}$ are then searched in the class hierarchy: `P` and $\texttt{P}^{-1}$ have a common descendant (their *glb*, in fact) that is `EQ`. Using Property 1, we deduce that the conjunction of the two conditions P and P^{-1} is equivalent to the relation EQ. Thus, checking the relation EQ can be reduced to checking the two conditions P and P^{-1}.

Practically, the `verify-relation` methods of the classes `P` and $\texttt{P}^{-1}$ are used on the image regions. These methods respectively compute the conditions P, or P^{-1}, associated to the classes; if they succeed, an instance of the *glb* of `P` and $\texttt{P}^{-1}$ is created, i.e. an instance of `EQ` is created. If the `verify-relation` method of one of the classes fails then the `complement` attribute of this class can be used to find out which relation has been verified. For example, if the `verify-relation` method of `P` fails, the system can infer that the relation `Dx` holds (the value of the `complement` attribute of `P` is `Dx`, representing the condition Dx); the system finally creates an instance of the *glb* of the two classes: $\texttt{P}^{-1} \frown \texttt{Dx} = \texttt{PP}^{-1}$ (see Figure 1).

```
(defclass SPATIAL-RELATION
     (is-a . RELATION)
     (complement (a . SPATIAL-RELATION))
     (converse (a . SPATIAL-RELATION))
     (condition (a . CONDITION))
     (search-conditions (method) (...))
     (verify-relation (method) (O1 O2) (...))
     (transitivity (method) (...))
     .... )

(defclass O
     (is-a . SPATIAL-RELATION)
     (converse (value . O))
     (complement (value . DR))
     (condition (value . CO))
     .... )

(defclass DR
     (is-a . DxetDy)
     (converse (value . DR))
     (condition (value . CDR))
     .... )

(defclass EC
     (is-a .  (DR DxetDyetA))
     (converse (value . EC))
     .... )
```

Figure 2: Classes representing the elements of the lattices $\mathcal{T}_{\mathcal{G}}$ or $\mathcal{T}_{\mathcal{G}\mathcal{E}}$. The properties of the relations are represented within attributes and methods.

4.2 Classification of spatial structures for landscape analysis

We have defined a number of models describing the main global structures of village territories in the Lorraine region (East of France) [9], such as `TERRITORY-EC-FOREST`, `TERRITORY-BETWEEN--FORESTS`, `TERRITORY-WITH-CROP-GROUP`, `TERRITORY-WITH-CROP-EC-FOREST`, `TERRITORY-WITH-MEADOW`, `TERRITORY-WITH-COVERING-MEADOW`, ... These models are linked to the models `CROP`, `CROP-GROUP`, `CROP-EC-FOREST`, `MEADOW`, `COVERING-MEADOW`, ... that are elements of the lattice describing the models of crop fields and meadows. This lattice itself contains about 140 elements.

The classification of a spatial structure requires the classification of a set of related spatial structures and the corresponding relations, as it is illustrated in the following simplified example. The system first recognizes a territory entity on the image thanks to its label. The system represents this entity as an instance, say `t1`, of the model class `TERRITORY`. The system tries then to classify the instance `t1` into a more specific class, for example the `TERRITORY-EC-FOREST` class. It checks therefore whether the instance `t1` matches the properties of `TERRITORY-EC-FOREST`: i.e. being externally connected with a forest. it searches for forest entities in the neighborhood of `t1` in the image, and computes the topological relation between each forest entity and the territory, until it finds one forest entity externally connected to `t1`. If it succeeds, then the external connection property is verified, and the instance `t1` is classified into the `TERRITORY-EC-FOREST` class. If it fails, the system tries to classify `t1` into another subclass of `TERRITORY`, e.g. `TERRITORY-DC-FOREST`. The classification process goes on down the hierarchy of spatial structures until all classes have been checked.

5 Discussion and conclusion

The system LoLA and the associated research work can be enhanced and continued in a number of directions. Lattices such as the boolean lattice $\mathcal{T}_\mathcal{P}$ have been used in systems based on the mereotopology for representing spatial knowledge and for spatial reasoning [16, 4]. By contrast, lattices combining computation and reasoning purposes, such as the lattice $\mathcal{T}_\mathcal{G}$, have not yet been used in the field of spatial representation and reasoning (at least to our knowledge). In addition to this "double competence", this lattice has the main advantage of being minimal in terms of memory space (34 elements for $\mathcal{T}_\mathcal{G}$ *versus* 256 elements for $\mathcal{T}_\mathcal{P}$) and of being easily extensible: indeed, it is possible to modify, extend or reduce the set of computational primitives or the set of base relations without loosing the properties associated to the Galois connection theory. Moreover, other Galois lattices may be designed in the same way [11].

Lattices are well adapted to classification-based reasoning. The classification mechanism that we have developed for our application manages two related lattices: the first one is a lattice of relation classes based on $\mathcal{T}_\mathcal{G}$, and the second one is a lattice of classes representing the models of spatial structures. In this context, there is a number of research perspectives among which the study of the complexity of qualitative spatial reasoning using the $\mathcal{T}_\mathcal{G}$ lattice (a first study is carried out in [10]), the possible extensions of the $\mathcal{T}_\mathcal{G}$ lattice to ensure a complete reasoning –there is a tradeoff between completeness of reasoning and the size of the lattice, and, finally, an extension to hierarchical case-based reasoning for landscape analysis (first elements in this research direction are given in [14]).

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