# Homogenization of the 1D Vlasov-Maxwell equations

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**Abstract:** In this report we investigate the homogenization of the one dimensional Vlasov-Maxwell system. We indicate the rate of convergence towards the limit solution. In the non relativistic case we compute explicitly the limit solution. The theoretical results are illustrated by some numerical simulations.

**Key-words:** Vlasov-Maxwell equations, homogenization, mild solutions

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# Homogénéisation des équations de Vlasov-Maxwell

**Résumé :** Dans ce rapport nous analysons l'homogénéisation des équations de Vlasov-Maxwell 1D. Dans le cas non relativiste nous calculons explicitement la solution du modèle limite. Les résultats théoriques sont illustrés par quelques simulations numériques.

**Mots-clés :** Equations de Vlasov-Maxwell, homogénéisation, solutions par caractéristiques

# 1 Introduction

We consider a population of electrons (with mass $m_e$ and charge $-e$, $e > 0$) interacting through their self-consistent electro-magnetic field. We denote by $f$ the electronic density, depending on the time $t \in \mathbb{R}^+$, position $x \in \mathbb{R}^3$ and momentum $p \in \mathbb{R}^3$ and by $(E, B)$ the electro-magnetic field. The notation $\rho_{\text{ext}}(x)$ stands for the charge density of the background ion distribution, which are supposed to be at rest. The unknown $(f, E, B)$ satisfy the Vlasov-Maxwell system

$$\partial_t f + v(p) \cdot \nabla_x f - e(E(t,x) + v(p) \wedge B(t,x)) \cdot \nabla_p f = 0, \tag{1}$$

$$\partial_t E - c^2 \mathrm{rot} B = \frac{e}{\varepsilon_0} \int_{\mathbb{R}^3} v(p) f(t,x,p)\, dp, \;\; \partial_t B + \mathrm{rot} E = 0, \tag{2}$$

$$\mathrm{div} E = \frac{1}{\varepsilon_0} \left( \rho_{\text{ext}}(x) - e \int_{\mathbb{R}^3} f(t,x,p)\, dp \right), \;\; \mathrm{div} B = 0, \tag{3}$$

where $c$ is the light speed in the vacuum and $\varepsilon_0$ is the dielectric permittivity of the vacuum. Here $v(p)$ is the velocity associated to the momentum $p$. This function is given by $v(p) = \frac{p}{m_e}$ in the non relativistic case (NR) and by $v(p) = \frac{p}{m_e} \left( 1 + \frac{|p|^2}{m_e^2 c^2} \right)^{-1/2}$ in the relativistic case (R). We prescribe initial data

$$f(0,x,p) = f_0(x,p), \;\; (x,p) \in \mathbb{R}^3 \times \mathbb{R}^3, \;\; (E,B)(0,x) = (E_0, B_0)(x), \;\; x \in \mathbb{R}^3, \tag{4}$$

satisfying the compatibility constraints

$$\mathrm{div} E_0 = \frac{1}{\varepsilon_0} \left( \rho_{\text{ext}}(x) - e \int_{\mathbb{R}^3} f_0(x,p)\, dp \right), \;\; \mathrm{div} B_0 = 0, \;\; x \in \mathbb{R}^3, \tag{5}$$

and the global neutrality condition

$$e \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_0(x,p)\, dp\, dx = \int_{\mathbb{R}^3} \rho_{\text{ext}}(x)\, dx. \tag{6}$$

There are several approaches for studying the Vlasov-Maxwell system (1), (2), (3), (4): classical solutions have been investigated in [16], [13], [14], [15], [19], [8]; the existence of weak solutions has been studied in [11], [22], [18], [4].

Neglecting the magnetic field $B$ and the relativistic corrections in the Vlasov equation leads to the Vlasov-Poisson system

$$\partial_t f + \frac{p}{m_e} \cdot \nabla_x f - eE(t,x) \cdot \nabla_p f = 0,$$

$$\mathrm{rot} E = 0, \;\; \mathrm{div} E = \frac{1}{\varepsilon_0} \left( \rho_{\text{ext}}(x) - e \int_{\mathbb{R}^3} f(t,x,p)\, dp \right),$$

which were studied by many authors, cf. [1], [2], [17], [20], [21], [3]. The Vlasov-Poisson system can be justified as the limit of the Vlasov-Maxwell model when the light speed is much larger than the particle velocities, see [10], [6].

Another interesting problem concerns the homogenization of these equations. Results for the Vlasov-Poisson system with strong external magnetic field can be found in [12].

We investigate here the Vlasov-Maxwell equations in one dimension. Choosing physical units such that $m_e = 1$, $e = 1$, $\varepsilon_0 = 1, c = 1$, these equations become

$$\partial_t f + v(p)\partial_x f - E(t,x)\partial_p f = 0, \;\; (t,x,p) \in \mathbb{R}^+ \times \mathbb{R}^2, \tag{7}$$

$$\partial_t E = j(t,x), \;\; \partial_x E = \rho_{\text{ext}}(x) - \rho(t,x), \;\; (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{8}$$

where $\rho = \int_{\mathbb{R}} f \; dp$, $j = \int_{\mathbb{R}} v(p) f \; dp$. We supplement the above equations with the initial conditions

$$f(0,x,p) = f_0(x,p), \;\; (x,p) \in \mathbb{R}^2, \;\; E(0,x) = E_0(x), \;\; x \in \mathbb{R}. \tag{9}$$

We assume that the background density is constant $\rho_{\text{ext}} = n > 0$ and that the initial conditions verify the hypotheses

H1) there is a bounded function $g_0$ non decreasing on $\mathbb{R}^-$ and non increasing on $\mathbb{R}^+$ such that $0 \leq f_0(x,p) \leq g_0(p)$, $\forall \; (x,p) \in \mathbb{R}^2$ and which belongs to $L^1(\mathbb{R}; dp)$ in the R case and to $L^1(\mathbb{R}; |p| \; dp)$ in the NR case;
H2) $E_0$ belongs to $L^\infty(\mathbb{R})$ such that $E_0' = n - \rho_0$, where $\rho_0 = \int_{\mathbb{R}} f_0 \; dp$.

Notice that $f_0$ is supposed to be only locally integrable with respect to the space variable. We work in the space periodic setting. Assume that $(f_0, E_0)$ are 1-periodic functions with respect to $x$. Notice that the solvability of $E_0' = n - \rho_0$ in the class of 1-periodic functions is equivalent to the neutrality condition $n = \int_0^1 \int_{\mathbb{R}} f_0(x,p) dp \; dx$. Moreover the solution is unique up to an additive constant. For any $\varepsilon > 0$ we consider the $\varepsilon$-periodic functions given by

$$f_0^\varepsilon(x,p) = f_0\left(\frac{x}{\varepsilon}, p\right), \; (x,p) \in \mathbb{R}^2, \;\; E_0^\varepsilon(x) = \varepsilon E_0\left(\frac{x}{\varepsilon}\right) + K, \;\; x \in \mathbb{R},$$

where $K \in \mathbb{R}$ is a fixed constant. Observe that $(f_0^\varepsilon, E_0^\varepsilon)$ satisfy H1, H2. When $\varepsilon$ goes to 0 we expect that the family of solutions $(f^\varepsilon, E^\varepsilon)_{\varepsilon>0}$ associated to $(f_0^\varepsilon, E_0^\varepsilon)_{\varepsilon>0}$ converges towards some functions not depending on $x$. Therefore we consider also the space homogeneous 1D Vlasov-Maxwell system

$$\partial_t f - E(t)\partial_p f = 0, \;\; (t,p) \in \mathbb{R}^+ \times \mathbb{R}, \tag{10}$$

$$\frac{dE}{dt} = \int_{\mathbb{R}} v(p) f(t,p) \; dp =: j(t), \;\; t \in \mathbb{R}^+, \tag{11}$$

with the initial conditions

$$f(0,p) = f_i(p) := \int_0^1 f_0(x,p)\,dx,\ p \in \mathbb{R},\ \ E(0) = E_i := K. \tag{12}$$

Our main result describes the behavior of the sequence $(f^\varepsilon, E^\varepsilon)_{\varepsilon>0}$ for small $\varepsilon$ (see the second section for the definition of $g_0^R$).

**Theorem 1.1** *Assume that $(f_0, E_0)$ are 1-periodic in $x$ and satisfy H1,H2. Then for any $\varepsilon > 0$ we have the inequality*

$$\|E^\varepsilon(t) - E(t)\|_{L^\infty(\mathbb{R})} \quad \leq \quad \varepsilon(\|E_0\|_{L^\infty(\mathbb{R})} + 4\|g_0\|_{L^1(\mathbb{R})}) \exp(2te^t \|g_0\|_{L^1(\mathbb{R})}),\ \ t \in \mathbb{R}^+.$$

*Moreover, for any $T > 0$ and $\varphi \in L^1(\mathbb{R}^2; g_0^{R(T)}(p)\,dp\,dx)$ we have*

$$\lim_{\varepsilon \searrow 0} \sup_{t\in[0,T]} \int_{\mathbb{R}}\int_{\mathbb{R}} (f^\varepsilon(t,x,p) - f(t,p))\varphi(x,p)\,dp\,dx = 0,$$

*where $R(T) = Ta_\alpha^K(T), \alpha \in \{\mathrm{R}, \mathrm{NR}\}$, $a_R^K(T) = \|E_0\|_{L^\infty(\mathbb{R})} + |K| + 2T\|g_0\|_{L^1(\mathbb{R})}$, $a_{NR}^K(T) = (\|E_0\|_{L^\infty(\mathbb{R})} + |K| + 2T\|\,|p|g_0\|_{L^1(\mathbb{R})}) \exp(2T^2\|g_0\|_{L^1(\mathbb{R})})$.*

Our paper is organized as follows. In Section 2 we recall the main existence and uniqueness results for the 1D Vlasov-Maxwell equations. We establish estimates for the electric field and its derivatives. In Section 3 we prove the Theorem 1.1. The main tools are the formulation by characteristics of the Vlasov problem combined with standard arguments in the homogenization theory. We indicate the convergence rate for the electric fields and establish weak convergence for the particle densities. The last section is devoted to numerical simulations.

## 2 The 1D Vlasov-Maxwell system

We start with existence and uniqueness results for the 1D Vlasov-Maxwell equations.

**Theorem 2.1** *Assume that $(f_0, E_0)$ verify H1,H2. Then there is a unique mild solution $(f, E)$ (i.e., $E$ is Lipschitz continuous function and $f$ is solution by characteristics) of (7), (8), (9) satisfying $E \in W^{1,\infty}(]0,T[\times\mathbb{R})$, $\rho, j \in L^\infty(]0,T[\times\mathbb{R})$, $\forall\, T > 0$.*

**Proof.** The arguments follow the lines in [5] (see also [9]) with minor changes. The main difference here is that we construct particle distributions which are only locally integrable in space, in view of the homogenization process of space periodic solutions. We do not give all the details. Let us explain how to obtain bounds for the electric field and its derivatives. These estimates will be useful for our further computations. Let us introduce the system of characteristics for (7)

$$\frac{dX}{ds} = v(P(s)),\ \ \frac{dP}{ds} = -E(s, X(s)),\ \ X(t) = x,\ \ P(t) = p. \tag{13}$$

We denote by $(X(s;t,x,p),P(s;t,x,p))$ the solution of (13). For any $\varphi \in L^1(\mathbb{R})$ we have by the first equation in (8) after the change of variables along the characteristics

$$\begin{aligned}\left|\int_{\mathbb{R}}(E(t,x)-E_0(x))\varphi(x)\,dx\right| &= \left|\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} f(s,x,p)v(p)\varphi(x)\,dp\,dx\,ds\right|\\ &= \left|\int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)\int_x^{X(t;0,x,p)}\varphi(u)\,du\,dp\,dx\right| \\ &\le \int_{\mathbb{R}}|\varphi(u)|\int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)\mathbf{1}_{\{|x-u|\le|X(t;0,x,p)-x|\}}\,dp\,dx\,du.\end{aligned} \tag{14}$$

In the R case we have $|X(t;0,x,p)-x|\le t$ and thus

$$\left|\int_{\mathbb{R}}(E(t,x)-E_0(x))\varphi(x)\,dx\right| \le 2t\,\|g_0\|_{L^1(\mathbb{R})}\|\varphi\|_{L^1(\mathbb{R})},$$

implying that

$$\|E(t)\|_{L^\infty(\mathbb{R})}\le\|E_0\|_{L^\infty(\mathbb{R})}+2t\,\|g_0\|_{L^1(\mathbb{R})}=:a_R(t). \tag{15}$$

In the NR case we have

$$|X(t;0,x,p)-x|\le\int_0^t\left(|p|+\int_0^s\|E(\tau)\|_{L^\infty}\right)\,ds\le t|p|+tR(t),$$

where $R(t)=\int_0^t\|E(s)\|_{L^\infty}\,ds$. Therefore we obtain

$$\left|\int_{\mathbb{R}}(E(t,x)-E_0(x))\varphi(x)\,dx\right|\le\int_{\mathbb{R}}|\varphi(u)|\int_{\mathbb{R}}\int_{\mathbb{R}}g_0(p)\mathbf{1}_{\{|x-u|\le t|p|+tR(t)\}}\,dp\,dx\,du,$$

implying that

$$\|E(t)\|_{L^\infty(\mathbb{R})}\le\|E_0\|_{L^\infty(\mathbb{R})}+2t\,\|\,|p|g_0\|_{L^1(\mathbb{R})}+2t\,\|g_0\|_{L^1(\mathbb{R})}R(t).$$

By Gronwall lemma one gets

$$\|E(t)\|_{L^\infty(\mathbb{R})}\le(\|E_0\|_{L^\infty(\mathbb{R})}+2t\,\|\,|p|g_0\|_{L^1(\mathbb{R})})\exp(2t^2\|g_0\|_{L^1(\mathbb{R})})=:a_{NR}(t). \tag{16}$$

In order to estimate the derivatives of $E$ consider for any $R>0$ the function $g_0^R(p)$ given by $g_0(p\mp R)$ if $\pm p>R$ and $g_0(0)$ if $|p|\le R$. Observing that $|P(t;0,x,p)-p|\le R(t)$ we obtain easily by using the monotonicity of $g_0$ that $g_0(P(0;t,x,p))\le g_0^{R(t)}(p)$ and thus we have

$$\begin{aligned}\rho(t,x) &= \int_{\mathbb{R}} f_0(X(0;t,x,p),P(0;t,x,p))\,dp\\ &\le \int_{\mathbb{R}} g_0^{R(t)}(p)\,dp\\ &= \|g_0\|_{L^1(\mathbb{R})}+2R(t)\|g_0\|_{L^\infty(\mathbb{R})}\\ &\le \|g_0\|_{L^1(\mathbb{R})}+2ta_\alpha(t)\|g_0\|_{L^\infty(\mathbb{R})}=:b_\alpha(t),\end{aligned} \tag{17}$$

where $\alpha \in \{\mathrm{R}, \mathrm{NR}\}$. In the R case we have also $|j(t,x)| \le d_R(t) := b_R(t)$. In the NR case we can write as before

$$\begin{aligned} |j(t,x)| &\le \int_{\mathbb{R}} |p| g_0^{R(t)}(p)\, dp \\ &= \| \, |p| g_0 \|_{L^1(\mathbb{R})} + R(t) \|g_0\|_{L^1(\mathbb{R})} + R^2(t) \|g_0\|_{L^\infty(\mathbb{R})} \\ &\le \| \, |p| g_0 \|_{L^1(\mathbb{R})} + t a_{NR}(t) \|g_0\|_{L^1(\mathbb{R})} + (t a_{NR}(t))^2 \|g_0\|_{L^\infty(\mathbb{R})} =: d_{NR}(t). \end{aligned} \tag{18}$$

It is easily seen by (8) that $\|\partial_x E(t)\|_{L^\infty(\mathbb{R})} \le \max\{n, b_\alpha(t)\} =: c_\alpha(t)$, $\alpha \in \{\mathrm{R}, \mathrm{NR}\}$, $t \in \mathbb{R}^+$ and $\|\partial_t E(t)\|_{L^\infty(\mathbb{R})} = \|j(t)\|_{L^\infty(\mathbb{R})} \le d_\alpha(t)$, $\alpha \in \{\mathrm{R}, \mathrm{NR}\}$, $t \in \mathbb{R}^+$. $\square$

**Corollary 2.1**
*1) Assume that $(f_0, E_0)$ are 1-periodic in $x$ and satisfy $H1, H2$. Then there is a unique 1-periodic mild solution of (7), (8), (9).*
*2) Assume that $f_0 = f_0(p)$ satisfies $H1$ and let $E_0 \in \mathbb{R}$. Then there is a unique mild solution of (10), (11) with the initial conditions $(f_0, E_0)$.*

# 3 Homogenization of the 1D Vlasov-Maxwell equations

Consider $(f_0, E_0)$ verifying H1, H2 and assume that these functions are 1-periodic in $x$. We denote by $(f_0^\varepsilon, E_0^\varepsilon)$ the $\varepsilon$-periodic functions

$$f_0^\varepsilon(x,p) = f_0\left(\frac{x}{\varepsilon}, p\right), \ (x,p) \in \mathbb{R}^2, \ \ E_0^\varepsilon(x) = \varepsilon E_0\left(\frac{x}{\varepsilon}\right) + K, \ \ x \in \mathbb{R},$$

where $K \in \mathbb{R}$. Note that $(f_0^\varepsilon, E_0^\varepsilon)$ satisfy H1, H2 and thus by Corollary 2.1, for any $\varepsilon > 0$ there is a unique $\varepsilon$-periodic mild solution $(f^\varepsilon, E^\varepsilon)$ for the 1D Vlasov-Maxwell system with the initial conditions $(f_0^\varepsilon, E_0^\varepsilon)$. Since the family $(E_0^\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty(\mathbb{R})$ we deduce by the estimates in Theorem 2.1 that $(E^\varepsilon)_{\varepsilon>0}$ is bounded in $W^{1,\infty}(]0,T[\times\mathbb{R})$ and $(\rho^\varepsilon := \int_{\mathbb{R}} f^\varepsilon \, dp, j^\varepsilon := \int_{\mathbb{R}} v(p) f^\varepsilon \, dp)_{\varepsilon>0}$ are bounded in $L^\infty(]0,T[\times\mathbb{R})$, $\forall \ T > 0$. Observing that $f_i = \int_0^1 f_0(x,\cdot)\, dx$ satisfies H1 (with the same function $g_0$) we deduce by Corollary 2.1 that there is a unique mild solution $(f = f(t,p), E = E(t))$ of (10), (11), (12) verifying $E \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+)$ and $\rho(\cdot) := \int_{\mathbb{R}} f(\cdot,p)\, dp, j(\cdot) := \int_{\mathbb{R}} v(p) f(\cdot,p)\, dp \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+)$. For any $\varepsilon > 0$ we intend to compare the solutions $(f^\varepsilon, E^\varepsilon)$ and $(f, E)$.

**Proof.** (of Theorem 1.1) We scale the solution $(f^\varepsilon, E^\varepsilon)$ by introducing the fast variable $\frac{x}{\varepsilon}$

$$f^\varepsilon(t,x,p) = g^\varepsilon\left(t, \frac{x}{\varepsilon}, p\right), \ \ E^\varepsilon(t,x) = F^\varepsilon\left(t, \frac{x}{\varepsilon}\right).$$

Then the functions $(g^\varepsilon, F^\varepsilon)$ are 1-periodic in $x$ and solve the problem

$$\partial_t g^\varepsilon + \frac{v(p)}{\varepsilon} \partial_x g^\varepsilon - F^\varepsilon(t,x) \partial_p g^\varepsilon = 0, \ \ (t,x,p) \in \mathbb{R}^+ \times \mathbb{R}^2, \tag{19}$$

$$\partial_t F^\varepsilon = \int_{\mathbb{R}} v(p) g^\varepsilon(t,x,p)\;dp,\;\; \frac{1}{\varepsilon}\partial_x F^\varepsilon = n - \int_{\mathbb{R}} g^\varepsilon(t,x,p)\;dp,\;\; (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{20}$$

$$g^\varepsilon(0,x,p) = f_0(x,p),\;(x,p) \in \mathbb{R}^2,\;\; F^\varepsilon(0,x) = \varepsilon E_0(x) + K,\;\; x \in \mathbb{R}. \tag{21}$$

Since $\partial_x f = \partial_x E = 0$, observe that $(f, E)$ solve also the problem

$$\partial_t f + \frac{v(p)}{\varepsilon}\partial_x f - E\partial_p f = 0,\;\; (t,x,p) \in \mathbb{R}^+ \times \mathbb{R}^2, \tag{22}$$

$$\partial_t E = \int_{\mathbb{R}} v(p) f\;dp,\;\; (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{23}$$

$$f(0,x,p) = f_i(p),\;(x,p) \in \mathbb{R}^2,\;\; E(0,x) = E_i,\; x \in \mathbb{R}. \tag{24}$$

For any $(t,x,p) \in \mathbb{R}^+ \times \mathbb{R}^2$ we denote by $(X^\varepsilon(\cdot;t,x,p), P^\varepsilon(\cdot;t,x,p))$ the characteristics of (19)

$$\frac{dX^\varepsilon}{ds} = \frac{v(P^\varepsilon(s;t,x,p))}{\varepsilon},\;\; \frac{dP^\varepsilon}{ds} = -F^\varepsilon(s, X^\varepsilon(s;t,x,p)),\;\; s \in \mathbb{R}^+, \tag{25}$$

verifying the conditions $X^\varepsilon(s=t;t,x,p) = x,\;\; P^\varepsilon(s=t;t,x,p) = p$. Similarly consider $(X(\cdot;t,x,p), P(\cdot;t,x,p))$ the characteristics of (22)

$$\frac{dX}{ds} = \frac{v(P(s;t,x,p))}{\varepsilon},\;\; \frac{dP}{ds} = -E(s),\;\; s \in \mathbb{R}^+, \tag{26}$$

verifying the conditions $X(s=t;t,x,p) = x,\;\; P(s=t;t,x,p) = p$. Surely, the characteristics in (26) depend also on $\varepsilon$ but in order to avoid the confusion with the characteristics in (25) we use the simplified notation $(X,P)$. We check immediately that for any $(s,t,x,p) \in (\mathbb{R}^+)^2 \times \mathbb{R}^2$

$$P(s;t,x,p) = p - \int_t^s E(\tau)\;d\tau =: P(s;t,p), \tag{27}$$

$$X(s;t,x,p) = x + \frac{1}{\varepsilon}\int_t^s v(P(\tau;t,p))\;d\tau =: x + \frac{1}{\varepsilon}Y(s;t,p). \tag{28}$$

Let us estimate $F^\varepsilon(t,\cdot) - E(t)$. For this take $\varphi \in L^1(\mathbb{R})$ and observe that by (20), (23) we have

$$
\begin{aligned}
\int_{\mathbb{R}} (F^\varepsilon(t,x) - E(t))\varphi(x)\,dx &= \int_{\mathbb{R}} (F^\varepsilon(0,x) - E(0))\varphi(x)\,dx \\
&+ \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} g^\varepsilon(s,x,p) v(p)\varphi(x)\,dp\,dx\,ds \\
&- \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} f(s,p) v(p)\varphi(x)\,dp\,dx\,ds \\
&= \int_{\mathbb{R}} (F^\varepsilon(0,x) - E(0))\varphi(x)\,dx \\
&+ \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} g^\varepsilon(0, X^\varepsilon(0;s,x,p), P^\varepsilon(0;s,x,p)) v(p)\varphi\,dp\,dx\,ds \\
&- \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} f_i(P(0;s,x,p)) v(p)\varphi(x)\,dp\,dx\,ds. \qquad (29)
\end{aligned}
$$

By changing the variables along the characteristics (recall that these changes are measure preserving) we obtain

$$
\int_{\mathbb{R}} (F^\varepsilon(t,x) - E(t))\varphi(x)\,dx \quad = \quad \int_{\mathbb{R}} (F^\varepsilon(0,x) - E_i)\varphi(x)\,dx + \mathcal{I}(g_0^\varepsilon) - \mathcal{I}(f_i), \qquad (30)
$$

where

$$
\mathcal{I}(g_0^\varepsilon) = \varepsilon \int_{\mathbb{R}} \int_{\mathbb{R}} g^\varepsilon(0,x,p) \int_x^{X^\varepsilon(t;0,x,p)} \varphi(u)\,du\,dp\,dx, \;\; \mathcal{I}(f_i) = \varepsilon \int_{\mathbb{R}} \int_{\mathbb{R}} f_i(p) \int_x^{X(t;0,x,p)} \varphi(u)\,du\,dp\,dx.
$$

In order to continue our computations we need to estimate $(X^\varepsilon - X, P^\varepsilon - P)$. Observe that in both R and NR cases we have

$$
\frac{d}{ds}|X^\varepsilon - X| \le \frac{1}{\varepsilon}|P^\varepsilon - P|, \;\; \frac{1}{\varepsilon}\frac{d}{ds}|P^\varepsilon - P| \le \frac{1}{\varepsilon}\|F^\varepsilon(s) - E(s)\|_{L^\infty(\mathbb{R})},
$$

and we find easily by Gronwall lemma that for any $(t,x,p) \in \mathbb{R}^+ \times \mathbb{R}^2$

$$
\left(|X^\varepsilon - X| + \frac{1}{\varepsilon}|P^\varepsilon - P|\right)(t;0,x,p) \le \frac{1}{\varepsilon}\int_0^t \|F^\varepsilon(s) - E(s)\|_{L^\infty(\mathbb{R})}\,ds\,e^t. \qquad (31)
$$

Since $g^\varepsilon(0,\cdot,\cdot) = f_0$ we can write

$$
|\mathcal{I}(g_0^\varepsilon) - \mathcal{I}(f_i)| \le |I_1| + |I_2| \qquad (32)
$$

where

$$
I_1 = \varepsilon \int_{\mathbb{R}} \int_{\mathbb{R}} (f_0(x,p) - f_i(p)) \int_x^{X(t;0,x,p)} \varphi(u)\,du\,dp\,dx, \;\; I_2 = \varepsilon \int_{\mathbb{R}} \int_{\mathbb{R}} f_0(x,p) \int_{X(t;0,x,p)}^{X^\varepsilon(t;0,x,p)} \varphi(u)\,du\,dp\,dx.
$$

The first integral $I_1$ can be written

$$\begin{aligned} I_1 &= \varepsilon \int_{\mathbb{R}}\int_{\mathbb{R}} (f_0(x,p) - f_i(p))\varphi(u)(\mathbf{1}_{\{x<u<X(t;0,x,p)\}} - \mathbf{1}_{\{x>u>X(t;0,x,p)\}})\;du\;dp\;dx \\ &= \int_{\mathbb{R}} \varphi(u)\{h_1^\varepsilon(u) - h_2^\varepsilon(u)\}\;du, \end{aligned} \tag{33}$$

where

$$h_1^\varepsilon(u) = \varepsilon \int_{\mathbb{R}}\int_{\mathbb{R}} \{f_0(x,p) - f_i(p)\}\mathbf{1}_{\{u-\frac{1}{\varepsilon}Y(t;0,p)<x<u\}}\;dp\;dx,$$

and

$$h_2^\varepsilon(u) = \varepsilon \int_{\mathbb{R}}\int_{\mathbb{R}} \{f_0(x,p) - f_i(p)\}\mathbf{1}_{\{u<x<u-\frac{1}{\varepsilon}Y(t;0,p)\}}\;dp\;dx.$$

Since $f_0(\cdot,p)$ is 1-periodic and $f_i(p)$ is its average we deduce easily that for any $p \in \mathbb{R}$ we have

$$-2f_i(p) \le \int_{\mathbb{R}} \{f_0(x,p) - f_i(p)\}\mathbf{1}_{\{u-\frac{1}{\varepsilon}Y(t;0,p)<x<u\}}\;dx \le 2f_i(p),$$

and similarly

$$-2f_i(p) \le \int_{\mathbb{R}} \{f_0(x,p) - f_i(p)\}\mathbf{1}_{\{u<x<u-\frac{1}{\varepsilon}Y(t;0,p)\}}\;dx \le 2f_i(p).$$

After integration with respect to $p \in \mathbb{R}$ one gets

$$\max\{\|h_1^\varepsilon\|_{L^\infty(\mathbb{R})}, \|h_2^\varepsilon\|_{L^\infty(\mathbb{R})}\} \le 2\varepsilon \int_0^1\int_{\mathbb{R}} f_0(x,p)\;dp\;dx \le 2\varepsilon\|g_0\|_{L^1(\mathbb{R})},$$

and therefore we deduce by (33) that

$$|I_1| \le 4\varepsilon\|g_0\|_{L^1(\mathbb{R})}\|\varphi\|_{L^1(\mathbb{R})}. \tag{34}$$

The estimate of the second integral $I_2$ follows by using (31)

$$\begin{aligned} |I_2| &\le \varepsilon \int_{\mathbb{R}} |\varphi(u)| \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)\mathbf{1}_{\{|x+\frac{1}{\varepsilon}Y(t;0,p)-u|\le\frac{e^t}{\varepsilon}\int_0^t\|F^\varepsilon(s)-E(s)\|_{L^\infty(\mathbb{R})}\;ds\}}\;dp\;dx\;du \\ &\le \varepsilon \int_{\mathbb{R}} |\varphi(u)| \int_{\mathbb{R}} g_0(p) \int_{\mathbb{R}} \mathbf{1}_{\{|x+\frac{1}{\varepsilon}Y(t;0,p)-u|\le\frac{e^t}{\varepsilon}\int_0^t\|F^\varepsilon(s)-E(s)\|_{L^\infty(\mathbb{R})}\;ds\}}\;dx\;dp\;du \\ &\le 2\|\varphi\|_{L^1(\mathbb{R})}\|g_0\|_{L^1(\mathbb{R})}\;e^t\int_0^t \|F^\varepsilon(s) - E(s)\|_{L^\infty(\mathbb{R})}\;ds. \end{aligned} \tag{35}$$

Combining (30), (32), (34), (35) yields

$$\|F^\varepsilon(t) - E(t)\|_{L^\infty(\mathbb{R})} \le \varepsilon(\|E_0\|_{L^\infty(\mathbb{R})} + 4\|g_0\|_{L^1(\mathbb{R})}) + 2e^t\|g_0\|_{L^1(\mathbb{R})}\int_0^t \|F^\varepsilon(s) - E(s)\|_{L^\infty(\mathbb{R})}\;ds,$$

which implies by Gronwall lemma

$$\|E^\varepsilon(t) - E(t)\|_{L^\infty(\mathbb{R})} \le \varepsilon(\|E_0\|_{L^\infty(\mathbb{R})} + 4\|g_0\|_{L^1(\mathbb{R})}) \exp\left(2te^t\|g_0\|_{L^1(\mathbb{R})}\right), \ \ t \in \mathbb{R}^+. \tag{36}$$

As in the proof of Theorem 2.1 we have for any $t \in [0,T]$, $\varepsilon \in ]0,1]$ the estimates

$$\max\{\|E(t)\|_{L^\infty(\mathbb{R})}, \|E^\varepsilon(t)\|_{L^\infty(\mathbb{R})}\} \le \|E_0\|_{L^\infty(\mathbb{R})} + |K| + 2T\ \|g_0\|_{L^1(\mathbb{R})} = a_R^K(T),$$

in the R case and

$$\begin{aligned}\max\{\|E(t)\|_{L^\infty(\mathbb{R})}, \|E^\varepsilon(t)\|_{L^\infty(\mathbb{R})}\} &\le (\|E_0\|_{L^\infty(\mathbb{R})} + |K| + 2T\ \|\ |p|g_0\|_{L^1(\mathbb{R})}) \\ &\times \ \exp(2T^2\|g_0\|_{L^1(\mathbb{R})}) = a_{NR}^K(T),\end{aligned}$$

in the NR case. Therefore we deduce that $f^\varepsilon(t,x,p) \le g_0^{R(T)}(p)$, $f(t,p) \le g_0^{R(T)}(p)$ for any $(t,x,p) \in [0,T] \times \mathbb{R}^2$, $\varepsilon \in ]0,1]$ where $R(T) = Ta_R^K(T)$ in the R case and $R(T) = Ta_{NR}^K(T)$ in the NR case. Take $\varphi \in L^1(\mathbb{R}^2; g_0^{R(T)}\ dp\ dx)$ and $\varphi_\eta \in C_c^1(\mathbb{R}^2)$ such that $\int_{\mathbb{R}}\int_{\mathbb{R}} |\varphi - \varphi_\eta| g_0^{R(T)}(p)\ dp\ dx < \eta$. Observe that

$$\left|\int_{\mathbb{R}}\int_{\mathbb{R}} (f^\varepsilon(t,x,p) - f(t,p))\varphi(x)\ dp\ dx\right| \le \left|\int_{\mathbb{R}}\int_{\mathbb{R}} (f^\varepsilon(t,x,p) - f(t,p))\varphi_\eta(x)\ dp\ dx\right| + 2\eta,$$

and therefore it is sufficient to consider only test functions in $C_c^1(\mathbb{R}^2)$. For such a test function $\varphi$ we can write

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} (f^\varepsilon(t,x,p) - f(t,p))\varphi\ dp\ dx &= \varepsilon \int_{\mathbb{R}}\int_{\mathbb{R}} (g^\varepsilon(t,x,p) - f(t,p))\varphi(\varepsilon x, p)\ dp\ dx \\ &= \varepsilon \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)\varphi(\varepsilon X^\varepsilon(t;0,x,p), P^\varepsilon(t;0,x,p))\ dp\ dx \\ &- \varepsilon \int_{\mathbb{R}}\int_{\mathbb{R}} f_i(p)\varphi(\varepsilon X(t;0,x,p), P(t;0,x,p))\ dp\ dx \\ &= I_3^\varepsilon + I_4^\varepsilon,\end{aligned}$$

where

$$I_3^\varepsilon = \varepsilon \int_{\mathbb{R}}\int_{\mathbb{R}} f_0\{\varphi(\varepsilon X^\varepsilon(t;0,x,p), P^\varepsilon(t;0,x,p)) - \varphi(\varepsilon X(t;0,x,p), P(t;0,x,p))\}\ dp\ dx,$$

and

$$I_4^\varepsilon = \varepsilon \int_{\mathbb{R}}\int_{\mathbb{R}} \{f_0(x,p) - f_i(p)\}\varphi(\varepsilon X(t;0,x,p), P(t;0,x,p))\ dp\ dx.$$

By (31), (36) we have for some constant $C$ depending on $T$, $E_0$, $g_0$

$$\left(\varepsilon|X^\varepsilon - X| + |P^\varepsilon - P|\right)(t;0,x,p) \le C\varepsilon, \ \ (t,x,p) \in [0,T] \times \mathbb{R}^2.$$

We need to estimate the support sizes of the functions $\varphi(\varepsilon X^\varepsilon(t;0,\cdot,\cdot), P^\varepsilon(t;0,\cdot,\cdot))$, $\varphi(\varepsilon X(t;0,\cdot,\cdot), P(t;0,\cdot,\cdot))$ with respect to the space variable. Assume that $\varphi(x,p) = 0$ for any $|x| > A, p \in \mathbb{R}$. In the R case we have for any $|x| > \frac{A+T}{\varepsilon}$, $p \in \mathbb{R}$

$$\varepsilon \min(|X^\varepsilon|, |X|)(t;0,x,p) \geq \varepsilon \left( |x| - \frac{T}{\varepsilon} \right) > A,$$

and thus $\varphi(\varepsilon X^\varepsilon(t;0,x,p), P^\varepsilon(t;0,x,p)) = \varphi(\varepsilon X(t;0,x,p), P(t;0,x,p)) = 0$. In the NR case we have for any $|x| > \frac{A+T(R(T)+|p|)}{\varepsilon}$, $p \in \mathbb{R}$

$$\varepsilon \min(|X^\varepsilon|, |X|)(t;0,x,p) \geq \varepsilon \left( |x| - \frac{T}{\varepsilon}(|p| + R(T)) \right) > A,$$

and thus $\varphi(\varepsilon X^\varepsilon(t;0,x,p), P^\varepsilon(t;0,x,p)) = \varphi(\varepsilon X(t;0,x,p), P(t;0,x,p)) = 0$. In the R case we deduce that

$$\begin{aligned}
|I_3^\varepsilon| &\leq \varepsilon \mathrm{Lip}(\varphi) \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)(\varepsilon|X^\varepsilon - X| + |P^\varepsilon - P|)(t;0,x,p)\mathbf{1}_{\{\varepsilon|x| \leq A+T\}}\, dp\, dx \\
&\leq C\varepsilon^2 \mathrm{Lip}(\varphi) \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)\mathbf{1}_{\{\varepsilon|x| \leq A+T\}}\, dp\, dx \\
&\leq 2C(A+T)\mathrm{Lip}(\varphi)\|g_0\|_{L^1(\mathbb{R})}\, \varepsilon.
\end{aligned}$$

In the NR case we have

$$\begin{aligned}
|I_4^\varepsilon| &\leq C\varepsilon^2 \mathrm{Lip}(\varphi) \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)\mathbf{1}_{\{\varepsilon|x| \leq A+T(R(T)+|p|)\}}\, dp\, dx \\
&\leq 2C\varepsilon \mathrm{Lip}(\varphi) \int_{\mathbb{R}} g_0(p)\{A + T(R(T)+|p|)\}\, dp \\
&\leq 2C\{(A+TR(T))\|g_0\|_{L^1(\mathbb{R})} + T\, \||p|g_0\|_{L^1(\mathbb{R})}\}\mathrm{Lip}(\varphi)\, \varepsilon.
\end{aligned}$$

In both cases we obtain that $\lim_{\varepsilon \searrow 0} I_3^\varepsilon = 0$. Let us analyze the term $I_4^\varepsilon$. Using (27), (28) one gets

$$\begin{aligned}
I_4^\varepsilon &= \varepsilon \int_{\mathbb{R}}\int_{\mathbb{R}} (f_0(x,p) - f_i(p))\varphi(\varepsilon x + Y(t;0,p), P(t;0,p))\, dp\, dx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \left\{ f_0\left(\frac{x}{\varepsilon}, p\right) - f_i(p) \right\} \varphi(x + Y(t;0,p), P(t;0,p))\, dp\, dx.
\end{aligned}$$

Since for any $p \in \mathbb{R}$ we have the convergence $f_0\left(\frac{\cdot}{\varepsilon}, p\right) \rightharpoonup \int_0^1 f_0(x,p)\, dx = f_i(p)$ weakly $\star$ in $L^\infty(\mathbb{R})$ we deduce

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \left\{ f_0\left(\frac{x}{\varepsilon}, p\right) - f_i(p) \right\} \varphi(x + Y(t;0,p), P(t;0,p))\, dx = 0, \;\; p \in \mathbb{R}.$$

Observing that $\varphi(x+Y(t;0,p),P(t;0,p)) = \varphi(\varepsilon X(t;0,\frac{x}{\varepsilon},p),P(t;0,\frac{x}{\varepsilon},p)) = 0$ if $|x| > A+T$ in the R case and if $|x| > A + T(|p| + R(T))$ in the NR case we deduce that

$$\left| \int_{\mathbb{R}} \left\{ f_0\left(\frac{x}{\varepsilon},p\right) - f_i(p) \right\} \varphi(x+Y(t;0,p),P(t;0,p))\,dx \right| \leq 4g_0(p)(A+T\,S(T,p))\|\varphi\|_{L^\infty},$$

where $S(T,p) = 1$ in the R case and $S(T,p) = |p|+R(T)$ in the NR case. Using the Lebesgue convergence theorem yields $\lim_{\varepsilon\searrow 0} I_4^\varepsilon = 0$ and thus

$$\lim_{\varepsilon\searrow 0} \int_{\mathbb{R}}\int_{\mathbb{R}} (f^\varepsilon(t,x,p) - f(t,p))\varphi(x,p)\,dp\,dx = 0, \text{ uniformly in } t \in [0,T].$$

□

**Corollary 3.1** *Under the hypotheses of Theorem 1.1 we have the convergences*

$$\lim_{\varepsilon\searrow 0} \int_{\mathbb{R}} \rho^\varepsilon(t,x)\varphi(x)\,dx = n\int_{\mathbb{R}}\varphi(x)\,dx, \text{ uniformly with respect to } t \in [0,T],$$

$$\lim_{\varepsilon\searrow 0} \int_{\mathbb{R}} j^\varepsilon(t,x)\varphi(x)\,dx = j(t)\int_{\mathbb{R}}\varphi(x)\,dx, \text{ uniformly with respect to } t \in [0,T],$$

*for any function* $\varphi \in L^1(\mathbb{R})$.

# 4 Numerical simulations

This section is devoted to the numerical validation of the asymptotic behavior for small $\varepsilon > 0$ predicted in Theorem 1.1. The numerical approximation of the 1D Vlasov-Maxwell problem (7), (8), (9) is performed by using a particle-in-cell method. Let us present briefly this numerical scheme. Assume for the moment that the initial density has finite mass $f_0 \in L^1(\mathbb{R}^2)$. For any $(n,i,j) \in \mathbb{N}\times\mathbb{Z}^2$ let $t^n = n\Delta t, x_i = i\Delta x, p_j = j\Delta p$ with $\Delta t, \Delta x, \Delta p > 0$ and denote by $f_{ij}^0$ the mean density over the cell around $(x_i,p_j)$

$$f_{ij}^0 = \frac{1}{\Delta x\Delta p}\int_{|x-x_i|<\frac{\Delta x}{2}}\int_{|p-p_j|\leq\frac{\Delta p}{2}} f_0(x,p)\,dp\,dx.$$

Consider $\psi$ a shape function (*i.e.*, $\psi \geq 0$, compactly supported, $\int_{\mathbb{R}}\psi(u)\,du = 1$). If $(f,E)$ is the exact solution of the 1D Vlasov-Maxwell system with the initial conditions $(f_0,E_0)$ we have (see (14)) for any $n \in \mathbb{N}$ and $\varphi \in L^1(\mathbb{R})$

$$\int_{\mathbb{R}} (E(t^n,x) - E_0(x))\varphi(x)\,dx = \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)\int_x^{X(t^n;0,x,p)}\varphi(u)\,du\,dp\,dx,$$

where $(X,P)$ are the characteristics associated to $E$. Taking now $\varphi_y(\cdot) = \frac{1}{\Delta x}\psi\left(\frac{\cdot - y}{\Delta x}\right)$ we obtain the formula

$$\begin{aligned} E(t^n,y) - E_0(y) &\approx \int_{\mathbb{R}} (E(t^n,x) - E_0(x)) \frac{1}{\Delta x}\psi\left(\frac{x-y}{\Delta x}\right)\,dx \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)\int_x^{X(t^n;0,x,p)} \frac{1}{\Delta x}\psi\left(\frac{u-y}{\Delta x}\right)\,du\,dp\,dx \\ &\approx \sum_{(i,j)\in\mathbb{Z}^2} \Delta p f^0_{ij}\int_{x_i}^{X(t^n;0,x_i,p_j)} \psi\left(\frac{u-y}{\Delta x}\right)\,du. \end{aligned}$$

Motivated by the above approximations we define our numerical scheme as follows
1) let $(X^0_{ij}, P^0_{ij}) = (x_i, p_j),\ \ (i,j)\in\mathbb{Z}^2$;
2) for any $n\in\mathbb{N}$ compute $(X^{n+1}_{ij}, P^{n+1}_{ij})_{(i,j)\in\mathbb{Z}^2}$ according to

$$X^{n+1}_{ij} = X^n_{ij} + \Delta t\ v(P^n_{ij}),\ \ P^{n+1}_{ij} = P^n_{ij} - \Delta t\ E^n(X^n_{ij}),$$

where

$$E^n(\cdot) = E_0(\cdot) + \sum_{(i,j)\in\mathbb{Z}^2} \Delta p f^0_{ij}\int_{x_i}^{X^n_{ij}} \psi\left(\frac{u-\cdot}{\Delta x}\right)\,du.$$

For the practical implementation of this scheme we restrict the summation in the expression of $E^n$ over a finite subset $D = \{(i,j)\in\mathbb{Z}^2\ :\ |i|\le I, |j|\le J\}$ by neglecting $f^0_{ij}$ for $(i,j)\notin D$. Indeed, observe that the contribution of the pairs $(i,j)\in\mathbb{Z}^2\cap\complement D$ is arbitrarily small if $I$ and $J$ are large enough since

$$\begin{aligned} \sum_{(i,j)\in\mathbb{Z}^2\cap\complement D} \Delta p f^0_{ij}\ \left|\int_{x_i}^{X^n_{ij}} \psi\left(\frac{u-\cdot}{\Delta x}\right)\,du\right| &\le \sum_{(i,j)\in\mathbb{Z}^2\cap\complement D} \Delta x\Delta p\ f^0_{ij} \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f_0(x,p)\mathbf{1}_{\{|x|>(I+1/2)\Delta x \text{ or } |p|>(J+1/2)\Delta p\}}\,dp\,dx. \end{aligned}$$

The above scheme adapts easily in the framework of 1-periodic solutions. Assume that the shape function $\psi$ has the support in $[-1/2,1/2]$. For any $\Delta x < 1$ let us denote by $\psi_{\Delta x}$ the function given by $\psi_{\Delta x}(\cdot) = \frac{1}{\Delta x}\psi\left(\frac{\cdot}{\Delta x}\right)$ and consider $\tilde{\psi}_{\Delta x}$ its periodic extension (*i.e.*, $\tilde{\psi}_{\Delta x}$ is 1-periodic and coincides with $\psi_{\Delta x}$ on $[-1/2,1/2]$). For any 1-periodic test function $\varphi\in L^1_{\text{loc}}(\mathbb{R})$ we have

$$\int_0^1 (E(t^n,x) - E_0(x))\varphi(x)\,dx = \int_0^1\int_{\mathbb{R}} f_0(x,p)\int_x^{X(t^n;0,x,p)} \varphi(u)\,du\,dp\,dx.$$

Taking now $\Delta x$ and $I$ such that $I\Delta x = 1$ and replacing $\varphi$ by $\tilde{\psi}_{\Delta x}(\cdot - y)$ yields

$$E(t^n,y) - E_0(y) \approx \sum_{i=0}^{I-1}\sum_{j\in\mathbb{Z}} \Delta p f^0_{ij}\int_{x_i}^{X(t^n;0,x_i,p_j)} \tilde{\psi}_{\Delta x}(u-y)\,du,\ \forall\ y\in\mathbb{R},$$

and the numerical scheme for the periodic case follows in obvious way. A complete analysis of this scheme can be found in [7].

We perform the numerical computations in the NR case since in this case the solution $(f, E)$ of (10), (11), (12) can be computed explicitly. Indeed, multiplying (10) by $p$ and integrating with respect to $p \in \mathbb{R}$ yields

$$\frac{dj}{dt} + E(t)\rho(t) = 0, \quad t \in \mathbb{R}^+, \tag{37}$$

where $\rho(t) = \int_{\mathbb{R}} f(t,p)\,dp$. Observe also, by integrating (10) with respect to $p$ that $\rho\,'(t) = 0$ and thus

$$\rho(t) = \rho(0) = \int_{\mathbb{R}} f_i(p)\,dp = \int_0^1 \int_{\mathbb{R}} f_0(x,p)\,dp\,dx = n.$$

Clearly we obtain from (11), (37)

$$j(t) = \int_0^1 \int_{\mathbb{R}} p f_0(x,p)\,dp\,dx \cos(\sqrt{n}t) - K\sqrt{n}\sin(\sqrt{n}t), \tag{38}$$

$$E(t) = K\cos(\sqrt{n}t) + \frac{\int_0^1 \int_{\mathbb{R}} p f_0(x,p)\,dp\,dx}{\sqrt{n}} \sin(\sqrt{n}t). \tag{39}$$

We check immediately that $f(t,p) = f_i\left(p - \frac{j(t)-j(0)}{n}\right)$ together with $E$ given above solve the Vlasov-Maxwell problem (10), (11), (12). We recognize here the oscillations of a spatial homogeneous plasma with frequency proportional to $\sqrt{n}$. We fix the initial conditions

$$f_0(x,p) = \left(1 + \frac{1}{2}\cos(2\pi x)\right)\frac{n}{\sqrt{2\pi\theta}}e^{-\frac{p^2}{2\theta}}, \ (x,p) \in \mathbb{R}^2, \ E_0(x) = -\frac{n}{4\pi}\sin(2\pi x), \ x \in \mathbb{R}.$$

For any $\varepsilon > 0$ we consider the solution $(f^\varepsilon, E^\varepsilon)$ for the NR 1D Vlasov-Maxwell equations with the initial conditions

$$f_0^\varepsilon(x,p) = \left(1 + \frac{1}{2}\cos\left(2\pi\frac{x}{\varepsilon}\right)\right)\frac{n}{\sqrt{2\pi\theta}}e^{-\frac{p^2}{2\theta}}, \quad E_0^\varepsilon(x) = -\frac{\varepsilon n}{4\pi}\sin\left(2\pi\frac{x}{\varepsilon}\right) + \sqrt{n\theta}.$$

The limit solution in this case is given by (38), (39) with $K = \sqrt{n\theta}$

$$f(t,p) = \frac{n}{\sqrt{2\pi\theta}}\exp\left(-\frac{1}{2\theta}\left(p + \sqrt{\theta}\sin(\sqrt{n}t)\right)^2\right),$$

$$j(t) = -n\sqrt{\theta}\sin(\sqrt{n}t), \quad E(t) = \sqrt{n\theta}\cos(\sqrt{n}t).$$

The following figures illustrate the behavior of the numerical approximations of $(f^\varepsilon, E^\varepsilon)$ with small $\varepsilon > 0$ comparing to the analytical space homogeneous solution $(f, E)$. By Theorem 1.1 and Corollary 3.1 we know that

$$\lim_{\varepsilon \searrow 0} E^\varepsilon(t,x) = E(t), \ \text{ uniformly with respect to } \ (t,x) \in [0,T] \times \mathbb{R},$$

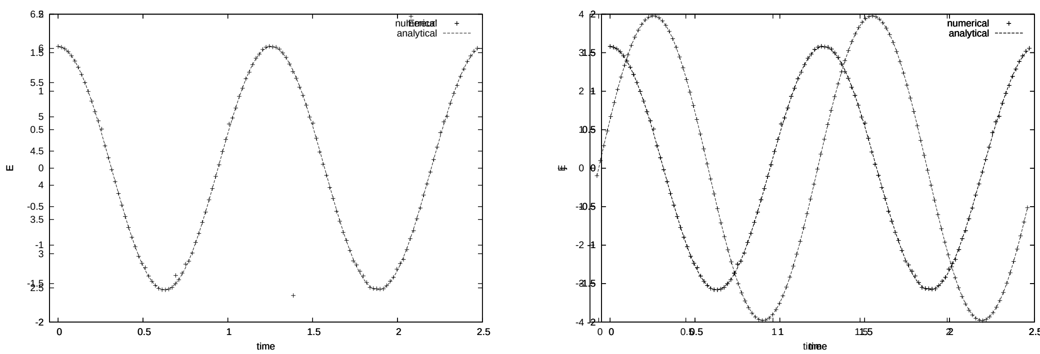



Figure 1: Time evolution of the electric field and the total current

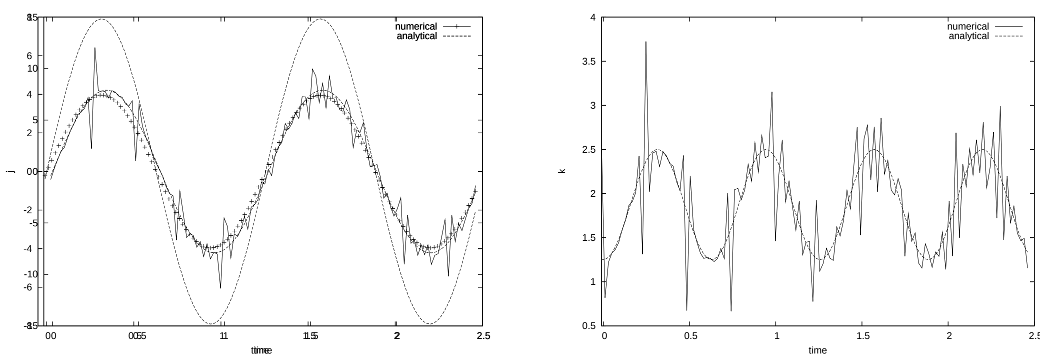
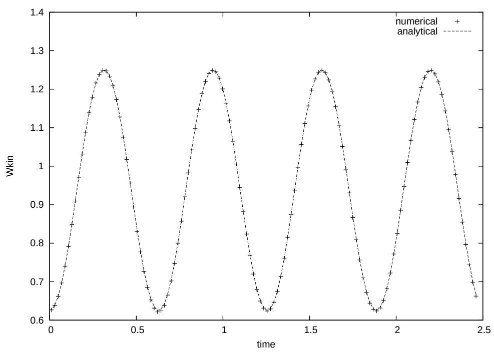



Figure 2: Time evolution of the electric current and the kinetic energy

and

$$\lim_{\varepsilon \searrow 0} \int_0^1 j^\varepsilon(t,x)\, dx = j(t), \text{ uniformly with respect to } t \in [0,T].$$

The previous convergences are emphasized in the Figure 1, the values of the parameter $n, \theta, \varepsilon$ for this simulation being $n = 25, \theta = 0.1, \varepsilon = 1/40$. The Figure 2 describes the time evolution of the electric current $j^\varepsilon(\cdot, x_0) = \int_{\mathbb{R}} p f^\varepsilon(\cdot, x_0, p)\, dp$ and the kinetic energy $k^\varepsilon(\cdot, x_0) = \int_{\mathbb{R}} \frac{p^2}{2} f^\varepsilon(\cdot, x_0, p)\, dp$ at some fixed space point $x_0$. We recognize here the weak convergences towards $j(\cdot) = \int_{\mathbb{R}} p f(\cdot, p)\, dp$ and $k(\cdot) = \int_{\mathbb{R}} \frac{p^2}{2} f(\cdot, p)\, dp$. The behaviors of the total kinetic energy $W^\varepsilon_{\text{kin}}(\cdot) = \int_0^1 \int_{\mathbb{R}} \frac{p^2}{2} f^\varepsilon(\cdot, x, p)\, dp\, dx$ and the total potential energy $W^\varepsilon_{\text{pot}}(\cdot) = \frac{1}{2} \int_0^1 E^\varepsilon(\cdot, x)^2\, dx$ are presented in Figure 3.



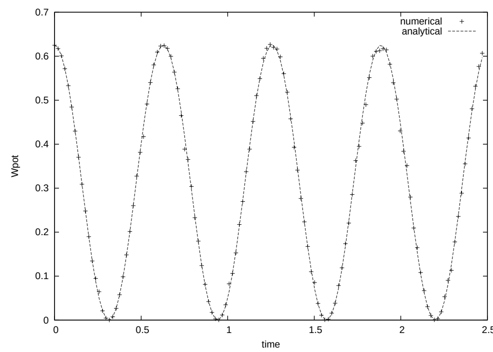




Figure 3: Time evolution of the total kinetic and potential energies

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