parameterizing Intersection of Time-varying Quadrics

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Abstract

This report addresses the problem of computing the parametrization of the intersection of deformable quadratic algebraic surfaces (quadrics) in projective space. It also presents an automatic method for describing the evolution in time of the topology of the intersection. The work is based on the results from [3, 4], which offer an exact parametrization of the intersection of two quadrics with rational coefficients of arbitrary size. This parametrization is rational when one exists, and its coefficients are almost as rational as possible [4].

Chapter 1

Introduction

The problem of computing a parametrization for the intersection of two surfaces can be regarded in an algebraic, or/and in a geometric manner. We are concerned, in our study, with the algebraic approach which we apply to quadrics. Computing the parametric expression of the intersection of two quadrics Q and P is reduced, roughly, to the following three main steps:

- find a parametrization $\mathbf{X}_Q$ for Q
- insert $\mathbf{X}_Q$ in the equation of P which results in a new equation Ω
- substitute in $\mathbf{X}_Q$ the solutions of equation Ω

Our work is based on the results from [3, 4]. The authors, L. Dupont, D. Lazard, S. Lazard, S. Petitjean have presented the first exact and efficient algorithm for computing a parametric representation of the intersection of two quadrics in three-dimensional real space given by implicit equations with rational coefficients [4]. Besides, they offer a complete analysis of the intersection curve, concerning its reducibility, planarity, and singularity.

Our interest is the study of the intersection of quadrics which can deform in time. Regarding this subject, we approach two problems, namely, the problem of describing the evolution in time of the topology of the intersection, and the problem of finding a well-defined parametrization as a function of time for the intersection in cause. The purpose of describing the topology is to understand how and when the intersection type changes, where the exception cases are, when it can be possible to predict the evolution of the intersection. This is a preliminary step in computing a parametrization. The main reason for studying the second problem is the advantage it presents: given a well-defined parametrization depending on time, there is no need to consider time as a discrete set, and for each of its values to compute a parametrization.

The methods we have developed work for two deformable quadrics which intersect in a smooth quartic for all values of time, with the exception of a finite number of moments. We mention that this is not necessarily a restriction, since in most situations quadrics intersect in a smooth quartic. Following the example from [3, 4], we denote this situation as the generic case. The contributions we bring consist in:

- describing the evolution of the topology of the intersection curve in the generic case. Formally, we give necessary conditions for the preservation/transformation of topology. Practically, using the results from [3, 4] we can detect automatically the type of intersection for any given instance of time.
- computing a parametrization on intervals of time for the intersection of two deformable quadrics. We are able to cover the whole axis of time with intervals for which we can compute the parametrization of the intersection curve. Our method proves that it is possible to parameterize the intersection of two deformable quadrics. The inconvenient is the fact that we do not have a bound over the number of intervals which cover the axis of time. The algorithm is sensitive to a choice we make at depart, in the sense that, sometimes, the number of intervals can be very large.

Outline

The present report consists of 5 chapters:

- Chapter 2 covers the work from [3, 4], concerning the intersection of two fixed quadrics with rational coefficients. This is the main ingredient of our work.
  It explains our choice for working in projective space and it presents the algebraic methods which are at the base of our study of quadrics. After these preliminary notions, in two distinct sections, it details the way one can refer to the topology and the rational parametrization of the intersection.

- Chapter 3 introduces our main contribution, which consists in adapting the previous work such that it would offer answers when one addresses the problem of time.
  After translating the previous framework such that it would reflect the notion of time, we detail, preserving an obvious parallelism, the two main problems: the topology and the intersection of deformable quadrics in the generic case.

- Chapter 4 illustrates our work through a case of study. We, therefore, take an example, and we describe the evolution of the topology of the intersection curbe. We also detail the way in which we can compute a parametrization for the intersection.

- Chapter 5 presents the conclusions and some further work, with possible applications.

Chapter 2

Quadrics

In this chapter, we present some algebraic basic notions on quadrics. We describe them in projective space, because of the advantages it has over working in affine space.

We mention, from the start, that all definitions are normally presented over an arbitrary field. In the following, we choose this field to be the set of real numbers, $\mathbb{R}$.

2.1 Projective Space Preliminaries

Projective three-dimensional space $\mathbb{P}^3(\mathbb{R})$ is the set of affine points of $\mathbb{R}^3$ plus the points at infinity.

Definition 2.1.1 *Projective three-dimentional space over $\mathbb{R}$ is the set of equivalence classes*

$$\mathbb{P}^3(\mathbb{R}) = \{[x_1 : x_2 : x_3 : x_4] \mid x_i \in \mathbb{R}, (x_1, x_2, x_3, x_4) \neq (0,0,0,0)\},$$

where $(x_1, x_2, x_3, x_4) \sim (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4)$, $\forall \lambda \in \mathbb{R} \setminus \{0\}$.

We call x_1, x_2, x_3, x_4 homogeneous coordinates for the point $\mathrm{P} = [x_1 : x_2 : x_3 : x_4]$. They are only determined up to a non-zero scalar factor: $[x_1 : x_2 : x_3 : x_4] = [\lambda x_1 : \lambda x_2 : \lambda x_3 : \lambda x_4]$.

As it can be seen, there is a bijection $(x_1,\ x_2,\ x_3,\ 1) \mapsto (x_1,\ x_2,\ x_3)$, meaning $\mathbb{R}^3 \subset \mathbb{P}^3(\mathbb{R})$. Because $(x_1,\ x_2,\ x_3,\ 0) \in \mathbb{P}^2(\mathbb{R})$ it follows that $\mathbb{P}^3(\mathbb{R})$ $= \mathbb{R}^3 \cup \mathbb{P}^2(\mathbb{R})$. $\mathbb{P}^2(\mathbb{R})$ represents the projective plane (which contains the points at infinity).

2.2 Quadrics and Associated Matrices

Definition 2.2.1 *A quadric Q in $\mathbb{P}^3(\mathbb{R})$ is the set of points* $X = (x_1, x_2, x_3, x_4)$ $\in \mathbb{P}^3(\mathbb{R})$ *which satisfy the equation:*

$$\sum_{i,j=1}^{4} a_{ij}x_ix_j = 0,$$

with $a_{ij} \in \mathbb{R}$, $i, j = \overline{1..4}$.

The associated matrix is

$$M_Q = \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} & \frac{1}{2}a_{13} & \frac{1}{2}a_{14} \\ \frac{1}{2}a_{12} & a_{22} & \frac{1}{2}a_{23} & \frac{1}{2}a_{24} \\ \frac{1}{2}a_{13} & \frac{1}{2}a_{23} & a_{33} & \frac{1}{2}a_{34} \\ \frac{1}{2}a_{14} & \frac{1}{2}a_{24} & \frac{1}{2}a_{34} & a_{44} \end{pmatrix}$$

Remarks

- a quadric Q is a homogeneous polynomial, namely $\sum_{i,j=1}^{4} a_{ij}x_ix_j = 0$ is of degree 2 in each of its terms.
- any point X belongs to the quadric Q iff ${}^tX \cdot M_Q \cdot X = 0$.

2.3 Transformations

New quadrics can be obtained by applying projective transformations. With the help of matrices associated to quadrics, we can express this as a matrix product. Given the associated matrix of a quadric, M_Q, and a transformation T, the resulting quadric has as associated matrix ${}^tT \cdot M_Q \cdot T$, where ${}^tT$ represents the transpose of T.

Definition 2.3.1 *Two matrices, M and M' are said to be projectively equivalent (or congruent) iff there exists a nonsingular real matrix T such that*

$$M' = {}^tT \cdot M \cdot T$$

The basic operation on quadratic forms is the change of variables [10]. If x is a point on a quadric Q, and y a new vector related to x through a nonsingular matrix, $x = Cy$, then the quadratic form becomes ${}^tC \cdot Q \cdot C$. Note that ${}^tC \cdot Q \cdot C$ is also a real symmetrical matrix, with the same number of positive (negative) eigenvalues as Q. It results, thus, that changing the coordinate system is a congruent transformation. The following example illustrates this operation.

Example

Let Q be a sphere of equation $\sum_{i=1}^{3}(x_i - c_iw)^2 - r^2w^2 = 0$ in projective space. We can perform a change of coordinates, such that the origin of the system is translated to the center of the sphere. Namely, we substitute $(x_i - c_iw) \mapsto x'_i \Rightarrow (Q')$: $\sum_{i=1}^{3} x'^2_i - r^2w^2 = 0$.

The associated matrix for Q, and the transformation matrix are:

$$M_Q = \begin{pmatrix} 1 & 0 & 0 & -c_1 \\ 0 & 1 & 0 & -c_2 \\ 0 & 0 & 1 & -c_3 \\ -c_1 & -c_2 & -c_3 & c_1^2 + c_2^2 + c_3^2 - r^2 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ c_1 & c_2 & c_3 & 1 \end{pmatrix}$$

It follows that ${}^tT \cdot M_Q \cdot T$ is

$$M_{Q'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -r^2 \end{pmatrix}$$

which is exactly the associated matrix for Q'.

2.4 Canonical Form

The associated matrix, M_Q, of a quadric, Q, is a real symmetric matrix, and so it follows that it has only real eigenvalues. Furthermore, there exists a transformation matrix, T, which sends M_Q into a diagonal form, D, namely, ${}^tT \cdot M_Q \cdot T = D$, and the elements of D are the eigenvalues of M_Q.

Theorem 2.4.1 *For each quadric Q of $\mathbb{P}^3(\mathbb{R})$ there exists a projective transformation which sends it into a coordinate system where its form is:*

$$D : \sum_{i=1}^{4} d_i x_i^2 = 0,$$

with $d_i \in \mathbb{R}$. Such an equation is called the canonical form of the quadric.

Canonical forms make it possible to have a classification of quadrics. We present, in the following, the notion which we need in order to classify quadrics in projective space.

Definition 2.4.2 *Given a matrix M_Q, associated to the quadric Q let n_p (resp. n_n) be the number of positive (resp. negative) eigenvalues. Then the inertia for a quadric Q is defined as the pair $(max\{n_p, n_n\}, min\{n_p, n_n\})$.*

We mention that, in order to classify quadrics in $\mathbb{R}^3$, one needs the inertias for the associated matrices, and, moreover, the inertias for the upper left submatrices. This is because, in affine space, despite the projective space, there exist 3 classes of canonical forms, which lead to a classification of 17 types. As Theorem 2.4.1 says, there is only one class of canonical forms for quadrics in $\mathbb{P}^3$. This implies that knowing the inertias is sufficient in order to classify quadrics, and thus, the inertia replaces the notion of type in affine space.

It follows that, finding the type (the inertia) of a quadric, Q, in $\mathbb{P}^3$, is equivalent to finding the type (the inertia) of its canonical form, D. This is true because ${}^tT \cdot M_Q \cdot T = D$ (where T is the projective transformation which sends Q into a canonical form, D), which means that M_Q and D are projectively congruent. By "Sylvester's Law of Inertia", M_Q and D have the same inertia, and thus, the same type.

Table 2.1, presented in [3, 4], illustrates all possible types for projective quadrics.

projective type	canonical equation	affine quadrics and inertia of the upper left 3×3 block	
$(4,0)$	$ax_1^2 + bx_2^2 + cx_3^2 + dx_4^2 = 0$	(3,0)	$\emptyset$
$(3,1)$	$ax_1^2 + bx_2^2 + cx_3^2 - dx_4^2 = 0$	(3,0) (2,1) (2,0)	ellipsoid hyperboloid of 2 sheets elliptic paraboloid
$(2,2)$	$ax_1^2 + bx_2^2 - cx_3^2 - dx_4^2 = 0$	(2,1) (2,0)	hyperboloid of 1 sheet hyperbolic paraboloid
$(3,0)$	$ax_1^2 + bx_2^2 + cx_3^2 = 0$	(3,0) (2,0)	point $\emptyset$
$(2,1)$	$ax_1^2 + bx_2^2 - cx_3^2 = 0$	(2,1) (2,0) (1,1)	cone elliptic cylinder parabolic cylinder
$(2,0)$	$ax_1^2 + bx_2^2 = 0$	(2,0) (1,0)	line $\emptyset$
$(1,1)$	$ax_1^2 - bx_2^2 = 0$	(1,0) (1,1) (0,0)	parallel planes secant planes simple plane
$(1,0)$	$ax_1^2 = 0$	(1,0) (0,0)	double plane $\emptyset$
$(0,0)$	$0 = 0$	(0,0)	$\mathbb{R}^3$

Table 2.1: Projective types for the quadrics, possible corresponding affine types $a, b, c, d \in \mathbb{R}^+ \setminus \{0\}$

As we have at most 4 real non-zero eigenvalues for a symmetric real matrix, it follows that we have 9 possible projective types for quadrics (the number of pairs (r,s) with $0 \leq s \leq r \leq 4 - s$). It can be clearly seen from Table 2.1 that to one projective type corresponds more affine types. We illustrate this by taking a particular example.

Example
Let Q be the equation of an ellipsoid in affine space, namely $\alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 = 1$, $\alpha_i > 0 \Rightarrow$ inertia(Q)=(3,0)
Let P be the equation of an elliptic paraboloid in affine space, namely $\alpha_1 x_1^2 + \alpha_2 x_2^2 = x_3$, $\alpha_i > 0$, $\Rightarrow$ inertia(P)=(2,0)
In projective space, both quadrics have the same equation (E): $\alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 - \alpha_4 x_4^2 = 0$, $\alpha_i > 0$, and so, inertia(P) = inertia(Q) = (3,1).

We have seen that with the help of inertia we can classify quadrics in projective space. The sign of the determinant for the associated matrix of a quadric is also useful in characterizing a quadric. We make the following observation concerning the relation between inertia and the sign of the determinant.

Remark
Given a quadric, Q, let M_Q be the associated matrix (real and symmetric). Then $M_Q = {}^tT \cdot D \cdot T$, where D is the diagonal matrix with its elements e_1, e_2, e_3, e_4 being the eigenvalues for M_Q, $\Rightarrow \det(M_Q) = \det({}^tT \cdot D \cdot T) = \det^2(T)\det(D) = \det^2(T)e_1e_2e_3e_4$, and consequently the sign of the determinant is

- negative only if one eigenvalue is negative and the rest are positive (inertia(M_Q) = (3, 1)).
- positive only if the number of negative eigenvalues is even and there are no zero eigenvalues (inertia(M_Q) $\in$ {(4, 0), (2, 2)}).
- zero only if there are zero eigenvalues (inertia(M_Q) $\in$ {(2, 1), (2, 0), (1, 1), (1, 0)}).

In function of the sign of the determinant for the associated matrix of a quadric we can say that if the determinant is 0 then the quadric is singular (nonsingular otherwise). Furthermore, in $\mathbb{P}^3$, if the inertia of a quadric is different from (3,1), the quadric is a ruled surface. A ruled surface is one that can be characterized as a collection of straight lines, for example a cylinder, or a cone. We are interested in ruled surfaces because they are easily parameterized. Note that because only quadrics with inertia (3,1) have a negative determinant, we can say also that a quadric is a ruled surface if the determinant is ≥ 0.

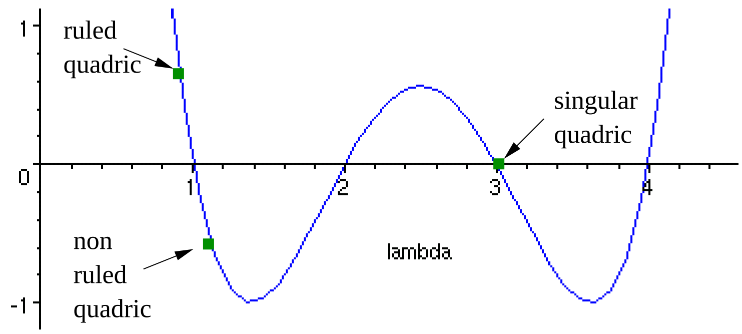

Figure 2.1: Determinantal equation

Figure 2.4 represents a given determinant, $D(\lambda) = (\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$. It can be seen that when $\lambda \in \{1, 2, 3, 4\}$ the determinant is zero, and so $R(\lambda)$ is a singular quadric. For $\lambda \in I = (-\infty, 1) \cup (2, 3) \cup (4, \infty)$ $D(\lambda) > 0$, thus the corresponding quadrics are ruled surfaces (their inertia is either (4,0), either (2,2)). For $\lambda \in \mathbb{R} - I$ the determinant is negative, or equivalently, $R(\lambda)$ have inertia (3,1).

We conclude this section by saying that, to this moment we know that if we work in projective space, we can classify each quadric in one of the nine possible categories. We know that each quadric can be transformed into an equivalent quadric which has an associated diagonal matrix. This diagonal matrix makes it easy to parameterize the quadric. We dedicate the following section to this subject.

2.5 Parametrization of a Quadric

As we have already mentioned in the introduction, finding a parameterization for the intersection curve of two quadrics implies finding a parameterization for one of the quadrics.

There are many ways in which a quadric can be parameterized. Our interest concerns optimal parametrizations, in the sense that they are linear in at least one parameter. This is a condition which we will see in a future section that is necessary for computing the parametrization of the intersection. Furthermore, in [3, 4], an optimal parametrization has the minimum possible number of square roots in its coefficients.

Because it can be proved that quadrics of inertia (3, 1) do not admit a parametrization linear in its parameters, the interest is focused on ruled surfaces. For these, we present the corresponding parametrizations in Table 2.2 as it appears in [3, 4].

inertia of M_Q	canonical equation $(a, b, c, d > 0)$	parametrization $\mathbf{X} = [x_1,x_2,x_3,x_4]$
$(4, 0)$	$ax_1^2 + bx_2^2 + cx_3^2 + dx_4^2 = 0$	Q is $\emptyset$
$(3, 0)$	$ax_1^2 + bx_2^2 + cx_3^2 = 0$	Q the point (0,0,0,1)
$(2, 2)$	$ax_1^2 + bx_2^2 - cx_3^2 - dx_4^2 = 0$	$\mathbf{X}=[\frac{ut+avs}{a}, \frac{us-bvt}{b}, \frac{ut-avs}{\sqrt{ac}}, \frac{us+bvt}{\sqrt{bd}}]$, (u,v), $(s,t) \in \mathbb{P}^1$
$(2, 1)$	$ax_1^2 + bx_2^2 - cx_3^2 = 0$	$\mathbf{X}=[uv, \frac{u^2-abv^2}{2b}, \frac{u^2+abv^2}{2\sqrt{bc}}, s]$, $(u,v,s) \in \mathbb{P}^{\star 2}$
$(2, 0)$	$ax_1^2 + bx_2^2 = 0$	$\mathbf{X}=[0,0,u,v]$, $(u,v) \in \mathbb{P}^1$
$(1, 1)$	$ax_1^2 - bx_2^2 = 0$	$\mathbf{X}_1=[u, \frac{\sqrt{ab}}{b}u,v,s]$, $\mathbf{X}_2= [u,-\frac{\sqrt{ab}}{b}u,v,s]$, $(u,v,s) \in \mathbb{P}^2$
$(1, 0)$	$ax_1^2 = 0$	$\mathbf{X}=[0,u,v,s]$, $(u,v,s) \in \mathbb{P}^2$

Table 2.2: Parametrization of projective quadrics of inertia different from (3,1). $\mathbb{P}^{\star 2}$ represents the 2-dimensional real quasi-projective space, which is the set of echivalence classes $\{[x_1, x_2, x_3] \mid x_i \in \mathbb{R}\} \setminus \{0\}$, where $(x_1, x_2, x_3) \sim (\lambda x_1, \lambda x_2, \lambda^2 x_3)$, $\forall \lambda \in \mathbb{R} \setminus \{0\}$

The parametrizations presented in Table 2.2 are all linear in at least one of the parameters. In this sense, they are optimal. They have also, in the worst case, an optimal number of radicals, meaning that, there are cases when the number of square roots of the parametrizations of Table 2.2 are required. Further optimality issues are discussed in detail in [3, 4].

We conclude this section by recalling that we have covered the main aspects on quadrics taken as individual objects. In the following we proceed by presenting how they can be characterized together. We therefore pass to the subject of a family of quadrics, namely, the pencil of quadrics, which is a preliminary and necessary step for our future sections on topology and parametrization.

2.6 Pencil of Quadrics

Given two quadrics, Q and P, we may construct a family of quadrics that are linear combinations of Q and P. This family is called the pencil of the two original quadrics.

Definition 2.6.1 *Let Q and P be two distinct quadrics. The pencil $\mathcal{F}(\mathcal{Q},\mathcal{P})$ is the set of quadrics*

$$\mathcal{F}(Q,P) = \{ \; R^1(\lambda) \mid \lambda Q + P, \lambda \in \mathbb{R} \cup \{\infty\}\}$$

One important property related to the notion of pencil says that the intersection of Q and P is the same with the intersection of any two quadrics from the pencil $\mathcal{F}(\mathcal{Q},\mathcal{P})$.

Theorem 2.6.2 *The intersection of two distinct quadrics from the same pencil is independent of the choice of these quadrics.*

Another significant property expresses the fact that in every pencil there exists a ruled quadric. We have mentioned in the introduction that, in order to compute the parametrization of the intersection of two quadrics Q and P, we need to compute the parametrization of Q or P. The above properties allow us to choose a ruled quadric (which is easy to parameterize) instead of Q or P (if these are not, already, ruled surfaces).

As quadrics, pencils can also be classified. In order to do this, we need to introduce the notion of determinantal equation.

Definition 2.6.3 *Given a pencil of quadrics, $R(\lambda) = \{\lambda Q + P\}$, the determinantal equation of the pencil, $\mathcal{D}(\lambda)$ is*

$$\mathcal{D}(\lambda) = det(R(\lambda))$$

In function of the multiplicity of the roots one can classify pencils as:

[1] For the rest of this report, we make the convention to denote the pencil of quadrics $R(\lambda) = \{\lambda Q + P \mid \lambda \in \mathbb{R}\}$ by the shortened notation $R(\lambda) = \{\lambda Q + P\}$.

Definition 2.6.4 *A pencil $R(\lambda)$ is:*

- *regular iff the determinantal equation has 4 distinct roots*[2]
- *singular iff the determinantal equation has at least one multiple root*
- *degenerate iff the determinantal equation has an infinite number of roots ($D(\lambda)$ is identically null)*

The classification of the pencils is in function of the multiplicity of the roots of the determinantal equation. The following theorem expresses the fact that the multiplicity of the roots is an invariant for the pencil, and thus, the classification is well-defined.

Theorem 2.6.5 *Given a pencil of quadrics, $R(\lambda) = \{\lambda Q + P\}$, we have that the nature of the roots of the determinantal equation and their multiplicity do not depend on the quadrics Q, P.*

We will see, in the following section, the way in which we can obtain, knowing the multiplicities of the roots, a more refined classification for pencils. This makes it possible to characterize the all types of intersection, or equivalently, to describe the topology of the intersection curve.

2.7 Topology of the Intersection Curve

The intersection of the quadrics Q and P is the set of solutions for the system:

$$\begin{cases} {}^tX \cdot M_Q \cdot X = 0 \\ {}^tX \cdot M_P \cdot X = 0 \end{cases}$$

We are concerned with the characterization (the type) of the intersection curve of two quadrics. This is motivated by the fact that knowing the intersection type makes it easier to find a parametrization.

The topology of the intersection curve concerns its reducibility, planarity, and singularity. Namely, the intersection of two quadrics is a space quartic. It can be reducible (if it contains some linear, conic, or cubic components, whose degree sum to 4), otherwise it is irreducible. When the curve is reducible, the components can be real or imaginary. A reducible curve can be planar (comprising lines or conics), or nonplanar (for example a line and a conic). The intersection curve is called singular if it contains a singular point (namely, a point at which the tangent is not uniquely defined). An irreducible intersection curve can be singular or nonsingular. If the curve is singular then the singular point can be of three types: acnode, crunode, or

[2] From now on, by abuse of notation, a solution of an equation $p(x) = 0$, where p is a polynomial, has the same meaning as the root of p

cusp. If the curve is nonsingular then it can have zero, one, or two connected components [14].

Characterizing the intersection implies determining its type. We have seen that the intersection is an invariant for the pencil which the input quadrics engender, thus classifying types of intersection is equivalent to classifying pencils.

We have presented, in the previous section, a rough classification for pencils, namely regular, singular and degenerate. For regular pencils the intersection is a smooth quartic (which can be also reduced to the empty set) [3, 4]. This is considered to be the generic case. As for the singular and degenerate pencils, there are more types for the intersection curve. Thus, the classification for these pencils can be further divided in distinct groups, each corresponding to a distinct type of intersection.

A classification for pencils defined over $\mathbb{P}^3(\mathbb{C})$ belongs to Segre. The main ingredient is the "Segre characteristic" for the determinant of the associated matrix of a quadric. Informally, this characteristic says whether $D(\lambda)$ has a multiple root λ_0, and, moreover, it says the rank, r_0, of $R(\lambda_0)$. Using "Segre characteristic" one can make an exhaustive classification for the pencils.

Our study concerns computing the intersection in real space. Thus we need a classification of pencils over $\mathbb{P}^3(\mathbb{R})$. As it is proved in [3, 4], such a classification is possible, by refining Segre's classification. The refinement is required because if, for a given Segre characteristic, there is only one type of interesection in the complex space, in real space there might be more than one. It is thus compulsory to find additional information to separate all cases. This information consists usually in the nature of the roots (real/complex) of the determinantal equation, and the inertia (sometimes the rank) for the quadrics corresponding to the roots.

The classification is synthesized in Table 2.7. Its correctness and completeness are proved in [3, 4]. The proof is based on the existence of the "Canonical Pair Forms"[11, 12] for pencils, which we detail in the appendix.

Segre String	roots of $D(\lambda)$	type of $R(\lambda_1)$	type of $R(\lambda_2)$	type of λ_2	s	real type of intersection
[112]	1 double root	(3,0)		real		point
[112]	1 double root	(2,1)		real	-	nodal quartic; isolated node
[112]	1 double root	(2,1)		real	+	nodal quartic; convex sing.
[112]	1 double root	rank 3		complex		nodal quartic;concave sing.
[11(11)]	1 double root	(2,0)		real	+	$\emptyset$
[11(11)]	1 double root	(2,0)		real	-	two points
[11(11)]	1 double root	(1,1)	(2,1)	real	-	two non-secant conics
[11(11)]	1 double root	(1,1)	(3,0)	real	-	$\emptyset$
[11(11)]	1 double root	(1,1)		real	+	two secant conics; convex sing.
[11(11)]	1 double root	rank 2		complex	-	conic
[11(11)]	1 double root	rank 2		complex	+	two secant conics; concave sing.

Segre String	roots of $D(\lambda)$	type of $R(\lambda_1)$	type of $R(\lambda_2)$	type of λ_2	s	real type of intersection
[13]	triple root	rank 3				cuspidal quartic
[1(21)]	triple root	(2,0)				double point
[1(21)]	triple root	(1,1)				two tangent conics
[1(111)]	triple root	rank 1	(2,1)			double conic
[1(111)]	triple root	rank 1	(3,0)			$\emptyset$
[4]	quadruple root	rank 3				cubic and tangent line
[(31)]	quadruple root	(1,1)			-	conic
[(31)]	quadruple root	(1,1)			+	conic and two lines crossing on the conic
[(22)]	quadruple root	(2,0)				double line
[(22)]	quadruple root	(1,1)			+	two single lines & a double line
[(211)]	quadruple root	rank 1			-	point
[(211)]	quadruple root	rank 1			+	two secant double lines
[(1111)]	quadruple root	rank 0				any smooth quadric of the pencil
[22]	2 double roots	rank 3	rank 3	real		cubic and secant line
[22]	2 double roots	rank 3	rank 3	complex		cubic and non-secant line
[2(11)]	2 double roots	(3,0)	rank 2	real		point
[2(11)]	2 double roots	(2,1)	rank 2	real	+	conic and two intersecting lines
[22]	2 double roots	(2,1)	rank 3	real	-	conic and a point
[(11)(11)]	2 double roots	(2,0)	(2,0)	real		$\emptyset$
[(11)(11)]	2 double roots	(2,0)	(1,1)	real		two points
[(11)(11)]	2 double roots	(1,1)	(2,0)	real		two points
[(11)(11)]	2 double roots	(1,1)	(1,1)	real		four skew lines
[(11)(11)]	2 double roots	rank 2	rank 2	complex		two secant lines

Table 2.3: Classification of pencils in the case when $D(\lambda)$ has a multiple root λ_1. λ_2 denotes another root. the "type" is the inertia (or the rank) of the matrix. s is an invariant, the sign for the determinant.

For the sake of clarity, we take an example from [4], namely, the Segre characteristic is [1(21)] and we detail the information from Table 2.7. [1(21)] says that the determinant, $D(\lambda)$, has one triple root, λ_1, and a simple one, λ_2. The group (21) says also that the rank for the associated matrix of the quadric corresponding to the triple root, $R(\lambda_1)$ is 2. We mention that for this characteristic, [1(21)], the complex type is "two tangent conics". In the real space, there are two possible types for the intersection. More precisely, we mention that there exists a projective transformation which sends simultaneously the quadrics $R(\lambda_1)$, $R(\lambda_2)$ in "Normal Canonical Forms" (see appendix for details). In the case of the characteristic [1(21)] they have the expressions:

$$\begin{cases} R'(\lambda_1): & x_1^2 + ax_4^2 = 0, \\ R'(\lambda_2): & x_1x_2 + x_3 = 0, \end{cases}$$

where $a = \pm 1$. We note that $R'(\lambda_2)$ represents the equation of a real cone, since its inertia is (2, 1). $R'(\lambda_1)$ is the equation of a pair of planes which is imaginary when a = 1 (the inertia is (2,0)), and real when a = -1 (the inertia is (1,1)). Thus there are 2 cases:

- when a = 1 the intersection is reduced to the real double point (0, 1, 0, 0)
- when a = -1 the intersection consists in two conics which have a common point (0, 1, 0, 0). This situation is illustrated in Figure 2.2[3] .

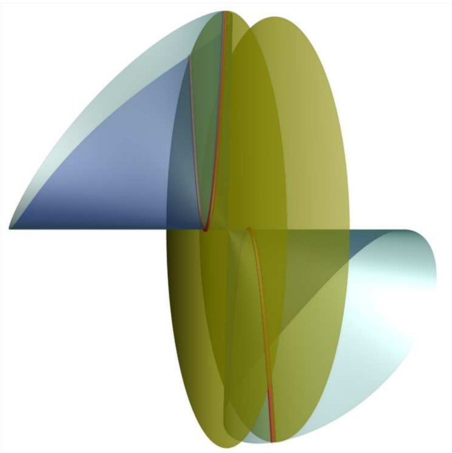

Figure 2.2: The intersection between a real projective cone and a real projective pair of planes

Table 2.7 says that, in order to compute the topology, it is necessary to find the multiplicity of the roots for the determinantal equation. There are 6 possible cases:

1. 4 simple, distinct roots
2. one double, and 2 simple ones
3. 2 double roots
4. a triple root, and a simple one
5. a quadruple root
6. an infinite number of roots (the determinantal equation is identically zero)

[3]The images in this report are realized with “Surf”, a free software for visualizing surfaces and curves [6]

We do not detail the last case. Specific and complete information can be found in [3, 4].

The first situation (4 distinct, simple roots), as we have already seen, represents the generic case. We mention that, in this situation, the intersection curve is a smooth quartic (complete proof appears in [3]). Furthermore, because we are concerned with the topology, we present the following theorem (which is proved in [3, 4]), giving information on whether the smooth quartic has one (two) finite (infinite) component(s).

Theorem 2.7.1 *Given 2 quadrics, Q and P in the generic case, the intersection curve (a smooth quartic) C can be classified as:*

- *if $D(\lambda)$ has 4 real roots, then C has either two real affinely finite connected components, either is empty*
- *if $D(\lambda)$ has 2 real roots and 2 complex roots, then C has one affinely finite connected component*
- *if $D(\lambda)$ has 4 complex roots, then C has two affinely infinite connected components.*

The cases 2-5 (when the determinantal equation has multiple roots) correspond to singular pencils. As we can see in Table 2.7, in order to determine the type of intersection in a singular pencil, we have to compute the inertias for the quadrics corresponding to the λ roots, sometimes the rank and the sign of the determinant, and we also have to determine, in some cases, the nature of the roots (whether they are real or complex). We dedicate the following two sections to the problem of finding the roots (and their multiplicity) for the determinantal equation, and to the problem of computing inertias (without being necessary to compute the eigenvalues).

2.7.1 Finding the Roots for the Determinantal Equation

Let a polynomial $p(x)$ have a root x_0 with multiplicity $m_0 > 1$. It follows that

$$p(x) = (x - x_0)^{m_0} q(x),$$

with degree(q) = degree(p) - m_0 (because $q(x_0) \neq 0$). Then

$$p'(x) = m_0(x - x_0)^{m_0-1} q(x) + (x - x_0)^{m_0} q'(x) = (x - x_0)^{m_0-1} r(x),$$

where $r(x) = (m_0 q(x) + (x - x_0) q'(x))$. It results that

$$\gcd(p, p') = (x - x_0)^{m_0-1} \text{ and degree(gcd)+1} = m_0.$$

Algorithm 1 Compute the multiplicity of roots

Require: A univariate polynomial (in our case, the determinant, $D(\lambda)$)
Ensure: the multiplicity of the roots of $D(\lambda)$
  compute the derivative of $\mathcal{D}(\lambda)$, let it be $\mathcal{D}$'(λ)
  $gcd \leftarrow$ the greatest common divisor of $\mathcal{D}(\lambda)$ and $\mathcal{D}'(\lambda)$
  if degree(gcd)=3 **then**
    print "D has a quadruple root"
  else if degree(gcd)=2 **then**
    $discrim \leftarrow$ discriminant(gcd)
    if $discrim = 0$ **then**
      print "D has a triple root"
    else
      print "D has two double roots"
    end if
  else if degree(gcd)=1 **then**
    print "D has one double root"
  end if

In conclusion, we can find out the multiplicity of the roots of a polynomial, by computing the greatest common divisor between the polynomial and its derivative.

Based on the above observation, we describe what is done in practice: Thus we are able to compute the multiplicities for the roots of the determinantal equation. It rests the problem of finding the exact form of the roots. We make the following observation:

Remark
By solving $\gcd(D, D') = 0$ we find the value of the multiple root. The other roots of $D(\lambda)$ are obtained by solving $D/\gcd(D, D') = 0$.

We present all possible cases in Algorithm 2.
In Algorithm 2 *ct* denotes a constant.

2.7.2 Computing the Inertias for the Associated Matrices

Knowing the solution for the determinantal equation, and the multiplicities, depending on each case, we have to compute the inertias for the matrices corresponding to the roots. We recall that the inertia is the pair of positive/negative eigenvalues. We make the remark that the exact computation of the eigenvalues is unnecessary when one needs to know only the signs of the roots. In this sense, "Descartes Rule of Signs" represents a solution. This method offers a bound on the number of positive roots of a polynomial, by counting the change of signs in the coefficients of the polynomial.

Algorithm 2 Compute the exact form of roots

Require: An univariate polynomial $D(\lambda)$ (in our case the determinant)
Ensure: The exact form of the roots

$gcd(\lambda) \leftarrow \mathrm{gcd}(D(\lambda), D'(\lambda))$
$rem(\lambda) \leftarrow \frac{D(\lambda)}{gcd(\lambda)}$
if degree(gcd) $= 3$ **then**
  print $D(\lambda) = ct(\lambda - \lambda_0)^4$, $gcd(\lambda) = (\lambda - \lambda_0)^3$
  print $rem(\lambda) = ct(\lambda - \lambda_0)$, a polynomial of degree 1 in λ
  the quadruple root $\lambda_0 \leftarrow$ the solution of $rem(\lambda) = 0$
end if
if degree(gcd) $= 2$ **then**
  $discrim \leftarrow$ discriminant(gcd)
  if $discrim = 0$ **then**
    print $D(\lambda) = ct(\lambda - \lambda_0)^3(\lambda - \lambda_1)$, $gcd(\lambda) = (\lambda - \lambda_0)^2$
    print $rem(\lambda) = ct(\lambda - \lambda_0)(\lambda - \lambda_1)$, a polynomial of degree 2 in λ
    the triple root and the simple root $\lambda_0, \lambda_1 \leftarrow$ the solutions $rem(\lambda)$=0
  else
    print $D(\lambda) = ct(\lambda - \lambda_0)^2(\lambda - \lambda_1)^2$
    print $gcd(\lambda) = (\lambda - \lambda_0)(\lambda - \lambda_1)$, a polynomial of degree 2
    the double roots $\lambda_0, \lambda_1 \leftarrow$ the solutions of $gcd(\lambda)$=0
  end if
end if
if degree(gcd) $= 1$ **then**
  print $D(\lambda) = ct(\lambda - \lambda_0)^2\ (\lambda - \lambda_1)(\lambda - \lambda_2)$
  print $gcd(\lambda) = (\lambda - \lambda_0)$, a polynomial of degree 1 in λ
  the double root $\lambda_0 \leftarrow$ is the solution of $gcd(\lambda)$=0
  print $rem(\lambda) = ct(\lambda - \lambda_1)(\lambda - \lambda_2)$, a polynomial of degree 2 in λ
  the simple roots $\lambda_1, \lambda_2 \leftarrow$ are the solutions of $rem(\lambda) = 0$
end if

Definition 2.7.2 *The sign of a real number a, sign(a) is defined as*

$$sign(a) = \begin{cases} 0, & \text{if } a = 0 \\ 1, & \text{if } a > 0 \\ -1, & \text{if } a < 0 \end{cases}$$

The number of changes of sign in a list of real numbers, $a = (a_1, a_2, ..., a_k)$*, is denoted by* $V(a)$ *and is recurrently defined by:*

$$V(a_1) = 0, \quad V(a_1, ..., a_k) = \begin{cases} V(a_1, ..., a_{k-1}) + 1, & \text{if } sign(a_{k-1}a_k) = -1 \\ V(a_1, ..., a_{k-1}), & \text{otherwise} \end{cases}$$

After the above convention of notation we present "Descartes Rule of Signs". More details and proofs can be found in [2, 8].

Theorem 2.7.3 *(Descartes Rule of Signs) Let* $p(x) = \sum_{i=0}^{n} a_i x^i$ *a polynomial in* $\mathbb{R}[x]$*. Let* $V(p(x))$ *be the number of changes of sign in the list* $(a_0, ..., a_n)$*, and* $pos(p(x))$ *the number of positive roots of p. Then* $pos(p(x)) \leq V(p(x))$ *and* $V(p(x)) - pos(p(x))$ *is even.*

Corollary 2.7.4
Let $neg(p(x))$ *be the number of negative roots. Then* $neg(p(x)) = pos(p(-x))$*.*

We recall that our problem consists in finding the number of positive (negative) eigenvalues for the associated matrix of a quadric. This is equivalent to finding the number of positive (negative) roots for the characteristical polynomial of the associated matrix. Given a quadric, Q, the associated matrix M_Q is in $Sym_{4\times4}(\mathbb{R})$. Let p be its characteristical polynomial, which is of degree 4:

$$p(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

Proposition 2.7.5 *Given the characteristical polynomial,* $p(x) = \sum_{i=0}^{4} a_i x_i$*, of the associated matrix,* M_Q*, for the quadric,* Q*, we have that* $pos(p(x)) = V(p(x))$ *and* $neg(p(x)) = V(p(-x))$*.*

Proof
Because M_Q real and symmetric, it follows that $p(x)$ has 4 real roots, and thus

$$pos(p(x)) + neg(p(x)) + n_0(p(x)) = 4 \quad (*),$$

where $n_0(p(x))$ is the number of roots of p which are equal to 0.
$p(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, and thus $V(p(x))$ is the number of negative elements from the list $l_1 = (a_4a_3,\ a_3a_2,\ a_2a_1,\ a_1a_0)$.
Similarly, $p(-x) = a_4x^4$ - $a_3x^3 + a_2x^2$ - $a_1x + a_0$, and thus $V(p(-x))$ is the number of negative elements from the list $l_2 = (a_4(-a_3),\ (-a_3)a_2,\ a_2(-a_1),\ (-a_1)a_0)$, which is, in fact, $-l_1$.
It is clear that for each negative number in l_1 it corresponds a positive one in l_2.
We have two cases, in function of whether $p(x)$ has 0 coefficients.

I. $p(x)$ does not have any 0 coefficients
In this case we have that $n_0(p(x)) = 0 \quad (1)$ and l_1 has no 0 elements. This means that the total number of negative elements from $l_1 \cup l_2$ is 4. Thus

$$V(p(x)) + V(p(-x)) = 4 \quad (2)$$

From "Descartes Rules of Signs" it follows that $V(p(x)) \geq pos(p(x))$ and $V(p(-x)) \geq neg(p(x))$.
Assume $V(p(x)) > pos(p(x))$ or $V(p(-x)) > neg(p(x))$. This implies

$$V(p(x)) - pos(p(x)) + V(p(-x)) - neg(p(x)) > 0 \quad (3)$$

From (*), (1) we have $pos(p(x)) + neg(p(x)) = 4$. Substituting in (3) this equality and (2) we have $0 > 0$, which is absurd. Thus our assumtions are false.

II. $p(x)$ has any 0 coefficients

1. $n_0 > 0$
It follows that $p(x) = x^{n_0} p_1(x)$, where degree(p_1) = 4 - n_0. We have that

$$\begin{array}{cc} pos(p(x)) = pos(p_1(x)) & neg(p(x)) = neg(p_1(x)) \\ V(p(x)) = V(p_1(x)) & V(p(-x)) = V(p_1(-x)) \end{array}$$

$$V(p_1(x)) + V(p_1(-x)) = pos(p_1(x)) + neg(p_1(x)) = 4 - n_0$$

We apply the reasoning from I. to p_1 and obtain the same contradiction $0 > 0$, when assuming that $V(p(x)) \neq pos(p(x))$ or $V(p(-x)) \neq neg(p(x))$

2. $n_0 = 0$
In this case $pos(p(x)) + neg(pos(x)) = 4$ (1), and $a_0 \neq 0$
Let $a_i = 0$, where i is between 1 and 4.
We reconstruct the lists l_1, l_2 in the following manner: we eliminate from both lists $a_{i-1}a_i$, $a_i a_{i+1}$ and we add $a_{i-1}a_{i+1}$. In this moment, the list l_1 contains 2 elements $a_j a_{j+1}$ for which it corresponds a $-a_j a_{j+1}$ in l_2, and an element $a_{i-1}a_{i+1}$ which is also common to l_2. We can have at most 4 negative elements in $l_1 \cup l_2$ (this happens when the 2 elements $a_j a_{j+1}$ and $a_{i-1}a_{i+1}$ are negative). This implies that $V(p(x)) + V(p(-x)) \leq 4$ (2). Assuming $V(p(x)) \neq pos(p(x))$ or $V(p(-x)) \neq neg(p(x))$ and considering (1), (2), it follows

$$4 = pos(p(x)) + neg(p(x)) < V(p(x)) + V(p(-x)) \leq 4$$

which is a contradiction, thus our assumptions are false.

We conclude by saying that we have presented the way in which one can compute efficiently the multiplicity and the exact form of the roots of the determinantal equation for a given pencil, and the inertias for the associated matrices of the quadrics from the pencil. Knowing these values, one can determine the type of the intersection curve of two quadrics simply by looking in Table 2.7 the corresponding line. We are now able to determine the topology of the intersection curve, and therefore we proceed to the next main issue which concerns the way in which one can parameterize this intersection.

2.8 Parametrization of the Intersection Curve

This section is concerned with the issue of finding an efficient and optimal (in the number of square roots) parametrization for the intersection of two quadrics.

We have seen, in the section concerning the study of the topology, that there is one generic case, where the intersection is a smooth quadric, and 47 singular cases. In the following, we are concerned with the generic case, which we present first, but for the sake of completeness, we will present, in the end, the main idea for the singular ones.

We describe in the following the basic idea for finding a parametrization of the intersection curve. We have mentioned in a previous section that for a given ruled quadric, there exists a parametrization linear in at least one of its parameters. Thus, by inserting this parametrization in the equation of the second quadric we obtain an equation of degree 2 at least in one of its parameters. Namely, given Q, a ruled quadric and P, a second quadric, we let T be the transformation that sends Q in the canonical form, and $\mathbf{X}$ its corresponding parametrization, then $\Omega : {}^t(T\mathbf{X}) \cdot P \cdot (T\mathbf{X}) = 0$ is the parametric equation of degree 2 in at least one of the parameters. By solving it for this parameter in function of the others and by substituing it in $T\mathbf{X}$,we can compute the parametrization for the intersection curve.
More formally, the method of parameterizing the intersection curve in the generic case is described in algorithm 3.

In order to prove the correctness of the algorithm, we state the necessary theorems, without presenting their proofs. Complete information is found in [3, 4].

First of all, Step 1. consists in finding a ruled quadric in the generic case. We have already mentioned this issue in the section dedicated to the pencils of quadrics. We present it more formally in the following theorem.

Theorem 2.8.1 *In a pencil generated by 2 distinct quadrics, the set of ruled quadrics is not empty.*

For the case when the inertia is (4, 0) the intersection is empty. This is because the quadric does not have any real points, given that its canonical

Algorithm 3 Computing the parameterization of the intersection for a pencil in the generic case

Require: A pencil $R(\lambda)$ in the generic case ($D(\lambda)$ has 4 distinct roots)
Ensure: The parametrization of the intersection

```
Find a ruled quadric R(λ0) {inertia(R(λ0)) is (4, 0), (2, 2)}
if R(λ0) is (4, 0) then
  return empty intersection
else
  {inertia(R(λ0)) is (2,2)}
  let p be a real point on R(λ0), and p' its rational approximation
  find a quadric R(λ1) of inertia (2, 2) with p' on it
  let T a transformation such that tT · R(λ1) · T is x^2 + y^2 - z^2 - δw^2 = 0
  let the parameterization of tT · R(λ1) · T be X
  insert the parametrization TX in the equation of the second quadric
  solve the new equation, which is of degree 2
  X_sol ← substitute the solutions in TX
  return X_sol
end if
```

equation is $ax^2 + by^2 + cz^2 + dw^2 = 0$, with a, b, c, $d \geq 0$. This result is known as a consequence of the theorem of Finsler:

Theorem 2.8.2 *The intersection curve of a pencil in $\mathbb{P}^n(\mathbb{R})$ is reduced to empty set iff there exists at least one quadric of inertia (n+1,0) in the pencil.*

For the case when the quadric $R(\lambda_0)$ has inertia (2,2), it is known that the canonical form is $ax_1^2 + bx_2^2 - cx_3^2 - dx_4^2$, where $a, b, c, d \in \mathbb{Q}^+$. The corresponding parametrization contains 2 square roots in its coefficients (if $\sqrt{ac}, \sqrt{bd} \notin \mathbb{Q}$):

$$\mathbf{X} = [\tfrac{ut+avs}{a},\ \tfrac{us-bvt}{b},\ \tfrac{ut-avs}{\sqrt{ac}},\ \tfrac{us+bvt}{\sqrt{bd}}]$$

In [3, 4] the authors have found a condition for which there exists a projective transformation which sends a quadric of inertia (2, 2) in the diagonal form (1,1,1,δ), $\delta = \sqrt{abcd}$. This condition refers to the existence of rational point on the quadric of inertia (2,2). In this case, the number of two square roots from the above parametrization is reduced to one (or even to zero, if $\sqrt{abcd} \in \mathbb{Q}$):

$$\mathbf{X} = [ut + vs,\ us - vt,\ ut - vs,\ \tfrac{us+vt}{\sqrt{\delta}}]$$

Thus, for the sake of optimality in the number of square roots one might want not only to find a quadric of inertia (2,2) in the pencil, but moreover, to find a quadric of inertia (2,2) which contains a rational point. We reproduce the solution presented in [3, 4] with respect to the definitions from this section. The solution consists in picking up a real point p on $R(\lambda_0)$ by intersecting the quadric with a line, for example, and consider p' as the rational approximation of p. By solving ${}^t p' \cdot R(\lambda) \cdot p' = 0$, we can choose from the solutions a particular λ_1 such that $R(\lambda_1)$ is of inertia (2,2). Such a value exists, as the following theorem proves it.

Theorem 2.8.3 *If a pencil contains a quadric of inertia (2, 2), then it contains a very close quadric of the same inertia with a rational point on it.*

Thus we are able to find a quadric (2, 2), $R(\lambda_1)$, with a rational point on it, p'. The authors in [3, 4] prove that, in this situation, there exists a transformation matrix T which sends the quadric into the canonical form: $x^2 + y^2 - z^2 - \delta w^2 = 0$. Finding this transformation T consists in:

- finding a second rational point on $R(\lambda_1)$, p''
- finding a transformation T_0 which sends p' and p'' in $[1, \pm 1, 0, 0]$
- apply Gauss reduction to ${}^t T_0 \cdot R(\lambda_1) \cdot T_0$, with the canonical form $x^2 - y^2 + \alpha z^2 + \beta w^2 = 0$ as result. Let G be the transformation matrix.
- apply the transformation matrix T_1

$$\frac{1}{2}\begin{pmatrix} 1+\alpha & 0 & 1-\alpha & 0 \\ 1-\alpha & 0 & 1+\alpha & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2\alpha \end{pmatrix}$$

  to ${}^t G \cdot {}^t T_0 \cdot R(\lambda_1) \cdot T_0 \cdot G$ which transforms it in $x^2 + y^2$ - z^2 - $\delta w^2 = 0$.

Then $T = T_0 \cdot G \cdot T_1$, and the parametrization, $\mathbf{X}$, obtained for ${}^t T \cdot R(\lambda_1) \cdot T$ contains only one square root, $\sqrt{\delta}$, in the coefficient for the last term.

By inserting $T\mathbf{X}$ in the equation of the second quadric, it results that the parametric equation Ω is biquadratic in the variables $\tau = (s, t)$, $\xi = (u, v)$. We can find (s, t) in function of (u, v), by computing the discriminant, $\Delta(u, v)$, which is a polynomial of degree 4 in u, v. Thus the parametrization is in $\mathbb{Q}(\sqrt{\delta})[\xi, \sqrt{\Delta}]$, with $\Delta \in \mathbb{Q}(\sqrt{\delta})[\xi]$. We illustrate this by taking an example.

Example
Let the equations of two quadrics Q and P be:

$$\begin{cases} Q: & x^2 + y^2 - z^2 - w^2 = 0 \\ P: & x^2 + yw - z^2 = 0 \end{cases}$$

For the sake of clarity, we have considered one of the quadrics to be already in the diagonal form (1,1,1,δ), with $\delta = 1$.

$\mathbf{X} = [ut + vs,\ us - vt,\ ut - vs,\ us + vt]$ is an optimal parametrization for Q. By inserting it in the equation of P we obtain

$$\Omega : u^2 s^2 + 4uvst - v^2 t^2 = 0.$$

Its discriminant with respect to the variable $(s,\, t)$ is:

$$\Delta_{s,t}(u,v) = 20u^2v^2$$

It follows that

$$(s,\, t) = (4uv \pm \sqrt{\Delta_{s,t}(u,v)}, 2u^2) = (uv(2 \pm \sqrt{5}), 2u^2)$$

It rests to substitute (s,t) in the expression $\mathbf{X}$ in order to obtain a parametrization for the intersection curve. Because Δ is positive, it follows that the intersection curve has 2 components, one is parameterized by

$$\mathbf{X}_1 = [(2+\sqrt{5})(-2u^2 + u^2\sqrt{5} + v^2), vu(1+\sqrt{5}), -(2+\sqrt{5})(2u^2 - u^2\sqrt{5} + v^2), vu(3+\sqrt{5})]$$

and the other one by

$$\mathbf{X}_2 = [(2-\sqrt{5})(-2u^2 - u^2\sqrt{5} + v^2), vu(1-\sqrt{5}), -(2-\sqrt{5})(2u^2 + u^2\sqrt{5} + v^2), vu(3-\sqrt{5})]$$

We have presented the way in which one can optimally parameterize the intersection curve in the generic case.

As for the singular cases, we give some ideas why the algorithm from the generic case is not optimal. For the case when the quadric is singular, in the generic case, the parametrization of the intersection is in $\mathbb{Q}(\sqrt{\delta})[\xi, \sqrt{\Delta}]$, where ξ is a variable of degree 2 in Ω, and Δ is the discriminant for Ω. Nevertheless, it has been proved that sometimes $\sqrt{\Delta}$ can be avoided. Namely, when the intersection curve is singular, it can be parameterized by rational functions. In order to minimize the number of square roots, or equivalently to find a rational parametrization (when the number of square roots is 0), there have been searched some other algorithms (for each of the 47 singular cases). Complete reference and proofs can be found in [3, 4].

We have, in the previous sections, described the main work which is fundamental for presenting our contribution. We have motivated, after some preliminary notions on quadrics, the choice of working in projective space, which is mainly the existence of a reduced diagonal form for the quadrics. We have explained why it is better to work with pencils of quadrics than with quadrics by their own. We have mentioned that there exists a transformation which sends simultaneously quadrics in a block diagonal matrix. The advantages that come with this transformation consist in theoretical results refering the complete classification of types for the intersection curve (1 generic case and 47 singular ones). We have shown how these theoretical results are used in practice, and also we have given some details regarding the implementation. After presenting the way in which we can characterize the topology of the intersection, we have proceed with the description of the method which parameterizes the intersection in the generic case. This parametrization is efficient, robust, and near-optimal in the number of square roots. We recall that more information, complete proofs and examples can be found in [3, 4], which consitute our start point and foundation basis.

Chapter 3

Intersection of Time-Varying Quadrics

Our contribution consists in adapting the work from [3, 4] such that it would offer satisfactory answers when one asks the question of time. In such a dynamic framework, the quadrics are in a continuous deformation. Our main interest concerns the evolution of the topology of the intersection curve and the possibility of obtaining a parametrization depending on time for it.

3.1 Preliminaries

We add, in the general framework of quadrics in projective space, the notion of time. In this sense, the equation of a quadric is rewritten as:

$$(Q(\theta)) \colon \sum_{i,j=1}^{4} f_{ij}(\theta) x_i x_j = 0,$$

with f_{ij} polynomials $\in \mathbb{R}[\theta]$ $i, j = \overline{1..4}$.

With the above considerations, the associated matrix changes to the following form:

$$M_Q(\theta) = \begin{pmatrix} f_{11}(\theta) & \frac{1}{2} f_{12}(\theta) & \frac{1}{2} f_{13}(\theta) & \frac{1}{2} f_{14}(\theta) \\ \frac{1}{2} f_{12}(\theta) & f_{22}(\theta) & \frac{1}{2} f_{23}(\theta) & \frac{1}{2} f_{24}(\theta) \\ \frac{1}{2} f_{13}(\theta) & \frac{1}{2} f_{23}[\theta) & f_{33}(\theta) & \frac{1}{2} f_{34}(\theta) \\ \frac{1}{2} f_{14}(\theta) & \frac{1}{2} f_{24}(\theta) & \frac{1}{2} f_{34}(\theta) & f_{44}(\theta) \end{pmatrix}$$

where f_{ij} are polynomials in $\mathbb{R}[\theta]$.

Given $Q(\theta)$ and $P(\theta)$, two quadrics depending on time, the pencil which engenders these quadrics is:

$$\mathcal{F}(Q, P)(\theta) = \{\ \mathcal{R}(\lambda, \theta) = \lambda Q(\theta) + P(\theta) \mid \lambda \in \mathbb{R} \cup \{\infty\}, \theta \in \mathbb{R}\}$$

We denote $\mathcal{F}(\theta)$ the "time pencil". We also make the convention of denoting by "static pencil" $\mathcal{F}(\theta_{fixed})$, namely a pencil for a given value of time. The connection between these notions is that a time pencil is a family of static pencils. As time goes on, one static pencil $\mathcal{F}(\theta_1)$ evolves to another $\mathcal{F}(\theta_2)$, and so does the intersection curve.

The determinant changes from a univariate polynomial in λ to a bivariate one, in λ, θ

$$D(\lambda, \theta) = det(R(\lambda, \theta)),$$

of degree 4 in λ, which we consider as

$$f_1(\theta)\lambda^4 + f_2(\theta)\lambda^3 + f_3(\theta)\lambda^2 + f_4(\theta)\lambda + f_5(\theta).$$

where $f_i(\theta)$ are functions of time.

After performing these necessary translations from a static framework to a dynamic one, we proceed to the next section, which is concerned with the study of the evolution in time of the topology for the intersection curve.

3.2 The Evolution in Time of the Intersection

Our interest is to determine the exact value of time for which the topology changes, and the intervals of time where the topology is preserved. We give necessary and sufficient conditions for the existence of these two situations. In this sense we consider the discriminant of the determinantal equation with respect to the λ variable as $\delta(D(\lambda,\ \theta))$, a univariate polynomial in θ.

We recall that the discriminant of a polynomial is defined as the product of the squares of the differences of the polynomial roots up to a constant ($\prod_{1\leq i<j}^{n}(\lambda_i - \lambda_j)^2$, where λ_i, λ_j are roots, and n is the degree of the polynomial, and consequently, the number of roots). We can see, from this definition, that a multiple root, λ_i is also a root for the discriminant. The discriminant can also be defined (up to a constant) as the resultant of the polynomial and its derivative. We recall that a resultant gives the common roots of two polynomials. In the case of a polynomial and its derivative, it is clear that the common roots are represented by the multiple roots. Thus it results even from this second definition that multiple roots are zeros for the discriminant.

We denote the discriminant of the determinant $D(\lambda,\theta)$ in a shorter form as $\delta(\theta)$. We let $\delta(\theta) = 0$ be the "time equation", and we will see that the preservation/change of topology is correlated with the roots of this equation.

We mention that the generic case in this dynamic framework means that for all values of time (except a finite set) the determinantal equation has 4 distinct roots in λ, or equivalently, that the intersection curve is a smooth quartic. This is the same to the fact that the discriminant is not identically null.

Proposition 3.2.1 *Given a time pencil $R(\lambda, \theta)$, if the discriminant for the determinantal equation $\delta(\theta)$ is not identically null, then the intersection curve is a smooth quartic for all values of the time, except a finite set.*

Proof

$\delta(\theta)$ is a continuous function (a polynomial in θ) and is not identically null. This implies that $\delta(\theta)$ has a finite number of roots, and that on each interval I of consecutive roots $\delta(\theta)$ is either positive or negative Thus $D(\lambda, \theta_i)$ has 4 distinct, non-zero roots (complex or/and real), $\forall \theta_i \in I$, and so, the pencil is in the generic case, where the intersection is a smooth quartic, for all time, except a finite number of values, which are the roots of the discriminant. ■

Formally, we treat the preservation/change of topology in two propositions. The first one presents the condition for which the topology does not change.

Proposition 3.2.2 *For any value of time between 2 consecutive roots θ_0, θ_1 of $\delta(\theta)$, the topology of the intersection is preserved.*

Proof

We are in the generic case and thus there is a $\theta' \in (\theta_0, \theta_1)$ such that $D(\lambda, \theta')$ has 4 distinct roots. In order to prove that the topology is preserved, we need to prove that for $\forall \theta \in (\theta_0, \theta_1)$ $D(\theta)$ has 4 distinct roots, and that the nature of the roots does not change (see Theorem 2.7.1).

(I.) Assume there exists a $\theta'' \in (\theta_0, \theta_1)$ such that $D(\lambda, \theta'')$ has a multiple root, λ', with multiplicity m', $m' > 1$. We have mentioned that a multiple root for a polynomial is a root for the discriminant of the polynomial. It follows $\delta(\theta) = 0$, which contradicts the fact that θ_0, θ_1 are consecutive roots.

We thus have proved that the multiplicity of the roots is invariant on (θ_0, θ_1). Namely, for any value of time between 2 consecutive roots for the discriminant, the time equation has 4 distinct roots. Equivalently, for the interval of 2 consecutive roots, the pencil rests regular.

(II). We now prove by contradiction that the nature of the λ roots cannot change between two consecutive time roots, θ_0, θ_1.

Assume there exists $\theta', \theta'' \in [\theta_0, \theta_1]$ such that $D(\lambda, \theta')$ has 4 real roots and $D(\lambda, \theta'')$ has also 2 complex roots and 2 real roots $\Rightarrow \delta(\theta') > 0$ and $\delta(\theta'') < 0$. This is a contradiction because the sign of the discriminant for the interval of 2 consecutive roots is constant.

Now assume there exists a $\theta', \theta'' \in [\theta_0, \theta_1]$ such that $D(\lambda, \theta')$ has 4 complex roots and $D(\lambda, \theta'')$ has also 2 complex roots and 2 real roots $\Rightarrow \delta(\theta') > 0$ and $\delta(\theta'') < 0$. It is the same contradiction as in the previous case.

The last possible case is when there exists a $\theta', \theta'' \in [\theta_0, \theta_1]$ such that $D(\lambda, \theta')$ has 4 real roots and $D(\lambda, \theta'')$ has 4 complex roots $\Rightarrow \delta(\theta') > 0$ and $\delta(\theta'') > 0$. We can write $D(\lambda, \theta)$ as the product of two polynomials of degree 2 in λ. Let them be $p_1(\lambda, \theta)$, $p_2(\lambda, \theta)$. It follows that either

p_1, or p_2 have 2 real roots when evaluated in θ' and similarly they have complex roots when evaluated in θ''. Thus we obtain again a contradiction, because on $[\theta_0, \theta_1]$ the discriminant of either p_1, either p_2 changes signs (positive in θ', negative in θ'')

In consequence, the nature of the roots is preserved. This means that the intersection curve either has one finite (resp. infinite) component, either has 2 finite (resp. infinite) components. It is not possible that the intersection has different types or numbers of components for distinct values of time belonging to an interval of cosecutive roots. ■

Symmetrically, the following proposition presents the condition for which the topology changes.

Proposition 3.2.3 *Given an instance of time, θ_0, the topology of the intersection changes at θ_0 iff $\delta(\theta_0) = 0$.*

Proof
We are in the generic case, meaning that for a given value of time θ_s, $D(\lambda, \theta_s)$ has 4 distinct λ roots. It follows that its discriminant is different from 0 and that the intersection is a quartic with no singularities. If, for θ_0, $\delta(\theta_0)$=0 $\Rightarrow \exists \lambda_0, m_0 > 1$, such that $(\lambda - \lambda_0)^{m_0}$ divides $D(\lambda,\ \theta_0)$, and so, λ_0 is a multiple root and so, we are in the case of a singular pencil. Thus the intersection cannot be a quartic with no singularities. ■

For a better intuition of the evolution of the topology, we present and explain Figure 3.1. The horizontal axis represents the time line. For each fixed value of time, θ_f, we have marked (in blue) the λ roots of the determinantal equation $D(\lambda,\ \theta_f)$. The determinant is of degree 4 in λ, and so it can have either 4 real roots, either 2 real and 2 complex roots, or 4 complex roots. It follows that at each moment of time, θ_f, we can plot 4 or 2 points. The t_i, $i = \overline{0..3}$ represent the roots of the time equation, meaning $\delta(t_i) = 0$. For each of them we have drawn a green vertical line, in order to make it obvious that for these values the topology changes.

We say that a change in the topology is equivalent with a change of the type of intersection, which, at its own, is equivalent with a change of the multiplicity, nature (complex/real) of the λ roots for the determinantal equation. Thus Figure 3.1 describes indeed the evolution of the topology.

For example, if we compute the roots of the determinant $D(\lambda,\ \theta_f)$, where θ_f is any value in the interval [-∞, t_0), then we obtain 2 real roots and 2 complex (on the graphic we can only see the real roots). It follows we are in the generic case. In θ_0 we see that the 2 complex roots appear in the real space, and represent a double real root. We are, at this particular moment, in a singular case (the Segre characteristic is [112] or [11(11)]). For $\theta \in (t_0,\ t_1)$ the determinantal equation has 4 real roots, and so, we are again in

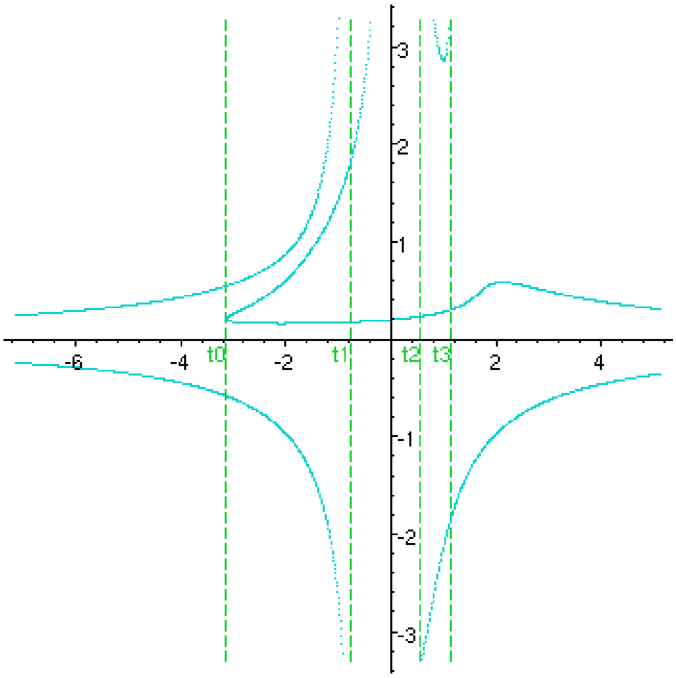

Figure 3.1: An Example for the Evolution of the Topology for the Intersection Curve

the generic case. In t_1 we have 2 double real roots, and again we are in a singular case. For $\theta \in (t_1, t_2)$ each pair of the double root disappears (it becomes complex), and thus we are in the generic case (2 real roots and 2 complex). And so on.

3.3 Computing the Topology

As we have already seen, given an instance of time, θ_0, two scenarios are possible.

If θ_0 is not a root for the time equation, consider θ_s, θ_f as two consecutive roots for the time equation such that $\theta_0 \in (\theta_s, \theta_f)$. Let this interval be I_{θ_0}. We know from Proposition 3.2.2 that the topology is preserved on I_{θ_0}, and so, we rest in the generic case. We are thus able to say that the intersection is a smooth quartic, furthermore, in function of the nature of the roots, real or complex, we can say if the intersection consists in one (or two) finite (infinite) component(s).

If θ_0 is a root of the time equation, then we know that $D(\lambda, \theta_0)$ has a multiple root in λ, and thus we are in the case of a singular pencil. We recall the algorithm for computing the topology in the case of singular pencils.

The problem we have to confront with, is the algebraic nature of the

Algorithm 4 Compute the topology for a pencil of quadrics, $R(\lambda, \theta_0)$

Require: A pencil $R(\lambda, \theta_0)$

Ensure: The type of intersection

compute $gcd := \gcd(D(\lambda,\ \theta_0),\ D'(\lambda,\ \theta_0))$, where $D' = \frac{dD}{d\lambda}$.
if degree(gcd)=3 **then**
  print " D has a quadruple root, λ' "
  compute the inertia for $R(\lambda', \theta_0)$, and for another nonsingular quadric from the pencil, plus some invariants in two cases
  return type from Table 2.7
else if degree(gcd)=2 **then**
  if discriminant(gcd) $= 0$ **then**
    print " D has a triple root, λ', and a simple one, λ'' "
    compute the inertia for $R(\lambda', \theta_0)$, $R(\lambda'', \theta_0)$.
    return type from Table 2.7
  else
    print " D has two double roots, λ', λ'' "
    compute the inertia for $R(\lambda', \theta_0)$, $R(\lambda'', \theta_0)$, and some invariants.
    return type from Table 2.7
  end if
else if degree(gcd)=1 **then**
  print " D has one double root, λ', and two simple roots, λ'', λ''' "
  compute the inertia for $R(\lambda', \theta_0)$, $R(\lambda'', \theta_0)$, $R(\lambda''', \theta_0)$, and for another nonsingular quadric in some cases
  return type from Table 2.7
end if

roots for the time equation. Thus, the above θ_0 is, in most cases, irrational. In consequence, we have to adapt the algorithms described in [3, 4] in order to be able to describe the topology of the intersection of two quadrics with irrational coefficients. In particular, we have as input quadrics in an algebraic field and we should find methods which solve the problems of computing multiplicities of roots, greatest common divisors, and inertias for the refered quadrics. Our solution consists in using interval arithmetics as a filter and, afterwards, we perform exact computation.

Interval arithmetics is not a novelty, but it makes sense applying it to our problem, as far as it offers answers. We dedicate a section in the appendix to a short description of interval arithmetics. Here, the important thing that we have to mention is that we represent an algebraic number x as a pair, (I, p), where I is an interval which bounds the value of x, $x \in I$, and p is a polynomial for which x is a solution, $p(x) = 0$.

We recall that we are interested in computing the topology of the intersection curve. In order to do this, Algorithm 4 requires the computation of the multiplicities for the roots of the determinantal equation, and the computation of inertias for different matrices. We describe the way in which we can solve algorithmically these two problems.

A. Finding the multiplicities for the λ roots of the determinant

As we have presented in a previous section, finding the multiplicity of a root for the determinantal equation $D(\lambda, \theta_0)$, means computing the greatest common divisor between D and its derivative with respect to the λ variable, which we denote by D'. We consider the case when θ_0 is an algebraic number.

We recall that the determinant is a bivariate polynomial in λ and θ, which we can regard as a polynomial of degree 4 in λ with polynomials in θ as coefficients:

$$D(\lambda,\, \theta) = f_1(\theta)\lambda^4 + f_2(\theta)\lambda^3 + f_3(\theta)\lambda^2 + f_4(\theta)\lambda + f_5(\theta).$$

It follows that its derivative with respect to λ is a polynomial of degree 3 in λ, with polynomials in θ as coefficients:

$$D'_\lambda(\lambda,\, \theta) = 4f_1(\theta)\lambda^3 + 3f_2(\theta)\lambda^2 + 2f_3(\theta)\lambda + f_4(\theta).$$

We are interested in finding out, if, for a given instance of time θ_0, $D(\lambda, \theta_0)$ and $D'_\lambda(\lambda,\, \theta_0)$ have a common divisor. We make the remark that it is possible that the 2 polynomials do not have a proper ($\neq 1$) common divisor for all θ, but they can have for particular values for θ.

The algorithm is an adaptation of the Euclid algorithm. We recall it for the sake of clarity. In our case, D, D' are bivariate polynomials, and so, the last non-zero remainder, r_n, from the sequence of Euclidean divisions, is also bivariate. The main idea in our approach is that if r_n has θ_0 as a root, then the precedent remainder which is not 0 when evaluated in θ_0 is the greatest common divisor for D and D'.

Algorithm 5 Euclid Algorithm

Require: Two polynomials p and q

Ensure: $gcd(p,q)$

```
r_0 ← p mod q
i ← 0
while r_i ≠ 0 do
   i ← i + 1
   p ← q
   q ← r
   r_i ← p mod q
end while
return r_{i-1}
```

Algorithm 6 Compute the multiplicity of the roots for $D(\lambda, \theta_0)$

Require: $D(\lambda, \theta)$, θ_0 as a pair (I_{θ_0}, p), where I_{θ_0} is the isolation interval, and $\delta(\theta)$ for which θ_0 is a root

Ensure: The multiplicity of the roots for $D(\lambda, \theta_0)$

```
p ← D(λ, θ)
q ← D'_λ(λ, θ)
r_0 ← p mod q
i ← 0
while r_i ≠ 0 do
   i ← i + 1
   p ← q
   q ← r
   r_i ← p mod q
end while
if θ_0 is a root for r_{i-1} then
   j ← i − 2
   while θ_0 is a root for r_j do
      j ← j − 1
   end while
   return degree(r_j,λ)
else
   return degree(r_{i-1},λ)
end if
```

In order to prove the correctness of Algorithm 6, we state the following proposition, which is, in fact, a generalization of our case (where the polynomials are of degree 4 in the λ variable).

Proposition 3.3.1 *Given a bivariate polynomial, $p(x, y)$, and $y_0 \in \mathbb{R}$, with x_0 a root of $p(x, y_0)$ of multiplicity $m_0 > 1$. Let r_n be the last remainder (obtained by the Euclidian division of $p(x, y)$ by $p'(x, y)$, its derivative) different from 0. Let $r_{n'}$ be the first previous remainder which is not 0 when evaluated in y_0. Then $r_{n'}(x_0, y_0) = 0$ and $m_0 = degree(r_{n'}(x, y_0)) + 1$.*

Proof
We have two possible situations:
I. $r_n(x, y_0) = 0$
Because $r_{n'}(x, y)$ is the first remainder such that $r_{n'}(x, y_0)$ is not identically null it follows that $r_{n'}(x, y) = gcd(p(x, y_0), p'(x, y_0))$. x_0 is a multiple root, and thus, it is also a root for the *gcd*. This implies $r_{n'}(x_0, y_0) = 0$. Obviously, being that $r_{n'}$ is the greatest common divisor between p and its derivative, it follows that its degree is the multiplicity of m_0 - 1.
II. $r_n(x, y_0) \neq 0$
In this case r_n is not identically null, and $n' = n$. It follows that $gcd(p, p')$=r_n. x_0 is a multiple root for $p(x, y_0)$, and thus $gcd(x_0, y_0) = 0$, and $m_0 = degree(r_{n'}(x, y_0)) + 1$. ■

B. Computing the inertia for matrices with algebraic coefficients

As for the second problem which needs to be solved in order to make Algorithm 6 work is the computation of the inertia. We recall that the inertia of a matrix is the pair of positive and negative eigenvalues. As we have mentioned in a previous section, we can apply "Descartes' rule of signs" to the characteristical polynomial, p. We have seen that, in our case, the number of changes of sign for $p(x)$ gives the number of positive eigenvalues, and similarly, the changes of sign in $p(-x)$.

In our framework, p has algebraic coefficients. We recall our problem is to compute inertia of deformable quadrics, for given irrational values of time, namely, the roots of the time equation. We thus pose the following problem to which we describe the solution:

Problem
Given a quadric from the pencil, $R(\lambda_0, \theta_0)$, where θ_0 is a solution for the time equation, find the inertia of $R(\lambda_0, \theta_0)$.
Solution
The idea is to take advantage of the representation for θ_0 as the pair (I_{θ_0}, δ).Let p be the characteristic polynomial for $R(\lambda_0, \theta)$, which we recall is a

4×4 matrix. Thus p is a polynomial of degree 4, and its coefficients are polyinomials in θ, $p(\mu) = p_4(\theta)\mu^4 + p_3(\theta)\mu^3 + p_2(\theta)\mu^2 + p_1(\theta)\mu^1 + p_0(\theta)$;

θ_0 is represented by the pair $(I_{\theta_0}, \delta(\theta))$. Let I_{θ_0} be $[a, b]$.

We need to count the changes of sign for $p(I_{\theta_0})$ and this implies that we have to compute the signs for $p_i(I_{\theta_0}) = [c, d]$. If $c > 0$, then p_i is positive. If $d < 0$, then p_i is negative. It remains the case when $0 \in [c, d]$, meaning that θ_0 is an hypothetical root for p_i. As θ_0 is a solution for the time equation, we have to check if degree$(gcd(p_i, \delta(\theta))) > 0$. If yes, and if θ_0 is a root for the gcd, then $p_i(\theta_0)$ is 0, and thus the number of zero roots increments. If the degree is 0 or if θ_0 is not a root for the gcd, then we refine I_{θ_0}, because we are sure to reach an answer.

Before counting the changes of sign, we need to find the number of zero roots, n_0. Note that n_0 is the number of consecutive coefficients (starting from the last one) equal to 0, namely $p_i(\theta_0) = 0$, $i = \overline{0..n_0}$.

After determining n_0, we can simply apply Descartes rule of signs, as normally, to $p(\mu)/\mu^{n_0}$.

We formalize the above in the following algorithm.

Algorithm 7 Compute the inertia for a quadric with algebraic coefficients

Require: $R(\lambda_0, \theta)$, θ_0 an algebraic number
Ensure: Inertia for $R(\lambda_0, \theta_0)$

$p(\mu) \leftarrow$ CharacteristicPoly$(R(\lambda_0, \theta)) \leftarrow p_4(\theta)\mu^4 + p_3(\theta)\mu^3 + p_2(\theta)\mu^2 + p_1(\theta)\mu^1 + p_0(\theta)$
$i \leftarrow 0$; $n_0 \leftarrow 0$ $\{n_0$ is the number of zero roots for $p\}$
$[c, d] \leftarrow p_i(\theta_0)$
while $0 \in [c, d]$ **do**
  if θ_0 is a common root for p_i and $\delta(\theta_0)$ **then**
    $p_i(\theta) \leftarrow 0$; $n_0 \leftarrow n_0 + 1$
    $i \leftarrow i + 1$; $[c, d] \leftarrow p_i(\theta_0)$
  else
    refine $[c, d]$
  end if
end while
$p(\mu) \leftarrow p(\mu)/\mu^{n_0}$
for $i = 0$ to degree(p,μ) **do**
  $[c, d] \leftarrow p_i(\theta_0)$
  if $c > 0$ **then**
    $p_i(\theta_0) \leftarrow 1$
  else if $d < 0$ **then**
    $p_i(\theta_0) \leftarrow -1$
  end if
end for
return Descartes$(p(\mu))$

3.4 Parametrization of the Intersection Curve

The previous section was dedicated to the study of the topology for the intersection of two quadrics which evolve in time. Knowing the type of the intersection, we are concerned with finding a parametrization depending on time. This presents the advantage of avoiding computing a parametrization for each instance (discrete value) of time. In this sense, given that the topology of the intersection does not change between 2 consecutive roots for the discriminant of the time equation, we intend to prove that for some intervals of time we can find a near-optimal, valid parametrization for the intersection.

As we have seen in the section dedicated to the static quadrics, the basic idea in obtaining a parametrization for the intersection curve is to parametrize one quadric and insert its parametrization in the equation of the second quadric. This results in solving an equation of degree 2 in its parameters, ξ, τ, with Δ being its discriminant. As only ruled surfaces admit parametrizations which are linear in at least one parameter, it follows that, in order to optimally parameterize the intersection, it is needed to find a quadric with positive determinant. Because we have placed ourselves in the generic case, which means that the determinant has 4 distinct, simple roots, the inertia for this quadric is either (4, 0), either (2, 2). We have said that if the inertia is (4, 0) there is no sense in finding a parametrization, since the quadric does not have any real points and thus the intersection is empty. In consequence, it rests the case when the quadric has inertia (2, 2). We have seen that a quadric (2, 2) has two square roots in the parametrization. We have said that, if it is possible to find a rational point on it (or another quadric of inertia (2, 2) with a rational point on it), then we can reduce one square root in the parametrization, and thus, the parametrization for the intersection curve is in $\mathbb{Q}(\sqrt{\delta})[\xi, \sqrt{\Delta}]$, which is optimal.

In the case of a dynamic framework, we pose the same problems. Our intention is to find an interval of time for which we can be sure that the quadrics are of inertia (2, 2) and that they contain a rational point. For these intervals we prove that a parametrization depending on time is feasible. Further, we are concerned with the possibility of being able to cover the whole axis of time with such intervals. In this sense, we will prove that we can find an interval of time, I, for which $R(\lambda, \theta)$ is of inertia (2,2), $\forall \theta \in I$. It follows that it is possible to find a worst-case optimal parametrization for the intersection curve on I. We present an algorithm that describes how to cover the whole axis of time with such intervals. This algorithm has two drawbacks. The first one is represented by the fact we do not have a bound for the number of intervals (which can be very large). The second one consists in the fact there are situations when the number of square roots can be reduced. Namely, we prove that if we can find a rational point on a quadric (2,2) then we can reduce the number of square roots from the parametrization of the intersection curve up to one. This will be the goal of the first theorem. Then we will prove that covering the time axis with such intervals is possible, though it is not always optimal. This is formalized in our second theorem.

We start by proving that, given a pair $(\lambda_0,\ \theta_0)$ such that $D(\lambda_0,\ \theta_0)$ is positive, then we can find an interval of time, I, with $\theta_0 \in I$, and for which there exists a set of quadrics, $R(\lambda(\theta), \theta)$ of inertia (2, 2) depending on time, with a rational point, $P(\lambda(\theta))$ on each of them.

We know that in a static pencil there exist ruled surfaces (Theorem 2.8.1). As a time pencil is a family of static pencils, it follows that for each instance of time, θ_i, there exists a ruled quadric $R(\lambda, \theta_i)$. Furthermore, we are in the generic case, thus the ruled quadrics are of inertia (4,0) or (2,2). When the inertia is (4,0) the intersection is empty and thus this case does not present any challenge. Given λ_0, θ_0 such that $R(\lambda_0, \theta_0)$ is a quadric (2,2) we have an interval I, for which $R(\lambda_0, \theta)$ remains (2,2), $\forall \theta \in I$.

Lemma 3.4.1 *Given a time pencil $R(\lambda, \theta)$, λ_0, θ_0 such that the static quadric $R(\lambda_0,\ \theta_0)$ is of inertia (2,2), then there exists an interval of time, I, such that $R(\lambda_0,\ \theta)$ has inertia (2,2), $\forall \theta \in I$.*

Proof

$R(\lambda_0,\ \theta_0)$ is of inertia (2,2) $\Rightarrow D(\lambda_0, \theta_0) > 0$.

Let θ_1 be the first root of $D(\lambda_0, \theta)$ such that $\theta_0 < \theta_1$. Then $D(\lambda_0, \theta)$ does not change sign for $I = [\theta_0, \theta_1)$, and thus it remains positive. It follows that $R(\lambda_0,\ \theta)$ is of inertia (2,2). ■

Thus it is possible to parametrize $R(\lambda_0,\ \theta)$ on the interval I and to obtain a worst-case optimal (in the number of square roots) for the intersection curve. More precisely, $R(\lambda_0,\ \theta)$ can be written in the canonical form as

$$f_1(\theta)x^2 + f_2(\theta)y^2 - f_3(\theta)z^2 - f_4(\theta)w^2 = 0.$$

Accordingly, it admits the parametrization

$$\mathbf{XR}(\theta) = \left[\ \frac{ut+f_1(\theta)vs}{f_1(\theta)},\quad \frac{us-f_2(\theta)vt}{f_2(\theta)},\quad \frac{ut-f_1(\theta)vs}{\sqrt{f_1(\theta)f_3(\theta)}},\quad \frac{us+f_2(\theta)vt}{\sqrt{f_2(\theta)f_4(\theta)}}\ \right].$$

In a similar manner as for static quadrics, the insertion of the parametrization $\mathbf{XR}(\theta)$ in the equation of the second quadric leads to an equation of degree 2 in its variables, and with coefficients as polynomials of θ. We find one variable in function of the other, insert it in $\mathbf{XR}(\theta)$ thus obtaining a parametrization defined on I for the intersection quadric.

Given that it is possible to find a parametrization of the intersection for an interval of time, we pose the problem of covering the whole axis of time with such intervals. In this sense, we propose Algorithm 8 which outputs a non-optimal paramterization for the intersection curve, defined on the whole axis of time.

Algorithm 8 A parametrization depending on time for the intersection curve

Require: a time pencil, $R(\lambda, \theta)$, and a pair (λ_0, θ_0) such that $D(\lambda_0, \theta_0)$ is positive

Ensure: A parametrization for the intersection curve

1: Start with λ_0, θ_0 such that the corresponding quadric is (2,2)
2: $i \leftarrow 0$
3: **while** $D(\lambda_i, \theta)$ has real roots greater than θ_i **do**
4:     Let θ_{i+1} be the smallest root of $D(\lambda_i, \theta)$ such that $\theta_i < \theta_{i+1}$. Then $I_i = [\theta_i, \theta_{1+1})$ and $\mathbf{X}_i(\theta)$ is a non optimal parameterization for the intersection, $\theta \in I_i$
5:     λ_{i+1} is chosen such that $D(\lambda_{i+1}, \theta_{i+1})$ is (2,2)
6:     $i \leftarrow i + 1$
7: **end while**
8: **return** $\mathbf{X}$

Algorithm 8 offers a parametrization for a given moment of time, with no need in effectively computing it. Moreover, as time covers the whole axis of real numbers, it is clear that we are in the position of determining a parametrization for the intersection of two quadrics with algebraic coefficients. This is important, as far as we are not aware of methods which can find an exact parametrization for the intersection of surfaces with irrational coefficients.

Nevertheless, Algorithm 8 has two drawbacks. The first consists in the fact that there is no bound on the number of intervals I_i. The second concerns the number of square roots which can be reduced if rational points are found.

Regarding the first drawback we say that we do not know yet how to find the minimal number of intervals. We consider Figure 3.2. For the sake of consistency, it is the same graphic as in the previous section. We have only added some red '+' signs, which represent the regions where $D(\lambda, \theta) > 0$, and some '-' signs, for the regions where the determinant is negative. We recall that the blue lines represent the pairs (λ, θ) for which the determinant is 0. It follows that for '-' regions the inertia of the corresponding quadrics is (3,1) and for the '+' regions the inertias are either (4,0), either (2,2).

In Step 1 our algorithm has to make a choice, namely, it has to pick up a λ such that, for a given value of time, the inertia is (2,2) and thus the determinant is positive. If we consider the graphic 3.2 it can be clearly seen that taking as starting pair $P2$ we can obtain a parametrization defined on only one interval which covers the axis of time. Instead, if we take $P1$ as the starting pair, then we obtain 3 intervals. Thus the number of intervals is sensitive to the choice of the λ_i such that $D(\lambda_i, \theta_i) > 0$.

We can improve the algorithm, by considering the following heuristic:

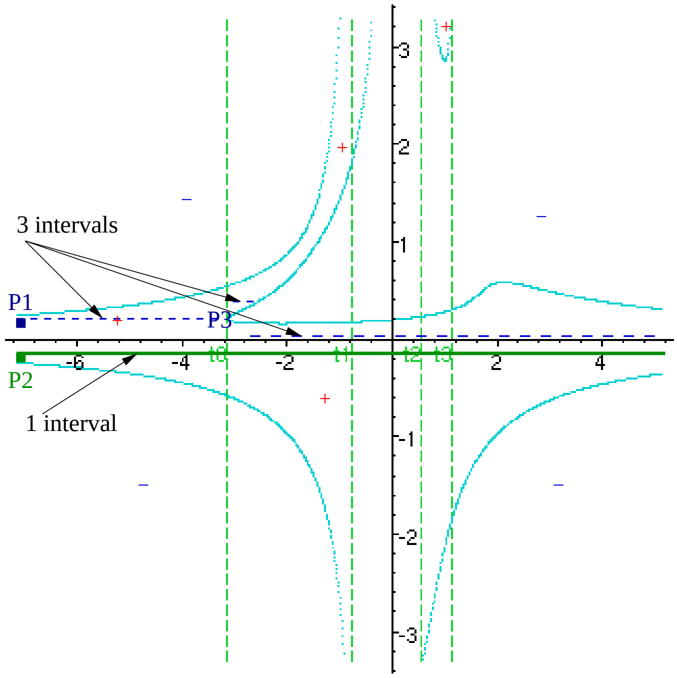

Figure 3.2: Covering the axis of time with intervals

"Whenever it is possible, choose λ_i in the largest interval."

More precisely, we recall that, given θ_i a value for the time variable, we search for a λ_i such that $D(\lambda_i,\, \theta_i) > 0$. Solving $D(\lambda,\, \theta_i)$ gives us at most 4 λ roots. For each interval of consecutive λ roots, the determinant has a constant sign. We choose a λ_i in an interval for which the determinant is positive. Our heuristic says that it is better to choose λ_i in a larger interval. Applying it to our graphic, we notice that the decision taken at $P3$ would luckily choose the green line and not the blue dotted one.

As for the second drawback, we present in the following the way in which we can reduce the number of square roots up till one.

Lemma 3.4.2 *Given a time pencil, $R(\lambda, \theta)$, with a static quadric $R(\lambda_0, \theta_0)$ of inertia $(2,\ 2)$, then there exists a rational point which belongs to all quadrics $R(f(\theta), \theta)$, where $f(\theta)$ is a polynomial in time, $f :\in \mathbb{R} \mapsto I$, where $\forall\theta \in I\ R(f(\theta), \theta)$ is of inertia (2,2).*

Proof

Let P be the rational point on $R(\lambda_0, \theta_0)$.

By solving ${}^tP \cdot R(\lambda, \theta_0) \cdot P = 0$, we obtain a λ as a function of θ, such that $\forall\theta\ \exists f(\theta)$ with ${}^tP \cdot R(f(\theta), \theta) \cdot P = 0$.

If $D(f(\theta), \theta) = 0$ does not have any solution, then $I = \mathbb{R}$. Otherwise, let

θ_1 be the smallest root such that $\theta_0 < \theta_1$. Then $I = [\theta_0, \theta_1)$ and $\forall \theta \in I$ $R(f(\theta),\theta)$ is of inertia (2,2). ∎

This lemma means that if P is a rational point on a quadric $R(\lambda_0, \theta_0)$, then there exists an interval, I, for which P is a common point for the set of quadrics (2,2) $R(f(\theta), \theta)$, $\theta \in I$. Furthermore, for each particular θ_f we can also find a second rational point on each $R(f(\theta_f), \theta_f)$.

Lemma 3.4.3 *Given a time pencil, $R(\lambda, \theta)$, and a rational point, P_1, belonging to the time pencil, $\exists P_2(\theta)$ such that $P_1, P_2(\theta) \in R(f(\theta), \theta)$.*

Proof
Let $R(f(\theta), \theta)$ be the quadrics to which P_1 is common. Let L be a line which contains P_1. Then by solving ${}^tL \cdot R(f(\theta), \theta) \cdot L = 0$, we obtain $P_2(\theta)$ as the second intersection point of the line L with each quadric $R(f(\theta_f), \theta_f)$, where θ_f is a fixed value of time. ∎

Proposition 3.4.4 *Given a time pencil, $R(\lambda, \theta)$, with a static quadric $R(\lambda_0, \theta_0)$ of inertia $(2, 2)$, and P_1 a rational point on it, there exists an interval I and a polynomial $f : \mathbb{R} \mapsto I$ such that P_1 is a common point to all quadrics (2,2) $R(f(\theta), \theta)$, which have an optimal parameterization (in the number of square roots) depending on time.*

Proof
The above lemmas prove the existence of the polynomial $f : \mathbb{R} \mapsto I$ such that $R(f(\theta),\theta)$ have P as a common rational point, and each $R(f(\theta_f),\theta_f)$ has another rational point $P_2(\theta_f)$. We prove that, in these conditions, the parametrization has at most one square root in its parameters.
Recall the parametrization for quadrics (2,2) is:

$$\mathbf{X} = \left[\begin{array}{cccc} \frac{ut+f_1(\theta)vs}{f_1(\theta)}, & \frac{us-f_2(\theta)vt}{f_2(\theta)}, & \frac{ut-f_1(\theta)vs}{\sqrt{f_1(\theta)f_3(\theta)}}, & \frac{us+f_2(\theta)vt}{\sqrt{f_2(\theta)f_4(\theta)}} \end{array} \right]$$

We remark the fact that if $f_1(\theta) = f_2(\theta) = 1$, then the parametrization is in $\mathbb{Q}[\sqrt{\xi_1}]$ and is linear in all u, v, s, t, and by all means, it is optimal.
It follows that we look for a transformation which sends a quadric of (2,2) in a canonical form with $f_1(\theta) = f_1(\theta) = 1$.
Let $TP(\theta)$ be the projective transformation that sends $P_1, P_2(\theta)$ in $[1, \pm 1, 0, 0]$ Then apply Gauss reduction on ${}^tTP(\theta) \cdot R(\lambda(\theta), \theta) \cdot TP(\theta)$. Let $TG(\theta)$ be the correspondent transformation matrix.
Prove that ${}^tTG(\theta) \cdot^t TP(\theta) \cdot R(\lambda(\theta), \theta) \cdot TP(\theta) \cdot TG(\theta)$ is of the form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \alpha(\theta) & 0 \\ 0 & 0 & 0 & \beta(\theta) \end{pmatrix},$$

where $\alpha(\theta)\beta(\theta) < 0$, $\forall\theta \in I$.
Consider the transformation matrix $TT(\theta)$:

$$\frac{1}{2}\begin{pmatrix} 1+\alpha(\theta) & 0 & 1-\alpha(\theta) & 0 \\ 1-\alpha(\theta) & 0 & 1+\alpha(\theta) & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2\alpha(\theta) \end{pmatrix}$$

After performing the calculations, we can see that ${}^tTT(\theta) \cdot {}^t TG(\theta) \cdot {}^t TP(\theta) \cdot R(\lambda(\theta), \theta) \cdot TP(\theta) \cdot TG(\theta) \cdot TT(\theta)$ is in the form we are looking for:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \gamma(\theta) \end{pmatrix},$$

where $\gamma(\theta) = -\alpha(\theta)\beta(\theta)$, thus $\gamma(\theta) > 0$.
Thus, there exists a transformation $T(\theta) = TP(\theta) \cdot TG(\theta) \cdot TT(\theta)$ such that the corresponding equation for ${}^tT(\theta) \cdot R(\lambda(\theta), \theta) \cdot T(\theta)$ is

$$x^2 + y^2 - z^2 - \gamma(\theta)w^2 = 0.$$

Thus $R(\lambda(\theta), \theta)$ can be parameterized by:

$$\mathbf{X} = [ut + vs, us - vt, ut - vs, \tfrac{us+vt}{\sqrt{\gamma(\theta)}}]$$

which is valid, because $\gamma(\theta) > 0$, for $\theta \in I$, and optimal. ■

Consequently, algorithm 8 changes to:

Algorithm 9 A parametrization depending on time for the intersection

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Require: a time pencil, $R(\lambda, \theta)$, a pair (λ_0, θ_0) such that $D(\lambda_0, \theta_0)$ is positive, a rational point P_1 on $R(\lambda_0, \theta_0)$
Ensure: A parametrization for the intersection curve which has at most one square root in its parameters
  $f \leftarrow$ solve $({}^tP_1 \cdot R(\lambda, \theta_0) \cdot P_1 = 0)$
  $i \leftarrow 0$
  while $D(f(\theta_i), \theta)$ has real roots greater than θ_i **do**
    Let θ_{i+1} be the smallest root of $D(\lambda_i, \theta)$ such that $\theta_i < \theta_{i+1}$. Then $I_i = [\theta_i, \theta_{1+1})$ and $\mathbf{X}_i(\theta)$ is a non optimal parameterization for the intersection, $\theta \in I_i$
    λ_{i+1} is chosen such that $D(\lambda_{i+1}, \theta_{i+1})$ is (2,2)
    $i \leftarrow i + 1$
  end while
  return X

Chapter 4

Case of Study

In order to illustrate our results we take the case of the following two time-varying quadrics[1]:

$$\begin{cases} P: & x^2 + 2(\theta - 4)xw + y^2 + z^2 - 2zw + w^2(\theta^2 - 8\theta + 8) = 0 \\ Q: & x^2 - 6\theta xw + 9y^2 - z^2 + (9\theta^2 - 25\theta)w^2 = 0 \end{cases}$$

We mention that P represents a sphere of center $(x_0, y_0, z_0)=(4,0,1)$ and radius $r_0=3$ which is translated on the $(Ox$ axis by applying the transformation T:

$$T = \begin{pmatrix} 1 & 0 & 0 & \theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4.1 Studying the Topology

We can describe the evolution of the topology of the intersection curve with the help of the determinantal equation $D(\lambda, \theta)$ and the time equation $\delta(\theta)$. In our case,

$$D(\lambda,\theta) = -9\lambda^4 + (16\theta^2 - 57\theta - 66)\lambda^3 + (128\theta^2 - 481\theta + 127)\lambda^2 + (-144\theta^2 + 313\theta - 72)\lambda + 225\theta$$

and its discriminant, $\delta(\theta)$, is a univariate polynomial of degree 12 in θ:

$$\delta(\theta) = (655360\theta^8 - 11386888\theta^7 + 73802224\theta^6 - 215477712\theta^5 + 265014760\theta^4 - 92037000\theta^3 - 9033270\theta^2 + 1450440\theta - 4032)(180\theta^2 - 610\theta + 981)^2$$

We have already proved that the topology is preserved on the open interval of two consecutive time roots of $\delta(\theta)$, and that it changes only for θ_i such that $\delta(\theta_i) = 0$.

[1] For simplicity, we abuse the notation, and denote by a quadric Q the set of quadrics $Q(\theta)$

We can see that the discriminant is a product of two polynomials, one of degree 8 and the other of degree 4, which does not have any real solutions. Thus the roots of the discriminant are the roots of the polynomial of degree 8, which is irreductible. The best we can do, in order to have an information of the roots, is to isolate them. After performing computations, it follows that the discriminant has 4 real roots, which are represented by the intervals

$$r_\theta = \left(\left[\frac{-77}{512}, \frac{-153}{1024} \right], \left[\frac{709}{1024}, \frac{355}{512} \right], \left[\frac{1051}{512}, \frac{2103}{1024} \right], \left[\frac{4489}{1024}, \frac{2245}{512} \right] \right)$$

We mention that these intervals can be refined, if needed. Figure 4.1 plots the determinantal equation and in addition the approximate values for the roots.

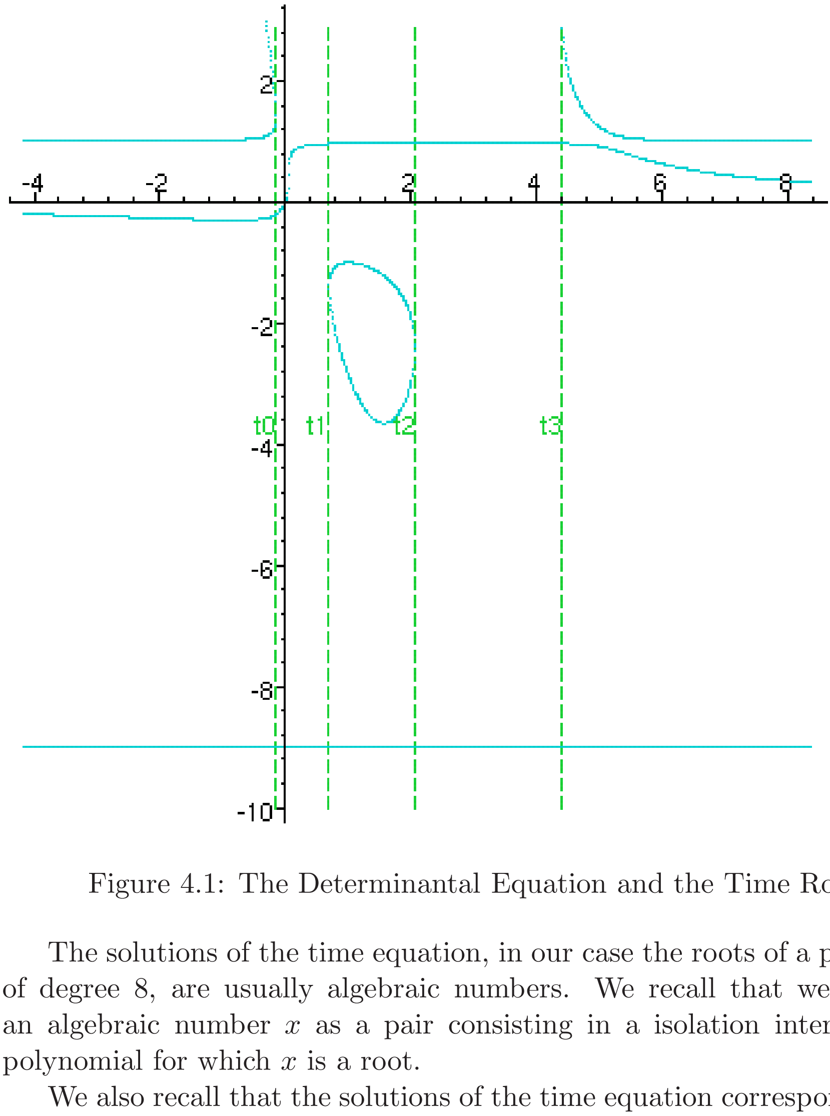

Figure 4.1: The Determinantal Equation and the Time Roots

The solutions of the time equation, in our case the roots of a polynomial of degree 8, are usually algebraic numbers. We recall that we represent an algebraic number x as a pair consisting in a isolation interval and a polynomial for which x is a root.

We also recall that the solutions of the time equation correspond to multiple λ roots of the determinantal equation, thus to singular static pencils.

When $\theta \in (r_\theta[i], r_\theta[i+1])$ we know that the intersection is a smooth quartic, and furthermore, in function of the nature of the roots we can say whether it has one/two, affinely finite/infinite connected components (Theorem 2.7.1). For example, when $\theta = 5$, $R(\lambda, 5)$ has 4 positive eigenvalues, $\forall \lambda \in \mathbb{R}$, and thus the intersection is empty. In fact, $\forall \theta \in (r_\theta[3], \infty)$ there cannot be any collision detection between Q and P. Furthermore, we have the following cases:

- if $\theta \in (-\infty, r_\theta[0]) \cup (r_\theta[1], r_\theta[2])$, then $D(\lambda, \theta)$ has 4 real roots, thus the intersection is a smooth quartic with two affinely finite components
- if $\theta \in (r_\theta[0], r_\theta[1]) \cup (r_\theta[2], r_\theta[3])$, then $D(\lambda, \theta)$ has 2 real and 2 complex roots, thus the intersection is a smooth quartic with one affinely finite component

At this moment, we know the type of intersection between consecutive roots $r_\theta[i]$. We are interested, in what it follows, to exactly determine the type of intersection when time is equal to r_θ. We perform computations as we have described in a previous section, and we find out, as it can also be seen in Figure 4.1 that for all time roots there corresponds a double and 2 simple λ roots for the determinantal equation. Furthermore, we compute some invariants and looking in (Table 2.7) we can find out the type of intersection for

- $\theta = r_\theta[0]$ the inertia for the double root is (3,0), thus the intersection type is a "point"
- $\theta = r_\theta[1]$ the inertia for the double root is (2,1) and the simple roots are real, thus the intersection type is "nodal quartic without singular points"
- $\theta = r_\theta[2]$ the inertia of the double root is (1,1) and the inertia of one of the simple roots is (3,0), thus the intersection is empty
- $\theta = r_\theta[3]$ the inertia for the double root is (3,0), thus the intersection type is a "point"

We can assemble all the information from above in a complete description of the evolution of the topology of the intersection curve. Q and P intersect in a quartic with 2 components until they become tangent, at $r_\theta[0]$. While the value of time is smaller than $r_\theta[1]$ the intersection is a smooth quartic with one finite component. At $r_\theta[1]$ the smooth quartic becomes a nodal quartic, which after this moment is a smooth quartic with one component until time reaches $r_\theta[2]$. This component becomes smaller and smaller until it dissapears. This happens at $r_\theta[2]$ when the intersection is empty. As soon as time goes on towards $r_\theta[3]$ the intersection is again a smooth quartic with two components until Q and P are tangent again. As time goes at ∞ the two quadrics do not intersect anymore.

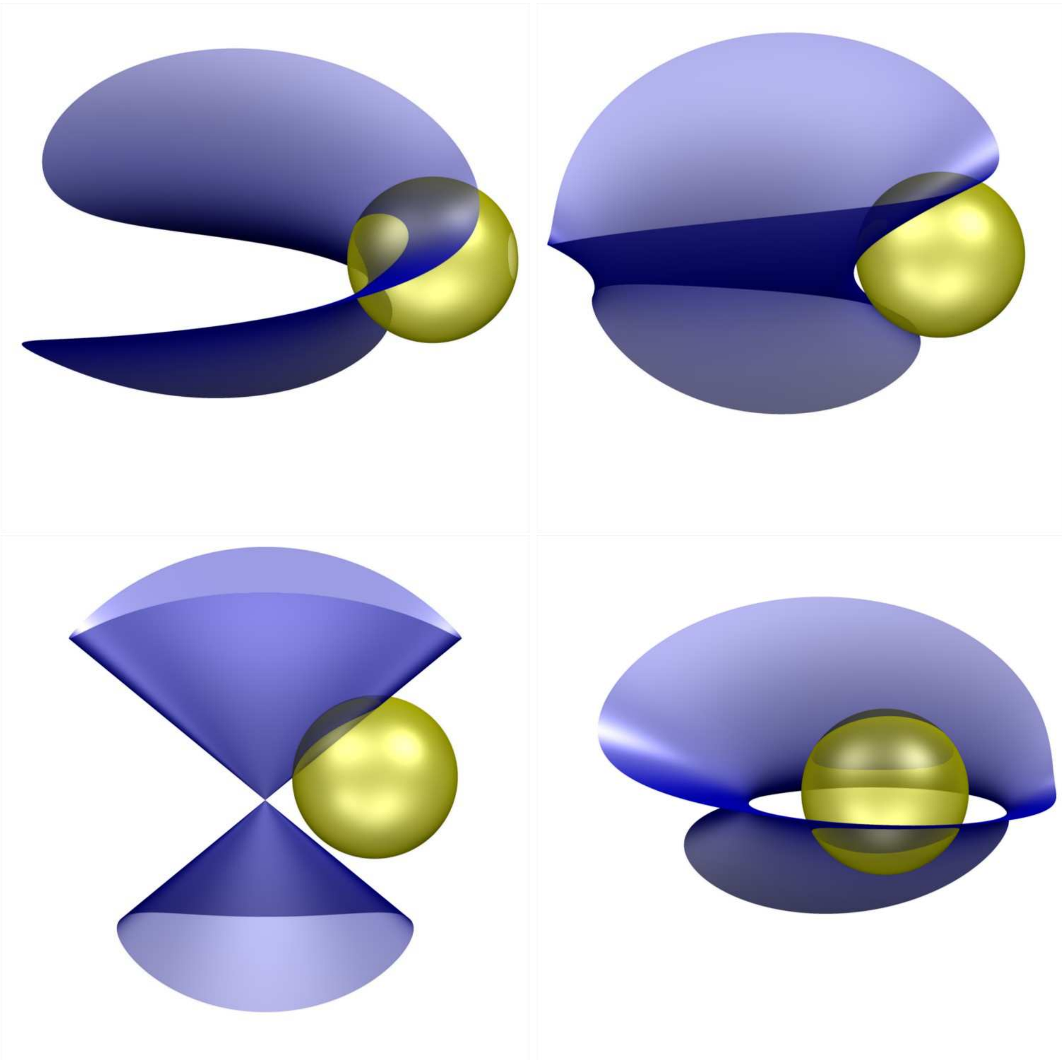

Figure 4.2: The intersections when $\theta \in \{-2, -1, 0, 1\}$ in order, from left to right: when $\theta \in \{-1, -2, 1\}$ the intersection is a smooth quartic with 2 components, and when $\theta = 0$ the intersection is a smooth quartic with one component

4.2 Computing a parametrization

Our problem is to find a parameterization depending on time for the intersection curve of P and Q. In order to do this, as we have already mentioned, we need to parameterize a quadric from the pencil $R(\lambda,\theta)=\lambda\ P + Q$ and insert its parametrization in the equation of another quadric. We look in the pencil for a ruled quadric in order to be easy to parameterize. Because P is not a ruled quadric (it is a sphere, and spheres have inertia (3,1)), its parameterization is not linear in at least one of its parameters, and thus it does not present any interest in our case.

Nevertheless, if we look carefully to the quadric Q, we can see its expression can be rewritten as:

$$(x - 3\theta w)^2 + 9y^2 - z^2 - 25\theta w^2 = 0.$$

Thus by applying a change of coordinates, we can transform Q into a projectively equivalent quadric of inertia (2,2). We let T be the transformation matrix which sends the quadric[2] Q into Q':

$$T = \begin{pmatrix} 1 & 0 & 0 & 3\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad Q' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -25\theta \end{pmatrix}$$

Q' is a quadric (2,2) as long as the time is positive ($\theta > 0$). Moreover, [0,1,1,0], [0,1,-1,0] $\in Q'$, and so Q' is a quadric of inertia (2,2)[3] and it has rational points on it. We know that a quadric (2,2) admits a parameterization $\mathbf{X}$

$$\mathbf{X} = \left[\frac{ut + avs}{a}, \frac{us - bvt}{b}, \frac{ut - avs}{\sqrt{ac}}, \frac{us + bvt}{\sqrt{bd}}\right]$$

where a, b, c, d are the elements from the diagonal of Q'.

Thus, after computing, we say that the quadric Q' admits the parametrization

$$\mathbf{X}_1(u, v, s, t) + \sqrt{\theta}\mathbf{X}_2(u, v, s, t),$$

where

$$\mathbf{X}_1(u, v, s, t) = \left[\frac{(us + 9vt)\theta}{5}, 0, 0, \frac{us + 5vt}{15}\right]$$

$$\mathbf{X}_2(u, v, s, t) = \left[ut + vs, \frac{us}{9} - vt, -ut + vs, 0\right]$$

By plugging $\mathbf{X}$ in P we obtain an equation Ω of degree 2 in (u, v) and (s, t). We solve Ω for (u, v) in function of (s, t) and inserting the solutions in $\mathbf{X}$ gives us a parametrization for the intersection:

[2]We make the convention of denoting by the same symbol the quadric and its associated matrix

[3]From now on we consider time as a positive variable, $\theta \in \mathbb{R}^+$

$$\mathbf{X}_1(s,t) + \sqrt{\theta}\mathbf{X}_2(s,t),$$

with

$$\mathbf{X}_1(s,t) = \begin{pmatrix} 4s(18s^2(\theta^2 - 4\theta + 2) + 9t^2(18\theta^2 - 341\theta - 36)) \\ -36t(8s^2(2\theta - 4\theta + 1) + 225t) \\ 2s(s^2(144\theta^2 - 263\theta + 72) + t^2(1296\theta^2 + 1233\theta + 648)) \\ -18s(s^2(4\theta - 5) - 9t^2(4\theta - 3)) \end{pmatrix} \pm \begin{pmatrix} t \\ s/9 \\ -t \\ 0 \end{pmatrix} \sqrt{\Delta}$$

and

$$\mathbf{X}_2(s,t) = 60t\theta(s^2 + 81t^2) \begin{pmatrix} 3\theta \\ 0 \\ 0 \\ 1 \end{pmatrix} \pm s/5 \begin{pmatrix} \theta \\ 0 \\ 0 \\ 1/3 \end{pmatrix} \sqrt{\Delta}$$

where

$$\Delta = -8100(\theta(144\theta^2 - 166\theta)s^4 + 81(144\theta^2 - 310\theta + 63)t^4 + 18\theta(176\theta^2 + 98\theta + 151)s^2t^2 + \sqrt{\theta}(480\theta(5\theta - 4)s^3t + 4320\theta(5\theta - 6)st^3))$$

Chapter 5

Conclusions and Further Work

In this report we have focused on two main problems regarding time-varying quadrics. Namely, we have presented methods which make it possible to study the evolution of the type of the intersection of time-varying quadrics. Furthermore, we have proved that it is possible to parameterize the intersection curve in function of time.

More precisely, we have analyzed the preservation/transformation of the topology for the intersection curve of two quadrics. We have presented necessary and sufficient conditions for conserving the topology and, similarly, we have detected when exactly the topology changes.

Being that we can detect for any value of time the type of intersection automatically, we have posed the problem of obtaining a parameterization depending on time for the changing intersection curve. In a first phase we have proved that, given a particular point, we can find an interval of time on which the parameterization is possible. Being able to accomplish this step, we have proceeded to determine if we can cover the whole axis of time with such intervals. We have presented an algorithm which proves that this is possible. The drawback of our method is that depending on a initial choice we make, the number of intervals which cover the axis of time can vary significantly. Nevertheless, our method presents advantages. One advantage is clearly represented by the fact that we do not need to compute a parametrization for each value of time, and especially for those values that are algebraic. The second one is significant because it permits us to have an exact parametrization for the intersection curve of two quadrics with algebraic coefficients.

As future work we mention that the most important problem is to find a way to control the number of intervals on which we parameterize the intersection curve. Another direction in which it is required to bring improvements is the computation of the canonical form of a quadric. One simple solution (not necessarily an efficient one) is Gauss reduction, which we have used. Nevertheless, computing the canonical form by a Gauss transformation can strongly augment the degree of the time variable which appears in the coefficients of a quadric. This implies that the expressions we work with become complicated and that the computation time increases. In this sense, one way to start would be the method presented in [9].

As possible directions of application of our results, we mention that we are interested in problems like continuous collision detection (CCD). There are some recent results regarding CCD for bounding boxes, articulated models, moving ellipsoids. One interesting paper is [13], which the authors renewed it recently, namely, they can detect collisions for moving ellipsoids under affine deformation in real time. We think that our framework would produce good results not only for affine transformations, and not only for particular types of surfaces.

Useless to mention that quadrics, as the simplest implicit surfaces, constitute the base of more complicated structures. Managing to control the objects which represent the foundation, is one step forward. Thus, we can imagine applications in each domain where implicit surfaces are an important ingredient: protein modelling, character animation, motion analysis.

Appendix A

Canonical Form for Pencils

A.1 Preliminaries

The notion of pencil, $R(\lambda)$, is defined as a family of quadrics, $\{\lambda P + Q \mid P, Q$ quadrics$\}$. Quadrics have associated real symmetric matrices, and so, the pencil of quadrics can be thought of as a pencil of symmetric matrices, $\{\lambda M_P + M_Q \mid M_P, M_Q \in Sym_{4\times 4}(\mathbb{R})\}$. We have mentioned that for symmetric real matrices there exists a transformation which can make them diagonal, with their eigenvalues as elements. This diagonal form represents, in fact, the canonical form for a quadric.

We now pose the problem of finding a canonical form for the pencil of two quadrics. If finding the canonical form for a quadric boils down to finding the eigenvalues for the associated matrix (the standard eigenvalue problem), finding the canonical form for a pencil is strongly connected to the generalized eigenvalue problem. Under one condition, this problem can be translated to the standard eigenvalue problem, namely, if M_P is nonsingular, it follows that $\lambda M_P + M_Q = M_P(M_P^{-1}M_Q + \lambda I)$. Finding the eigenvalues for $M_P^{-1}M_Q$, we can obtain a canonical form for $R(\lambda)$, which is not necessarily diagonal, taking in consideration that $M_P^{-1}M_Q$ is not symmetric.

It has been proved that there exists a matrix that sends simultaneously both quadrics engendered by a pencil in an almost diagonal matrix. More precisely, with the help of Jordan Forms, these quadrics become projectively equivalent to block diagonal matrices.

A.2 Jordan Forms

Though the notions we are about to present are defined for the n-dimensional case, for the sake of simplicity, we will introduce them for our given framework, namely the family of quadrics, thus square matrices of dimension 4.

Given a pencil $R(\lambda)= \{\lambda P + Q\}$, we are interested in finding a canonical form for it, namely a transformation that sends simultaneously both quadrics which engender the pencil, in a simplified form. We recall that this is related to the generalized eigenvalue problem, and that it is reduced to finding eigenvalues for $M_P^{-1}M_Q$. This is not a symmetric matrix, so a diagonal form does not always exists. Instead, as the next theorem states, there exists a transformation which can send it into a block diagonal matrix, which is called "real Jordan normal form".

Theorem A.2.1 *Every real square matrix A is similar to a block diagonal matrix $diag(J_1, .., J_k)$, called real Jordan form of A, in which each J_i is a Jordan block associated with an eigenvalue of A.*

Given λ a real eigenvalue for A, with multiplicity m, we associate a number of "real" blocks with it. The number of these blocks is exactly the number of eigenvectors corresponding to λ. This number is at most m, and it depends on the rank of the matrix. The form of a block associated with a real eigenvalue is one of the following:

$$(\lambda) \qquad \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

If λ is complex, $\lambda = \alpha + \mathrm{i}\beta$, the sum of all blocks associated with λ is always twice the multiplicity (because the conjugate is also a root), and the form of one of these blocks is:

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \qquad \begin{pmatrix} \alpha & -\beta & 1 & 0 & & & & 0 \\ \beta & \alpha & 0 & 1 & & & & \\ & & \ddots & \ddots & & & & \\ & & & \ddots & \ddots & & 1 & 0 \\ & & & & \ddots & & 0 & 1 \\ & & & & & & \alpha & \beta \\ 0 & & & & & & \beta & \alpha \end{pmatrix}$$

Having the Jordan form of a matrix we can compute the "Segre characteristic", which is a sequence of numbers (or group of numbers) associated to each eigenvalue. Namely, if for an eigenvalue λ, we have, in the Jordan form, only one block associated, which must be of dimension m, where m is the multiplicity for λ, then in the sequence it appears m as the number corresponding to λ. If for λ there are associated 2 blocks, one of dimension m and the other of dimension n, then we can conclude that the multiplicity

of λ is $m+n$, and that this eigenvalue is put in correspondence with the group (mn) in the Segre characteristic.
For example, if the Jordan form of a matrix looks like

$$\begin{pmatrix}
\alpha & 1 & & & & & & & & \\
 & \alpha & & & & & & & & \\
 & & \alpha & & & & & & & \\
 & & & \beta & 1 & & & & & \\
 & & & & \beta & 1 & & & & \\
 & & & & & \beta & & & & \\
 & & & & & & \gamma & & & \\
 & & & & & & & \delta & 1 & \\
 & & & & & & & & \delta & \\
 & & & & & & & & & \delta
\end{pmatrix}$$

then there are 4 eigenvalues, α, β, δ of multiplicity 3, and γ of multiplicity 1. The Segre characteristic is $[(21)31(21)]$, as there are 2 blocks associated with α, one of dimension 2 and the other of dimension 1, for β there is associated only one block of dimension 3, for γ a block of dimension 1, and finally, for δ, there are 2 blocks associated, one of dimension 2 and the other of dimension 1.

The Segre characteristic is a useful notion which makes it possible to exhaustively classify quadrics in $\mathbb{P}^3(\mathbb{C})$.

In our case, the matrix $M_P^{-1}M_Q$ is 4×4, and so it has 4 eigenvalues, with multiplicity at most 4, if the eigenvalue is real, or at most 2, if the eigenvalue is complex. We illustrate the above, by taking 2 examples:

Example

1. $\det(M_P^{-1}M_Q - \lambda I) = (\lambda - \lambda_0)(\lambda - \lambda_1)(\lambda$ - $(\alpha$ + i$\beta))(\lambda$ - $(\alpha$ - i$\beta)) \Rightarrow$ $M_P^{-1}M_Q$ has 2 simple real eigenvalues, and one which is complex, and it can be reduced to the following block diagonal matrix:

$$\begin{pmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \alpha & -\beta \\ 0 & 0 & \beta & \alpha \end{pmatrix}$$

2. $\det(M_P^{-1}M_Q - \lambda I) = (\lambda - \lambda_0)^3(\lambda$ - $\lambda_1) \Rightarrow M_P^{-1}M_Q$ has 1 simple real eigenvalue, and one triple, and it can be reduced to one of the following forms, depending on the rang of $M_P^{-1}M_Q$:

$$\begin{pmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_0 & 0 & 0 \\ 0 & 0 & \lambda_0 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} \quad \begin{pmatrix} \lambda_0 & 1 & 0 & 0 \\ 0 & \lambda_0 & 0 & 0 \\ 0 & 0 & \lambda_0 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} \quad \begin{pmatrix} \lambda_0 & 1 & 0 & 0 \\ 0 & \lambda_0 & 1 & 0 \\ 0 & 0 & \lambda_0 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$$

so for the triple eigenvalue we might have 3 blocks of size 1 (rank=1), one block of size 2 and one of size 1 (rank=2), or just one block of size 3 (rank=3).

After seeing how the block diagonal matrix looks like, we proceed and present the main theorem which refers the simultaneous reducing of the pair of quadrics engendered by a given pencil.

Theorem A.2.2 ***Canonical Pair Form***. *Let M_Q and M_P be two real symmetric matrices, with M_P nonsingular. Let $M_P^{-1}M_Q$ have real Jordan normal form diag(J_1,..,J_k,J_{k+1},..,J_m), where J_i, $i=\overline{1..k}$, are real Jordan blocks corresponding to real eigenvalues of $M_P^{-1}M_Q$, and J_i, $i = \overline{k+1..m}$ are complex Jordan blocks corresponding to complex eigenvalues of $M_P^{-1}M_Q$. Then M_P and M_Q are simultaneously congruent by a real congruence transformation to*

$$diag(\epsilon_1 E_1, .., \epsilon_r E_r, E_{r+1}, .., E_m)$$

and

$$diag(\epsilon_1 E_1 J_1, .., \epsilon_r E_r J_r, E_{r+1} J_{r+1}, .., E_m J_m),$$

respectively, where $\epsilon_i = \pm 1$ and E_i denotes the square matrix

$$\begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix}$$

of the same size as J_i.

We illustrate the above written by completing the previous examples.

Example

1. $M_P^{-1}M_Q$ has 2 simple real eigenvalues, and one which is complex $\Rightarrow$

$$M_P^{-1}M_Q \equiv \begin{pmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \alpha & -\beta \\ 0 & 0 & \beta & \alpha \end{pmatrix}$$

and M_P, respectively M_Q can be reduced to the following block diagonal matrices:

$$M_P \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad M_Q \equiv \begin{pmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \beta & \alpha \\ 0 & 0 & \alpha & -\beta \end{pmatrix}$$

2. $M_P^{-1}M_Q$ has 1 simple real eigenvalue, and one triple, and it can be reduced to one of the following forms, depending on the rang of $M_P^{-1}M_Q$:

$$\begin{pmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_0 & 0 & 0 \\ 0 & 0 & \lambda_0 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} \quad \begin{pmatrix} \lambda_0 & 1 & 0 & 0 \\ 0 & \lambda_0 & 0 & 0 \\ 0 & 0 & \lambda_0 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} \quad \begin{pmatrix} \lambda_0 & 1 & 0 & 0 \\ 0 & \lambda_0 & 1 & 0 \\ 0 & 0 & \lambda_0 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$$

   and M_P, respectively M_Q can be reduced to the following block diagonal matrices, for each of the three cases:

$$\text{rang}(M_P^{-1}M_Q)\text{=}1: \quad M_P \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad M_Q \equiv \begin{pmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_0 & 0 & 0 \\ 0 & 0 & \lambda_0 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$$

$$\text{rang}(M_P^{-1}M_Q)\text{=}2: \quad M_P \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad M_Q \equiv \begin{pmatrix} 0 & \lambda_0 & 0 & 0 \\ \lambda_0 & 1 & 0 & 0 \\ 0 & 0 & \lambda_0 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$$

$$\text{rang}(M_P^{-1}M_Q)\text{=}3: \quad M_P \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad M_Q \equiv \begin{pmatrix} 0 & 0 & \lambda_0 & 0 \\ 0 & \lambda_0 & 1 & 0 \\ \lambda_0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$$

The existence of the above block diagonal matrices makes it easier to compute the type of intersection between quadrics. We do not detail this issue here, as complete information can be found in [3, 4].

Appendix B

Interval Arithmetics

Interval arithmetics is an extension of the usual arithmetics on numbers to intervals. We mention, as it appears in [1] that an interval is a closed, convex set of a totally ordered field, in our case $\mathbb{R}$. An interval is denoted by the notation [x] = $[\underline{x}, \overline{x}]$, where $\underline{x}$ represents the lower bound, and $\overline{x}$, the upper bound. The main operations from exact arithmetics are extended by the inclusion property: $f([x]) = [f(x)] = \{f(x) \mid x \in [x]\}$. Precisely, given 2 intervals [x] = $[\underline{x}, \overline{x}]$, [y] = $[\underline{y}, \overline{y}]$, we have:

$$[x] + [y] = [\underline{x} + \underline{y},\ \overline{x} + \overline{y}]$$

$$[x] - [y] = [\underline{x} - \overline{y},\ \overline{x} + \underline{y}]$$

$$[x] \times [y] = [\min\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{xy}\}, \max\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{xy}\}]$$

$$[x]\ /\ [y] = [\mathrm{x}] \cdot [1/\overline{y},\ 1/\underline{y}], \text{ if } 0 \notin [y]$$

Remark

x - [x] $\neq 0$

x / [x] $\neq 1$

In order to develop our study, we need to add to the above basic operations, the possibility of comparing two intervals and the power operator:

$$[x] < [y] \text{ iff } \underline{x} < \underline{y} \text{ and } \overline{x} < \overline{y}$$

$$[x]^r = [\underline{x}^r, \overline{x}^r],\ r \in \mathbb{Q}.$$

Remark We do not need unbounded intervals, and for the sake of simplicity, we do not present them here. Nethertheless, more information can be found in [1, 5].

Applications of interval arithmetics in computational geometry are described in [7]. As an informative title, we add that interval arithmetics is used as a formal foundation for proofs by approximation [], in the computation of the predicates, where there is no time for spending on exact arithmetics.

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