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# An example on approximation by fat arcs and fat biarcs

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## Abstract

Fat arcs form bounding boxes for planar curves. An example on approximation by fat arcs, provided by Qun and Rokne, is corrected. The third derivative of the given curve segment under the polar coordinate system is increasing only at the beginning of the curve segment, and then decreasing, rather than monotonically increasing for the whole curve segment. Some numerical data are corrected as well.

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Circular arcs play a very important role in CAD (Computer-Aided Design) and CNC (Computerized Numerical Control) [1,4,5 et al.]. Currently, fat arcs are used to define bounding boxes for planar curves [2,3]. In [2], an example is given to illustrate the approximation algorithm by fat arcs. Here, the example is corrected.

In the example, the given curve is

$$t(\alpha) = e^{0.2(2\pi+\alpha)}$$

under a polar coordinate system  $(t, \alpha)$ . Under the Cartesian coordinate system with the same center point as the polar coordinate system and with the  $x$ -axis being the polar axis,  $t(\alpha)$  becomes

$$\mathbf{S}(\alpha) = (\mathbf{S}_x(\alpha), \mathbf{S}_y(\alpha)) = (e^{0.2(2\pi+\alpha)} \cos(\alpha), e^{0.2(2\pi+\alpha)} \sin(\alpha)),$$

where the notations  $\mathbf{S}_x(\alpha)$  and  $\mathbf{S}_y(\alpha)$  follow the rule, in [2], that uses the symbols ' $\mathbf{P}_x$ ' and ' $\mathbf{P}_y$ ' to represent the  $x$  and  $y$  components, respectively, of an arbitrary point (or curve)  $\mathbf{P}$ . The rule is followed below as well. Let  $\alpha_{-1} = -\pi/8$ ,  $\alpha_0 = 0$  and  $\alpha_1 = \pi/8$ . We obtain three points

$$\mathbf{P}_{-1} = \mathbf{S}(\alpha_{-1}) = (3.001, -1.243),$$

$$\mathbf{P}_0 = \mathbf{S}(\alpha_0) = (3.514, 0), \quad \mathbf{P}_1 = \mathbf{S}(\alpha_1) = (3.511, 1.454),$$

and three distances

$$l_0 = \|\mathbf{P}_{-1} - \mathbf{P}_0\|_2 = 1.345, \quad l_1 = \|\mathbf{P}_1 - \mathbf{P}_0\|_2 = 1.454,$$

$$l = \|\mathbf{P}_{-1} - \mathbf{P}_1\|_2 = 2.745.$$

Here, the numerical values of  $\mathbf{P}_1$  and  $l$  are different from those in [2], which gives:

$$\mathbf{P}_1 = (5.511, 1.454) \quad \text{and} \quad l = 2.645.$$

The unsigned angle between the  $x$ -axis and the line segment  $\mathbf{P}_{-1}\mathbf{P}_0$  is

$$\beta_{-1} = \text{tg}^{-1} \left( \left| \frac{(\mathbf{P}_0)_y - (\mathbf{P}_{-1})_y}{(\mathbf{P}_0)_x - (\mathbf{P}_{-1})_x} \right| \right) = 1.18,$$

and the unsigned angle between the  $x$ -axis and the line segment  $\mathbf{P}_1\mathbf{P}_0$  is

$$\beta_0 = \text{tg}^{-1} \left( \left| \frac{(\mathbf{P}_0)_y - (\mathbf{P}_1)_y}{(\mathbf{P}_0)_x - (\mathbf{P}_1)_x} \right| \right) = 1.57,$$

where  $\text{tg}^{-1}(\cdot)$  represents the inverse function of  $\text{tg}(\cdot) = \sin(\cdot)/\cos(\cdot)$ . Thus, we obtain:

$$\varphi_0 = \pi - \beta_{-1} - \beta_0 = 0.393.$$

Ref. [2] provides the wrong formulae for  $\beta_{-1}$ ,  $\beta_0$  and  $\varphi_0$ .

The points  $\mathbf{P}_{-1}$ ,  $\mathbf{P}_0$  and  $\mathbf{P}_1$  determine a unique circle with the center point

$$\mathbf{O} = (-8.604 \times 10^{-15}, 0.722)$$

and the radius  $r_0 = \|\mathbf{P}_{-1} - \mathbf{O}\|_2 = 3.587$  (see the dashed circle in Fig. 1). Fig. 1(b) gives a magnified view of Fig. 1(a). In Fig. 1, the solid curve from  $\mathbf{P}_{-1}$  to  $\mathbf{P}_1$  is the given curve  $t(\alpha)$  or  $\mathbf{S}(\alpha)$  under the Cartesian coordinate system. It is very close to the dashed circle as shown in Fig. 1.

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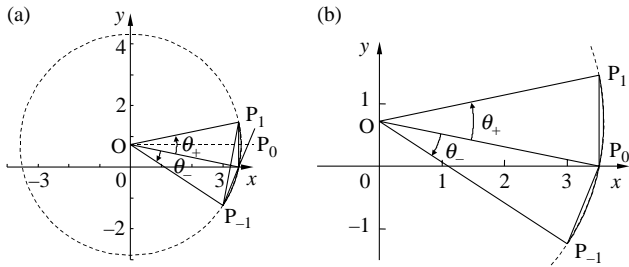


Fig. 1. The given curve  $t(\alpha)$  (solid) and the circle (dashed).

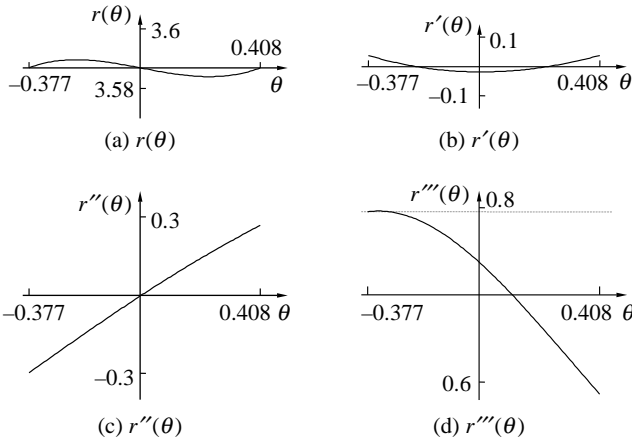


Fig. 2.  $r(\theta)$  and its first three derivatives.

Let

$$\theta_{-1} = \text{tg}^{-1} \left( \left| \frac{(\mathbf{P}_{-1})_y - \mathbf{O}_y}{(\mathbf{P}_{-1})_x - \mathbf{O}_x} \right| \right) = 0.580,$$

$$\theta_0 = \text{tg}^{-1} \left( \left| \frac{(\mathbf{P}_0)_y - \mathbf{O}_y}{(\mathbf{P}_0)_x - \mathbf{O}_x} \right| \right) = 0.203$$

and

$$\theta_1 = \text{tg}^{-1} \left( \left| \frac{(\mathbf{P}_1)_y - \mathbf{O}_y}{(\mathbf{P}_1)_x - \mathbf{O}_x} \right| \right) = 0.206$$

be the unsigned angles between the  $x$ -axis and the line segments  $\mathbf{OP}_{-1}$ ,  $\mathbf{OP}_0$  and  $\mathbf{OP}_1$ , respectively.

Table 1  
Comparison of the new results with the results in [1]

Lines	Results in [2]	New results
4	$\mathbf{P}_1 = (5.511, 1.454)$	$\mathbf{P}_1 = (3.511, 1.454)$
6	$l = 2.645$	$l = 2.745$
7	$\beta_{-1} = \frac{(\mathbf{P}_{-1})_y}{(\mathbf{P}_0)_x - (\mathbf{P}_{-1})_x} = 1.57$	$\beta_{-1} = \text{tg}^{-1} \left( \left  \frac{(\mathbf{P}_0)_y - (\mathbf{P}_{-1})_y}{(\mathbf{P}_0)_x - (\mathbf{P}_{-1})_x} \right  \right) = 1.18$
8	$\beta_0 = \frac{(\mathbf{P}_1)_y}{(\mathbf{P}_0)_x - (\mathbf{P}_1)_x} = 1.18$	$\beta_0 = \text{tg}^{-1} \left( \left  \frac{(\mathbf{P}_0)_y - (\mathbf{P}_1)_y}{(\mathbf{P}_0)_x - (\mathbf{P}_1)_x} \right  \right) = 1.57,$
9	$\varphi_0 = \pi - \beta_0 - \beta_1 = 0.393$	$\varphi_0 = \pi - \beta_0 - \beta_{-1} = 0.393$
14	$R'''(\Theta) = [\min_{-\theta_- \leq \theta \leq \theta_+} r(\theta), \max_{-\theta_- \leq \theta \leq \theta_+} r(\theta)]$	$R'''(\Theta) = [\min_{-\theta_- \leq \theta \leq \theta_+} r'''(\theta), \max_{-\theta_- \leq \theta \leq \theta_+} r'''(\theta)]$
15	$r'''$ is monotonically increasing	$r'''(\theta)$ is increasing at the beginning, and then decreasing for the rest part of the curve
16	$R'''(\Theta) \subseteq [0.00434, 0.00509]$	$R'''(\Theta) = [0.586, 0.796]$
18	$(\Theta + \theta_-)\Theta(\Theta - \theta_+) = [0.251, 0.233]$	$(\Theta + \theta_-)\Theta(\Theta - \theta_+) = [-0.252, 0.233]$
20	$[r_{0L}, r_{0R}] = [3.585, 3.589]$	$[r_{0L}, r_{0R}] = [3.554, 3.618]$
22–23	A fat curve of width $3.589 - 3.585 = 0.004$	A fat curve of width $3.618 - 3.554 = 0.064$

We build a new polar coordinate system with the center point being  $\mathbf{O}$  and the pole axis being the ray starting from  $\mathbf{O}$  and passing through the point  $\mathbf{P}_0$ . Under the new polar coordinate system, the curve  $t(\alpha)$  becomes  $r(\theta)$ , where the polar radius is equal to the distance from  $t(\alpha)$  to the new center point  $\mathbf{O}$ , i.e.

$$r(\alpha) = \sqrt{(e^{0.2(2\pi+\alpha)} \cos(\alpha) - \mathbf{O}_x)^2 + (e^{0.2(2\pi+\alpha)} \sin(\alpha) - \mathbf{O}_y)^2},$$

and the polar angle is:

$$\theta(\alpha) = \text{tg}^{-1} \left( \frac{e^{0.2(2\pi+\alpha)} \sin(\alpha) - \mathbf{O}_y}{e^{0.2(2\pi+\alpha)} \cos(\alpha) - \mathbf{O}_x} \right) + \theta_0.$$

Let

$$\theta_- = \theta_{-1} - \theta_0 = 0.377 \quad \text{and} \quad \theta_+ = \theta_0 + \theta_1 = 0.408.$$

Then the parametric domain of the curve segment  $r(\theta)$  from  $\mathbf{P}_{-1}$  to  $\mathbf{P}_1$  is:

$$\Theta = [-\theta_-, \theta_+] = [-0.377, 0.408].$$

Thus, we have:

$$(\Theta + \theta_-)\Theta(\Theta - \theta_+) = [-0.252, 0.233].$$

For convenience, let  $r'(\theta)$ ,  $r''(\theta)$  and  $r'''(\theta)$  be the first, the second and the third derivatives of  $r(\theta)$  with respect to  $\theta$ , respectively. Similarly, let  $r'(\alpha)$ ,  $r''(\alpha)$  and  $r'''(\alpha)$  be the first, the second and the third derivatives of  $r(\alpha)$  with respect to  $\alpha$ , respectively, and let  $\theta'(\alpha)$ ,  $\theta''(\alpha)$  and  $\theta'''(\alpha)$  be the first, the second and the third derivatives of  $\theta(\alpha)$  with respect to  $\alpha$ , respectively. We have

$$r'(\theta) = \frac{r'(\alpha)}{\theta'(\alpha)},$$

$$r''(\theta) = \frac{r''(\alpha)}{(\theta'(\alpha))^2} - \frac{r'(\alpha)\theta''(\alpha)}{(\theta'(\alpha))^3},$$

and

$$r'''(\theta) = \frac{r'''(\alpha)}{(\theta'(\alpha))^3} - \frac{3r''(\alpha)\theta''(\alpha)}{(\theta'(\alpha))^4} - \frac{r'(\alpha)\theta'''(\alpha)}{(\theta'(\alpha))^4} + \frac{3r'(\alpha)(\theta''(\alpha))^2}{(\theta'(\alpha))^5},$$

which are shown in Fig. 2(b–d), respectively.

As shown in Fig. 2(d),  $r'''(\theta)$  is increasing only at the beginning, and then monotonically decreasing for the rest part of the curve, rather than monotonically increasing for the whole curve segment as declared in [2]. Hence, the range of  $r'''(\theta)$  from  $\mathbf{P}_{-1}$  to  $\mathbf{P}_1$  is:

$$R'''(\Theta) = [0.586, 0.796].$$

Therefore, the radius range of the fat arc is

$$\begin{aligned} [r_{0L}, r_{0R}] &= r_0 + \frac{1}{3!} R'''(\Theta)(\Theta + \theta_-)\Theta(\Theta - \theta_+) \\ &= [3.554, 3.618] \end{aligned}$$

according to Formula (11) in [2], and the width of the radius range is  $3.618 - 3.554 = 0.064$  rather than 0.004 given by [2]. The comparison of the above results with those in [2] is summarized in Table 1. In Table 1, the numbers in the ‘Lines’ column are the line numbers of the corresponding results in the left column of Page 971 of [2].

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