On the Axiomatisation of Boolean Categories with and without Medial

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Abstract. In its most general meaning, a Boolean category is to categories what a Boolean algebra is to posets. In a more specific meaning a Boolean category should provide the abstract algebraic structure underlying the proofs in Boolean Logic, in the same sense as a Cartesian closed category captures the proofs in intuitionistic logic and a *-autonomous category captures the proofs in linear logic. However, recent work has shown that there is no canonical axiomatisation of a Boolean category. In this work, we will see a series (with increasing strength) of possible such axiomatisations, all based on the notion of *-autonomous category. We will particularly focus on the medial map, which has its origin in an inference rule in KS, a cut-free deductive system for Boolean logic in the calculus of structures. Finally, we will present a category proof nets as a particularly well-behaved example of a Boolean category.

1 Introduction

The questions *"What is a proof?"* and *"When are two proofs the same?"* are fundamental for proof theory. But for the most prominent logic, Boolean (or classical) propositional logic, we still have no satisfactory answers.

This is not only embarrassing for proof theory itself, but also for computer science, where Boolean propositional logic plays a major role in automated reasoning and logic programming. Also the design and verification of hardware is based on Boolean logic. Every area in which proof search is employed can benefit from a better understanding of the concept of proof in Boolean logic, and the famous NP-versus-coNP problem can be reduced to the question whether there is a short (i.e., polynomial size) proof for every Boolean tautology [CR79].

Usually proofs are studied as syntactic objects within some deductive system (e.g., tableaux, sequent calculus, resolution, ...). This paper takes the point of view that these syntactic objects (also known as proof trees) should be considered as concrete representation of certain abstract proof objects, and that such an abstract proof object can be represented by a resolution proof tree and a sequent calculus proof tree, or even by several different sequent calculus proof trees.

Under this point of view the motivation for this work is to provide an abstract algebraic theory of proofs. Already Lambek [Lam68,Lam69] observed that such an algebraic treatment can be provided by category theory. For this, it is necessary to accept the following postulates about proofs:

- for every proof f of conclusion B from hypothesis A (denoted by $f\colon A \to B$) and every proof g of conclusion C from hypothesis B (denoted by $g\colon B \to C$) there is a uniquely defined composite proof $g \circ f$ of conclusion C from hypothesis A (denoted by $g \circ f\colon A \to C$),
- this composition of proofs is associative,
- for each formula A there is an identity proof $1_A\colon A \to A$ such that for $f\colon A \to B$ we have $f \circ 1_A = f = 1_B \circ f$.

Under these assumptions the proofs are the arrows in a category whose objects are the formulas of the logic. What remains is to provide the right axioms for the "category of proofs".

It seems that finding these axioms is particularly difficult for the case of Boolean logic. For intuitionistic logic, Prawitz [Pra71] proposed the notion of *proof normalization* for identifying proofs. It was soon discovered that this notion of identity coincides with the notion of identity that results from the axioms of a Cartesian closed category (see, e.g., [LS86]). In fact, one can say that the proofs of intuitionistic logic are the arrows in the free (bi-)cartesian closed category generated by the set of propositional variables. An alternative way of representing the arrows in that category is via terms in the simply-typed λ-calculus: arrow composition is normalization of terms. This observation is well-known as Curry-Howard-correspondence [How80].

In the case of linear logic, the relation to *-autonomous categories [Bar79] was noticed immediately after its discovery [Laf88,See89]. In the sequent calculus linear logic proofs are identified when they can be transformed into each other via "trivial" rule permutations [Laf95]. For multiplicative linear logic this coincides with the proof identifications induced by the axioms of a *-autonomous category [Blu93,SL04]. Therefore, we can safely say that a proof in multiplicative linear logic is an arrow in the free *-autonomous category generated by the propositional variables [BCST96,LS04,Hug05a].

But for classical logic no such well-accepted category of proofs exists. We can distinguish two main reasons. First, if we start from a Cartesian closed category and add an involutive negation[1], we get the collapse into a Boolean algebra, i.e., any two proofs $f, g\colon A \to B$ are identified. For every formula there would be at most one proof (see, e.g., [LS86] or the appendix of [Gir91] for details). Alternatively, starting from a *-autonomous category and adding natural transformations $A \to A \wedge A$ and $A \to \mathbf{t}$, i.e., the proofs for weakening and contraction, yields the same collapse.[2]

The second reason is that cut elimination in the sequent calculus for classical logic is not confluent. Since cut elimination is the usual way of composing proofs, this means that there is no canonical way of composing two proofs, let alone associativity of composition.

Consequently, for avoiding these two problems, we have to accept that (i) cartesian closed categories do not provide an abstract algebraic axiomatisation for proofs in classical logic, and that (ii) the sequent calculus is not the right framework for investigating the identity of proofs in in classical logic.

Recently, several authors [DP04,FP04c,LS05a] have shown that proof theory for classical logic is possible outside the restrictive world of cartesian closed categories. However, the fact that all three proposals considerably differ from each other suggests that there might be no canonical way of giving a categorical axiomatisation for proofs in classical logic.

In this paper we will provide a series of possible such axiomatisations with increasing strength. They will all build on the structure of a *-autonomous category in which every object has a monoid (and a comonoid) structure. In this respect it will closely follow the work of [FP04c] and [LS05a], but will differ from [DP04].

The main proof theoretical inspiration for this work comes from system SKS [BT01], which is a deductive system for Boolean logic within the formalism of the calculus of structures [Gug02b,GS01,BT01]. A remarkable feature of the cut-free version of SKS, which is called KS, is that it can (cut-free) polynomially simulate not only sequent calculus and tableaux systems but also resolution and Frege-Hilbert systems [Gug04a]. This means that if a tautology has a polynomial size proof in any of these systems, then it has a cut-free proynomial size proof in KS. This ability of KS is a consequence of two features:

1. *Deep inference*: Instead of decomposing the formulae along their root connectives into subformulae during the construction of a proof, in KS inference rules are applied deep inside formulae in the same way as we know it from term rewriting.
2. The two inference rules *switch* and *medial*, which look as follows:

$$\mathsf{s}\,\frac{F\{(A \vee B) \wedge C\}}{F\{A \vee (B \wedge C)\}} \qquad \text{and} \qquad \mathsf{m}\,\frac{F\{(A \wedge B) \vee (C \wedge D)\}}{F\{(A \vee C) \wedge (B \vee D)\}} \quad , \tag{1}$$

   where $F\{\ \}$ stands for an arbitrary (positive) formula context and A, B, C, and D are formula variables.

[1] i.e., a natural isomorphism between A and the double-negation of A (in this paper denoted by $\bar{\bar{A}}$)

[2] Since we are dealing with Boolean logic, we will use the symbols $\wedge$ and $\mathbf{t}$ for the tensor operation (usually $\otimes$) and the unit (usually $\mathbf{1}$ or $\mathbf{I}$) in a *-autonomous category.

From deep inference to algebra

Deep inference allows to establish the relationship between proof theory and algebra in a much cleaner way than this is possible with shallow inference formalisms like the sequent calculus. The reason is that from a derivation in a deep inference formalism one can directly "read off the morphisms". Take for example the following derivation in system KS:

$$\mathsf{m}\frac{\mathsf{r}\dfrac{(A' \wedge B) \vee (C \wedge D)}{(A \wedge B) \vee (C \wedge D)}}{(A \vee C) \wedge (B \vee D)} \tag{2}$$

where A, A', B, C, and D are arbitrary formulas, and r is any inference rule taking A' to A. In category theoretical language this would be written as a composition of maps:

$$(A' \wedge B) \vee (C \wedge D) \xrightarrow{(\mathsf{r} \wedge B) \vee (C \wedge D)} (A \wedge B) \vee (C \wedge D) \xrightarrow{\mathsf{m}_{A,B,C,D}} (A \vee C) \wedge (B \vee D)$$

where $\mathsf{m}_{A,B,C,D} \colon (A \wedge B) \vee (C \wedge D) \to (A \vee C) \wedge (B \vee D)$ is called the *medial map*, and $\mathsf{r} \colon A' \to A$ is the map corresponing to the rule r. System KS also allows the derivation

$$\mathsf{r}\frac{\mathsf{m}\dfrac{(A' \wedge B) \vee (C \wedge D)}{(A' \vee C) \wedge (B \vee D)}}{(A \vee C) \wedge (B \vee D)} \tag{3}$$

From the proof theoretical point of view it makes perfectly sense to identify the two derivations in (2) and (3) because they do "essentially" the same. This is what Guglielmi calles *bureaucracy of type B* [Gug04c]. In the language of category theory, the identification of (2) and (3) is saying that the diagram

$$\begin{tikzcd}
(A' \wedge B) \vee (C \wedge D) \arrow[r, "\mathsf{m}_{A',B,C,D}"] \arrow[d, "(\mathsf{r} \wedge B) \vee (C \wedge D)"'] & (A' \vee C) \wedge (B \vee D) \arrow[d, "(\mathsf{r} \vee C) \wedge (B \vee D)"] \\
(A \wedge B) \vee (C \wedge D) \arrow[r, "\mathsf{m}_{A,B,C,D}"'] & (A \vee C) \wedge (B \vee D)
\end{tikzcd} \tag{4}$$

has to commute, which exactly means that the medial map has to be natural.

For deep inference, Guglielmi also introduces the notion of *bureaucracy of type A* [Gug04b], which is the formal distinction between the derivations

$$\mathsf{r}_1\frac{\mathsf{r}_2\dfrac{A' \wedge B'}{A' \wedge B}}{A \wedge B} \qquad \text{and} \qquad \mathsf{r}_2\frac{\mathsf{r}_1\dfrac{A' \wedge B'}{A \wedge B'}}{A \wedge B} \tag{5}$$

where rule r_1 takes A' to A, and rule r_2 takes B' to B. Proof theoretically, the two derivations in (5) are "essentially" the same, so it makes sense to identify them. Translating this into category theory means to say that the operation $\wedge$ is a bifunctor.

However, it is not always the case that the demands of algebra and proof theory coincide so nicely. Sometimes they contradict each other, which causes "creative tensions" [LS04]. One example is the treatment of units. Proof theoretically it might be desirable to distinguish between the following two proofs in the sequent calculus (here $\mathbf{t}$ stands for "truth" and $\mathbf{f}$ for "falsum"):

$$\mathsf{weakening}\frac{\mathsf{axiom(true)}\dfrac{}{\vdash \mathbf{t}}}{\vdash \mathbf{t}, \mathbf{f}} \qquad \text{and} \qquad \mathsf{axiom(identity)}\frac{}{\vdash \mathbf{t}, \mathbf{f}} \tag{6}$$

This distinction is made, for example, by the proof nets presented in [LS05b]. From the algebraic point of view, this causes certain difficulties: In [LS05a] the concept of weak units has been introduced in order to give a clean algebraic treatment to the distinction in (6). However, in this paper we will depart from this and use proper units instead. This is from the algebraic point of view more reasonable and simplifies the theory considerably. But it forces the identification of the two proofs in (6).

Some remarks about switch and medial

The inference rule switch in (1), or the *switch map* $\mathsf{s}_{A,B,C} \colon (A \vee B) \wedge C \to A \vee (B \wedge C)$ has already been well investigated from the viewpoint of proof theory [Gug02b], as well as from the viewpoint of category theory, where it is also called *weak distributivity* [HdP93,CS97b], *linear distributivity*, or *dissociativity* [DP04]. On the other hand, the medial rule or *medial map* $\mathsf{m}_{A,B,C,D} \colon (A \wedge B) \vee (C \wedge D) \to (A \vee C) \wedge (B \vee D)$ has not yet been so thoroughly investigated. Only very recently Lamarche [Lam05] started to study the consequences of the presence of the medial map in a *-autonomous category.

Seen from the deductive point of view, the two rules switch and medial have certain similarities:

- switch allows the reduction of the identity rule and the cut rule to atomic form, and medial allows the reduction of the contraction rule (and the cocontration rule) to atomic form (see [BT01] for details),
- switch and medial are both self-dual, and
- they look similar, as can bee seen in (1). In fact, recent work shows that they can both be seen as instance of a single more general inference rule [Gug02a,Gug05].

However, from the algebraic point of view, they are quite different: Switch is a consequence of more primitive properties, namely the associativity of $\wedge$ and $\vee$ and the de Morgan duality between the two operations[3], whereas medial has to be put as additional primitive, if we want it in the category.[4]

Outline of the paper

In this work we will present a series of axioms that seem reasonable (from the proof theoretical as well as from the algebraic points of view) to have in a Boolean category. While introducing axioms, we will also show their consequences. Some of the axioms presented here coincide with axioms given in [Lam05] which has been written at the same time as this paper. But while [Lam05] works in the minimal setting of a *-autonomous category with medial (or with "linear logic plus medial"), we assume from the beginning full classical propositional logic, i.e., the presence of weakening and contraction. But the more important difference to [Lam05] is that we are staying in the realm of syntax, whereas [Lam05] is primarily concerned with the construction of concrete models for classical proofs.

In the end of the paper, we will use a concrete example of a Boolean category (namely a variation of the proof nets of [LS05b]) to show that the axioms presented here do not lead to the collapse into a Boolean algebra. This last section can be read independently by the reader interested only in proof nets.

This paper is another attempt of making the straddle of being accessible to the category theorist *and* the proof theorist. Since it is mainly about algebra, we use here the language of category theory. Nonetheless, the seasoned proof theorist might find it easier to understand if he substitutes everywhere *"object"* by *"formula"* and *"map"/"morphism"/"arrow"* by *"proof"*. Every commuting diagram in the paper is nothing but an equation between proofs written in a deep inference formalism. In order to make the paper easier accessible to proof theorists, all statements are proved in more detail than the seasoned category theorist might find appropriate.

2 What is a Boolean Category ?

In its most general sense, a Boolean category should be for categories, what a Boolean algebra is for posets. This leads to the following definition:

[3] Nonetheless it has been investigated from the category theoretic viewpoint under the assumption that negation (and therefore the de Morgan duality) is absent in [CS97b].

[4] This fact raises an open problem: can we find simple primitives from which medial arises naturally, in the same way as switch arises naturally from associativity and duality?

2.1 Definition We say a category $\mathscr{C}$ is a $\mathsf{B0}$-*category* if there is a Boolean algebra $\mathscr{B}$ and a mapping $F\colon \mathscr{C} \to \mathscr{B}$ from objects of $\mathscr{C}$ to elements of $\mathscr{B}$, such that for all objects A and B in $\mathscr{C}$, we have $F(A) \leq F(B)$ in $\mathscr{B}$ if and only if there is an arrow $f\colon A \to B$ in $\mathscr{C}$.

In other words, a $\mathsf{B0}$-category is a category whose image under the forgetful functor from the category of categories to the category of posets is a Boolean algebra.

This definition is neither enlightening nor useful. It is necessary to add some additional structure in order to obtain a "nicely behaved" theory of Boolean categories. However, as already mentioned in the introduction, the naive approach of adding structure, namely adding the structure of a bicartesian closed category (also called Heyting category) with an involutive negation leads to collapse: Every Boolean category in that strong sense is a Boolean algebra. The hom-sets are either singletons or empty (see, e.g., [LS86,Gir91]).

This means that we have here two extremes that are both not very interesting, neither for proof theory nor for category theory. But there is a whole universe between the two, which we will start to investigate now. Before, let us make the following (trivial) observation.

2.2 Observation In a $\mathsf{B0}$-category, we can for any pair of objects A and B, provide objects $A \wedge B$ and $A \vee B$ and $\bar{A}$, and there are objects $\mathbf{t}$ and $\mathbf{f}$, such that there are maps

$$
\begin{array}{cc}
\hat{\alpha}_{A,B,C}\colon A \wedge (B \wedge C) \to (A \wedge B) \wedge C & \check{\alpha}_{A,B,C}\colon A \vee (B \vee C) \to (A \vee B) \vee C \\
\hat{\sigma}_{A,B}\colon A \wedge B \to B \wedge A & \check{\sigma}_{A,B}\colon A \vee B \to B \vee A \\
\hat{\varrho}_{A}\colon A \wedge \mathbf{t} \to A & \check{\varrho}_{A}\colon A \vee \mathbf{f} \to A \\
\hat{\lambda}_{A}\colon \mathbf{t} \wedge A \to A & \check{\lambda}_{A}\colon \mathbf{f} \vee A \to A \\
\hat{\mathsf{i}}_{A}\colon A \wedge \bar{A} \to \mathbf{f} & \check{\mathsf{i}}_{A}\colon \mathbf{t} \to \bar{A} \vee A \\
\multicolumn{2}{c}{\mathsf{s}_{A,B,C}\colon (A \vee B) \wedge C \to A \vee (B \wedge C)} \\
\multicolumn{2}{c}{\mathsf{m}_{A,B,C,D}\colon (A \wedge B) \vee (C \wedge D) \to (A \vee C) \wedge (B \vee D)} \\
\Delta_{A}\colon A \to A \wedge A & \nabla_{A}\colon A \vee A \to A \\
\Pi^{A}\colon A \to \mathbf{t} & \amalg^{A}\colon \mathbf{f} \to A
\end{array}
\tag{7}
$$

for all objects A, B, and C. This can easily be shown by verifying that all of them correspond to valid implications in Boolean logic. Conversely, a category in which every arrow can be given as a composite of the ones given above by using only the operations of $\wedge$, $\vee$, and the usual arrow composition, is a $\mathsf{B0}$-category. This is a consequence of the completeness of system SKS [BT01], which is a deep inference deductive system for Boolean logic incorporating the maps in (7) as inference rules.

3 *-Autonomous categories

Let us stress the fact that in a plain $\mathsf{B0}$-category there is no relation between the maps listed in (7). In particular, there is no functoriality of $\vee$ and $\wedge$, no naturality of $\hat{\alpha}$, $\hat{\sigma}$, ..., and no deMorgan duality. Adding this structure means exactly adding the structure of a *-autonomous category [Bar79].

Since we are working in classical logic, we will here use the symbols $\wedge, \vee, \mathbf{t}, \mathbf{f}$ for the usual $\otimes, \parr, 1, \bot$.

3.1 Definition A $\mathsf{B0}$-category $\mathscr{C}$ is *symmetric $\wedge$-monoidal* if the operation $- \wedge - : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is a bifunctor and the maps $\hat{\alpha}_{A,B,C}, \hat{\sigma}_{A,B}, \hat{\varrho}_{A}, \hat{\lambda}_{A}$ in (7) are natural isomorphisms that obey the following equations:

$$
\begin{tikzcd}[column sep=small]
A \wedge (B \wedge (C \wedge D)) \arrow[rr, "A\wedge\hat{\alpha}_{B,C,D}"] \arrow[d, "\hat{\alpha}_{A,B,C\wedge D}"'] & & A \wedge ((B \wedge C) \wedge D) \arrow[d, "\hat{\alpha}_{A,B\wedge C,D}"] \\
(A \wedge B) \wedge (C \wedge D) \arrow[dr, "\hat{\alpha}_{A\wedge B,C,D}"'] & & (A \wedge (B \wedge C)) \wedge D \arrow[dl, "\hat{\alpha}_{A,B,C}\wedge D"] \\
& ((A \wedge B) \wedge C) \wedge D &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \wedge (B \wedge C) \arrow[r, "A\wedge\hat{\sigma}_{B,C}"] \arrow[d, "\hat{\alpha}_{A,B,C}"'] & A \wedge (C \wedge B) \arrow[d, "\hat{\alpha}_{A,C,B}"] \\
(A \wedge B) \wedge C \arrow[d, "\hat{\sigma}_{A\wedge B,C}"'] & (A \wedge C) \wedge B \arrow[d, "\hat{\sigma}_{A,C}\wedge B"] \\
C \wedge (A \wedge B) \arrow[r, "\hat{\alpha}_{C,A,B}"'] & (C \wedge A) \wedge B
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \wedge (\mathbf{t} \wedge B) \arrow[rr, "\hat{\alpha}_{A,\mathbf{t},B}"] \arrow[dr, "A\wedge\hat{\lambda}_B"'] & & (A \wedge \mathbf{t}) \wedge B \arrow[dl, "\hat{\varrho}_A\wedge B"] \\
& A \wedge B &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \wedge B \arrow[rr, "\hat{\sigma}_{A,B}"] \arrow[dr, "1_{A\wedge B}"'] & & B \wedge A \arrow[dl, "\hat{\sigma}_{B,A}"] \\
& A \wedge B &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{t} \wedge A \arrow[rr, "\hat{\sigma}_{\mathbf{t},A}"] \arrow[dr, "\hat{\lambda}_A"'] & & A \wedge \mathbf{t} \arrow[dl, "\hat{\varrho}_A"] \\
& A &
\end{tikzcd}
$$

The notion of *symmetric* $\vee$*-monoidal* is defined in a similar way.

An important property of symmetric monoidal categories is the coherence theorem [Mac63], which says that every diagram containing only maps composed of $\hat{\alpha}$, $\hat{\sigma}$, $\hat{\varrho}$, and $\hat{\lambda}$, via $\wedge$ and $\circ$ must commute (for details, see [Mac71]).

As a consequence of the coherence theorem, we can omit certain parentheses to ease the reading. For example, we will write $A \wedge B \wedge C \wedge D$ for $(A \wedge B) \wedge (C \wedge D)$ as well as for $A \wedge ((B \wedge C) \wedge D)$. This can be done because there is a uniquely defined "coherence isomorphism" between any two of these objects.

Let us now turn our attention to a very important feature of Boolean logic: the duality between $\wedge$ and $\vee$. We can safely say that it is reasonable to ask for this duality also in a Boolean category. That means, we are asking for $\bar{\bar{A}} \cong A$ and $\overline{A \wedge B} \cong \bar{A} \vee \bar{B}$. At the same time we ask for the possibility of transposition (or currying): The proofs of $A \wedge B \to C$ are in one-to-one correspondence with the proofs of $A \to \bar{B} \vee C$. This is exactly what makes a monoidal category *-autonomous.

3.2 Definition A $\mathsf{B0}$-category $\mathscr{C}$ is **-autonomous* if it is symmetric $\wedge$-monoidal and is equipped with a contravariant functor $\overline{(-)} : \mathscr{C} \to \mathscr{C}$, such that $\overline{\overline{(-)}} \colon \mathscr{C} \to \mathscr{C}$ is a natural isomorphism and such that for any three objects A, B, C there is a natural bijection

$$
\mathrm{Hom}_{\mathscr{C}}(A \wedge B, C) \quad \cong \quad \mathrm{Hom}_{\mathscr{C}}(A, \bar{B} \vee C) \quad . \tag{$\star$}
$$

where the bifunctor $- \vee -$ is defined via $A \vee B = \overline{\bar{B} \wedge \bar{A}}$.[5] We also define $\mathbf{f} = \bar{\mathbf{t}}$.

Clearly, if a $\mathsf{B0}$-category $\mathscr{C}$ is **-autonomous*, then it is also $\vee$-monoidal with $\check{\alpha}_{A,B,C} = \overline{\hat{\alpha}_{\bar{C},\bar{B},\bar{A}}}$, $\check{\sigma}_{A,B} = \overline{\hat{\sigma}_{\bar{B},\bar{A}}}$, $\check{\varrho}_A = \overline{\hat{\lambda}_{\bar{A}}}$, $\check{\lambda}_A = \overline{\hat{\varrho}_{\bar{A}}}$.

[5] Although we live in the commutative world, we invert the order of the arguments when taking the negation.

Let us continue with stating some well-known facts about *-autonomous categories (for proofs of these facts, see e.g. [LS04]). Via the bijection ($\star$) we can assign to every map $f\colon A \to B \vee C$ a map $g\colon A \wedge \bar{B} \to C$, and vice versa. We say that f and g are *transposes* of each other if they determine each other via ($\star$). We will use the term "transpose" in a very general sense: given objects A, B, C, D, E such that $D \cong A \wedge B$ and $E \cong \bar{B} \vee C$, then any $f\colon D \to C$ uniquely determines a $g\colon A \to E$, and vice versa. Also in that general case we will say that f and g are transposes of each other. For example, $\hat{\lambda}_A\colon \mathbf{t} \wedge A \to A$ and $\check{\varrho}_A\colon A \to A \vee \mathbf{f}$ are transposes of each other, and another way of transposing them yields the maps

$$\check{\imath}_A\colon \mathbf{t} \to \bar{A} \vee A \qquad \text{and} \qquad \hat{\imath}_A\colon A \wedge \bar{A} \to \mathbf{f} \quad .$$

If we have $f\colon A \to B \vee C$ and $b\colon B' \to B$, then

$$A \wedge B' \xrightarrow{A \wedge b} A \wedge B \xrightarrow{f} C \quad \text{is transpose of} \quad A \xrightarrow{g} \bar{B} \vee C \xrightarrow{\bar{b} \vee C} \overline{B'} \vee C \tag{8}$$

where g is transpose of f.

Let us now transpose the identity $1_{B \vee C}\colon B \vee C \to B \vee C$. This yields the *evaluation map* $\mathsf{eval}\colon (B \vee C) \wedge \bar{C} \to B$. Taking the $\wedge$ of this with $1_A\colon A \to A$ and transposing back determines a map $\mathsf{s}_{A,B,C}\colon A \wedge (B \vee C) \to (A \wedge B) \vee C$ that is natural in all three arguments, and that we call the *switch map* [Gug02b,BT01][6]. In a similar fashion we obtain maps $(A \vee B) \wedge C \to A \vee (B \wedge C)$ and $A \wedge (B \vee C) \to B \vee (A \wedge C)$ and $(A \vee B) \wedge C \to (A \wedge C) \vee B$. Alternatively these maps can be obtained from s by composing with $\hat{\sigma}$ and $\check{\sigma}$. For this reason we will use the term "switch" for all of them, and denote them by $\mathsf{s}_{A,B,C}$ if it is clear from context which one is meant, as for example in the two diagrams

$$\begin{tikzcd}
(A \vee B) \wedge (C \vee D) \arrow[r, "\mathsf{s}_{A,B,C\vee D}"] \arrow[d, "\mathsf{s}_{A\vee B,C,D}"'] & A \vee (B \wedge (C \vee D)) \arrow[d, "A \vee \mathsf{s}_{B,C,D}"] \\
((A \vee B) \wedge C) \vee D \arrow[r, "\mathsf{s}_{A,B,C} \vee D"'] & A \vee (B \wedge C) \vee D
\end{tikzcd} \tag{9}$$

and

$$\begin{tikzcd}
A \wedge (B \vee C) \wedge D \arrow[r, "A \wedge \mathsf{s}_{B,C,D}"] \arrow[d, "\mathsf{s}_{A,B,C} \wedge D"'] & A \wedge (B \vee (C \wedge D)) \arrow[d, "\mathsf{s}_{A,B,C\wedge D}"] \\
((A \wedge B) \vee C) \wedge D \arrow[r, "\mathsf{s}_{A\wedge B,C,D}"'] & (A \wedge B) \vee (C \wedge D)
\end{tikzcd} \tag{10}$$

which commute in any *-autonomous category. Sometimes we will denote the map defined by (9) by $\hat{\mathsf{t}}_{A,B,C,D}\colon (A \vee B) \wedge (C \vee D) \to A \vee (B \wedge C) \vee D$, called the *tensor map*[7] and the one of (10) by $\check{\mathsf{t}}_{A,B,C,D}\colon A \wedge (B \vee C) \wedge D \to (A \wedge B) \vee (C \wedge D)$, called the *cotensor map*.

Note that the switch map is self-dual, while the two maps $\hat{\mathsf{t}}$ and $\check{\mathsf{t}}$ are dual to each other, i.e.,

$$\begin{tikzcd}
\overline{(A \wedge B) \vee C} \arrow[r, "\overline{\mathsf{s}_{A,B,C}}"] \arrow[d, "\cong"'] & \overline{A \wedge (B \vee C)} \arrow[d, "\cong"] \\
\bar{C} \wedge (\bar{B} \vee \bar{A}) \arrow[r, "\mathsf{s}_{\bar{C},\bar{B},\bar{A}}"'] & (\bar{C} \wedge \bar{B}) \vee \bar{A}
\end{tikzcd} \tag{11}$$

and

$$\begin{tikzcd}
\overline{(A \wedge B) \vee (C \wedge D)} \arrow[r, "\overline{\check{\mathsf{t}}_{A,B,C,D}}"] \arrow[d, "\cong"'] & \overline{A \wedge (B \vee C) \wedge D} \arrow[d, "\cong"] \\
(\bar{D} \vee \bar{C}) \wedge (\bar{B} \vee \bar{A}) \arrow[r, "\hat{\mathsf{t}}_{\bar{D},\bar{C},\bar{B},\bar{A}}"'] & \bar{D} \vee (\bar{C} \wedge \bar{B}) \vee \bar{A}
\end{tikzcd} \tag{12}$$

[6] To category theorists it is probably better known under the names *weak distributivity* [HdP93,CS97b] or *linear distributivity*. However, strictly speaking, it is not a form of distributivity. An alternative is the name *dissociativity* [DP04].

[7] This map describes precisely the tensor rule in the sequent system for linear logic.

where the vertical maps are the canonical isomophisms determined by the *-autonomous structure. Another property of switch that we will use later is the commutativity of the following diagrams:

$$
\begin{tikzcd}
(A \vee B) \wedge \mathbf{t} \arrow[d, "\mathsf{s}_{A,B,\mathbf{t}}"'] \arrow[dr, "\hat{\varrho}_{A\vee B}"] & \\
& A \vee B \\
A \vee (B \wedge \mathbf{t}) \arrow[ur, "A\vee\hat{\varrho}_B"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& (\mathbf{f} \vee A) \wedge B \arrow[dd, "\mathsf{s}_{\mathbf{f},A,B}"] \\
A \wedge B \arrow[ur, "\check{\lambda}_A^{-1}\wedge B"] \arrow[dr, "\check{\lambda}_{A\vee B}^{-1}"'] & \\
& \mathbf{f} \vee (A \wedge B)
\end{tikzcd}
\tag{13}
$$

4 Some remarks on mix

In this section we will recall what it means for a *-autonomous category to have mix. Although most of the material of this section can also be found in [CS97a], [FP04a], [DP04], and [Lam05], we give here a complete survey since the main result (Corollary 4.3) is rather crucial for the following sections.

4.1 Theorem *Let $\mathscr{C}$ be a *-autonomous category and $e\colon \mathbf{f} \to \mathbf{t}$ be a map in $\mathscr{C}$. Then*

$$
\begin{tikzcd}
\mathbf{f} \wedge \mathbf{f} \arrow[r, "e\wedge\mathbf{f}"] \arrow[d, "\mathbf{f}\wedge e"'] & \mathbf{t} \wedge \mathbf{f} \arrow[d, "\hat{\lambda}_{\mathbf{f}}"] \\
\mathbf{f} \wedge \mathbf{t} \arrow[r, "\hat{\varrho}_{\mathbf{f}}"'] & \mathbf{f}
\end{tikzcd}
\tag{14}
$$

if and only if

$$
\begin{tikzcd}
\mathbf{t} \arrow[r, "\check{\lambda}_{\mathbf{t}}^{-1}"] \arrow[d, "\check{\varrho}_{\mathbf{t}}^{-1}"'] & \mathbf{f} \vee \mathbf{t} \arrow[d, "e\vee\mathbf{t}"] \\
\mathbf{t} \vee \mathbf{f} \arrow[r, "\mathbf{t}\vee e"'] & \mathbf{t} \vee \mathbf{t}
\end{tikzcd}
\tag{15}
$$

if and only if

$$
\begin{tikzcd}
A \wedge B \arrow[r, "A\wedge\check{\lambda}_B^{-1}"] \arrow[d, "\check{\varrho}_A^{-1}\wedge B"'] & A \wedge (\mathbf{f} \vee B) \arrow[r, "\mathsf{s}_{A,\mathbf{f},B}"] & (A \wedge \mathbf{f}) \vee B \arrow[d, "(A\wedge e)\vee B"] \\
(A \vee \mathbf{f}) \wedge B \arrow[d, "\mathsf{s}_{A,\mathbf{f},B}"'] & & (A \wedge \mathbf{t}) \vee B \arrow[d, "\hat{\varrho}_A\vee B"] \\
A \vee (\mathbf{f} \wedge B) \arrow[r, "A\vee(e\wedge B)"'] & A \vee (\mathbf{t} \wedge B) \arrow[r, "A\vee\hat{\lambda}_B"'] & A \vee B
\end{tikzcd}
\tag{16}
$$

for all objects A and B.

Proof: First we show that (14) implies (16). For this, chase

$$\begin{tikzcd}[column sep=tiny]
 & & & & A \wedge B \arrow[dl, "\check\varrho_A^{-1}"'] \arrow[dr, "\check\lambda_B^{-1}"] & & & & \\
 & & & (A \vee \mathbf{f}) \wedge B \arrow[dl, "\mathsf{s}"'] \arrow[dr, "\check\lambda_B^{-1}"] & & A \wedge (\mathbf{f} \vee B) \arrow[dl, "\check\varrho_A^{-1}"'] \arrow[dr, "\mathsf{s}"] & & & \\
 & & A \vee (\mathbf{f} \wedge B) \arrow[dl, "e"'] \arrow[dr, "\check\lambda_B^{-1}"] & & (A \vee \mathbf{f}) \wedge (\mathbf{f} \vee B) \arrow[dl, "\mathsf{s}"'] \arrow[dr, "\mathsf{s}"] & & (A \wedge \mathbf{f}) \vee B \arrow[dl, "\check\varrho_A^{-1}"'] \arrow[dr, "e"] & & \\
 & A \vee (\mathbf{t} \wedge B) \arrow[dr, "\check\lambda_B^{-1}"] \arrow[dddd, "\hat\lambda_B"'] & & A \vee (\mathbf{f} \wedge (\mathbf{f} \vee B)) \arrow[dl, "e"'] \arrow[dr, "\mathsf{s}"] & & ((A \vee \mathbf{f}) \wedge \mathbf{f}) \vee B \arrow[dl, "\mathsf{s}"'] \arrow[dr, "e"] & & (A \wedge \mathbf{t}) \vee B \arrow[dl, "\check\varrho_A^{-1}"'] \arrow[dddd, "\hat\varrho_A"] & \\
 & & A \vee (\mathbf{t} \wedge (\mathbf{f} \vee B)) \arrow[dr, "\mathsf{s}"] \arrow[dd, "\hat\lambda_{\mathbf{f} \vee B}"'] & & A \vee (\mathbf{f} \wedge \mathbf{f}) \vee B \arrow[dl, "e"'] \arrow[dr, "e"] & & ((A \vee \mathbf{f}) \wedge \mathbf{t}) \vee B \arrow[dl, "\mathsf{s}"'] \arrow[dd, "\hat\varrho_{A \vee \mathbf{f}}"] & & \\
 & & & A \vee (\mathbf{t} \wedge \mathbf{f}) \vee B \arrow[dl, "\hat\lambda_{\mathbf{f}}"'] & & A \vee (\mathbf{f} \wedge \mathbf{t}) \vee B \arrow[dr, "\hat\varrho_{\mathbf{f}}"] & & & \\
 & & A \vee \mathbf{f} \vee B \arrow[rrrr, equal] \arrow[drr, "\check\lambda_B"] & & & & A \vee \mathbf{f} \vee B \arrow[dll, "\check\varrho_A"'] & & \\
 & A \vee B \arrow[ur, "\check\lambda_B^{-1}"] \arrow[rrr, equal] & & & A \vee B \arrow[rrr, equal] & & & A \vee B \arrow[ul, "\check\varrho_A^{-1}"'] &
\end{tikzcd} \tag{17}$$

The big triangle at the center is an application of (14). The two little triangles next to it are (variations of) (13), and the triangles at the bottom are trivial. The topmost square is functoriality of $\wedge$, the square in the center is (9), and all other squares commute because of naturality of s, $\hat\lambda$, $\hat\varrho$, $\check\lambda$, and $\check\varrho$. Now observe that (16) commutes if and only if

$$\begin{tikzcd}
A \wedge B \arrow[r, "A \wedge \check\lambda_B^{-1}"] \arrow[d, "\check\varrho_A^{-1} \wedge B"'] & A \wedge (\mathbf{f} \vee B) \arrow[r, "A \wedge (e \vee B)"] & A \wedge (\mathbf{t} \vee B) \arrow[d, "\mathsf{s}_{A,\mathbf{t},B}"] \\
(A \vee \mathbf{f}) \wedge B \arrow[d, "(A \vee e) \wedge B"'] & & (A \wedge \mathbf{t}) \vee B \arrow[d, "\hat\varrho_A \vee B"] \\
(A \vee \mathbf{t}) \wedge B \arrow[r, "\mathsf{s}_{A,\mathbf{t},B}"'] & A \vee (\mathbf{t} \wedge B) \arrow[r, "A \vee \hat\lambda_B"'] & A \vee B
\end{tikzcd} \tag{18}$$

commutes (because of naturalitiy of switch), and that the diagonals of (16) and (18) are the same map $\mathsf{mix}_{A,B} \colon A \wedge B \to A \vee B$. Note that by the dual of (17) we get that (15) implies (18). Therefore we also get that (15) implies (16). Now we show that (18) implies (15). We will do this by showing that

$$\begin{tikzcd}
 & \mathbf{t} \arrow[dl, "\check\varrho_{\mathbf{t}}^{-1}"'] \arrow[d, "\hat\varrho_{\mathbf{t}}^{-1} = \hat\lambda_{\mathbf{t}}^{-1}"] \arrow[dr, "\check\lambda_{\mathbf{t}}^{-1}"] & \\
\mathbf{t} \vee \mathbf{f} \arrow[dr, "\mathbf{t} \vee e"'] & \mathbf{t} \wedge \mathbf{t} \arrow[d, "\mathsf{mix}_{\mathbf{t},\mathbf{t}}"] & \mathbf{f} \vee \mathbf{t} \arrow[dl, "e \vee \mathbf{t}"] \\
 & \mathbf{t} \vee \mathbf{t} &
\end{tikzcd} \tag{19}$$

commutes. For this, consider

$$
\begin{tikzcd}
\mathbf{t} \arrow[dr, "\hat{\varrho}_{\mathbf{t}}^{-1}"] \arrow[dd, equal] & \\
& \mathbf{t} \wedge \mathbf{t} \arrow[dl, "\hat{\varrho}_{\mathbf{t}}"] \arrow[dd, "\check{\varrho}_{\mathbf{t}}^{-1} \wedge \mathbf{t}"] \\
\mathbf{t} \arrow[dd, "\check{\varrho}_{\mathbf{t}}^{-1}"'] & \\
& (\mathbf{t} \vee \mathbf{f}) \wedge \mathbf{t} \arrow[dl, "\hat{\varrho}_{\mathbf{t} \vee \mathbf{f}}"] \arrow[dd, "(\mathbf{t} \vee e) \wedge \mathbf{t}"] \\
\mathbf{t} \vee \mathbf{f} \arrow[dd, "\mathbf{t} \vee e"'] & \\
& (\mathbf{t} \vee \mathbf{t}) \wedge \mathbf{t} \arrow[dl, "\hat{\varrho}_{\mathbf{t} \vee \mathbf{t}}"] \arrow[dd, "\mathsf{s}_{\mathbf{t},\mathbf{t},\mathbf{t}}"] \\
\mathbf{t} \vee \mathbf{t} \arrow[dd, equal] & \\
& \mathbf{t} \vee (\mathbf{t} \wedge \mathbf{t}) \arrow[ul, "\mathbf{t} \vee \hat{\varrho}_{\mathbf{t}}"] \arrow[dl, "\mathbf{t} \vee \hat{\lambda}_{\mathbf{t}}"] \\
\mathbf{t} \vee \mathbf{t} &
\end{tikzcd}
\tag{20}
$$

which says that the left triangle in (19) commutes because the right down path in (20) is exactly the lower left path in (18). Similarly we obtain the commutativity of the right triangle in (19). In the same way we show that (16) implies (14), which completes the proof. □

Therefore, in a *-autonomous category every map $e\colon \mathbf{f} \to \mathbf{t}$ obeying (14) uniquely determines a map $\mathsf{mix}_{A,B}\colon A \wedge B \to A \vee B$ which is natural in A and B. It can be shown that this *mix map* goes well with the twist, associativity, and switch maps:

4.2 Proposition *The map* $\mathsf{mix}_{A,B}\colon A \wedge B \to A \vee B$ *obtained from* (16) *is natural in both arguments and obeys the equations*

$$
\begin{tikzcd}
A \wedge B \arrow[r, "\mathsf{mix}_{A,B}"] \arrow[d, "\hat{\sigma}_{A,B}"'] & A \vee B \arrow[d, "\check{\sigma}_{A,B}"] \\
B \wedge A \arrow[r, "\mathsf{mix}_{B,A}"'] & B \vee A
\end{tikzcd}
\tag{mix-$\hat{\sigma}$}
$$

and

$$
\begin{tikzcd}
A \wedge (B \wedge C) \arrow[r, "A \wedge \mathsf{mix}_{B,C}"] \arrow[d, "\hat{\alpha}_{A,B,C}"'] & A \wedge (B \vee C) \arrow[r, "\mathsf{mix}_{A,B \vee C}"] \arrow[d, "\mathsf{s}_{A,B,C}"] & A \vee (B \vee C) \arrow[d, "\check{\alpha}_{A,B,C}"] \\
(A \wedge B) \wedge C \arrow[r, "\mathsf{mix}_{A \wedge B,C}"'] & (A \wedge B) \vee C \arrow[r, "\mathsf{mix}_{A,B} \vee C"'] & (A \vee B) \vee C
\end{tikzcd}
\tag{mix-$\hat{\alpha}$}
$$

Proof: Naturality of mix follows immediately from the naturality of switch. Equation (mix-$\hat{\sigma}$) follows immediately from the definition of switch, and (mix-$\hat{\alpha}$) can be shown with a similar diagram as (17). □

4.3 Corollary *In a *autonomous category there is a one-to-one correspondence between the maps* $e\colon \mathbf{f} \to \mathbf{t}$ *obeying* (14) *and the natural transformations* $\mathsf{mix}_{A,B}\colon A \wedge B \to A \vee B$ *obeying* (mix-$\hat{\sigma}$) *and* (mix-$\hat{\alpha}$).

Proof: Whenever we have a map $\mathsf{mix}_{A,B}\colon A \wedge B \to A \vee B$ for all A and B, we can form the map

$$
e\colon \mathbf{f} \xrightarrow{\hat{\varrho}_{\mathbf{f}}^{-1}} \mathbf{f} \wedge \mathbf{t} \xrightarrow{\mathsf{mix}_{\mathbf{f},\mathbf{t}}} \mathbf{f} \vee \mathbf{t} \xrightarrow{\check{\lambda}_{\mathbf{t}}} \mathbf{t}
\tag{21}
$$

One can now easily show that naturality of mix, as well as (mix-$\hat{\sigma}$) and (mix-$\hat{\alpha}$) are exactly what is needed to let the map $e\colon \mathbf{f} \to \mathbf{t}$ defined in (21) obey equation (14). We leave the details to the reader. Hint: Show that both maps of (14) are equal to

$$\mathbf{f} \wedge \mathbf{f} \xrightarrow{\mathsf{mix}_{\mathbf{f},\mathbf{f}}} \mathbf{f} \vee \mathbf{f} \xrightarrow{\check{\lambda}_{\mathbf{f}} = \check{\varrho}_{\mathbf{f}}} \mathbf{f} \quad .$$

It remains to show that plugging the map of (21) into (16) gives back the same natural transformation $\mathsf{mix}_{A,B} \colon A \wedge B \to A \vee B$ we started from. Similarly, plugging in the the mix defined via (16) into (21) gives back the same map $e \colon \mathbf{f} \to \mathbf{t}$ that has been plugged into (16). Again, we leave the details to the reader. ◻

Note that a *-autonomous category can have many different maps $e \colon \mathbf{f} \to \mathbf{t}$ with the property of Theorem 4.1, each of them defining its own natural mix obeying (mix-$\hat{\sigma}$) and (mix-$\hat{\alpha}$).

5 $\vee$-Monoids and $\wedge$-comonoids

The structure investigated so far is exactly the same as for proofs in linear logic (with or without mix). For classical logic, we need to provide algebraic structure for the maps $\nabla_A \colon A \vee A \to A$ and $\amalg^A \colon \mathbf{f} \to A$, as well as $\Delta_A \colon A \to A \wedge A$ and $\Pi^A \colon A \to \mathbf{t}$, which are listed in (7). This is done via monoids and comonoids.

5.1 Definition A $\mathsf{B0}$-category has *commutative $\vee$-monoids* if it is symmetric $\vee$-monoidal and for every object A, the maps ∇_A and $\amalg^A$ obey the equations

$$\begin{tikzcd}
A \vee (A \vee A) \arrow[r, "A \vee \nabla_A"] \arrow[d, "\check{\alpha}_{A,A,A}"'] & A \vee A \arrow[dr, "\nabla_A"] & \\
 & & A \\
(A \vee A) \vee A \arrow[r, "\nabla_A \vee A"'] & A \vee A \arrow[ur, "\nabla_A"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \vee A \arrow[dr, "\nabla_A"] \arrow[dd, "\check{\sigma}_{A,A}"'] & \\
 & A \\
A \vee A \arrow[ur, "\nabla_A"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \vee \mathbf{f} \arrow[dr, "\check{\varrho}_A"] \arrow[dd, "A \vee \amalg^A"'] & \\
 & A \\
A \vee A \arrow[ur, "\nabla_A"'] &
\end{tikzcd}
\tag{22}$$

Dually, we say that a $\mathsf{B0}$-category has *cocommutative $\wedge$-comonoids* if it is symmetric $\wedge$-monoidal and for every object A, the maps Δ_A and Π^A obey the equations

$$\begin{tikzcd}
 & A \wedge A \arrow[r, "\Delta_A \wedge A"] & (A \wedge A) \wedge A \arrow[dd, "\hat{\alpha}^{-1}_{A,A,A}"] \\
A \arrow[ur, "\Delta_A"] \arrow[dr, "\Delta_A"'] & & \\
 & A \wedge A \arrow[r, "A \wedge \Delta_A"'] & A \wedge (A \wedge A)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \wedge A \arrow[dd, "\hat{\sigma}^{-1}_{A,A}"] \\
A \arrow[ur, "\Delta_A"] \arrow[dr, "\Delta_A"'] & \\
 & A \wedge A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \wedge A \arrow[dd, "A \wedge \Pi^A"] \\
A \arrow[ur, "\Delta_A"] \arrow[dr, "\hat{\varrho}^{-1}_A"'] & \\
 & A \wedge \mathbf{t}
\end{tikzcd}
\tag{23}$$

Translated into the language of the sequent calculus (cf. [FP04c]), having the structure of a $\vee$-monoid, i.e., the equations in (22), means

(i) to force the identification of the two possible proofs of the shape

$$\mathsf{contraction}\,\dfrac{\mathsf{contraction}\,\dfrac{\Lambda \vdash A, A, A, \Gamma}{\Lambda \vdash A, A, \Gamma}}{\Lambda \vdash A, \Gamma}$$

(ii) to identify the two proofs

$$\mathsf{contraction}\,\dfrac{\mathsf{exchange}\,\dfrac{\Lambda \vdash A, A, \Gamma}{\Lambda \vdash A, A, \Gamma}}{\Lambda \vdash A, \Gamma} \qquad \text{and} \qquad \mathsf{contraction}\,\dfrac{\Lambda \vdash A, A, \Gamma}{\Lambda \vdash A, \Gamma}$$

(iii) to say that the derivation

$$\mathsf{contraction}\,\dfrac{\mathsf{weakening}\,\dfrac{\Lambda \vdash A, \Gamma}{\Lambda \vdash A, A, \Gamma}}{\Lambda \vdash A, \Gamma}$$

is the same as doing nothing (i.e., the identity).

The equations in (23), i.e., the structure of an $\wedge$-comonoid, forces the same identification on the left-hand side of the turnstile. See [FP04c] for a detailed discussion of this correspondence.

5.2 Remark The (co)associativity of the maps Δ_A and ∇_A allows us to use the notation $\Delta^2_A\colon A \to A \wedge A \wedge A$ and $\nabla^2_A\colon A \vee A \vee A \to A$.

5.3 Proposition *Let $\mathscr{C}$ be a category with commutative $\vee$-monoids, and let*

$$\begin{tikzcd}
A \vee \mathbf{f} \arrow[d, "A\vee f"'] \arrow[dr, "\check{\varrho}_A"] & \\
& A \\
A \vee A \arrow[ur, "\nabla_A"'] &
\end{tikzcd}$$

commute for some $f\colon \mathbf{f} \to A$. Then $f = \amalg^A$.

Proof: This is a well-known fact from algebra: in a monoid the unit is uniquely defined. Written as diagram, the standard proof looks as follows:

$$\begin{tikzcd}
& & \mathbf{f} \arrow[dll, equal] \arrow[drr, equal] \arrow[d, "\check{\lambda}^{-1}_{\mathbf{f}} = \check{\varrho}^{-1}_{\mathbf{f}}"] & & \\
\mathbf{f} \arrow[dd, equal] & & \mathbf{f} \vee \mathbf{f} \arrow[ll, "\check{\lambda}_{\mathbf{f}}"] \arrow[rr, "\check{\varrho}_{\mathbf{f}}"'] \arrow[dl, "\mathbf{f}\vee f"'] \arrow[dr, "\amalg^A \vee \mathbf{f}"] & & \mathbf{f} \arrow[dd, equal] \\
& \mathbf{f} \vee A \arrow[r, "\amalg^A \vee A"] \arrow[dr, "\check{\lambda}_A"'] & A \vee A \arrow[d, "\nabla_A"] & A \vee \mathbf{f} \arrow[l, "A\vee f"'] \arrow[dl, "\check{\varrho}_A"] & \\
\mathbf{f} \arrow[rr, "f"'] & & A & & \mathbf{f} \arrow[ll, "\amalg^A"]
\end{tikzcd}$$

Note that in the same way it follows that the counit in a comonoid is uniquely defined. ☐

Athough the operations $\wedge$ and $\vee$ are *not* the product and coproduct in the category theoretical sense, we use the notation:

$$\langle f, g\rangle = (f \wedge g) \circ \Delta_A \colon A \to C \wedge D \quad \text{and} \quad [f, h] = \nabla_C \circ (f \vee h)\colon A \vee B \to C \tag{24}$$

where $f\colon A \to C$ and $g\colon A \to D$ and $h\colon B \to C$ are arbitrary maps.

Another helpful notation (see [LS05a]) is the following:

$$\begin{aligned}
\Pi^B_{A\square} &= \hat{\varrho}_A \circ (A \wedge \Pi^B)\colon A \wedge B \to A & \qquad \Pi^A_{\square B} &= \hat{\lambda}_B \circ (\Pi^A \wedge B)\colon A \wedge B \to B \\
\amalg^B_{A\square} &= (A \vee \amalg^B) \circ \check{\varrho}^{-1}_A \colon A \to A \vee B & \qquad \amalg^A_{\square B} &= (\amalg^A \vee B) \circ \check{\lambda}^{-1}_B \colon B \to A \vee B
\end{aligned} \tag{25}$$

Note that

$$\nabla_A \circ \amalg^A_{\square A} = 1_A = \nabla_A \circ \amalg^A_{A\square} \qquad \text{and} \qquad \Pi^A_{\square A} \circ \Delta_A = 1_A = \Pi^A_{A\square} \circ \Delta_A \tag{26}$$

5.4 Definition Let $f\colon A \to B$ be a map in a $\mathsf{B0}$-category with commutative $\vee$-monoids and cocommutative $\wedge$-comonoids. Consider the following four diagrams:

$$
\begin{tikzcd}
A \vee A \arrow[r, "f \vee f"] \arrow[d, "\nabla_A"'] & B \vee B \arrow[d, "\nabla_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbf{f} \arrow[dl, "\amalg^A"'] \arrow[dr, "\amalg^B"] & \\
A \arrow[rr, "f"'] & & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\Pi^A"'] & & B \arrow[dl, "\Pi^B"] \\
& \mathbf{t} &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\Delta_A"'] & B \arrow[d, "\Delta_B"] \\
A \wedge A \arrow[r, "f \wedge f"'] & B \wedge B
\end{tikzcd}
$$

We say that

- f *preserves the* $\vee$*-multiplication* if the left square commutes,
- f *preserves the* $\vee$*-unit* if the left triangle commutes,
- f *preserves the* $\wedge$*-counit* if the right triangle commutes,
- f *preserves the* $\wedge$*-comultiplication* if the right square commutes,
- f is a $\vee$*-monoid morphism* if the two left diagrams commute,
- f is a $\wedge$*-comonoid morphism* if the two right diagrams commute,
- f is a *quasientropy* if both triangles commute,
- f is *clonable* if both squares commute,
- f is *strong* if all four diagrams commute.

5.5 Definition A $\mathsf{B1}$*-category* is a $\mathsf{B0}$-category that is *-autonomous and has cocommutative $\wedge$-comonoids.

Clearly, a $\mathsf{B1}$-category does also have commutative $\vee$-monoids with ∇ dual to Δ, and $\amalg$ dual to Π.

5.6 Remark Definition 5.5 exhibits another "creative tension" between algebra and proof theory. From the algebraic point of view one should add the phrase "and all isomorphisms preserve the $\wedge$-comonoid structure" because in a semantics of proofs this will probably be inevitable. But here we do not assume it from the beginning, but systematically give conditions that will ensure it in the end (cf. Theorem 7.19 and Remark 7.20). From the proof theoretical view point this is more interesting because when seen sytactically, these conditions are more primitive. The reason is that in syntax the morphisms (i.e., proofs) come after the objects (i.e., formulas), and the formulas can always be decomposed into subformulas, whereas in semantics we have no access to the outermost connective. Furthermore, forcing all isomorphisms to preserve the $\wedge$-comonoid structure can cause identifications of proofs that might not necessarily be wanted by every proof theorist (see, e.g., Proposition 7.14).

5.7 Remark For each object A in a $\mathsf{B1}$-category $\mathscr{C}$, the identity map $1_A\colon A \to A$ is strong, and all kinds of maps defined in Definition 5.4 are closed under composition. Therefore, each kind defines a wide subcategory (i.e., a subcategory that has all objects) of $\mathscr{C}$, e.g., the wide subcategory of quasientropies, or the wide subcategory of $\vee$-monoid morphisms.

In a $\mathsf{B1}$-category we have two canonical maps $\mathbf{f} \to \mathbf{t}$, namely $\Pi^{\mathbf{f}}$ and $\amalg^{\mathbf{t}}$. Because of the $\wedge$-comonoid structure on $\mathbf{f}$ and the $\vee$-monoid structure on $\mathbf{t}$, we have

$$
\begin{tikzcd}
\mathbf{f} \vee \mathbf{t} \arrow[r, "\amalg^{\mathbf{t}} \vee \mathbf{t}"] \arrow[dr, "\check{\lambda}_{\mathbf{t}}"'] & \mathbf{t} \vee \mathbf{t} \arrow[d, "\nabla_{\mathbf{t}}"] & \mathbf{t} \vee \mathbf{f} \arrow[l, "\mathbf{t} \vee \amalg^{\mathbf{t}}"'] \arrow[dl, "\check{\varrho}_{\mathbf{t}}"] \\
& \mathbf{t} &
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
\mathbf{t} \wedge \mathbf{f} & \mathbf{f} \wedge \mathbf{f} \arrow[l, "\Pi^{\mathbf{f}} \wedge \mathbf{f}"'] \arrow[r, "\mathbf{f} \wedge \Pi^{\mathbf{f}}"] & \mathbf{f} \wedge \mathbf{t} \\
& \mathbf{f} \arrow[ul, "\hat{\lambda}_{\mathbf{f}}^{-1}"] \arrow[u, "\Delta_{\mathbf{f}}"'] \arrow[ur, "\hat{\varrho}_{\mathbf{f}}^{-1}"'] &
\end{tikzcd}
$$

(which even hold if the (co)monoids are not (co)commutative.) Since $\check{\lambda}_{\mathbf{t}}$, $\check{\varrho}_{\mathbf{t}}$, $\hat{\lambda}_{\mathbf{f}}$, and $\hat{\varrho}_{\mathbf{f}}$ are isomorphisms, we immediately can conclude that the following two diagrams commute (cf. [FP04a]):

$$
\begin{tikzcd}
\mathbf{t} \arrow[r, "\check{\lambda}_{\mathbf{t}}^{-1}"] \arrow[d, "\check{\varrho}_{\mathbf{t}}^{-1}"'] & \mathbf{f} \vee \mathbf{t} \arrow[d, "\amalg^{\mathbf{t}} \vee \mathbf{t}"] \\
\mathbf{t} \vee \mathbf{f} \arrow[r, "\mathbf{t} \vee \amalg^{\mathbf{t}}"'] & \mathbf{t} \vee \mathbf{t}
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
\mathbf{f} \wedge \mathbf{f} \arrow[r, "\mathbf{f} \wedge \Pi^{\mathbf{f}}"] \arrow[d, "\Pi^{\mathbf{f}} \wedge \mathbf{f}"'] & \mathbf{f} \wedge \mathbf{t} \arrow[d, "\hat{\varrho}_{\mathbf{f}}"] \\
\mathbf{t} \wedge \mathbf{f} \arrow[r, "\hat{\lambda}_{\mathbf{f}}"'] & \mathbf{f}
\end{tikzcd}
$$

By Section 4, this gives us two different mix maps $A \wedge B \to A \vee B$, and motivates the following definition:

5.8 Definition A $\mathsf{B1}$-category is called *single-mixed* if $\Pi^{\mathbf{f}} = \Pi^{\mathbf{t}}$.

In a single-mixed $\mathsf{B1}$-category we have, as the name says, a single canonical mix map $\mathsf{mix}_{A,B} \colon A \wedge B \to A \vee B$ obeying (mix-$\hat{\sigma}$) and (mix-$\hat{\alpha}$). The naturality of mix, i.e., the commutativity of

$$
\begin{tikzcd}
A \wedge B \arrow[r, "\mathsf{mix}_{A,B}"] \arrow[d, "f \wedge g"'] & A \vee B \arrow[d, "f \vee g"] \\
C \wedge D \arrow[r, "\mathsf{mix}_{C,D}"'] & C \vee D
\end{tikzcd}
\tag{27}
$$

for all maps $f \colon A \to C$ and $g \colon B \to D$, uniquely determines a map $f \between g \colon A \wedge B \to C \vee D$. Then, for every $f, g \colon A \to B$ we can define

$$
f + g = \nabla_B \circ (f \between g) \circ \Delta_A \colon A \to B \quad .
$$

It follows from (co)-associativity and (co)-commutativity of Δ and ∇, along with naturality of mix, that the operation $+$ on maps is associative and commutative. This gives us for $\mathrm{Hom}(A, B)$ a commutative semigroup structure.

Note that in general the semigroup structure on the Hom-sets is not an enrichment, e.g., $(f+g)h$ is in general not the same as $fh + gh$.

5.9 Definition Let $\mathscr{C}$ be a single-mixed $\mathsf{B1}$-category. Then $\mathscr{C}$ is called *idempotent* if for every A and B, the semigroup on $\mathrm{Hom}(A, B)$ is idempotent, i.e., for every $f \colon A \to B$ we have $f + f = f$.

In an idempotent $\mathsf{B1}$-category the semigroup structure on $\mathrm{Hom}(A, B)$ is in fact a sup-semilattice structure, given by $f \leq g$ iff $f + g = g$.

One can argue that the structure of $\mathsf{B1}$-categories is in some sense the minimum of algebraic structure that a Boolean category should have: *-autonomous categories provide the right structure for linear logic proofs, and the $\vee$-monoids and $\wedge$-comonoids seem to be exactly what is needed to "model contraction and weakening" in classical logic. There are certainly reasons to argue against that since it is by no means God-given that the proofs in classical logic obey the bijection ($\star$) nor that "contraction is associative". But let us, for the time being, assume that proofs in classical logic form a $\mathsf{B1}$-category. Then it is desirable that there is some more structure. This can be, for example, an agreement between the $\wedge$-monoidal structure (Definition 3.1) and the $\wedge$-comonoid structure (Definition 5.1), or, a more sophisticated condition like the commutativity of the diagram

$$
\begin{tikzcd}
((A \wedge B) \vee (A \wedge B)) \wedge (A \vee B) \arrow[r, "\nabla_{A \wedge B} \wedge (A \vee B)"] \arrow[d, "\mathsf{s}_{A \wedge B, A \wedge B, A \vee B}"'] & A \wedge B \wedge (A \vee B) \arrow[d, "\cong"] \\
(A \wedge B) \vee (A \wedge B \wedge (A \vee B)) \arrow[d, "\cong"'] & A \wedge (B \vee A) \wedge B \arrow[d, "\check{\mathsf{t}}_{A,B,A,B}"] \\
(A \wedge B) \vee (A \wedge (B \vee A) \wedge B) \arrow[d, "(A \wedge B) \vee \check{\mathsf{t}}_{A,B,A,B}"'] & (A \wedge B) \vee (A \wedge B) \arrow[d, "\nabla_{A \wedge B}"] \\
(A \wedge B) \vee (A \wedge B) \vee (A \wedge B) \arrow[r, "\nabla^2_{A \wedge B}"'] & A \wedge B
\end{tikzcd}
\tag{28}
$$

for all objects A and B. We now start to add the axioms for this.

5.10 Proposition *Let $\mathscr{C}$ be a* $\mathsf{B1}$*-category in which the equation*

$$\Pi^{\mathbf{t}} = 1_{\mathbf{t}} \colon \mathbf{t} \to \mathbf{t} \tag{B2a}$$

holds. Then we have that

(i) $\Delta_{\mathbf{t}} = \hat{\varrho}_{\mathbf{t}}^{-1} \colon \mathbf{t} \to \mathbf{t} \wedge \mathbf{t}$
(ii) *For all objects A, the map Π^A is a $\wedge$-comonoid morphism.*

Proof: The equation $\Delta_{\mathbf{t}} = \hat{\varrho}_{\mathbf{t}}^{-1}$ follows immediately from $\Pi^{\mathbf{t}} = 1_{\mathbf{t}}$ and the definition of $\wedge$-comonoids. That Π^A preserves the $\wedge$-counit is trivial and that it preserves the $\wedge$-comultiplication follows from

$$\begin{tikzcd}
A \arrow[rr, "\Pi^A"] \arrow[dr, "\hat{\varrho}_A^{-1}"'] \arrow[dd, "\Delta_A"'] & & \mathbf{t} \arrow[dd, "\Delta_{\mathbf{t}} = \hat{\varrho}_{\mathbf{t}}^{-1}"] \\
& A \wedge \mathbf{t} \arrow[dr, "\Pi^A \wedge \mathbf{t}"] & \\
A \wedge A \arrow[ur, "A \wedge \Pi^A"] \arrow[rr, "\Pi^A \wedge \Pi^A"'] & & \mathbf{t} \wedge \mathbf{t}
\end{tikzcd}$$

where the left triangle is the definition of $\wedge$-comonoids, the lower triangle is functoriality of $\wedge$ and the big "triangle" is naturality of $\hat{\varrho}$. $\square$

5.11 Lemma *If a* $\mathsf{B1}$*-category is single-mixed and obeys* ($\mathsf{B2a}$)*, then*

$$1_{\mathbf{t}} + 1_{\mathbf{t}} = 1_{\mathbf{t}} \qquad \textit{and} \qquad 1_{\mathbf{f}} + 1_{\mathbf{f}} = 1_{\mathbf{f}} \tag{29}$$

Proof: First, we show that

$$\Pi^{\mathbf{f}}_{\square\mathbf{f}} = \nabla_{\mathbf{f}} \circ \mathsf{mix}_{\mathbf{f},\mathbf{f}} \colon \mathbf{f} \wedge \mathbf{f} \to \mathbf{f} \tag{30}$$

This is done by chasing the diagram

$$\begin{tikzcd}
\mathbf{f} \wedge \mathbf{f} \arrow[r, "\check{\lambda}_{\mathbf{f}}^{-1} \wedge \mathbf{f}"] \arrow[d, "\Pi^{\mathbf{f}} \wedge \mathbf{f}"'] & (\mathbf{f} \vee \mathbf{f}) \wedge \mathbf{f} \arrow[d, "(\mathbf{f} \wedge \Pi^{\mathbf{f}}) \wedge \mathbf{f}"] \\
\mathbf{t} \wedge \mathbf{f} \arrow[r, "\check{\lambda}_{\mathbf{t}}^{-1} \wedge \mathbf{f}"] \arrow[dr, "\check{\lambda}_{\mathbf{t} \wedge \mathbf{f}}^{-1}"'] \arrow[dd, equal] & (\mathbf{f} \vee \mathbf{t}) \wedge \mathbf{f} \arrow[d, "\mathsf{s}_{\mathbf{f},\mathbf{t},\mathbf{f}}"] \\
& \mathbf{f} \vee (\mathbf{t} \wedge \mathbf{f}) \arrow[dl, "\check{\lambda}_{\mathbf{t} \wedge \mathbf{f}}"'] \arrow[d, "\mathbf{f} \vee \hat{\lambda}_{\mathbf{f}}"] \\
\mathbf{t} \wedge \mathbf{f} \arrow[d, "\hat{\lambda}_{\mathbf{f}}"'] & \mathbf{f} \vee \mathbf{f} \arrow[dl, "\check{\lambda}_{\mathbf{f}}"'] \arrow[d, "\nabla_{\mathbf{f}}"] \\
\mathbf{f} \arrow[r, equal] & \mathbf{f}
\end{tikzcd} \tag{31}$$

The right-down path is $\nabla_{\mathbf{f}} \circ \mathsf{mix}_{\mathbf{f},\mathbf{f}}$ and the left down path is $\Pi^{\mathbf{f}}_{\square\mathbf{f}}$. The two squares commute because of naturality of $\check{\lambda}$, the upper triangle holds because (13), the big triangle in the center is trivial, and that the lower triangle commutes follows from (the dual of) Proposition 5.10 (i). Now we can proceed:

$$1_{\mathbf{f}} = \Pi^{\mathbf{f}}_{\square\mathbf{f}} \circ \Delta_{\mathbf{f}} = \nabla_{\mathbf{f}} \circ \mathsf{mix}_{\mathbf{f},\mathbf{f}} \circ \Delta_{\mathbf{f}} = 1_{\mathbf{f}} + 1_{\mathbf{f}}$$

The equation $1_{\mathbf{t}} = 1_{\mathbf{t}} + 1_{\mathbf{t}}$ follows by duality. $\square$

Note that Lemma 5.11 is a consequence of having proper units. In the case of weak units (see [LS05b,LS05a]) it does not hold.

5.12 Proposition *In a* B1*-category that is single-mixed and obeys* (B2a)*, we have*

$$f + \amalg^A \;=\; f \tag{32}$$

for all maps $f\colon A \to \mathbf{t}$*. Dually, we have*

$$g + \amalg^B \;=\; g \tag{33}$$

for all maps $g\colon \mathbf{f} \to B$*.*

Proof: Chase the diagram

$$\begin{tikzcd}
A \arrow[r, "\Delta_A"] \arrow[d, equal] & A \wedge A \arrow[d, "A\wedge\amalg^A"] \\
A \arrow[d, "f"'] & A \wedge \mathbf{t} \arrow[l, "\hat{\varrho}_A"'] \arrow[d, "f\wedge\mathbf{t}"] \\
\mathbf{t} \arrow[dd, equal] \arrow[dr, "\Delta_{\mathbf{t}}"'] & \mathbf{t} \wedge \mathbf{t} \arrow[l, "\hat{\varrho}_{\mathbf{t}}"'] \arrow[d, equal] \\
 & \mathbf{t} \wedge \mathbf{t} \arrow[d, "\mathsf{mix}_{\mathbf{t},\mathbf{t}}"] \\
\mathbf{t} & \mathbf{t} \vee \mathbf{t} \arrow[l, "\nabla_{\mathbf{t}}"]
\end{tikzcd} \tag{34}$$

The first square is the comonoid equation, the second one is naturality of $\hat{\varrho}$, the triangle commutes because of Proposition 5.10 (i), and the lower quadrangle is (29). □

5.13 Proposition *In a* B1*-category obeying* (B2a)*, the equation*

$$\begin{tikzcd}
 & A \wedge B \arrow[dl, "\amalg^A\wedge\amalg^B"'] \arrow[dr, "\amalg^{A\wedge B}"] & \\
\mathbf{t} \wedge \mathbf{t} \arrow[rr, "\hat{\varrho}_{\mathbf{t}}"'] & & \mathbf{t}
\end{tikzcd} \tag{B2b}$$

holds if and only if

(i) $\amalg^{\mathbf{t}\wedge\mathbf{t}} = \hat{\varrho}_{\mathbf{t}}\colon \mathbf{t}\wedge\mathbf{t} \to \mathbf{t}$ *and*

(ii) *the maps that preserve the* $\wedge$*-counit are closed under* $\wedge$*.*

Proof: We see that (i) follows from (B2a) and (B2b) by plugging in $\mathbf{t}$ for A and B in (B2b). That (ii) holds follows from

$$\begin{tikzcd}
A \wedge B \arrow[rr, "f\wedge g"] \arrow[dr, "\amalg^A\wedge\amalg^B"] \arrow[ddr, "\amalg^{A\wedge B}"'] & & C \wedge D \arrow[dl, "\amalg^C\wedge\amalg^D"'] \arrow[ddl, "\amalg^{C\wedge D}"] \\
 & \mathbf{t} \wedge \mathbf{t} \arrow[d, "\hat{\varrho}_{\mathbf{t}}"] & \\
 & \mathbf{t} &
\end{tikzcd} \tag{35}$$

where $f\colon A \to B$ and $g\colon C \to D$ are maps that preserve the $\wedge$-counit. Conversely, it follows from (ii) and Proposition 5.10 that $\amalg^A \wedge \amalg^B$ preserves the $\wedge$-counit. With (i) this yields (B2b). □

5.14 Proposition *In a* B1*-category obeying* (B2a) *and* (B2b) *the maps* $\hat{\alpha}_{A,B,C}$, $\hat{\sigma}_{A,B}$, $\hat{\varrho}_A$, $\hat{\lambda}_A$, Π^A, $\Pi^A_{B\llbracket}$, *and* $\Pi^A_{\rrbracket B}$ *all preserve the* $\wedge$*-counit. And dually, the maps* $\check{\alpha}_{A,B,C}$, $\check{\sigma}_{A,B}$, $\check{\varrho}_A$, $\check{\lambda}_A$, $\amalg^A$, $\amalg^A_{B\llbracket}$, *and* $\amalg^A_{\rrbracket B}$ *all preserve the* $\vee$*-unit.*

Proof: We show the case for $\hat{\sigma}_{A,B}$:

$$\begin{tikzcd}
A\wedge B \arrow[rrrr, "\hat\sigma_{A,B}"] \arrow[dr, "\Pi^A\wedge\Pi^B"'] \arrow[ddrr, "\Pi^{A\wedge B}"'] & & & & B\wedge A \arrow[dl, "\Pi^B\wedge\Pi^A"] \arrow[ddll, "\Pi^{B\wedge A}"] \\
& \mathbf{t}\wedge\mathbf{t} \arrow[rr, "\hat\sigma_{\mathbf{t},\mathbf{t}}"] \arrow[dr, "\hat\varrho_{\mathbf{t}}"] & & \mathbf{t}\wedge\mathbf{t} \arrow[dl, "\hat\varrho_{\mathbf{t}}"'] & \\
& & \mathbf{t} & &
\end{tikzcd}$$

The quadrangle in naturality of $\hat{\sigma}$ and the commutativity of triangle in the center is a consequence of the coherence theorem for monoidal categories. The two slim triangles are just (B2b). The cases for $\hat{\alpha}_{A,B,C}$, $\hat{\varrho}_A$, $\hat{\lambda}_A$ are similar. For Π^A, it follows directly from (B2a) and for $\Pi^A_{B\llbracket}$ and $\Pi^A_{\rrbracket B}$ from Proposition 5.13 (ii) and from (25). $\square$

5.15 Proposition *If a* B1*-category obeys* (B2a) *and the equation*

$$\begin{tikzcd}
& A\wedge B \arrow[dl, "\Delta_A\wedge\Delta_B"'] \arrow[dr, "\Delta_{A\wedge B}"] & \\
A\wedge A\wedge B\wedge B \arrow[rr, "A\wedge\hat\sigma_{A,B}\wedge B"'] & & A\wedge B\wedge A\wedge B
\end{tikzcd} \tag{B2c}$$

then

(i) *also the equation* (B2b) *holds,*
(ii) *for every* A*, the map* Δ_A *is a* $\wedge$*-comonoid morphism, and*
(iii) *the maps that preserve the* $\wedge$*-comultiplication are closed under* $\wedge$*.*

Proof: (i) For showing that (B2b) holds, consider the diagram

$$\begin{tikzcd}
A\wedge B \arrow[rrr, equal] \arrow[dr, "\Delta_A\wedge\Delta_B"] \arrow[dd, "\Delta_{A\wedge B}"'] & & & A\wedge B \arrow[dl, "\hat\varrho_A^{-1}\wedge\hat\varrho_B^{-1}"'] \arrow[dd, "\hat\varrho^{-1}_{A\wedge B}"] \\
& A\wedge A\wedge B\wedge B \arrow[r, "A\wedge\Pi^A\wedge B\wedge\Pi^B"] \arrow[dl, "A\wedge\hat\sigma_{A,B}\wedge B"] & A\wedge\mathbf{t}\wedge B\wedge\mathbf{t} \arrow[d, "A\wedge\hat\sigma_{\mathbf{t},B}\wedge\mathbf{t}"] & \\
A\wedge B\wedge A\wedge B \arrow[rr, "A\wedge B\wedge\Pi^A\wedge\Pi^B"'] & & A\wedge B\wedge\mathbf{t}\wedge\mathbf{t} \arrow[r, "A\wedge B\wedge\hat\varrho_{\mathbf{t}}"'] & A\wedge B\wedge\mathbf{t}
\end{tikzcd}$$

The triangle on the left is (B2c), the upper quadrangle is the comonoid equation, the lower quadrangle is naturality of $\hat{\sigma}$ and the quadrangle on the right commutes because of the coherence in monoidal categories. The outer square says that $\hat{\varrho}_{\mathbf{t}} \circ (\Pi^A \wedge \Pi^B)$ is $\wedge$-counit for $\Delta_{A\wedge B}$. By Proposition 5.3 (uniqueness of units) it must therefore be equal to $\Pi^{A\wedge B}$. (ii) That Δ_A preserves the $\wedge$-comultiplication follows from

$$\begin{tikzcd}
A \arrow[rr, "\Delta_A"] \arrow[dd, "\Delta_A"'] & & A\wedge A \arrow[dl, "\Delta_A\wedge\Delta_A"'] \arrow[dd, "\Delta_{A\wedge A}"] \\
& A\wedge A\wedge A\wedge A \arrow[dr, "A\wedge\hat\sigma_{A,A}\wedge A"] & \\
A\wedge A \arrow[rr, "\Delta_A\wedge\Delta_A"'] & & A\wedge A\wedge A\wedge A
\end{tikzcd} \tag{36}$$

where the pentagon commutes because of the coassociativity and cocommutativity of $\Delta_A \colon A \to A \wedge A$. For showing that Δ_A preserves the $\wedge$-counit, consider the diagram

$$
\begin{tikzcd}
A \arrow[rr, "\Delta_A"] \arrow[dd, "\Pi^A"'] & & A \wedge A \arrow[dl, "\Pi^A \wedge \Pi^A"] \arrow[dd, "\Pi^{A \wedge A}"] \\
& \mathbf{t} \wedge \mathbf{t} \arrow[dr, "\hat{\varrho}_{\mathbf{t}}"] & \\
\mathbf{t} \arrow[ur, "\Delta_{\mathbf{t}}"] \arrow[rr, equal] & & \mathbf{t}
\end{tikzcd}
$$

The big and the lower triangle commute by Proposition 5.10, and the left triangle is ($\mathsf{B2b}$) which has been shown before. For (iii) chase

$$
\begin{tikzcd}
A \wedge B \arrow[rrr, "f \wedge g"] \arrow[dr, "\Delta_A \wedge \Delta_B"] \arrow[dd, "\Delta_{A \wedge B}"'] & & & C \wedge D \arrow[dl, "\Delta_C \wedge \Delta_D"'] \arrow[dd, "\Delta_{C \wedge D}"] \\
& A \wedge A \wedge B \wedge B \arrow[r, "f \wedge f \wedge g \wedge g"] \arrow[dl, "A \wedge \hat{\sigma}_{A,B} \wedge B"] & C \wedge C \wedge D \wedge D \arrow[dr, "C \wedge \hat{\sigma}_{C,D} \wedge D"'] & \\
A \wedge B \wedge A \wedge B \arrow[rrr, "f \wedge g \wedge f \wedge g"'] & & & C \wedge D \wedge C \wedge D
\end{tikzcd}
$$

where $f \colon A \to B$ and $g \colon C \to D$ are maps preserving the $\wedge$-comultiplication. $\square$

5.16 Proposition *In a* $\mathsf{B1}$*-category obeying* ($\mathsf{B2a}$) *and* ($\mathsf{B2c}$) *the maps* $\hat{\alpha}_{A,B,C}$, $\hat{\sigma}_{A,B}$, $\hat{\varrho}_A$, $\hat{\lambda}_A$, Π^A, $\Pi^B_{A\square}$, *and* $\Pi^A_{\square B}$ *all preserve the* $\wedge$*-comultiplication. Dually, the maps* $\check{\alpha}_{A,B,C}$, $\check{\sigma}_{A,B}$, $\check{\varrho}_A$, $\check{\lambda}_A$, $\amalg^A$, $\amalg^B_{A\square}$, *and* $\amalg^A_{\square B}$ *all preserve the* $\vee$*-multiplication.*

Proof: Again, we show the case only for $\hat{\sigma}$:

$$
\begin{tikzcd}
A \wedge B \arrow[rrr, "\hat{\sigma}_{A,B}"] \arrow[dr, "\Delta_A \wedge \Delta_B"] \arrow[dd, "\Delta_{A \wedge B}"'] & & & B \wedge A \arrow[dl, "\Delta_B \wedge \Delta_A"'] \arrow[dd, "\Delta_{B \wedge A}"] \\
& A \wedge A \wedge B \wedge B \arrow[r, "\hat{\sigma}_{A \wedge A, B \wedge B}"] \arrow[dl, "A \wedge \hat{\sigma}_{A,B} \wedge B"] & B \wedge B \wedge A \wedge A \arrow[dr, "B \wedge \hat{\sigma}_{B,A} \wedge A"'] & \\
A \wedge B \wedge A \wedge B \arrow[rrr, "\hat{\sigma}_{A,B} \wedge \hat{\sigma}_{A,B}"'] & & & B \wedge A \wedge B \wedge A
\end{tikzcd}
$$

The two triangles are ($\mathsf{B2c}$), the upper square is naturality of $\hat{\sigma}$ and the lower square commutes because of coherence in monoidal categories. For $\hat{\alpha}$, $\hat{\varrho}$, and $\hat{\lambda}$ the situation is similar. For Π^A it has been shown already in Proposition 5.10, and for $\Pi^B_{A\square}$, and $\Pi^A_{\square B}$ it follows from Proposition 5.15. $\square$

Propositions 5.10–5.15 give rise to the following definition:

5.17 Definition A $\mathsf{B2}$-*category* is a $\mathsf{B1}$-category which obeys equations ($\mathsf{B2a}$) and ($\mathsf{B2c}$) for all objects A and B.

The following theorem summarizes the properties of $\mathsf{B2}$-categories.

5.18 Theorem *In a* $\mathsf{B2}$*-category, the maps* $\hat{\alpha}_{A,B,C}$, $\hat{\sigma}_{A,B}$, $\hat{\varrho}_A$, $\hat{\lambda}_A$, Δ_A, Π^A, $\Pi^B_{A\square}$, *and* $\Pi^A_{\square B}$, *all are* $\wedge$*-comonoid morphisms, and the* $\wedge$*-comonoid morphisms are closed under* $\wedge$*. Dually, the maps* $\check{\alpha}_{A,B,C}$, $\check{\sigma}_{A,B}$, $\check{\varrho}_A$, $\check{\lambda}_A$, ∇_A, $\amalg^A$, $\amalg^B_{A\square}$, *and* $\amalg^A_{\square B}$, *all are* $\vee$*-monoid morphisms, and the* $\vee$*-monoid morphisms are closed under* $\vee$*.*

Proof: Propositions 5.10, 5.13, 5.14, 5.15, and 5.16. $\square$

5.19 Proposition *Let $f\colon A \to C$ and $g\colon A \to D$ and $h\colon B \to C$ and $a\colon A' \to A$ and $b\colon B' \to B$ and $c\colon C \to C'$ and $d\colon D \to D'$ be maps for some objects $A, B, C, D, A', B', C', D'$ in a* $\mathsf{B2}$*-category. As picture:*

$$\begin{tikzcd}
A' \arrow[r, "a"] & A \arrow[r, "f"] \arrow[dr, "g"'] & C \arrow[r, "c"] & C' \\
B' \arrow[r, "b"] & B \arrow[ur, "h"] & D \arrow[r, "d"] & D'
\end{tikzcd}$$

Then we have:

(i) $(c \wedge d) \circ \langle f, g\rangle = \langle c \circ f, d \circ g\rangle$.
(ii) *If a preserves the comultiplication, then $\langle f, g\rangle \circ a = \langle f \circ a, g \circ a\rangle$.*
(iii) *If g preserves the counit, then $\Pi^C_{\square D} \circ \langle f, g\rangle = f$.*
If f preserves the counit, then $\Pi^D_{C\square} \circ \langle f, g\rangle = g$.
(iv) $\langle \Pi^D_{C\square}, \Pi^C_{\square D}\rangle = 1_{C\wedge D}$.

Dually, we also have:

(i) $[f, h] \circ (a \vee b) = [f \circ a, h \circ b]$.
(ii) *If c preserves the multiplication, then $c \circ [f, h] = [c \circ f, c \circ h]$.*
(iii) *If h preserves the unit, then $[f, h] \circ \amalg^B_{A\square} = f$.*
If f preserves the unit, then $[f, h] \circ \amalg^A_{\square B} = h$.
(iv) $[\amalg^B_{A\square}, \amalg^A_{\square B}] = 1_{A\vee B}$.

Proof: Straightforward calculation. Note that (i)–(iii) hold already in a $\mathsf{B1}$-category, only for (iv) is the equation ($\mathsf{B2c}$) needed. ☐

As observed before, if a $\mathsf{B1}$-category is single-mixed then $\mathrm{Hom}(A, B)$ carries a semigroup structure. If we additionally have the structure of a $\mathsf{B2}$-category, then the bijection ($\star$) of Definition 3.2 preserves this semigroup structure:

5.20 Proposition *In a single-mixed* $\mathsf{B2}$*-category the bijection ($\star$) is a semigroup isomorphism.*

Proof: Let $f, g\colon A \wedge B \to C$ be two maps for some objects A, B, and C, and let $f', g'\colon A \to \bar{B} \vee C$ be their transposes. We have to show that $f' + g'$ is the transpose of $f + g$. First note, that in any *-autonomous category the map

$$A \wedge A \wedge B \wedge B \xrightarrow{A\wedge\hat{\sigma}_{A,B}\wedge B} A \wedge B \wedge A \wedge B \xrightarrow{f\wedge g} C \wedge C$$

is a transpose of

$$A \wedge A \xrightarrow{f'\wedge g'} (\bar{B} \vee C) \wedge (\bar{B} \vee C) \xrightarrow{\hat{\mathsf{t}}} \bar{B} \vee \bar{B} \vee (C \wedge C)$$

where $\hat{\mathsf{t}}$ is the canonical map obtained from two switches, cf. (9). Now, by definition, $f + g$ is the map

$$\begin{tikzcd}
A \wedge B \arrow[r, "\Delta_{A\wedge B}"] \arrow[dr, "\Delta_A\wedge\Delta_B"'] & A \wedge B \wedge A \wedge B \arrow[r, "f\wedge g"] & C \wedge C \arrow[r, "\mathsf{mix}_{C,C}"] & C \vee C \arrow[r, "\nabla_C"] & C \\
& A \wedge A \wedge B \wedge B \arrow[u, "A\wedge\hat{\sigma}_{A,B}\wedge B"'] & & &
\end{tikzcd}$$

By (8) and what has been said above, the transpose of the lower path is the outermost path of the following:

$$\begin{tikzcd}
A \arrow[r, "\Delta_A"] & A \wedge A \arrow[r, "f'\wedge g'"] & (\bar{B} \vee C) \wedge (\bar{B} \vee C) \arrow[r, "\hat{\mathsf{t}}"] \arrow[d, "\mathsf{mix}_{\bar{B}\vee C,\bar{B}\vee C}"'] & \bar{B} \vee \bar{B} \vee (C \wedge C) \arrow[d, "\bar{B}\vee\bar{B}\vee\mathsf{mix}_{C,C}"] \\
& & \bar{B} \vee C \vee \bar{B} \vee C \arrow[r, "\bar{B}\vee\hat{\sigma}_{C,\bar{B}}\vee C"] \arrow[dr, "\nabla_{\bar{B}\vee C}"'] & \bar{B} \vee \bar{B} \vee C \vee C \arrow[d, "\nabla_{\bar{B}}\vee\nabla_C"] \\
& & & \bar{B} \vee C
\end{tikzcd}$$

The innermost path is by definition $f' + g'$. The square commutes because of (mix-$\hat{\sigma}$) and (mix-$\hat{\alpha}$), and the triangle is the dual of ($\mathsf{B2c}$). $\square$

6 Order enrichment

In [FP04c], Führman and Pym equipped $\mathsf{B2}$-categories with an order enrichment, such that the proof identifications induced by the axioms are exactly the same as the proof identifications made by Gentzen's sequent calculus LK [Gen34], modulo "trivial rule permutations" (see [Laf95,Rob03]), and such that $f \preccurlyeq g$ if g is obtained from f via cut elimination (which is not confluent in LK).

6.1 Definition A $\mathsf{B2}$-category is called an LK-*category* if for every A, B, the set $\mathrm{Hom}(A, B)$ is equipped with a partial order structure $\preccurlyeq$ such that

(i) the arrow composition $\circ$, as well as the bifunctors $\wedge$ and $\vee$ are monotonic in both arguments,

(ii) for every map $f\colon A \to B$ we have

$$\Pi^B \circ f \ \preccurlyeq\ \Pi^A \tag{LK-Π}$$

$$\Delta_B \circ f \ \preccurlyeq\ (f \wedge f) \circ \Delta_A \tag{LK-Δ}$$

(iii) and the bijection ($\star$) of Definition 3.2 is an order isomorphism for $\preccurlyeq$.

In [FP04c,FP04b] Führmann and Pym use the term "classical category". We use here the term LK-categories because—as worked out in detail in [FP04c]—they provide a category theoretical axiomatisation of sequent calculus proofs in Gentzens's system LK [Gen34]. However, it should be clear that LK-categories are only one particular example of a wide range of possible category theoretical axiomatisations of proofs in classical logic.

6.2 Remark In [FP04c], Führmann and Pym give a different definition for LK-categories. Since they start from a weakly distributive category [CS97b] instead of a *-autonomous one, they do not have immediate access to transposition. For this reason, they have to give a larger set of inequalities, defining the order $\preccurlyeq$:

$$\begin{array}{ll}
\Delta_B \circ f \preccurlyeq (f \wedge f) \circ \Delta_A & f \circ \nabla_A \preccurlyeq \nabla_B \circ (f \vee f) \\
\Pi^B \circ f \preccurlyeq \Pi^A & f \circ \amalg^A \preccurlyeq \amalg^B \\
A \vee \Delta_B \preccurlyeq (\nabla_A \vee (B \wedge B)) \circ \hat{\mathsf{t}} \circ \Delta_{A \vee B} & A \wedge \nabla_B \preccurlyeq \Delta_{A \wedge B} \circ \check{\mathsf{t}} \circ (\Delta_A \wedge (B \vee B)) \\
A \vee \Pi^B \preccurlyeq (\amalg^A \vee \mathbf{t}) \circ \check{\lambda}_{\mathbf{t}}^{-1} \circ \Pi^{A \vee B} & A \wedge \amalg^B \preccurlyeq \amalg^{A \wedge B} \circ \hat{\lambda}_{\mathbf{f}} \circ (\Pi^A \wedge \mathbf{f})
\end{array} \tag{FP}$$

where $f\colon A \to B$ is an arbitrary map and $\hat{\mathsf{t}}\colon (A \vee B) \wedge (A \vee B) \to A \vee A \vee (B \wedge B)$ and $\check{\mathsf{t}}\colon A \wedge A \wedge (B \vee B) \to (A \wedge B) \vee (A \wedge B)$ are the tensor and cotensor map, cf. (9) and (10). One can now easily show that both definitions are equivalent: Clearly the inequations on the right in (FP) are just transposes of the ones on the left. The two top ones on the left are just (LK-Π) and (LK-Δ), and the two bottom ones follow as follows. If we transpose $A \vee B \xrightarrow{A \vee \Delta_B} A \vee (B \wedge B)$ we get the map

$$\bar{A} \wedge (A \vee B) \xrightarrow{\mathsf{eval}} B \xrightarrow{\Delta_B} B \wedge B$$

By (LK-Δ), this is smaller or equal to

$$\bar{A} \wedge (A \vee B) \xrightarrow{\Delta_{\bar{A} \wedge (A \vee B)}} \bar{A} \wedge (A \vee B) \wedge \bar{A} \wedge (A \vee B) \xrightarrow{\mathsf{eval} \wedge \mathsf{eval}} B \wedge B$$

By ($\mathsf{B2c}$) this is the same map as

$$\bar{A} \wedge (A \vee B) \xrightarrow{\Delta_{\bar{A}} \wedge \Delta_{A \vee B}} \bar{A} \wedge \bar{A} \wedge (A \vee B) \wedge (A \vee B) \xrightarrow{\cong} \bar{A} \wedge (A \vee B) \wedge \bar{A} \wedge (A \vee B) \xrightarrow{\mathsf{eval} \wedge \mathsf{eval}} B \wedge B$$

Transposing back yields

$$A \vee B \xrightarrow{\Delta_{A\vee B}} (A \vee B) \wedge (A \vee B) \xrightarrow{\hat{\mathsf{t}}} A \vee A \vee (B \wedge B) \xrightarrow{\nabla_A} A \vee (B \wedge B)$$

This shows the third inequation on the left in (FP). For the last one, we proceed similar: Transposing $A \vee B \xrightarrow{A \vee \Pi^B} A \vee \mathbf{t}$ yields

$$\bar{A} \wedge (A \vee B) \xrightarrow{\mathsf{eval}} B \xrightarrow{\Pi^B} \mathbf{t}$$

which is by (LK-Π) smaller or equal to

$$\bar{A} \wedge (A \vee B) \xrightarrow{\Pi^{\bar{A} \wedge (A \vee B)}} \mathbf{t}$$

which is by ($\mathsf{B2b}$) and (25) the same as

$$\bar{A} \wedge (A \vee B) \xrightarrow{\Pi^{\bar{A}}_{\Box A \vee B}} A \vee B \xrightarrow{\Pi^{A \vee B}} \mathbf{t}$$

If transpose back, we get

$$A \vee B \xrightarrow{\Pi^{A \vee B}} \mathbf{t} \xrightarrow{\amalg^{A}_{\Box \mathbf{t}}} A \vee \mathbf{t}$$

as desired. We do not show here the other direction because it is rather tedious: It is almost literally the same as the proof for showing that any weakly distributive category with negation is a *-autonomous category (see [CS97b,BCST96]).

The following theorem states the main properties of LK-categories. It has first been observed and proved by Führmann and Pym in [FP04a].

6.3 Theorem *Every* LK*-category is single-mixed and idempotent. Furthermore, for all maps* $f, g \colon A \to B$*, we have* $f \leq g$ *iff* $g \preccurlyeq f$*.*

Proof: Because of ($\mathsf{B2a}$) and (LK-Π) we have that $\amalg^{\mathbf{t}} = 1_{\mathbf{t}} \circ \amalg^{\mathbf{t}} = \Pi^{\mathbf{t}} \circ \amalg^{\mathbf{t}} \preccurlyeq \Pi^{\mathbf{f}}$. By duality, we also get $\Pi^{\mathbf{f}} \preccurlyeq \amalg^{\mathbf{t}}$. Therefore $\Pi^{\mathbf{f}} = \amalg^{\mathbf{t}}$, i.e., the category is single-mixed. Next, we show that $f + g \preccurlyeq f$ for all maps $f, g \colon A \to B$. For this, note that

$$A \wedge B \xrightarrow{\mathsf{mix}_{A,B}} A \vee B \quad \preccurlyeq \quad A \wedge B \xrightarrow{\Pi^{A}_{\Box B}} B \xrightarrow{\amalg^{A}_{\Box B}} A \vee B$$

because these are the transposes of

$$A \wedge \bar{A} \xrightarrow{\hat{\mathsf{i}}_A} \mathbf{f} \xrightarrow{\Pi^{\mathbf{f}}} \mathbf{t} \xrightarrow{\check{\mathsf{i}}_B} B \vee \bar{B} \quad \preccurlyeq \quad A \wedge \bar{A} \xrightarrow{\Pi^{A \wedge \bar{A}}} \mathbf{t} \xrightarrow{\check{\mathsf{i}}_B} B \vee \bar{B}$$

Now we can proceed as follows:

$$\begin{aligned}
f + g &= \nabla_B \circ (f \vee g) \circ \mathsf{mix}_{A,A} \circ \Delta_A \\
&\preccurlyeq \nabla_B \circ (f \vee g) \circ \amalg^{A}_{\Box A} \circ \Pi^{A}_{\Box A} \circ \Delta_A \\
&= \nabla_B \circ (f \vee g) \circ \amalg^{A}_{\Box A} \circ 1_A \\
&= \nabla_B \circ (f \vee B) \circ (A \vee g) \circ (A \vee \amalg^{A}) \circ \check{\varrho}_A \\
&= \nabla_B \circ (f \vee B) \circ (A \vee g \circ \amalg^{A}) \circ \check{\varrho}_A \\
&\preccurlyeq \nabla_B \circ (f \vee B) \circ (A \vee \amalg^{B}) \circ \check{\varrho}_A \\
&= \nabla_B \circ \amalg^{B}_{\Box B} \circ f \\
&= f
\end{aligned}$$

Similarly, we get $f + g \preccurlyeq g$. Now we show that $f \preccurlyeq f + f$ for $f\colon A \to B$. Let $\hat{f}\colon A \wedge \bar{B} \to \mathbf{f}$ be the transpose of f. Then we have

$$
\begin{aligned}
\hat{f} &= 1_{\mathbf{f}} \circ \hat{f} \\
&= (1_{\mathbf{f}} + 1_{\mathbf{f}}) \circ \hat{f} \\
&= \nabla_{\mathbf{f}} \circ \mathsf{mix}_{\mathbf{f},\mathbf{f}} \circ \Delta_{\mathbf{f}} \circ \hat{f} \\
&\preccurlyeq \nabla_{\mathbf{f}} \circ \mathsf{mix}_{\mathbf{f},\mathbf{f}} \circ (\hat{f} \wedge \hat{f}) \circ \Delta_{A \wedge \bar{B}} \\
&= \hat{f} + \hat{f} \\
&= \widehat{f + f}
\end{aligned}
$$

The second equation is Lemma 5.11, the third one is the definition of $+$, the fourth one is (LK-Δ), the fifth again the definition of $+$, and the last equation uses Proposition 5.20. By transposing back, we get $f \preccurlyeq f + f$. From this together with $f + f \preccurlyeq f$ we get idempotency. For showing that $f \leq g$ iff $g \preccurlyeq f$, we need to show that $g \preccurlyeq f$ iff $f + g = g$. Since $f + g \preccurlyeq f$, we have that $f + g = g$ implies $g \preccurlyeq f$. Now suppose $g \preccurlyeq f$. Then we have $g = g + g \preccurlyeq f + g$. This finishes the proof since $f + g \preccurlyeq g$ has been shown already. □

Note that the converse is not necessarily true. Not every single-mixed idempotent $\mathsf{B2}$-category is an LK-category. Nonetheless, because of Proposition 5.12, in every single-mixed idempotent $\mathsf{B2}$-category we have for every $f\colon A \to B$ that $\Pi^B \circ f + \Pi^A = \Pi^B \circ f$, and hence $\Pi^A \leq \Pi^B \circ f$ which is exactly (LK-Π). However, the inequality (LK-Δ) does not follow from idempotency. One can easily construct countermodels along the lines of [Str05] (see also Section 9).

7 The medial map and the nullary medial map

That LK-categories are idempotent means that they are already at the degenerate end of the spectrum of Boolean categories. On the other hand, $\mathsf{B2}$-categories have (apart from Theorem 5.18) very little structure. The question that arises now is therefore, how we can add additional structure to $\mathsf{B2}$-categories without getting too much collapse. In particular, can we extend the structure such that all the maps mentioned in Therorem 5.18 become $\vee$-monoid morphisms *and* $\wedge$-comonoid morphisms? This is where medial enters the scene.

7.1 Definition We say, a $\mathsf{B2}$-category $\mathscr{C}$ *has medial* if for all objects A, B, C, and D there is a map $\mathsf{m}_{A,B,C,D}\colon (A \wedge B) \vee (C \wedge D) \to (A \vee C) \wedge (B \vee D)$ with the following properties:

- it is natural in A, B, C and D,
- it is self-dual, i.e.,

$$
\begin{tikzcd}
\overline{(A \vee C) \wedge (B \vee D)} \arrow[r, "\overline{\mathsf{m}_{A,B,C,D}}"] \arrow[d, "\cong"'] & \overline{(A \wedge B) \vee (C \wedge D)} \arrow[d, "\cong"] \\
(\bar{D} \wedge \bar{B}) \vee (\bar{C} \wedge \bar{A}) \arrow[r, "\mathsf{m}_{\bar{D},\bar{B},\bar{C},\bar{A}}"'] & (\bar{D} \vee \bar{C}) \wedge (\bar{B} \vee \bar{A})
\end{tikzcd}
\tag{37}
$$

  commutes, where the vertical maps are the cononical isomophisms induced by Definition 3.2,
- and it obeys the equation

$$
\begin{tikzcd}
 & A \vee B \arrow[dl, "\Delta_A \vee \Delta_B"'] \arrow[dr, "\Delta_{A \vee B}"] & \\
(A \wedge A) \vee (B \wedge B) \arrow[rr, "\mathsf{m}_{A,A,B,B}"'] & & (A \vee B) \wedge (A \vee B)
\end{tikzcd}
\tag{B3c}
$$

  for all objects A and B.

The following equation is a consequence of (B3c) and the self-duality of medial.

$$
\begin{tikzcd}
(A \wedge B) \vee (A \wedge B) \arrow[rr, "\mathsf{m}_{A,B,A,B}"] \arrow[dr, "\nabla_{A \wedge B}"'] & & (A \vee A) \wedge (B \vee B) \arrow[dl, "\nabla_A \wedge \nabla_B"] \\
& A \wedge B &
\end{tikzcd}
\tag{B3c$'$}
$$

7.2 Theorem *Let $\mathscr{C}$ be a* B2*-category that has medial. Then*

(i) *The maps that preserve the $\wedge$-comultiplication are closed under $\vee$, and dually, the maps that preserve the $\vee$-multiplication are closed under $\wedge$.*

(ii) *For all maps $A \xrightarrow{f} C$, $A \xrightarrow{g} D$, $B \xrightarrow{h} C$, and $B \xrightarrow{k} D$, we have that*

$$[\langle f, g\rangle, \langle h, k\rangle] = \langle [f, h], [g, k]\rangle \colon A \vee B \to C \wedge D \quad .$$

(iii) *For all objects A, B, C, and D,*

$$
\begin{aligned}
\mathsf{m}_{A,B,C,D} &= \left[\left\langle \amalg^C_{A\Box} \circ \Pi^B_{A\Box}, \amalg^D_{B\Box} \circ \Pi^A_{\Box B} \right\rangle, \left\langle \amalg^A_{\Box C} \circ \Pi^D_{C\Box}, \amalg^B_{\Box D} \circ \Pi^C_{\Box D} \right\rangle \right] \\
&= \left\langle \left[\amalg^C_{A\Box} \circ \Pi^B_{A\Box}, \amalg^A_{\Box C} \circ \Pi^D_{C\Box} \right], \left[\amalg^D_{B\Box} \circ \Pi^A_{\Box B}, \amalg^B_{\Box D} \circ \Pi^C_{\Box D} \right] \right\rangle
\end{aligned}
$$

(iv) *For all objects A, B, C, and D, the following diagram commutes:*

$$
\begin{tikzcd}
& ((A \wedge B) \vee (C \wedge D)) \wedge ((A \wedge B) \vee (C \wedge D)) \arrow[dr, "(\Pi^B_{A\Box} \vee \Pi^D_{C\Box}) \wedge (\Pi^A_{\Box B} \vee \Pi^C_{\Box D})"] & \\
(A \wedge B) \vee (C \wedge D) \arrow[ur, "\Delta_{(A \wedge B) \vee (C \wedge D)}"] \arrow[dr, "(\amalg^C_{A\Box} \wedge \amalg^D_{B\Box}) \vee (\amalg^A_{\Box C} \wedge \amalg^B_{\Box D})"'] & & (A \vee C) \wedge (B \vee D) \\
& ((A \vee C) \wedge (B \vee D)) \vee ((A \vee C) \wedge (B \vee D)) \arrow[ur, "\nabla_{(A \vee C) \wedge (B \vee D)}"'] &
\end{tikzcd}
\tag{38}
$$

(v) *The horizontal diagonal of* (38) *is equal to $\mathsf{m}_{A,B,C,D}$.*

Proof: For (i), chase the following (compare with the proof of Proposition 5.15 (iii))

$$
\begin{tikzcd}
A \vee B \arrow[rrr, "f \vee g"] \arrow[dr, "\Delta_A \vee \Delta_B"] \arrow[dd, "\Delta_{A \vee B}"'] & & & C \vee D \arrow[dl, "\Delta_C \vee \Delta_D"'] \arrow[dd, "\Delta_{C \vee D}"] \\
& (A \wedge A) \vee (B \wedge B) \arrow[r, "(f \wedge f) \vee (g \wedge g)"] \arrow[dl, "\mathsf{m}_{A,A,B,B}"] & (C \wedge C) \vee (D \wedge D) \arrow[dr, "\mathsf{m}_{C,C,D,D}"'] & \\
(A \vee B) \wedge (A \vee B) \arrow[rrr, "(f \vee g) \wedge (f \vee g)"'] & & & (C \vee D) \wedge (C \vee D)
\end{tikzcd}
$$

For (ii) chase the diagram

$$
\begin{tikzcd}
A \vee B \arrow[rrr, "{[\langle f,g\rangle, \langle h,k\rangle]}"] \arrow[drr, "\langle f,g\rangle \vee \langle h,k\rangle"'] \arrow[dd, equal] & & & C \wedge D \arrow[dd, equal] \\
& (A \wedge A) \vee (B \wedge B) \arrow[r, "(f \wedge g) \vee (h \wedge k)"'] \arrow[dd, "\mathsf{m}_{A,A,B,B}"] & (C \wedge D) \vee (C \wedge D) \arrow[dr, "\nabla_{C \wedge D}"] \arrow[dd, "\mathsf{m}_{C,D,C,D}"] & \\
A \vee B \arrow[ur, "\Delta_A \vee \Delta_B"] \arrow[dr, "\Delta_{A \vee B}"'] \arrow[dd, equal] & & & C \wedge D \arrow[dd, equal] \\
& (A \vee B) \wedge (A \vee B) \arrow[r, "(f \vee h) \wedge (g \vee k)"] \arrow[drr, "{[f,h] \wedge [g,k]}"] & (C \vee C) \wedge (D \vee D) \arrow[ur, "\nabla_C \wedge \nabla_D"'] & \\
A \vee B \arrow[rrr, "{\langle [f,h], [g,k] \rangle}"'] & & & C \wedge D
\end{tikzcd}
$$

where the square in the center is naturality of medial, the two small triangles are (B3c) and (B3c′). The big triangles are just (24). Note the importance of naturality of medial in the two diagrams above. Let us now continue with (iv) and (v), which are proved by

$$
\begin{tikzcd}
(A \wedge B) \vee (C \wedge D) \arrow[rr, "\Delta_{(A\wedge B)\vee(C\wedge D)}"] \arrow[dr, "\Delta_{A\wedge B}\vee\Delta_{C\wedge D}"'] \arrow[dd, "(\Delta_A\wedge\Delta_B)\vee(\Delta_C\wedge\Delta_D)"'] & & ((A \wedge B) \vee (C \wedge D)) \wedge ((A \wedge B) \vee (C \wedge D)) \arrow[dd, "(\Pi^B_{A\Box}\vee\Pi^D_{C\Box})\wedge(\Pi^A_{\Box B}\vee\Pi^C_{\Box D})"] \\
& (A \wedge B \wedge A \wedge B) \vee (C \wedge D \wedge C \wedge D) \arrow[ur, "\mathsf{m}"'] \arrow[dd, "(\Pi^B_{A\Box}\wedge\Pi^A_{\Box B})\vee(\Pi^D_{C\Box}\wedge\Pi^C_{\Box D})"] & \\
(A \wedge A \wedge B \wedge B) \vee (C \wedge C \wedge D \wedge D) \arrow[ur, "\cong"'] \arrow[dr, "(\Pi^A_{A\Box}\wedge\Pi^B_{\Box B})\vee(\Pi^C_{C\Box}\wedge\Pi^D_{\Box D})"'] & & (A \vee C) \wedge (B \vee D) \\
& (A \wedge B) \vee (C \wedge D) \arrow[ur, "\mathsf{m}"'] &
\end{tikzcd}
$$

and (26) and the self-duality of medial. It remains to show (iii). For this consider

$$
\begin{tikzcd}
(A \wedge B) \vee (C \wedge D) \arrow[r, "\Delta_{(A\wedge B)\vee(C\wedge D)}"] \arrow[d, "\mathsf{m}_{A,B,C,D}"'] & ((A \wedge B) \vee (C \wedge D)) \wedge ((A \wedge B) \vee (C \wedge D)) \arrow[dl, "(\Pi^B_{A\Box}\vee\Pi^D_{C\Box})\wedge(\Pi^A_{\Box B}\vee\Pi^C_{\Box D})"] \arrow[d, "\begin{array}{l}(\amalg^C_{A\Box}\circ\Pi^B_{A\Box}\vee\amalg^A_{\Box C}\circ\Pi^D_{C\Box})\wedge\\ \quad(\amalg^D_{B\Box}\circ\Pi^A_{\Box B}\vee\amalg^B_{\Box D}\circ\Pi^C_{\Box D})\end{array}"] \\
(A \vee C) \wedge (B \vee D) \arrow[r, "(\amalg^C_{A\Box}\vee\amalg^A_{\Box C})\wedge(\amalg^D_{B\Box}\vee\amalg^B_{\Box D})"'] \arrow[d, "(\amalg^A_{\Box A}\vee\amalg^C_{C\Box})\wedge(\amalg^B_{\Box B}\vee\amalg^D_{D\Box})"'] & (A \vee C \vee A \vee C) \wedge (B \vee D \vee B \vee D) \arrow[d, "\nabla_{A\vee C}\wedge\nabla_{B\vee D}"] \\
(A \vee A \vee C \vee C) \wedge (B \vee B \vee D \vee D) \arrow[ur, "(A\vee\check{\sigma}_{A,C}\vee C)\wedge(B\vee\check{\sigma}_{B,D}\vee D)"'] \arrow[r, "(\nabla_A\vee\nabla_C)\wedge(\nabla_B\vee\nabla_D)"'] & (A \vee C) \wedge (B \vee D)
\end{tikzcd}
$$

The topmost triangle is (v), the two middle ones are trivial, and the bottommost triangle is (B2c). Note that the first-right-then-down path is

$$
\left\langle \left[\amalg^C_{A\Box} \circ \Pi^B_{A\Box}, \amalg^A_{\Box C} \circ \Pi^D_{C\Box} \right], \left[\amalg^D_{B\Box} \circ \Pi^A_{\Box B}, \amalg^B_{\Box D} \circ \Pi^C_{\Box D} \right] \right\rangle
$$

by definition, and the first-down-then-right path is $\mathsf{m}_{A,B,C,D}$ because of (26). We get (iii) by self-duality of medial. $\Box$

7.3 Remark Because of (iii) and (v) in Theorem 7.2, we could obtain a *weak medial map* by adding (iv) or (ii) as axiom to a B2-category. This weak medial map would be self-dual. By also adding Theorem 7.2 (i) as axiom, we could even recover equations (B3c) and (B3c′), as the following diagram shows:

$$
\begin{tikzcd}
A \vee B \arrow[r, "\Delta_{A\vee B}"] \arrow[d, "\Delta_A\vee\Delta_B"'] & (A \vee B) \wedge (A \vee B) \arrow[r, equal] \arrow[dr, "(\Delta_A\vee\Delta_B)\wedge(\Delta_A\vee\Delta_B)"'] & (A \vee B) \wedge (A \vee B) \\
(A \wedge A) \vee (B \wedge B) \arrow[rr, "\Delta_{(A\wedge A)\vee(B\wedge B)}"'] & & ((A \wedge A) \vee (B \wedge B)) \wedge ((A \wedge A) \vee (B \wedge B)) \arrow[u, "(\Pi^A_{A\Box}\vee\Pi^B_{B\Box})\wedge(\Pi^A_{\Box A}\vee\Pi^B_{\Box B})"']
\end{tikzcd}
$$

where the left square says that $\Delta_A \vee \Delta_B$ preserves the $\wedge$-comultiplication. However, by doing this, we would *not* get naturality of medial, which is crucial for algebraic as well as proof theoretical reasons (see also the introduction).

7.4 Definition We say, a B2-category $\mathscr{C}$ *has nullary medial* if there is a map $\check{\mathsf{nm}} \colon \mathbf{t} \vee \mathbf{t} \to \mathbf{t}$ (called the *nullary medial map*) such that for all objects A, B, the following holds:

$$
\begin{tikzcd}
& A \vee B \arrow[dl, "\Pi^A\vee\Pi^B"'] \arrow[dr, "\Pi^{A\vee B}"] & \\
\mathbf{t} \vee \mathbf{t} \arrow[rr, "\check{\mathsf{nm}}"'] & & \mathbf{t}
\end{tikzcd}
\tag{B3b}
$$

Clearly, if a a $\mathsf{B2}$-category has nullary medial, then $\mathsf{n}\breve{\mathsf{m}} = \Pi^{\mathbf{t}\vee\mathbf{t}}$. This can be seen by plugging in $\mathbf{t}$ for A and B in ($\mathsf{B3b}$). By duality $\amalg^{\mathbf{f}\wedge\mathbf{f}} = \mathsf{n}\hat{\mathsf{m}} \colon \mathbf{f} \to \mathbf{f} \wedge \mathbf{f}$ (the *nullary comedial map*) obeys the dual of ($\mathsf{B3b}$).

7.5 Proposition *In a $\mathsf{B2}$-category $\mathscr{C}$ that has nullary medial, we have that*

(i) *The maps that preserve the $\wedge$-counit are closed under $\vee$, and dually, the maps that preserve the $\vee$-unit are closed under $\wedge$.*
(ii) *For all objects A, B, C, the map $\mathsf{s}_{A,B,C}$ is a quasientropy.*

Proof: For showing the first statement, replace in (35) every $\wedge$ by an $\vee$, and $\hat{\varrho}_{\mathbf{t}}$ by $\mathsf{n}\breve{\mathsf{m}}$. The second statement is shown by

$$\begin{tikzcd}
(A \vee B) \wedge C \arrow[rr, "\mathsf{s}_{A,B,C}"] \arrow[d, "(\Pi^A \vee \Pi^B) \wedge \Pi^C"'] & & A \vee (B \wedge C) \arrow[d, "\Pi^A \vee (\Pi^B \wedge \Pi^C)"] \\
(\mathbf{t} \vee \mathbf{t}) \wedge \mathbf{t} \arrow[rr, "\mathsf{s}_{\mathbf{t},\mathbf{t},\mathbf{t}}"] \arrow[d, "\mathsf{n}\breve{\mathsf{m}} \wedge \mathbf{t}"'] \arrow[dr, "\hat{\varrho}_{\mathbf{t}\vee\mathbf{t}}"'] & & \mathbf{t} \vee (\mathbf{t} \wedge \mathbf{t}) \arrow[dl, "\mathbf{t} \vee \hat{\varrho}_{\mathbf{t}}"] \\
\mathbf{t} \wedge \mathbf{t} \arrow[dr, "\hat{\varrho}_{\mathbf{t}}"'] & \mathbf{t} \vee \mathbf{t} \arrow[d, "\mathsf{n}\breve{\mathsf{m}}"] & \\
& \mathbf{t} &
\end{tikzcd}$$

where the left down-path is $\Pi^{(A\vee B)\wedge C}$ and the right down-path is $\Pi^{A\vee(B\wedge C)}$ (because of ($\mathsf{B2b}$) and ($\mathsf{B3b}$)). The two squares are naturality of s and $\hat{\varrho}$, and the triangle at the center is just (13). Hence, switch preserves the $\wedge$-counit, and by duality also the $\vee$-unit. $\square$

7.6 Proposition *Let $\mathscr{C}$ be a $\mathsf{B2}$-category with medial and nullary medial. Then $\mathscr{C}$ obeys the equation*

$$\begin{tikzcd}
(A \wedge \mathbf{t}) \vee (B \wedge \mathbf{t}) \arrow[r, "\mathsf{m}_{A,\mathbf{t},B,\mathbf{t}}"] \arrow[d, "\hat{\varrho}_A \vee \hat{\varrho}_B"'] & (A \vee B) \wedge (\mathbf{t} \vee \mathbf{t}) \arrow[d, "(A\vee B)\wedge \mathsf{n}\breve{\mathsf{m}}"] \\
A \vee B \arrow[r, "\hat{\varrho}^{-1}_{A\vee B}"'] & (A \vee B) \wedge \mathbf{t}
\end{tikzcd} \tag{m-$\hat{\varrho}$}$$

Proof: Chase

$$\begin{tikzcd}
(A \wedge \mathbf{t}) \vee (B \wedge \mathbf{t}) \arrow[rr, "\mathsf{m}_{A,\mathbf{t},B,\mathbf{t}}"] \arrow[dr, "\Delta_{(A\wedge\mathbf{t})\vee(B\wedge\mathbf{t})}"] \arrow[ddd, "\hat{\varrho}_A \vee \hat{\varrho}_B"'] & & (A \vee B) \wedge (\mathbf{t} \vee \mathbf{t}) \arrow[ddd, "(A\vee B)\wedge \mathsf{n}\breve{\mathsf{m}}"] \\
& ((A \wedge \mathbf{t}) \vee (B \wedge \mathbf{t})) \wedge ((A \wedge \mathbf{t}) \vee (B \wedge \mathbf{t})) \arrow[ur, "(\Pi^{\mathbf{t}}_{A\Box} \vee \Pi^{\mathbf{t}}_{B\Box}) \wedge (\Pi^A_{\Box\mathbf{t}} \vee \Pi^B_{\Box\mathbf{t}})"] \arrow[d, "(\hat{\varrho}_A\vee\hat{\varrho}_B)\wedge(\hat{\varrho}_A\vee\hat{\varrho}_B)"'] & \\
& (A \vee B) \wedge (A \vee B) \arrow[uur, "(A\vee B)\wedge(\Pi^A \vee \Pi^B)"'] \arrow[dr, "(A\vee B)\wedge(\Pi^{A\vee B})"] & \\
A \vee B \arrow[ur, "\Delta_{A\vee B}"'] \arrow[rr, "\hat{\varrho}^{-1}_{A\vee B}"'] & & (A \vee B) \wedge \mathbf{t}
\end{tikzcd}$$

The upper triangle is Theorem 7.2 (v), the lower triangle is the comonoid equation, the left square says that $\hat{\varrho}_A \vee \hat{\varrho}_B$ preserves the $\wedge$-comultiplication (Theorems 5.18 and 7.2 (i)), the triange on the right is ($\mathsf{B3b}$), and the triangle in the middle commutes because $\Pi^{\mathbf{t}}_{A\Box} = \hat{\varrho}_A$ and $\Pi^A_{\Box\mathbf{t}} = \Pi^A \circ \hat{\varrho}_A$, where the latter equation holds because of (25) and naturality of $\hat{\varrho}$. $\square$

7.7 Proposition *A* $\mathsf{B2}$*-category with medial and nullary medial the following are equivalent:*

(i) *We have*

$$\amalg^{\mathbf{t}\vee\mathbf{t}} = \mathsf{n\check{m}} = \nabla_{\mathbf{t}} \colon \mathbf{t}\vee\mathbf{t} \to \mathbf{t} \tag{B3a}$$

(ii) *For all objects A, the map $\hat{\varrho}_A$ preserves the $\vee$-multiplication.*

Proof: Chasing the diagram

$$\begin{tikzcd}
(A\wedge\mathbf{t})\vee(A\wedge\mathbf{t}) \arrow[rrr, "\hat{\varrho}_A\vee\hat{\varrho}_A"] \arrow[dr, "\mathsf{m}_{A,\mathbf{t},B,\mathbf{t}}"] \arrow[ddd, "\nabla_{A\wedge\mathbf{t}}"'] & & & A\vee A \arrow[ddd, "\nabla_A"] \\
& (A\vee A)\wedge(\mathbf{t}\vee\mathbf{t}) \arrow[r, "(A\vee A)\wedge\mathsf{n\check{m}}"] \arrow[d, "\nabla_A\wedge(\mathbf{t}\vee\mathbf{t})"] \arrow[ddl, "\nabla_A\wedge\nabla_{\mathbf{t}}"'] & (A\vee A)\wedge\mathbf{t} \arrow[ur, "\hat{\varrho}_{A\vee A}"'] \arrow[d, "\nabla_A\wedge\mathbf{t}"] & \\
& A\wedge(\mathbf{t}\vee\mathbf{t}) \arrow[r, "A\wedge\mathsf{n\check{m}}"'] \arrow[dl, "A\wedge\nabla_{\mathbf{t}}"] & A\wedge\mathbf{t} \arrow[dr, "\hat{\varrho}_A"] & \\
A\wedge\mathbf{t} \arrow[rrr, "\hat{\varrho}_A"'] & & & A
\end{tikzcd}$$

shows that in the presence of medial, nullary medial, and (B3a) the map $\hat{\varrho}_A$ preserves the $\vee$-multiplication. Note that in that diagram the uppermost square is (m-$\hat{\varrho}$) from the previous proposition. The lowermost square commutes because of (B3a), and the big left triangle is (B3c′). Conversely, consider the diagram

$$\begin{tikzcd}
\mathbf{t}\vee\mathbf{t} \arrow[rrr, "\Delta_{\mathbf{t}\vee\mathbf{t}}"] \arrow[dr, "\Delta_{\mathbf{t}}\vee\Delta_{\mathbf{t}}=\hat{\varrho}_{\mathbf{t}}^{-1}\vee\hat{\varrho}_{\mathbf{t}}^{-1}"] \arrow[ddr, "\amalg^{\mathbf{t}}_{\mathbf{t}\Box}\vee\amalg^{\mathbf{t}}_{\Box\mathbf{t}}"'] \arrow[ddd, equal] & & & (\mathbf{t}\vee\mathbf{t})\wedge(\mathbf{t}\vee\mathbf{t}) \arrow[ddd, "1_{\mathbf{t}\vee\mathbf{t}}\wedge\nabla_{\mathbf{t}}"] \\
& (\mathbf{t}\wedge\mathbf{t})\vee(\mathbf{t}\wedge\mathbf{t}) \arrow[urr, "\mathsf{m}_{\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{t}}"] \arrow[r, "p"'] \arrow[dr, "(\amalg^{\mathbf{t}}_{\mathbf{t}\Box}\wedge\mathbf{t})\vee(\amalg^{\mathbf{t}}_{\Box\mathbf{t}}\wedge\mathbf{t})"'] & ((\mathbf{t}\vee\mathbf{t})\wedge(\mathbf{t}\vee\mathbf{t}))\vee((\mathbf{t}\vee\mathbf{t})\wedge(\mathbf{t}\vee\mathbf{t})) \arrow[ur, "\nabla_{(\mathbf{t}\vee\mathbf{t})\wedge(\mathbf{t}\vee\mathbf{t})}"'] \arrow[d, "(1_{\mathbf{t}\vee\mathbf{t}}\wedge\nabla_{\mathbf{t}})\vee(1_{\mathbf{t}\vee\mathbf{t}}\wedge\nabla_{\mathbf{t}})"] & \\
& \mathbf{t}\vee\mathbf{t}\vee\mathbf{t}\vee\mathbf{t} \arrow[r, "\hat{\varrho}^{-1}_{\mathbf{t}\vee\mathbf{t}}\vee\hat{\varrho}^{-1}_{\mathbf{t}\vee\mathbf{t}}"'] \arrow[dl, "\nabla_{\mathbf{t}\vee\mathbf{t}}"'] & ((\mathbf{t}\vee\mathbf{t})\wedge\mathbf{t})\vee((\mathbf{t}\vee\mathbf{t})\wedge\mathbf{t}) \arrow[dr, "\nabla_{(\mathbf{t}\vee\mathbf{t})\wedge\mathbf{t}}"] & \\
\mathbf{t}\vee\mathbf{t} \arrow[rrr, "\hat{\varrho}^{-1}_{\mathbf{t}\vee\mathbf{t}}"'] & & & (\mathbf{t}\vee\mathbf{t})\wedge\mathbf{t}
\end{tikzcd}$$

where $p = (\amalg^{\mathbf{t}}_{\mathbf{t}\Box}\vee\amalg^{\mathbf{t}}_{\mathbf{t}\Box})\wedge(\amalg^{\mathbf{t}}_{\Box\mathbf{t}}\vee\amalg^{\mathbf{t}}_{\Box\mathbf{t}})$. The upper two triagles are (B3c) and Theorem 7.2 (v). The left triangle commutes because of Proposition 5.19 (iv), and the triangle at the center is the monoid equation. The triangle-shaped square is the naturality of $\hat{\varrho}$, and the rightmost square commutes because $1_{\mathbf{t}\vee\mathbf{t}}\wedge\nabla_{\mathbf{t}}$ preserves the $\vee$-multiplication, which follows from (the dual of) Proposition 5.15 (ii) and Theorem 7.2 (i). Finally, the lower square commutes because we assumed that $\hat{\varrho}_A$ preserved the $\vee$-multiplication. Note that the commutativity of the outer square says that $\nabla_{\mathbf{t}}$ is unit for $\Delta_{\mathbf{t}\vee\mathbf{t}}$. Therefore, by Proposition 5.3, we can conlude that $\mathsf{n\check{m}} = \amalg^{\mathbf{t}\vee\mathbf{t}} = \nabla_{\mathbf{t}}$. □

7.8 Definition A $\mathsf{B3}$-*category* is a $\mathsf{B2}$-category that obeys (B3a) and has medial and nullary medial.

7.9 Corollary *In a* $\mathsf{B3}$*-category, the maps $\hat{\varrho}_A$, $\hat{\lambda}_A$, $\check{\varrho}_A$, and $\check{\lambda}_A$ are clonable for all objects A.*

Proof: Theorem 5.18 and Proposition 7.7 suffice to show that $\hat{\varrho}_A$ is clonable. For $\hat{\lambda}_A$ it is similar, and for $\check{\varrho}_A$ and $\check{\lambda}_A$ it follows by duality. □

It has first been observed by Lamarche [Lam05] that the presence of a natural and self-dual map $\mathsf{m}_{A,B,C,D} \colon (A\wedge B)\vee(C\wedge D) \to (A\vee C)\wedge(B\vee D)$ in a *-autonomous category induces two canonical maps $e \colon \mathbf{f} \to \mathbf{t}$, namely

$$\mathsf{e}_1 \colon \mathbf{f} \xrightarrow{\check{\lambda}_{\mathbf{f}}^{-1}} \mathbf{f}\vee\mathbf{f} \xrightarrow{\hat{\varrho}_{\mathbf{f}}^{-1}\vee\hat{\lambda}_{\mathbf{f}}^{-1}} (\mathbf{f}\wedge\mathbf{t})\vee(\mathbf{t}\wedge\mathbf{f}) \xrightarrow{\mathsf{m}_{\mathbf{f},\mathbf{t},\mathbf{t},\mathbf{f}}} (\mathbf{f}\vee\mathbf{t})\wedge(\mathbf{t}\vee\mathbf{f}) \xrightarrow{\check{\lambda}_{\mathbf{t}}\wedge\check{\varrho}_{\mathbf{t}}} \mathbf{t}\wedge\mathbf{t} \xrightarrow{\hat{\varrho}_{\mathbf{t}}} \mathbf{t}$$

and

$$\mathsf{e}_2 \colon \mathbf{f} \xrightarrow{\check{\lambda}_{\mathbf{f}}^{-1}} \mathbf{f} \vee \mathbf{f} \xrightarrow{\hat{\lambda}_{\mathbf{f}}^{-1} \vee \hat{\varrho}_{\mathbf{f}}^{-1}} (\mathbf{t} \wedge \mathbf{f}) \vee (\mathbf{f} \wedge \mathbf{t}) \xrightarrow{\mathsf{m}_{\mathbf{t},\mathbf{f},\mathbf{f},\mathbf{t}}} (\mathbf{t} \vee \mathbf{f}) \wedge (\mathbf{f} \vee \mathbf{t}) \xrightarrow{\check{\varrho}_{\mathbf{t}} \wedge \check{\lambda}_{\mathbf{t}}} \mathbf{t} \wedge \mathbf{t} \xrightarrow{\hat{\varrho}_{\mathbf{t}}} \mathbf{t}$$

which are both self-dual (while $\Pi^{\mathbf{f}}$ and $\amalg^{\mathbf{t}}$ are dual to each other). By adding sufficient structure one can enforce that $\mathsf{e}_1 = \mathsf{e}_2$ and that this map has the properties of Theorem 4.1. In [Lam05], Lamarche shows how this can be done without the $\wedge$-comonoid and $\vee$-monoid structure for every object by using equation (m-$\hat{\sigma}$) that we will introduce in Proposition 7.13. In our case the structure of a $\mathsf{B2}$-category is sufficient to obtain that $\mathsf{e}_1 = \mathsf{e}_2$. But for letting this map have the properties of Theorem 4.1, as it is the case with $\Pi^{\mathbf{f}}$ and $\amalg^{\mathbf{t}}$, we need all the structure of a $\mathsf{B3}$-category. Then we have the following:

7.10 Theorem *In a* $\mathsf{B3}$*-category we have* $\Pi^{\mathbf{f}} = \mathsf{e}_1 = \mathsf{e}_2 = \amalg^{\mathbf{t}}$*, i.e., every* $\mathsf{B3}$*-category is single-mixed.*

Proof: We will first show that $\Pi^{\mathbf{f}} = \mathsf{e}_1$. For this, note that

$$\begin{aligned}
&(\check{\lambda}_{\mathbf{t}} \wedge \check{\varrho}_{\mathbf{t}}) \circ \mathsf{m}_{\mathbf{f},\mathbf{t},\mathbf{t},\mathbf{f}} \circ (\hat{\varrho}_{\mathbf{f}}^{-1} \vee \hat{\lambda}_{\mathbf{f}}^{-1}) \\
&= (\check{\lambda}_{\mathbf{t}} \wedge \check{\varrho}_{\mathbf{t}}) \circ \big\langle \big[\, \amalg^{\mathbf{t}}_{\mathbf{f}\square} \circ \Pi^{\mathbf{t}}_{\mathbf{f}\square}, \amalg^{\mathbf{f}}_{\square\mathbf{t}} \circ \Pi^{\mathbf{f}}_{\mathbf{t}\square} \,\big], \big[\, \amalg^{\mathbf{f}}_{\mathbf{t}\square} \circ \Pi^{\mathbf{f}}_{\square\mathbf{t}}, \amalg^{\mathbf{t}}_{\square\mathbf{f}} \circ \Pi^{\mathbf{t}}_{\square\mathbf{f}} \,\big] \big\rangle \circ (\hat{\varrho}_{\mathbf{f}}^{-1} \vee \hat{\lambda}_{\mathbf{f}}^{-1}) \\
&= \Big\langle \Big[\, \check{\lambda}_{\mathbf{t}} \circ \amalg^{\mathbf{t}}_{\mathbf{f}\square} \circ \Pi^{\mathbf{t}}_{\mathbf{f}\square} \circ \hat{\varrho}_{\mathbf{f}}^{-1}, \check{\lambda}_{\mathbf{t}} \circ \amalg^{\mathbf{f}}_{\square\mathbf{t}} \circ \Pi^{\mathbf{f}}_{\mathbf{t}\square} \circ \hat{\lambda}_{\mathbf{f}}^{-1} \,\Big], \Big[\, \check{\varrho}_{\mathbf{t}} \circ \amalg^{\mathbf{f}}_{\mathbf{t}\square} \circ \Pi^{\mathbf{f}}_{\square\mathbf{t}} \circ \hat{\varrho}_{\mathbf{f}}^{-1}, \check{\varrho}_{\mathbf{t}} \circ \amalg^{\mathbf{t}}_{\square\mathbf{f}} \circ \Pi^{\mathbf{t}}_{\square\mathbf{f}} \circ \hat{\lambda}_{\mathbf{f}}^{-1} \,\Big] \Big\rangle \\
&= \big\langle \big[\, \amalg^{\mathbf{t}}, \Pi^{\mathbf{f}} \,\big], \big[\, \Pi^{\mathbf{f}}, \amalg^{\mathbf{t}} \,\big] \big\rangle
\end{aligned}$$

The first equation is an application of Theorem 7.2 (iii), the second one uses Proposition 5.19 together with the fact that $\hat{\varrho}_{\mathbf{f}}$ and $\hat{\lambda}_{\mathbf{f}}$ preserve the $\wedge$-comultiplication (Theorem 5.18) and that these maps are closed under $\vee$ (Theorem 7.2 (i)). The third equation is an easy calculation, involving (24) and the naturality of $\hat{\varrho}$ and $\hat{\lambda}$. Before we proceed, notice that:

$$\Pi^{\mathbf{t}}_{\square\mathbf{t}} = \hat{\lambda}_{\mathbf{t}} = \hat{\varrho}_{\mathbf{t}} = \Pi^{\mathbf{t}}_{\mathbf{t}\square} \colon \mathbf{t} \wedge \mathbf{t} \to \mathbf{t} \qquad \text{and} \qquad \amalg^{\mathbf{f}}_{\square\mathbf{f}} = \hat{\lambda}_{\mathbf{f}}^{-1} = \hat{\varrho}_{\mathbf{f}}^{-1} = \amalg^{\mathbf{f}}_{\mathbf{f}\square} \colon \mathbf{f} \to \mathbf{f} \vee \mathbf{f} \tag{39}$$

Now we have:

$$\begin{aligned}
\mathsf{e}_1 &= \hat{\varrho}_{\mathbf{t}} \circ (\check{\lambda}_{\mathbf{t}} \wedge \check{\varrho}_{\mathbf{t}}) \circ \mathsf{m}_{\mathbf{f},\mathbf{t},\mathbf{t},\mathbf{f}} \circ (\hat{\varrho}_{\mathbf{f}}^{-1} \vee \hat{\lambda}_{\mathbf{f}}^{-1}) \circ \check{\lambda}_{\mathbf{f}}^{-1} \\
&= \hat{\varrho}_{\mathbf{t}} \circ \big\langle \big[\, \amalg^{\mathbf{t}}, \Pi^{\mathbf{f}} \,\big], \big[\, \Pi^{\mathbf{f}}, \amalg^{\mathbf{t}} \,\big] \big\rangle \circ \check{\lambda}_{\mathbf{f}}^{-1} \\
&= \hat{\varrho}_{\mathbf{t}} \circ \big\langle \big[\, \amalg^{\mathbf{t}}, \Pi^{\mathbf{f}} \,\big] \circ \check{\lambda}_{\mathbf{f}}^{-1}, \big[\, \Pi^{\mathbf{f}}, \amalg^{\mathbf{t}} \,\big] \circ \check{\lambda}_{\mathbf{f}}^{-1} \big\rangle \\
&= \Pi^{\mathbf{t}}_{\mathbf{t}\square} \circ \big\langle \big[\, \amalg^{\mathbf{t}}, \Pi^{\mathbf{f}} \,\big] \circ \amalg^{\mathbf{f}}_{\square\mathbf{f}}, \big[\, \Pi^{\mathbf{f}}, \amalg^{\mathbf{t}} \,\big] \circ \amalg^{\mathbf{f}}_{\mathbf{f}\square} \big\rangle \\
&= \Pi^{\mathbf{t}}_{\mathbf{t}\square} \circ \big\langle \Pi^{\mathbf{f}}, \Pi^{\mathbf{f}} \big\rangle \\
&= \Pi^{\mathbf{f}}
\end{aligned}$$

The first two equations are just the definition of e_1 and the previous calculation. The third equation uses Proposition 5.19 and the fact that $\check{\lambda}_{\mathbf{f}} = \check{\varrho}_{\mathbf{f}}$ preserves the $\wedge$-comultiplication (Corollary 7.9). The fourth equation applies (39), and the last two equations are again Proposition 5.19, together with the fact that $\amalg^{\mathbf{t}}$ preserves the $\vee$-unit and $\Pi^{\mathbf{f}}$ preserves the $\wedge$-counit (Theorem 5.18). Similarly, we show that $\mathsf{e}_2 = \Pi^{\mathbf{f}}$ and dually, we obtain $\mathsf{e}_1 = \mathsf{e}_2 = \amalg^{\mathbf{t}}$. ◻

7.11 Theorem *In a* $\mathsf{B3}$*-category, the strong maps (in fact, all types of maps defined in Definition 5.4) are closed under* $\wedge$ *and* $\vee$*. Furthermore, the maps* $\mathsf{m}_{A,B,C,D}$ *and* $\check{\mathsf{nm}}$ *and* $\hat{\mathsf{nm}}$ *are strong.*

Proof: By Propositions 5.13 and 5.15, the $\wedge$-comonoid morphisms are closed under $\wedge$, and by Proposition 7.5 and Theorem 7.2 they are closed under $\vee$. Dually, the $\vee$-monoid morphisms are closed under $\vee$ and $\wedge$, and therefore also the strong maps have this property. Since by Theorem 7.2 (v), medial is $((\Pi^B_{A\square} \vee \Pi^D_{C\square}) \wedge (\Pi^A_{\square B} \vee \Pi^C_{\square D})) \circ \Delta_{(A \wedge B) \vee (C \wedge D)}$ as well as $\nabla_{(A \vee C) \wedge (B \vee D)} \circ ((\amalg^C_{A\square} \wedge \amalg^D_{B\square}) \vee (\amalg^A_{\square C} \wedge \amalg^B_{\square D}))$, we have by Theorem 5.18 that it is a $\wedge$-comonoid morphism and a $\vee$-monoid morphism, and therefore strong. Since $\check{\mathsf{nm}} = \amalg^{\mathbf{t} \vee \mathbf{t}} = \nabla_{\mathbf{t}}$, we get again from Theorem 5.18 that it is a $\wedge$-comonoid morphism and a $\vee$-monoid morphism. Similarly for $\hat{\mathsf{nm}} = \amalg^{\mathbf{f} \wedge \mathbf{f}} = \Delta_{\mathbf{f}}$. ◻

7.12 Proposition *In a* $\mathsf{B3}$*-category the maps* $\check{\alpha}_{A,B,C}$*,* $\check{\sigma}_{A,B}$*,* $\check{\lambda}_A$*, and* $\check{\varrho}_A$ *preserve the* $\wedge$*-counit for all objects* A, B, C*. Dually, the maps* $\hat{\alpha}_{A,B,C}$*,* $\hat{\sigma}_{A,B}$*,* $\hat{\lambda}_A$*, and* $\hat{\varrho}_A$ *all preserve the* $\vee$*-unit.*

Proof: As before, the cases for $\check{\alpha}_{A,B,C}$ and $\check{\sigma}_{A,B}$ are similar. This time, we show the case for $\check{\alpha}_{A,B,C}$:

$$
\begin{tikzcd}
A \vee (B \vee C) \arrow[rr, "\check{\alpha}_{A,B,C}"] \arrow[d, "\Pi^A \vee (\Pi^B \vee \Pi^C)"'] & & (A \vee B) \vee C \arrow[d, "(\Pi^A \vee \Pi^B) \vee \Pi^C"] \\
\mathbf{t} \vee (\mathbf{t} \vee \mathbf{t}) \arrow[rr, "\check{\alpha}_{\mathbf{t},\mathbf{t},\mathbf{t}}"'] \arrow[d, "\mathbf{t} \vee \nabla_{\mathbf{t}}"'] & & (\mathbf{t} \vee \mathbf{t}) \vee \mathbf{t} \arrow[d, "\nabla_{\mathbf{t}} \vee \mathbf{t}"] \\
\mathbf{t} \vee \mathbf{t} \arrow[dr, "\nabla_{\mathbf{t}}"'] & & \mathbf{t} \vee \mathbf{t} \arrow[dl, "\nabla_{\mathbf{t}}"] \\
& \mathbf{t} &
\end{tikzcd}
$$

The square is naturality of $\check{\alpha}$, and the pentagon is associativity of ∇. The left down path is $\Pi^{A \vee (B \vee C)}$ and the right down path is $\Pi^{(A \vee B) \vee C}$ (because of ($\mathsf{B3b}$) and ($\mathsf{B3a}$)). For $\check{\varrho}_A$, chase

$$
\begin{tikzcd}
A \vee \mathbf{f} \arrow[rr, "\check{\varrho}_A"] \arrow[dr, "\Pi^A \vee \mathbf{f}"] \arrow[ddr, "\Pi^A \vee \Pi^{\mathbf{f}}"'] \arrow[ddd, "\Pi^{A \vee \mathbf{f}}"'] & & A \arrow[ddd, "\Pi^A"] \\
& \mathbf{t} \vee \mathbf{f} \arrow[d, "\mathbf{t} \vee \Pi^{\mathbf{t}}"'] \arrow[ddr, "\check{\varrho}_{\mathbf{t}}"] & \\
& \mathbf{t} \vee \mathbf{t} \arrow[dl, "\mathsf{n\check{m}}"'] \arrow[dr, "\nabla_{\mathbf{t}}"] & \\
\mathbf{t} \arrow[rr, equal] & & \mathbf{t}
\end{tikzcd}
$$

The upper right quadrangle is naturality of $\check{\varrho}$. The leftmost triangle is ($\mathsf{B3b}$). The one in the center next to it commutes because of functoriality of $\vee$ and $\Pi^{\mathbf{f}} = \Pi^{\mathbf{t}}$ (Theorem 7.10). The lower right triangle the the monoid equation and the triangle at the bottom is ($\mathsf{B3a}$). The case for $\check{\lambda}_A$ is similar. □

7.13 Proposition *In a* $\mathsf{B3}$*-category the following are equivalent:*

(i) *The equation*

$$
\begin{tikzcd}
(A \wedge B) \vee (C \wedge D) \arrow[r, "\hat{\sigma}_{A,B} \vee \hat{\sigma}_{C,D}"] \arrow[d, "\mathsf{m}_{A,B,C,D}"'] & (B \wedge A) \vee (D \wedge C) \arrow[d, "\mathsf{m}_{B,A,D,C}"] \\
(A \vee C) \wedge (B \vee D) \arrow[r, "\hat{\sigma}_{A \vee C, B \vee D}"'] & (B \vee D) \wedge (A \vee C)
\end{tikzcd}
\tag{m-$\hat{\sigma}$}
$$

holds for all objects A*,* B*,* C*, and* D*.*

(ii) *The map* $\hat{\sigma}_{A,B} \colon A \wedge B \to B \wedge A$ *preserves the* $\vee$*-multiplication.*

(iii) *The map* $\check{\sigma}_{A,B} \colon A \vee B \to B \vee A$ *preserves the* $\wedge$*-comultiplication.*

(iv) *The equation*

$$
\begin{tikzcd}
(A \wedge B) \vee (C \wedge D) \arrow[r, "\check{\sigma}_{A \wedge B, C \wedge D}"] \arrow[d, "\mathsf{m}_{A,B,C,D}"'] & (C \wedge D) \vee (A \wedge B) \arrow[d, "\mathsf{m}_{C,D,A,B}"] \\
(A \vee C) \wedge (B \vee D) \arrow[r, "\check{\sigma}_{A,C} \wedge \check{\sigma}_{B,D}"'] & (C \vee A) \wedge (D \vee B)
\end{tikzcd}
\tag{m-$\check{\sigma}$}
$$

holds for all objects A*,* B*,* C*, and* D*.*

Proof: Suppose (m-$\hat{\sigma}$) does hold. Then we have

$$
\begin{tikzcd}
(A\wedge B)\vee(A\wedge B) \arrow[r, "\hat{\sigma}_{A,B}\vee\hat{\sigma}_{A,B}"] \arrow[d, "\mathsf{m}_{A,B,A,B}"'] & (B\wedge A)\vee(B\wedge A) \arrow[d, "\mathsf{m}_{B,A,B,A}"] \\
(A\vee A)\wedge(B\vee B) \arrow[r, "\hat{\sigma}_{A\vee A,B\vee B}"] \arrow[d, "\nabla_A\wedge\nabla_B"'] & (B\vee B)\wedge(A\vee A) \arrow[d, "\nabla_B\wedge\nabla_A"] \\
A\wedge B \arrow[r, "\hat{\sigma}_{A,B}"'] & B\wedge A
\end{tikzcd}
$$

which together with (B3c$'$) says that $\hat{\sigma}_{A,B}$ preserves the $\vee$-multiplication. Conversely, we have

$$
\begin{tikzcd}
(A\wedge B)\vee(C\wedge D) \arrow[r, "\hat{\sigma}_{A,B}\vee\hat{\sigma}_{C,D}"] \arrow[d, "(\amalg^C_{A\square}\wedge\amalg^D_{B\square})\vee(\amalg^A_{\square C}\wedge\amalg^B_{\square D})"'] & (B\wedge A)\vee(D\wedge C) \arrow[d, "(\amalg^D_{B\square}\wedge\amalg^C_{A\square})\vee(\amalg^B_{\square D}\wedge\amalg^A_{\square C})"] \\
((A\vee C)\wedge(B\vee D))\vee((A\vee C)\wedge(B\vee D)) \arrow[r, "\hat{\sigma}_{A\vee C,B\vee D}\vee\hat{\sigma}_{A\vee C,B\vee D}"] \arrow[d, "\nabla_{(A\vee C)\wedge(B\vee D)}"'] & ((B\vee D)\wedge(A\vee C))\vee((B\vee D)\wedge(A\vee C)) \arrow[d, "\nabla_{(B\vee D)\wedge(A\vee C)}"] \\
(A\vee C)\wedge(B\vee D) \arrow[r, "\hat{\sigma}_{A\vee C,B\vee D}"'] & (B\vee D)\wedge(A\vee C)
\end{tikzcd}
$$

The upper square is naturality of $\hat{\sigma}$, and the lower square says that $\hat{\sigma}_{A,B}$ preserves the $\vee$-multiplication. Together with Theorem 7.2 (v), this is (m-$\hat{\sigma}$). Hence (i) and (ii) are equivalent. The other equivalences follow because of duality. $\square$

7.14 Proposition *In a* B3*-category the following are equivalent:*

(i) *The equation*

$$
\begin{tikzcd}
(A\wedge(B\wedge C))\vee(D\wedge(E\wedge F)) \arrow[r, "\hat{\alpha}_{A,B,C}\vee\hat{\alpha}_{D,E,F}"] \arrow[d, "\mathsf{m}_{A,B\wedge C,D,E\wedge F}"'] & ((A\wedge B)\wedge C)\vee((D\wedge E)\wedge F) \arrow[d, "\mathsf{m}_{A\wedge B,C,D\wedge E,F}"] \\
(A\vee D)\wedge((B\wedge C)\vee(E\wedge F)) \arrow[d, "(A\vee D)\wedge\mathsf{m}_{B,C,E,F}"'] & ((A\wedge B)\vee(D\wedge E))\wedge(C\vee F) \arrow[d, "\mathsf{m}_{A,B,D,E}\wedge(C\vee F)"] \\
(A\vee D)\wedge((B\vee E)\wedge(C\vee F)) \arrow[r, "\hat{\alpha}_{A\vee D,B\vee E,C\vee F}"'] & ((A\vee D)\wedge(B\vee E))\wedge(C\vee F)
\end{tikzcd}
\qquad (\mathsf{m}\text{-}\hat{\alpha})
$$

holds for all objects A, B, C, D, E, and F.

(ii) *The map $\hat{\alpha}_{A,B,C}\colon A\wedge(B\wedge C)\to(A\wedge B)\wedge C$ preserves the $\vee$-multiplication.*

Proof: Similar to the previous proposition. (Here the statements corresponding to (iii) and (iv) in Proposition 7.13 are omitted to save space, but obviously they hold accordingly.) $\square$

7.15 Remark This proposition allows us to speak of uniquely definied maps

$$\hat{\mathsf{m}}^2_{A,B,C,D,E,F}\colon(A\wedge B\wedge C)\vee(D\wedge E\wedge F)\to(A\vee D)\wedge(B\vee E)\wedge(C\vee F)$$

and dually

$$\check{\mathsf{m}}^2_{A,B,C,D,E,F}\colon(A\wedge B)\vee(C\wedge D)\vee(E\wedge F)\to(A\vee C\vee E)\wedge(B\vee D\vee F)$$

A more sophisticated and more general notation for composed variations of medial is introduced by Lamarche in [Lam05].

7.16 Proposition *In a a* B3*-category obeying* (m-$\hat{\sigma}$) *and* (m-$\hat{\alpha}$) *the following are equivalent:*

(i) *The equation*

$$\begin{tikzcd}
((A \wedge B) \vee (C \wedge D)) \wedge E \arrow[r, "\mathsf{s}_{A\wedge B, C\wedge D, E}"] \arrow[d, "\mathsf{m}_{A,B,C,D} \wedge E"'] & (A \wedge B) \vee (C \wedge D \wedge E) \arrow[d, "\mathsf{m}_{A,B,C,D\wedge E}"] \\
(A \vee C) \wedge (B \vee D) \wedge E \arrow[r, "(A\vee C)\wedge \mathsf{s}_{B,D,E}"'] & (A \vee C) \wedge (B \vee (D \wedge E))
\end{tikzcd} \tag{m-s}$$

holds for all objects A, B, C, D, *and* E.

(ii) *The map* $\mathsf{s}_{A,B,C} \colon A \wedge (B \vee C) \to (A \wedge B) \vee C$ *preserves the* $\wedge$*-comultiplication.*

Proof: First note that if the equations (m-$\hat{\sigma}$), (m-$\hat{\alpha}$), and (m-s) hold, we can compose them to get the commutativity of diagrams like

$$\begin{tikzcd}
((A \wedge B) \vee (C \wedge D)) \wedge E \wedge F \arrow[r] \arrow[d, "\mathsf{m}_{A,B,C,D} \wedge E \wedge F"'] & (A \wedge B) \vee (C \wedge E \wedge D \wedge F) \arrow[d, "\mathsf{m}_{A,B,C\wedge E, D\wedge F}"] \\
(A \vee C) \wedge (B \vee D) \wedge E \wedge F \arrow[r] & (A \vee (C \wedge E)) \wedge (B \vee (D \wedge F))
\end{tikzcd} \tag{40}$$

where the horizontal maps are the canonical maps (composed of twist, associativity, and switch) that are uniquely determined by the *-autonomous structure. Now chase

$$\begin{tikzcd}
(A \vee B) \wedge C \arrow[r, "\mathsf{s}_{A,B,C}"] \arrow[d, "(\Delta_A \vee \Delta_B)\wedge \Delta_C"'] & A \vee (B \wedge C) \arrow[d, "\Delta_A \vee (\Delta_B \wedge \Delta_C)"] \\
((A \wedge A) \vee (B \wedge B)) \wedge C \wedge C \arrow[r, "\mathsf{s}_{A\wedge A, B\wedge B, C\wedge C}"] \arrow[d, "\mathsf{m}_{A,A,B,B} \wedge C \wedge C"'] \arrow[dr] & (A \wedge A) \vee (B \wedge B \wedge C \wedge C) \arrow[d, "(A\wedge A)\vee (B \wedge \hat{\sigma}_{B,C} \wedge C)"] \\
(A \vee B) \wedge (A \vee B) \wedge C \wedge C \arrow[d, "(A\vee B)\wedge \hat{\sigma}_{A\vee B, C} \wedge C"'] \arrow[dr] & (A \wedge A) \vee (B \wedge C \wedge B \wedge C) \arrow[d, "\mathsf{m}_{A,A,B\wedge C, B\wedge C}"] \\
(A \vee B) \wedge C \wedge (A \vee B) \wedge C \arrow[r, "\mathsf{s}_{A,B,C} \wedge \mathsf{s}_{A,B,C}"'] & (A \vee (B \wedge C)) \wedge (A \vee (B \wedge C))
\end{tikzcd}$$

where the parallelogram is just (40), the upper square is naturality of switch and the two triangles are laws of *-autonomous categories. Note that, by (B2c) and (B3c), the vertical paths are just $\Delta_{(A\vee B)\wedge C}$ and $\Delta_{A\vee(B\wedge C)}$. Therefore switch preserves the $\wedge$-comultiplication. Conversely, consider the diagram

$$\begin{tikzcd}
((A \wedge B) \vee (C \wedge D)) \wedge E \arrow[r, "\mathsf{s}_{A\wedge B, C\wedge D, E}"] \arrow[d, "\Delta_{((A\wedge B)\vee(C\wedge D))\wedge E}"'] & (A \wedge B) \vee (C \wedge D \wedge E) \arrow[d, "\Delta_{(A\wedge B)\vee(C\wedge D \wedge E)}"] \\
((A \wedge B) \vee (C \wedge D)) \wedge E \wedge ((A \wedge B) \vee (C \wedge D)) \wedge E \arrow[r, "\mathsf{s}\wedge\mathsf{s}"] \arrow[d, "p"'] & ((A \wedge B) \vee (C \wedge D \wedge E)) \wedge ((A \wedge B) \vee (C \wedge D \wedge E)) \arrow[d, "q"] \\
((A \wedge \mathbf{t}) \vee (C \wedge \mathbf{t})) \wedge \mathbf{t} \wedge ((\mathbf{t} \wedge B) \vee (\mathbf{t} \wedge D)) \wedge E \arrow[r, "\mathsf{s}\wedge\mathsf{s}"] \arrow[d, "\cong"'] & ((A \wedge \mathbf{t}) \vee (C \wedge \mathbf{t} \wedge \mathbf{t})) \wedge ((\mathbf{t} \wedge B) \vee (\mathbf{t} \wedge D \wedge E)) \arrow[d, "\cong"] \\
(A \vee C) \wedge (B \vee C) \wedge E \arrow[r, "(A\vee C)\wedge \mathsf{s}_{B,D,E}"'] & (A \vee C) \wedge (B \vee (D \wedge E))
\end{tikzcd}$$

where

$$p = ((A \wedge \Pi^B) \vee (C \wedge \Pi^D)) \wedge \Pi^E \wedge ((\Pi^A \wedge B) \vee (\Pi^C \wedge D)) \wedge E$$

$$q = ((A \wedge \Pi^B) \vee (C \wedge \Pi^D \wedge \Pi^E)) \wedge ((\Pi^A \wedge B) \vee (\Pi^C \wedge D \wedge E))$$

Note that the left vertical map is $\mathsf{m}_{A,B,C,D} \wedge 1_E$ while the right vertical map is $\mathsf{m}_{A,B,C,D\wedge E}$. The upper square commutes because we assumed that switch preserves the $\wedge$-comultiplication, the middle one is naturality of switch, and the lower one commutes because the category is *-autonomous (the isomorphisms are just compositions of $\hat{\varrho}$ and $\hat{\lambda}$). $\square$

7.17 Definition A B4-*category* is a B3-category that obeys the equations (m-$\hat{\sigma}$), (m-$\hat{\alpha}$), and (m-s).

7.18 Remark Equivalently, one can define a B4-category as a B3-category in which $\hat{\sigma}$, $\hat{\alpha}$, and s are strong. We have chosen the form of Definition (7.17) to show the resemblance to the work [Lam05] where the equations (m-$\hat{\sigma}$), (m-$\hat{\alpha}$), and (m-s) are also considered as primitives.

7.19 Theorem *In a* B4-*category, the maps* $\hat{\alpha}_{A,B,C}$, $\hat{\sigma}_{A,B}$, $\hat{\varrho}_A$, $\hat{\lambda}_A$ *and* $\check{\alpha}_{A,B,C}$, $\check{\sigma}_{A,B}$, $\check{\varrho}_A$, $\check{\lambda}_A$, *as well as* $\mathsf{s}_{A,B,C}$ *and* $\mathsf{mix}_{A,B}$ *are all strong.*

Proof: That $\hat{\alpha}_{A,B,C}$, $\hat{\sigma}_{A,B}$, $\hat{\varrho}_A$, $\hat{\lambda}_A$ and $\check{\alpha}_{A,B,C}$, $\check{\sigma}_{A,B}$, $\check{\varrho}_A$, $\check{\lambda}_A$ are quasientropies follows from Theorem 5.18 and Proposition 7.12. That $\hat{\varrho}_A$, $\hat{\lambda}_A$ and $\check{\varrho}_A$, $\check{\lambda}_A$ are clonable has been said already in Corollary 7.9. For $\hat{\alpha}_{A,B,C}$, $\hat{\sigma}_{A,B}$ and $\check{\alpha}_{A,B,C}$, $\check{\sigma}_{A,B}$ this follows from Theorem 5.18 and Propositions 7.13 and 7.14 (and by duality). Hence, all these maps are strong. That $\mathsf{s}_{A,B,C}$ is strong follows from Proposition 7.5 and Proposition 7.16 (and self-duality of switch). For showing that $\mathsf{mix}_{A,B}$ is also strong it suffices to observe that mix is a composition of strong maps via $\circ$, $\wedge$, and $\vee$. See (16), Theorem 7.10, and Theorem 7.11. ☐

7.20 Remark Theorem 7.19 gives justification to the algebraic concern raised in Remark 5.6. In a B4-category all isomorphisms that are imposed by the B4-structure do preserve the $\vee$-monoid and $\wedge$-comonoid structure (and are therefore "proper isomorphisms"). Note that there might still be "improper isomorphisms" in a B4-category. But these live outside the B4-structure and are therefore not accessible to proof theoretical investigations.

It has first been observed by Lamarche in [Lam05] that the equation (m-mix) (see below) is a consequence of the equations (m-$\hat{\alpha}$), (m-$\hat{\sigma}$), and (m-s). Due to the presence of the $\vee$-monoids and $\wedge$-comonoids, we can give here a simpler proof of that fact:

7.21 Corollary *In a* B4-*category, the diagram*

$$\begin{tikzcd}
A \wedge B \wedge C \wedge D \arrow[r, "A\wedge\hat{\sigma}_{B,C}\wedge D"] \arrow[d, "\mathsf{mix}_{A\wedge B,C\wedge D}"'] & A \wedge C \wedge B \wedge D \arrow[d, "\mathsf{mix}_{A,C}\wedge\mathsf{mix}_{B,D}"] \\
(A \wedge B) \vee (C \wedge D) \arrow[r, "\mathsf{m}_{A,B,C,D}"'] & (A \vee C) \wedge (B \vee D)
\end{tikzcd} \qquad (\mathsf{m}\text{-}\mathsf{mix})$$

commutes.

Proof: Chase

$$\begin{tikzcd}
A \wedge B \wedge C \wedge D \arrow[rr, "A\wedge\hat{\sigma}_{B,C}\wedge D"] \arrow[dr, "\Delta_A\wedge\Delta_B\wedge\Delta_C\wedge\Delta_D"'] \arrow[ddd, equal] & & A \wedge C \wedge B \wedge D \arrow[ddl, "\Delta_A\wedge\Delta_C\wedge\Delta_B\wedge\Delta_D"] \arrow[ddd, equal] \\
& A \wedge A \wedge B \wedge B \wedge C \wedge C \wedge D \wedge D \arrow[d, "A\wedge A\wedge\hat{\sigma}_{B\wedge B,C\wedge C}\wedge D\wedge D"'] & \\
& A \wedge A \wedge C \wedge C \wedge B \wedge B \wedge D \wedge D \arrow[d, "\cong"'] \arrow[dr, "\Pi^{A}_{\Box A}\wedge\Pi^{C}_{\Box C}\wedge\Pi^{B}_{B\Box}\wedge\Pi^{D}_{D\Box}"] & \\
A \wedge B \wedge C \wedge D \arrow[r, "\Delta_{A\wedge B\wedge C\wedge D}"] \arrow[dd, "\mathsf{mix}_{A\wedge B,C\wedge D}"'] & A \wedge B \wedge C \wedge D \wedge A \wedge B \wedge C \wedge D \arrow[r, "(\Pi^{B}_{A\Box}\wedge\Pi^{D}_{C\Box})\wedge\Pi^{A}_{\Box B}\wedge\Pi^{C}_{\Box D}"'] \arrow[d, "\mathsf{mix}_{A\wedge B,C\wedge D}\wedge\mathsf{mix}_{A\wedge B,C\wedge D}"'] & A \wedge C \wedge B \wedge D \arrow[dd, "\mathsf{mix}_{A,C}\wedge\mathsf{mix}_{B,D}"] \\
& ((A \wedge B) \vee (C \wedge D)) \wedge ((A \wedge B) \vee (C \wedge D)) \arrow[dr, "(\Pi^{B}_{A\Box}\vee\Pi^{D}_{C\Box})\wedge(\Pi^{A}_{\Box B}\vee\Pi^{C}_{\Box D})"] & \\
(A \wedge B) \vee (C \wedge D) \arrow[ur, "\Delta_{(A\wedge B)\vee(C\wedge D)}"] \arrow[rr, "\mathsf{m}_{A,B,C,D}"'] & & (A \vee C) \wedge (B \vee D)
\end{tikzcd}$$

The topmost quadrangle commutes because of naturality of $\hat{\sigma}$. The pentagon below consists of several applications of ($\mathsf{B2c}$). The two triangles on the right are trivial. The quadrangle on the lower left commutes because mix preserves the $\wedge$-comultiplication, and the quadrangle on the lower right because of naturality of mix. Finally, the triangle on the bottom is Theorem 7.2 (v). $\square$

Obviously one can come up with more diagrams like (m-mix) or (m-$\hat{\varrho}$) and ask whether they commute, for example the following due to McKinley [McK05]:

$$
\begin{tikzcd}
(A \wedge \mathbf{f}) \vee (B \wedge C) \arrow[rr, "\mathsf{m}_{A,\mathbf{f},B,C}"] \arrow[d, "(A\wedge\Pi^{\mathbf{f}})\vee(B\wedge C)"'] & & (A \vee B) \wedge (\mathbf{f} \vee C) \arrow[d, "(A\vee B)\wedge\check{\lambda}_C"] \\
(A \wedge \mathbf{t}) \vee (B \wedge C) \arrow[dr, "\hat{\varrho}_A\vee(B\wedge C)"'] & & (A \vee B) \wedge C \arrow[dl, "\mathsf{s}_{A,B,C}"] \\
& A \vee (B \wedge C) &
\end{tikzcd}
\tag{41}
$$

It was soon discovered independently by several people that (41) is equivalent to:

$$
\begin{tikzcd}
(A \wedge B) \vee (C \wedge D) \arrow[dr, "\mathsf{mix}_{A,B}\vee(C\wedge D)"] \arrow[dd, "\mathsf{m}_{A,B,C,D}"'] & \\
& A \vee B \vee (C \wedge D) \\
(A \vee C) \wedge (B \vee D) \arrow[ur, "\hat{\mathsf{t}}_{A,C,B,D}"'] &
\end{tikzcd}
\tag{mix-m-$\hat{\mathsf{t}}$}
$$

Here are two other examples that do not contain the units:

$$
\begin{tikzcd}
((A \wedge B) \vee (C \wedge D)) \wedge (E \vee F) \arrow[r, "\mathsf{m}_{A,B,C,D}\wedge(E\vee F)"] \arrow[d, "\mathsf{s}_{A\wedge B,C\wedge D,E\wedge F}"'] & (A \vee C) \wedge (B \vee D) \wedge (E \vee F) \arrow[d, "\check{\mathsf{t}}_{A\vee C,B\vee D,E,F}"] \\
(A \wedge B) \vee (C \wedge D \wedge (E \vee F)) \arrow[d, "(A\wedge B)\vee\check{\mathsf{t}}_{C,D,E,F}"'] & ((A \vee C) \wedge F) \vee (E \wedge (B \vee D)) \arrow[d, "\mathsf{m}_{A\vee C,F,E,B\vee D}"] \\
(A \wedge B) \vee (C \wedge F) \vee (E \wedge D) \arrow[r, "\check{\mathsf{m}}^2_{A,B,C,F,E,D}"'] & (A \vee C \vee E) \wedge (F \vee B \vee D)
\end{tikzcd}
\tag{m-$\check{\mathsf{t}}$-s}
$$

$$
\begin{tikzcd}
& (A' \vee A) \wedge (B' \vee B) \wedge (C' \vee C) \wedge (D' \vee D) \arrow[dl, "p"'] \arrow[dr, "q"] & \\
((A' \vee B') \wedge (C' \vee D')) \vee (D \wedge C) \vee (B \wedge A) \arrow[d, "\check{\mathsf{m}}^2_{A'\vee B',C'\vee D',D,C,B,A}"'] & & ((A' \vee A) \wedge (B' \vee C)) \vee (B \wedge D') \vee (D \vee C') \arrow[d, "\check{\mathsf{m}}^2_{A'\vee A,B'\vee C,B,D',D,C'}"] \\
(A' \vee B' \vee B \vee D) \wedge (D' \vee C' \vee C \vee A) \arrow[dr, "\hat{\mathsf{t}}_{A'\vee B',B\vee D,D'\vee C',C\vee A}"] & & (A' \vee A \vee B \vee D) \wedge (D' \vee C' \vee B' \vee C) \arrow[dl, "\hat{\mathsf{t}}_{A'\vee A,B\vee D,D'\vee C',B'\vee C}"'] \\
& A' \vee B' \vee ((B \vee D) \wedge (D' \vee C')) \vee C \vee A &
\end{tikzcd}
\tag{$\check{\mathsf{m}}^2$-s-$\check{\mathsf{m}}^2$}
$$

where p and q are the canonical maps (composed of several switches, twists, and associativity) that are determined by the *-autonomous structure.

One usually speaks of "coherence" [Mac71] if all diagrams of this kind commute. Very often a "coherence theorem" is based on so-called "coherence graphs" [KM71,DP04]. In certain cases (see, e.g., [Str05]) the notion of coherence graph is too restricted. For this reason, in [LS05a], the notion of "graphicality" is introduced.

7.22 Definition Let $\mathscr{C}$ be a single-mixed $\mathsf{B1}$-category, and let $\mathscr{C}^{\between}$ be the category obtained from $\mathscr{C}$ by adding for each pair of objects A and B a map $\mathsf{mix}^{-1}_{A,B} \colon A \vee B \to A \wedge B$ which is invers to $\mathsf{mix}_{A,B}$ (i.e., the two bifunctors $- \wedge -$ and $- \vee -$ are naturally isomorphic in $\mathscr{C}^{\between}$). We say that $\mathscr{C}$ is *graphical* if the canonical forgetful functor $F \colon \mathscr{C} \to \mathscr{C}^{\between}$ is faithful.

7.23 Open Problem Let $\mathscr{E}$ be a set of equations and let $\mathscr{C}$ be the free $\mathsf{B1}$-category that is generated from a set $\mathscr{A}$ of generators (e.g., propositional variables) and that obeys all of $\mathscr{E}$. Is $\mathscr{C}$ graphical? Note that for example the freely generated *-autonomous category without units [LS05a,HHS05,DP05] is graphical. This can be shown by using traditional proof nets for multiplicative linear logic. However, the work of [LS04] can be used to show that the freely generated *-autonomous category with units is not graphical.

Clearly, in a graphical $\mathsf{B4}$-category the equations $(\mathsf{mix}\text{-}\mathsf{m}\text{-}\hat{\mathsf{t}})$, $(\mathsf{m}\text{-}\check{\mathsf{t}}\text{-}\mathsf{s})$, and $(\check{\mathsf{m}}^2\text{-}\mathsf{s}\text{-}\check{\mathsf{m}}^2)$ all hold. However, at the current state of the art it is not known whether they hold in every $\mathsf{B4}$-category.[8] But what can easily be shown is the following proposition.

7.24 Proposition *In every* $\mathsf{B4}$*-category*

(i) *the equation* (41) *holds if and only if equation* $(\mathsf{mix}\text{-}\mathsf{m}\text{-}\hat{\mathsf{t}})$ *holds, and*
(ii) *the equation* (28) *holds if and only if equation* $(\mathsf{m}\text{-}\check{\mathsf{t}}\text{-}\mathsf{s})$ *holds.*

Proof: Since we do not need this later, we leave the proof as an exercise to the reader. □

7.25 Definition A $\mathsf{B4}'$*-category* is a $\mathsf{B4}$-category that obeys equations $(\mathsf{mix}\text{-}\mathsf{m}\text{-}\hat{\mathsf{t}})$, $(\mathsf{m}\text{-}\check{\mathsf{t}}\text{-}\mathsf{s})$, and $(\check{\mathsf{m}}^2\text{-}\mathsf{s}\text{-}\check{\mathsf{m}}^2)$ for all objects.

The motivation for this definition is the following lemma which will be needed in the next section.

7.26 Lemma *In a* $\mathsf{B4}'$*-category the following equation holds for all objects* A, A', B, B', C, C', D, *and* D'*:*

$$
\begin{tikzcd}
 & (A' \vee A) \wedge (B' \vee B) \wedge (C' \vee C) \wedge (D' \vee D) \arrow[dl, "(A'\vee A)\wedge\hat{\mathsf{t}}_{B',B,C',C}\wedge(D'\vee D)"'] \arrow[dr, "\hat{\mathsf{t}}_{A',A,B',B}\wedge\hat{\mathsf{t}}_{C',C,D',D}"] & \\
(A' \vee A) \wedge (B' \vee (B \wedge C') \vee C) \wedge (D' \vee D) \arrow[d, "\mathsf{mix}_{A'\vee A,B'\vee(B\wedge C')\vee C}\wedge(D'\vee D)"'] & & (A' \vee B' \vee (A \wedge B)) \wedge (C' \vee D' \vee (C \wedge D)) \arrow[d, "\hat{\mathsf{t}}_{A'\vee B',A\wedge B,C'\vee D',C\wedge D}"] \\
(A' \vee B' \vee (B \wedge C') \vee C \vee A) \wedge (D' \vee D) \arrow[d, "\mathsf{s}_{A'\vee B'\vee C\vee A,B\wedge C',D'\vee D}"'] & & ((A' \vee B') \wedge (C' \vee D')) \vee (D \wedge C) \vee (B \wedge A) \arrow[d, "((A'\vee B')\wedge(C'\vee D'))\vee\mathsf{m}_{D,C,B,A}"] \\
A' \vee B' \vee (B \wedge (D' \vee D) \wedge C') \vee C \vee A \arrow[d, "A'\vee B'\vee\check{\mathsf{t}}_{B,D',D,C'}\vee C\vee A"'] & & ((A' \vee B') \wedge (C' \vee D')) \vee ((D \vee B) \wedge (C \vee A)) \arrow[d, "\mathsf{m}_{A'\vee B',C'\vee D',D\vee B,C\vee A}"] \\
A' \vee B' \vee (B \wedge D') \vee (D \wedge C') \vee C \vee A \arrow[dr, "A'\vee B'\vee\mathsf{m}_{B,D',D,C'}\vee C\vee A"] & & (A' \vee B' \vee B \vee D) \wedge (D' \vee C' \vee C \vee A) \arrow[dl, "\hat{\mathsf{t}}_{A'\vee B',B\vee D,D'\vee C',C\vee A}"'] \\
 & A' \vee B' \vee ((B \vee D) \wedge (D' \vee C')) \vee C \vee A &
\end{tikzcd}
$$

[8] The conjecture is that it is not the case, but so far no counterexample could be constructed.

Proof: Chase the following diagram (in which the indices of the maps have been omitted):

$$
\begin{tikzcd}[column sep=small]
(A'\vee A)\wedge(B'\vee(B\wedge C')\vee C)\wedge(D'\vee D) \arrow[dr, "\mathsf{s}"] \arrow[dd, "\mathsf{mix}"'] & & (A'\vee A)\wedge(B'\vee B)\wedge(C'\vee C)\wedge(D'\vee D) \arrow[ll, "\hat{\mathsf{t}}"'] \arrow[rr, "\hat{\mathsf{t}}\wedge\hat{\mathsf{t}}"] & & (A'\vee B'\vee(A\wedge B))\wedge(C'\vee D'\vee(C\wedge D)) \arrow[dd, "\hat{\mathsf{t}}"] \\
& (((A'\vee A)\wedge(B'\vee C))\vee(B\wedge C'))\wedge(D'\vee D) \arrow[rr, "\mathsf{s}"] \arrow[dl, "\mathsf{mix}"] \arrow[dd, "\mathsf{m}"] & & (((A'\vee A)\wedge(B'\vee C))\vee(B\wedge C'\wedge(D'\vee D)) \arrow[dd, "\check{\mathsf{t}}"'] & \\
(A'\vee B'\vee(B\wedge C')\vee C\vee A)\wedge(D'\vee D) \arrow[dd, "\mathsf{s}"'] & & & & ((A'\vee B')\wedge(C'\vee D'))\vee(D\wedge C)\vee(B\wedge A) \arrow[dd, "\mathsf{m}"] \\
& (A'\vee A\vee B)\wedge(B'\vee C\vee C')\wedge(D'\vee D) \arrow[ul, "\hat{\mathsf{t}}"'] \arrow[dd, "\check{\mathsf{t}}"] & & ((A'\vee A)\wedge(B'\vee C))\vee(B\wedge D')\vee(C'\wedge D) \arrow[dd, "\check{\mathsf{m}}^2"'] & \\
A'\vee B'\vee(B\wedge(D'\vee D)\wedge C')\vee C\vee A \arrow[dd, "\check{\mathsf{t}}"'] & & & & ((A'\vee B')\wedge(C'\vee D'))\vee((B\vee D)\wedge(C\wedge A)) \arrow[dd, "\mathsf{m}"] \\
& ((A'\vee A\vee B)\wedge D')\vee(D\wedge(B'\vee C\vee C')) \arrow[rr, "\mathsf{m}"] \arrow[dl, "\mathsf{s}\vee\mathsf{s}"] & & (A'\vee A\vee B\vee D)\wedge(D'\vee B'\vee C\vee C') \arrow[dl, "\hat{\mathsf{t}}"] & \\
A'\vee B'\vee(B\wedge D')\vee(D\wedge C')\vee C\vee A \arrow[rr, "\mathsf{m}"'] & & A'\vee B'\vee((B\vee D)\wedge(D'\vee C'))\vee C\vee A & & (A'\vee B'\vee B\vee D)\wedge(D'\vee C'\vee C\vee A) \arrow[ll, "\hat{\mathsf{t}}"]
\end{tikzcd}
$$

The little triangle in the upper left commutes because of (mix-$\hat{\alpha}$). The little triangle below it is just (mix-m-$\hat{\mathsf{t}}$), and the pentagon below that commutes because of the coherence in *-autonomous categories[9] [BCST96,LS04]. The big square in the center is (m-$\check{\mathsf{t}}$-s) and the small parallelogram at the bottom is just two applications of (m-s) plugged together, and the big horse-shoe shape on the left is ($\check{\mathsf{m}}^2$-s-$\check{\mathsf{m}}^2$). □

8 Beyond medial

The definition of monoidal categories settles how the maps $\hat{\alpha}_{A,B,C}$, $\hat{\sigma}_{A,B}$, $\hat{\varrho}_A$, and $\hat{\lambda}_A$ behave with respect to each other, and how the maps $\check{\alpha}_{A,B,C}$, $\check{\sigma}_{A,B}$, $\check{\varrho}_A$, and $\check{\lambda}_A$ behave with respect to each other. The notion of *-autonomous category then settles via the bijection ($\star$) how the two monoidal structures interact. Then, the structure of a B1-category adds $\vee$-monoids and $\wedge$-comonoids, and the structure of B2-categories allows the $\vee$-monoidal structure to go well with the $\vee$-monoids and the $\wedge$-monoidal structure to go well with the $\wedge$-comonoids. Finally, the structure of B4-categories ensures that *both* monoidal structures go well with the $\vee$-monoids *and* the $\wedge$-comonoids.

However, what has been neglected so far is how the $\vee$-monoids and the $\wedge$-comonoids go along with each other. Recall that in any B2-category the maps ∇ and $\amalg$ preserve the $\vee$-monoid structure and the maps Δ and Π preserve the $\wedge$-comonoid structure (Theorem 7.19).

8.1 Combatibility of $\vee$-monoids and $\wedge$-comonoids We have the following possibilities:

(i) The maps Π and $\amalg$ are quasientropies.
(ii) The maps Π and $\amalg$ are clonable.
(iii) The maps Δ and ∇ are quasientropies.
(iv) The maps Δ and ∇ are clonable.

Condition (i) says in particular that the following diagram commutes

$$
\begin{tikzcd}
& \mathbf{f} \arrow[dl, "\amalg^A"'] \arrow[dr, "\Pi^{\mathbf{f}}"] & \\
A \arrow[rr, "\Pi^A"'] & & \mathbf{t}
\end{tikzcd}
\tag{42}
$$

[9] It even commutes in the setting of weakly distributive categories.

Consequently, every $\mathsf{B1}$-category obeying ($\mathsf{B2a}$) and (42) is not only single-mixed but also for every object A the composition $\mathbf{f} \overset{\coprod^A}{\to} A \overset{\Pi^A}{\to} \mathbf{t}$ yields the same result. In [LS05a] the equation (42) was used as basic axiom, and the mix map was constructed from that without the use proper units.

The next observation to make is that (ii) and (iii) of 8.1 are equivalent, provided ($\mathsf{B3b}$) and ($\mathsf{B3a}$) are present:

8.2 Proposition *In a $\mathsf{B2}$-category with nullary medial and* ($\mathsf{B3a}$) *the following are equivalent for every object A:*

(i) *The map Π^A preserves the $\vee$-multiplication.*
(ii) *The map ∇_A preserves the $\wedge$-counit.*
(iii) *The map $\coprod^{\bar{A}}$ preserves the $\wedge$-comultiplication.*
(iv) *The map $\Delta_{\bar{A}}$ preserves the $\vee$-unit.*

Proof: The equivalence of (ii) and (i) follows from

$$\begin{tikzcd}
A \vee A \arrow[r, "\nabla_A"] \arrow[d, "\Pi^A \vee \Pi^A"'] \arrow[dr, "\Pi^{A \vee A}"] & A \arrow[d, "\Pi^A"] \\
\mathbf{t} \vee \mathbf{t} \arrow[r, "\nabla_{\mathbf{t}}"'] & \mathbf{t}
\end{tikzcd}$$

The lower triangle is ($\mathsf{B3b}$) together with ($\mathsf{B3a}$). The upper triangle is (ii), and the square is (i). The other equivalences follow by duality. $\square$

Condition 8.1 (iv) exhibits yet another example of a "creative tension" between algebra and proof theory. From the viewpoint of algebra, it makes perfectly sense to demand that the $\vee$-monoid structure and the $\wedge$-comonoid structure be compatible with each other, i.e., that 8.1 (i)–(iv) do all hold (see [Lam05]). However, from the proof theoretical point of view it is reasonable to make some fine distinctions: We have to keep in mind that in the sequent calculus it is the "contraction-contraction-case"

$$\mathsf{cut}\,\dfrac{\mathsf{cont}\,\dfrac{\overset{\pi_1}{\vdash \Gamma, A, A}}{\vdash \Gamma, A} \qquad \mathsf{cont}\,\dfrac{\overset{\pi_2}{\vdash \bar{A}, \bar{A}, \Delta}}{\vdash \bar{A}, \Delta}}{\vdash \Gamma, \Delta}$$

which spoils the confluence of cut elimination and which causes the exponential blow-up of the size of the proof. This questions 8.1 (iv), i.e., the commutativity of the diagram

$$\begin{tikzcd}
A \vee A \arrow[r, "\nabla_A"] \arrow[d, "\Delta_A \vee \Delta_A"'] & A \arrow[d, "\Delta_A"] \\
(A \wedge A) \vee (A \wedge A) \arrow[r, "\nabla_{A \wedge A}"'] & A \wedge A
\end{tikzcd} \tag{43}$$

and motivates the distinction made in the following definition.

8.3 Definition We say a $\mathsf{B1}$-category is *weakly smooth* if for every object A, the maps Π^A and $\coprod^A$ are strong and the maps Δ_A and ∇_A are quasientropies (i.e., 8.1 (i)–(iii) hold), and it is *smooth* if for every object A, the maps Π^A, $\coprod^A$, Δ_A and ∇_A are all strong (i.e., all of 8.1 (i)–(iv) do hold).

8.4 Corollary *A* $\mathsf{B3}$*-category is weakly smooth, if and only if* Π^A *is a* $\vee$*-monoid morphism for every object* A*.*

To understand the next (and final) axiom of this paper, recall that in every *-autonomous category we have

$$
\begin{tikzcd}
\mathbf{t} \arrow[r, "\check{\mathsf{i}}_A \wedge \check{\mathsf{i}}_A"] \arrow[d, "\check{\mathsf{i}}_A"'] & (\bar{A} \vee A) \wedge (\bar{A} \vee A) \arrow[d, "\hat{\mathsf{t}}"] \\
\bar{A} \vee A & \bar{A} \vee (A \wedge \bar{A}) \vee A \arrow[l, "\bar{A} \vee \hat{\mathsf{i}}_A \vee A"]
\end{tikzcd}
\tag{44}
$$

and that this equation is the reason why the cut elimination for multiplicative linear logic (proof nets as well as sequent calculus) works so well. The motivation for the following definition is to obtain something similar for classical logic (cf. [LS05a]).

8.5 Definition A $\mathsf{B1}$-category is *contractible* if the following diagram commutes for all objects A.

$$
\begin{tikzcd}
\mathbf{t} \arrow[r, "\check{\mathsf{i}}_A"] \arrow[dd, "\check{\mathsf{i}}_A"'] & \bar{A} \vee A \arrow[d, "\Delta_{\bar{A} \vee A}"] \\
 & (\bar{A} \vee A) \wedge (\bar{A} \vee A) \arrow[d, "\hat{\mathsf{t}}"] \\
\bar{A} \vee A & \bar{A} \vee (A \wedge \bar{A}) \vee A \arrow[l, "\bar{A} \vee \hat{\mathsf{i}}_A \vee A"]
\end{tikzcd}
\tag{45}
$$

The following theorem states one of the main results of this paper. It explains the deep reasons why the cut elimination for the proof nets of [LS05b] is not confluent in the general case. It also shows that the combination equations (43) and (45) leads to a certain collapse in a $\mathsf{B4}'$-category, which can be compared to the collapse made by an LK-category. Nonetheless, even with this collapse we can find reasonable models for proofs of Boolean logic, as it is shown in the next section.

8.6 Theorem *In a* $\mathsf{B4}'$*-category that is smooth and contractible, we have*

$$1_A + 1_A = 1_A$$

for all objects A*.*

Proof: The proof idea here is the same as in the proof of Theorem 2.4.7 in [LS05a]. The novelty is that here we do not need the sledge-hammer axiom of graphicality. Instead we make use of Lemma 7.26. We proceed by showing that $\check{\mathsf{i}}_A + \check{\mathsf{i}}_A = \check{\mathsf{i}}_A \colon \mathbf{t} \to \bar{A} \vee A$ for all objects A. From this the result follows by Proposition 5.20. Note that in particular we have that $\check{\mathsf{i}}_A + \check{\mathsf{i}}_A$ is the map

$$\mathbf{t} \xrightarrow{\check{\mathsf{i}}_{A \wedge A}} \bar{A} \vee \bar{A} \vee (A \wedge A) \xrightarrow{\bar{A} \vee \bar{A} \vee \mathsf{mix}_{A,A}} \bar{A} \vee \bar{A} \vee A \vee A \xrightarrow{\nabla_{\bar{A}} \vee \nabla_A} \bar{A} \vee A$$

which is because of (mix-$\hat{\alpha}$) and the *-autonomous structure the same as the left-most down path in the following diagram.

$$
\begin{tikzcd}
\mathsf{t} \arrow[rr, "\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A"] \arrow[dr, "\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A"] \arrow[ddd, "\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A"'] & & (\bar{A}\vee A)\wedge(\bar{A}\vee A)\wedge(\bar{A}\vee A)\wedge(\bar{A}\vee A) \arrow[d, "\hat{\mathsf{t}}_{\bar{A},A,\bar{A},A}"] \\
& (\bar{A}\vee A)\wedge(\bar{A}\vee A)\wedge(\bar{A}\vee A) \arrow[ur, "\check{\mathsf{i}}_A"] \arrow[d, "\hat{\mathsf{t}}_{\bar{A},A,\bar{A},A}"'] & (\bar{A}\vee A)\wedge(\bar{A}\vee(A\wedge\bar{A})\vee A)\wedge(\bar{A}\vee A) \arrow[d, "\mathsf{mix}_{\bar{A}\vee A,\bar{A}\vee(A\wedge\bar{A})\vee A}"] \\
& (\bar{A}\vee A)\wedge(\bar{A}\vee(A\wedge\bar{A})\vee A) \arrow[ur, "\check{\mathsf{i}}_A"] \arrow[dl, "\hat{\mathsf{i}}_A"'] \arrow[d, "\mathsf{mix}"'] & (\bar{A}\vee\bar{A}\vee(A\wedge\bar{A})\vee A\vee A)\wedge(\bar{A}\vee A) \arrow[d, "\mathsf{s}_{\bar{A}\vee\bar{A}\vee A\vee A,A\wedge\bar{A},\bar{A}\vee A}"] \\
(\bar{A}\vee A)\wedge(\bar{A}\vee A) \arrow[d, "\mathsf{mix}_{\bar{A}\vee A,\bar{A}\vee A}"'] & \bar{A}\vee\bar{A}\vee(A\wedge\bar{A})\vee A\vee A \arrow[r, "\check{\mathsf{i}}_A"'] \arrow[ur, "\check{\mathsf{i}}_A"] \arrow[dl, "\hat{\mathsf{i}}_A"'] & \bar{A}\vee\bar{A}\vee(A\wedge(\bar{A}\vee A)\wedge\bar{A})\vee A\vee A \arrow[d, "\check{\mathsf{t}}_{A,\bar{A},A,\bar{A}}"] \\
\bar{A}\vee\bar{A}\vee A\vee A \arrow[d, "\nabla_{\bar{A}}\vee\nabla_A"'] & \bar{A}\vee\bar{A}\vee(A\wedge\bar{A})\vee A\vee A \arrow[l, "\hat{\mathsf{i}}_A"] \arrow[d, "\nabla_{\bar{A}}\vee\nabla_A"'] & \bar{A}\vee\bar{A}\vee(A\wedge\bar{A})\vee(A\wedge\bar{A})\vee A\vee A \arrow[l, "\nabla_{A\wedge\bar{A}}"'] \arrow[d, "\mathsf{m}_{A,\bar{A},A,\bar{A}}"] \\
\bar{A}\vee A & \bar{A}\vee(A\wedge\bar{A})\vee A \arrow[l, "\hat{\mathsf{i}}_A"] & \bar{A}\vee\bar{A}\vee((A\vee A)\wedge(\bar{A}\vee\bar{A}))\vee A\vee A \arrow[l, "\nabla_{\bar{A}}\vee(\nabla_A\wedge\nabla_{\bar{A}})\vee\nabla_A"] \arrow[ul, "\nabla_A\wedge\nabla_{\bar{A}}"']
\end{tikzcd}
$$

The upper triangle commutes because of functoriality of $\wedge$, the square in the lower left corner because of functoriality of $\vee$, and the parallelograms because of naturality of mix and $\hat{\mathsf{t}}$. The quadrangle in the upper left commutes because of (44), and the little triangle in the right center is just (13) together with naturality of switch. The pentagon below it is just the dual of (45), and the two little triangles at the lower right corner are ($\mathsf{B3c}'$) and functoriality of $\vee$. Therefore, this diagram gives us a complicated way of writing just $\check{\mathsf{i}}_A + \check{\mathsf{i}}_A$. Similarly, the next diagram gives us a complicated way of writing $\check{\mathsf{i}}_A$:

$$
\begin{tikzcd}
\mathsf{t} \arrow[rr, "\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A"] \arrow[dr, "\check{\mathsf{i}}_A"'] \arrow[drr, "\check{\mathsf{i}}_{(A\wedge A)\vee(A\wedge A)}"] \arrow[ddddd, "\check{\mathsf{i}}_A"'] & & (\bar{A}\vee A)\wedge(\bar{A}\vee A)\wedge(\bar{A}\vee A)\wedge(\bar{A}\vee A) \arrow[d, "\hat{\mathsf{t}}_{\bar{A},A,\bar{A},A}\wedge\hat{\mathsf{t}}_{\bar{A},A,\bar{A},A}"] \\
& & (\bar{A}\vee\bar{A}\vee(A\wedge A))\wedge(\bar{A}\vee\bar{A}\vee(A\wedge A)) \arrow[d, "\hat{\mathsf{t}}_{\bar{A}\vee\bar{A},A\wedge A,\bar{A}\vee\bar{A},A\wedge A}"] \\
& \bar{A}\vee A \arrow[d, "\Delta_{\bar{A}}\vee\Delta_A"'] & ((\bar{A}\vee\bar{A})\wedge(\bar{A}\vee\bar{A}))\vee(A\wedge A)\vee(A\wedge A) \arrow[dl, "(\nabla_{\bar{A}}\wedge\nabla_{\bar{A}})\vee\nabla_{A\wedge A}"'] \arrow[d, "\mathsf{m}_{A,A,A,A}"] \\
& (\bar{A}\wedge\bar{A})\vee(A\wedge A) \arrow[d, "\mathsf{m}_{\bar{A},\bar{A},A,A}"'] & ((\bar{A}\vee\bar{A})\wedge(\bar{A}\vee\bar{A}))\vee((A\vee A)\wedge(A\vee A)) \arrow[l, "(\nabla_{\bar{A}}\wedge\nabla_{\bar{A}})\vee(\nabla_A\wedge\nabla_A)"] \arrow[d, "\mathsf{m}_{\bar{A}\wedge\bar{A},\bar{A}\vee\bar{A},A\vee A,A\vee A}"] \\
& (\bar{A}\vee A)\wedge(\bar{A}\vee A) \arrow[d, "\hat{\mathsf{t}}_{\bar{A},A,\bar{A},A}"'] & (\bar{A}\vee\bar{A}\vee A\vee A)\wedge(\bar{A}\vee\bar{A}\vee A\vee A) \arrow[l, "(\nabla_{\bar{A}}\vee\nabla_A)\wedge(\nabla_{\bar{A}}\vee\nabla_A)"] \arrow[d, "\hat{\mathsf{t}}_{\bar{A}\wedge\bar{A},A\vee A,\bar{A}\vee\bar{A},A\vee A}"] \\
\bar{A}\vee A & \bar{A}\vee(A\wedge\bar{A})\vee A \arrow[l, "\hat{\mathsf{i}}_A"] & \bar{A}\vee\bar{A}\vee((A\vee A)\wedge(\bar{A}\vee\bar{A}))\vee A\vee A \arrow[l, "\nabla_{\bar{A}}\vee(\nabla_A\wedge\nabla_{\bar{A}})\vee\nabla_A"]
\end{tikzcd}
$$

Here the big upper right "triangle" commutes because of the *-autonomous structure. The irregular quadrangle in the center is a transposed version of (43), the little triangle below it is ($\mathsf{B3c}'$), the two squares at the bottom are naturality of m and $\hat{\mathsf{t}}$, and the left-most part of the diagram commutes because of (45) and (B3c). Finally, we apply Lemma 7.26 to paste the two diagrams together, which yields $\check{\imath}_A + \check{\imath}_A = \check{\imath}_A$ as desired. □

In Figure 1 the basic idea of this proof is shown. The first four equations in that figure express the idea behind the first big diagram in the proof of Theorem 8.6, and the last three equations in Figure 1 express the idea of the second diagram. More explanations on this will follow in the next section.

8.7 Corollary *Let $\mathscr{A}$ be a set of propositional variables and let $\mathscr{C}$ be the free smooth and contractible* B4′*-category generated by $\mathscr{A}$. Then $\mathscr{C}$ is idempotent.*

9 A concrete example: proof nets

In this section we will construct a concrete example of a category which has almost all the properties discussed in this paper. Its existence shows that this paper actually makes sense: The equations presented here do not lead to the collapse into a Boolean algebra. In fact, this category was the main source of motivation for introducing the equations presented in Sections 3, 5, 7, and 8.

We are going to present two versions of proof nets:

1. The *simple proof nets* are a slight modification of the proof nets introduced in [LS05b,LS05a]. The difference is that the categories of proof nets defined in these papers had only weak units, while here we are assuming from the beginning that $\mathbf{t}$ and $\mathbf{f}$ are proper unit objects.
2. The *extended proof nets* have a richer structure than the simple nets. From the algebraic point of view the main difference to the simple nets is that the category of extended nets does not obey equation (43) and is not idempotent (and is therefore not an LK-category). From the proof theoretic point of view, the extended nets keep more information about the proofs. In particular the size of proofs can captured.

9.1 Definition Let $\mathscr{A}$ be a set of propositional variables. The set $\mathscr{F}$ of *formulas* is generated via

$$\mathscr{F} ::= \mathscr{A} \mid \bar{\mathscr{A}} \mid \mathbf{t} \mid \mathbf{f} \mid \mathscr{F} \wedge \mathscr{F} \mid \mathscr{F} \vee \mathscr{F} \quad .$$

A *sequent* is a finite list of formulas, separated by comma. A formula can be seen as a tree and a sequent as a forest whose leaves are labelled by elements of the set $\mathscr{A} \cup \bar{\mathscr{A}} \cup \{\mathbf{f}, \mathbf{t}\}$ and whose inner nodes are labelled by elements of $\{\wedge, \vee\}$. For a sequent Γ, let $\mathscr{L}(\Gamma)$ denote the set of its leaves. For a leaf $i \in \mathscr{L}(\Gamma)$ let $\ell(i) \in \mathscr{A} \cup \bar{\mathscr{A}} \cup \{\mathbf{f}, \mathbf{t}\}$ denote its labelling. A *linking* for a sequent Γ is a binary relation $P \subseteq \mathscr{L}(\Gamma) \times \mathscr{L}(\Gamma)$ such that

(i) for every $i \in \mathscr{L}(\Gamma)$ with $\ell(i) = \mathbf{t}$ we have $(i,i) \in P$, and
(ii) if $(i,j) \in P$, then one of the following cases must hold:
  - $i = j$ and $\ell(i) = \mathbf{t}$, or
  - $i \neq j$ and $\ell(i) = \bar{a}$ and $\ell(j) = a$ for some $a \in \mathscr{A}$.

A *simple prenet*[10] consists of a sequent Γ and a linking P for it. It will be denoted by $P \triangleright \Gamma$.

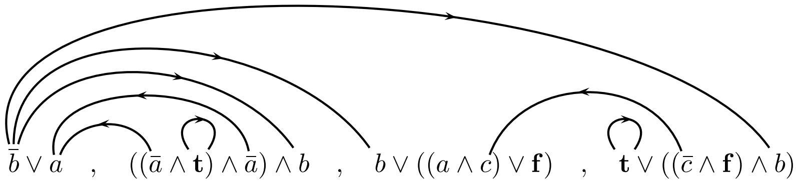
In this paper, we will write prenets by simply writing down the sequent and by putting the linking as (directed) graph above it, as in these two examples[11]:

$$\bar{b} \wedge a \quad , \quad \bar{a} \wedge \bar{b} \quad , \quad b \wedge a \quad , \quad \bar{a} \wedge b \tag{46}$$

[10] What we call *prenet* is in the literature sometimes also called a *proof structure*.

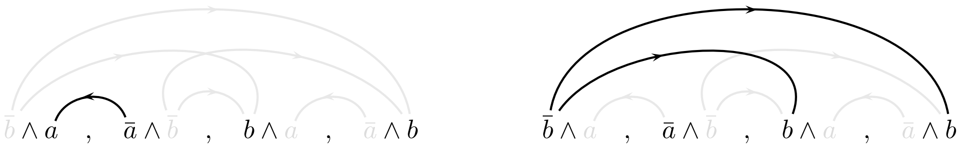
[11] Here we make two modifications to the proof nets used in [LS05b,LS05a]: (i) We force every $\mathbf{t}$ to be linked to itself and we do not allow links between $\mathbf{t}$ and $\mathbf{f}$. The reason is that we deal in this paper with proper units in the categorical sense, while [LS05b,LS05a] used "weak units" (see also the introduction). The observation that linking every $\mathbf{t}$ to itself and disallowing $\mathbf{t}$-$\mathbf{f}$-links is enough to get proper units is due to François Lamarche. (ii) We use here directed links between complementary pairs of atoms (instead of undirected links as in [LS05b,LS05a]). This brings a slight simplification of cut elimination via path composition. The idea for this has been taken from Dominic Hughes [Hug05b,Hug05a].

and

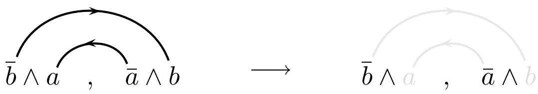


(47)

Now we will define when a simple prenet is correct, i.e., comes from an actual proof. A *conjunctive pruning* of a prenet $P \rhd \Gamma$ is the result of removing one of the two subformulas of each $\wedge$ in Γ and restricting the linking P accordingly. Here are two (of 16 possible) conjunctive prunings of (46):



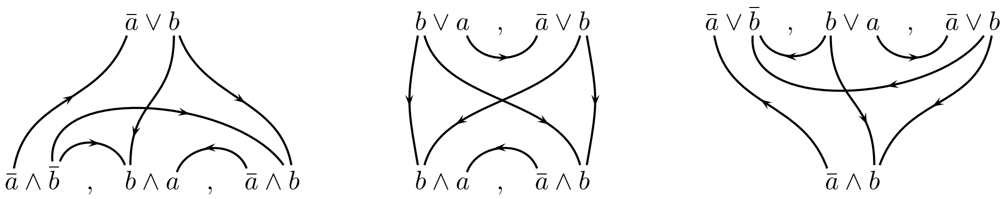
A simple prenet is *correct* if each of its conjunctive prunings contains at least one link (i.e, the linking is not empty). A *simple proof net* is a correct simple prenet. The two examples in (46) and (47) are proof nets. Here is a prenet, which is not a proof net because there is a pruning in which all links disappear:



Now we show how simple proof nets can be composed. As in [LS05b,LS05a,Str05] this is done via cut elimination. But we use here a notational trick to make it even more intuitive: we allow to write proof nets in two-sided form: Instead of putting the linking *above* a sequent $A_1, \dots, A_n, B_1, \dots, B_m$, we put it *in between* $\bar{A}_n, \dots, \bar{A}_1$ and $B_1, \dots, B_m$, where the negation $\bar{A}$ of a formula A is inductively defined as follows[12]:

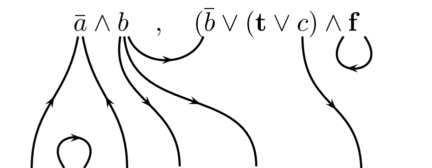
$$\bar{a} = \bar{a}\,, \quad \bar{\bar{a}} = a\,, \quad \bar{\mathbf{t}} = \mathbf{f}\,, \quad \bar{\mathbf{f}} = \mathbf{t}\,, \quad \overline{(A \wedge B)} = \bar{B} \vee \bar{A}\,, \quad \overline{(A \vee B)} = \bar{B} \wedge \bar{A}\,.$$

Here are three different ways of writing example (46) in two-sided form:



(48)

And here is a different way of writing example (47):

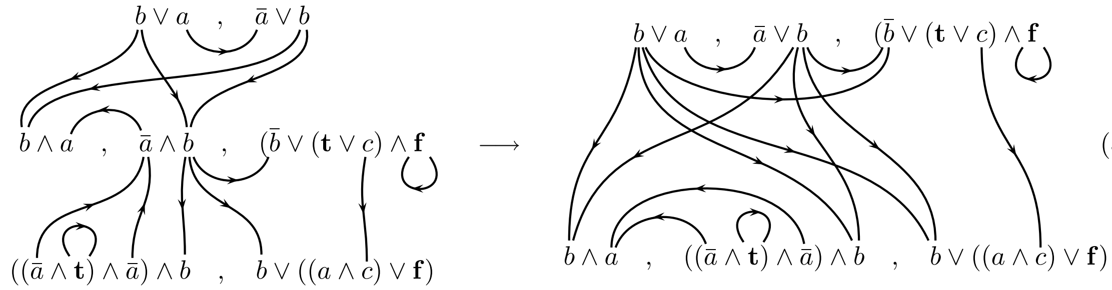


(49)

Note that for defining the direction of the links and for checking correctness we assume we had not taken the negation of the formulas on top. (This can be made formally precise by using polarities [Lam01].) What is important is the fact that the objects in (46) and in (48) denote *the same* net. Similarly, (47) and in (49) are just different ways of drawing the same proof net.

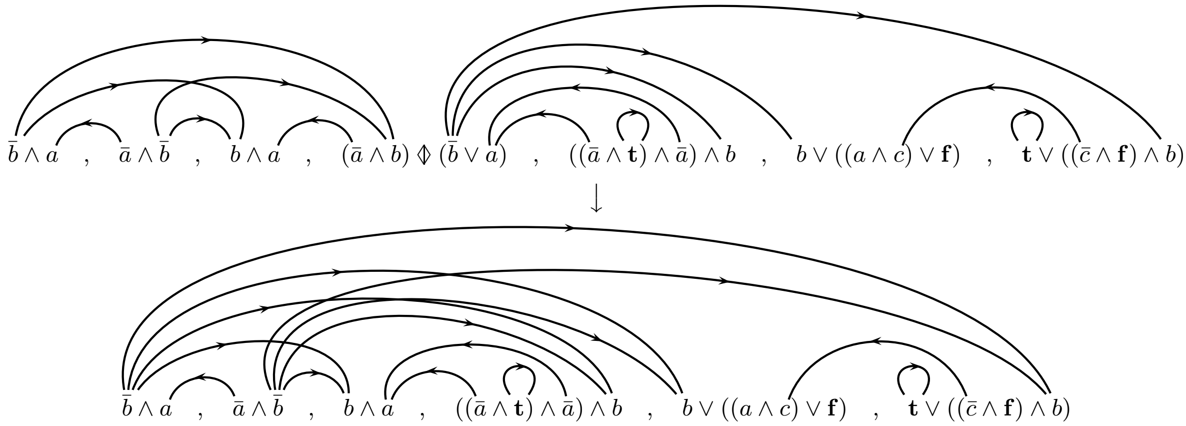
[12] We invert the order when taking the negation in order to reduce the number of crossings in the pictures.

Cut elimination can now be defined by plugging nets together, as in the following example which is a composition of (the middle) net in (48) and the one in (49):



(50)

There is a link in the resulting net if and only if there is a corresponding path in the nonreduced composition. Writing it in the two-sided version makes it more intuitive than in the one-sided version, where (50) would be written as:

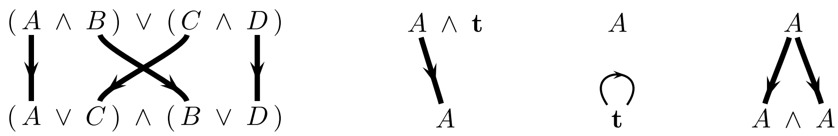


This is the way it has been done in [LS05b] where it has been shown that this operation is associative and preserves correctness. In [LS05b] it has also been shown how sequent calculus derivations are translated into proof nets. After what has been said here, it might be more intuitive to think of them as *flow graphs* [Bus91,Car97] of derivations in SKS [BT01]. From the historical perspective it should be mentioned that the basic idea of the simple proof nets discussed here appeared in the literature already in [And76] as *matings* and in [Bib81] as *matrix proofs*[13], and that this idea goes even back to the *coherence graphs* of [KM71].

If we now restrict ourselves to proof nets with only two conclusions, then we have a category[14]: the objects are the formulas and the arrows $A \to B$ are the proof nets $P \rhd \bar{A}, B$. Arrow composition is defined as above, and the identities are the trivial proof nets $P \rhd \bar{A}, A$. Let us call this category $\mathbf{SNet}(\mathscr{A})$ where $\mathscr{A}$ is the set of propositional variables from which we started.

9.2 Theorem *The category* $\mathbf{SNet}(\mathscr{A})$ *is a* $\mathsf{B4}'$*-category that is smooth and contractible.*

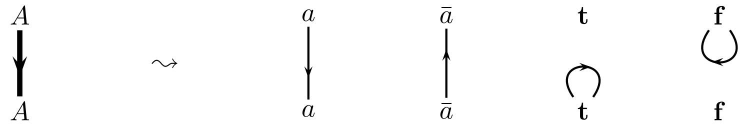
Proof: The maps $\hat{\alpha}$, $\hat{\sigma}$, $\hat{\varrho}$, $\hat{\lambda}$, s, m, Π, and Δ are given by the obvious nets. We show here as example the nets for $\mathsf{m}_{A,B,C,D}$, $\hat{\varrho}_A$, Π^A, and Δ_A (the others being similar, cf. [Str05]):



[13] Note, however, that there is a subtle but crucial difference between our simple nets and the work of [And76,Bib81]: in this early work links between atoms in a conjunction relation were not allowed (because they are irrelevant for correctness), but these links are crucial for obtaining an associative cut elimination operation (see [LS05b]).

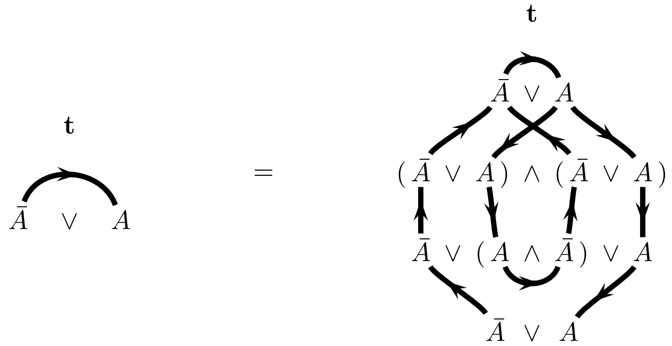
[14] Without the restriction to two formulas we would obtain a polycategory [Lam69,Sza75].

In these drawings the bold links between formulas represent bundles of several links, one for each leaf of the formula tree. Note that the arrows have to have the right direction, and that there are no links connecting $\mathbf{f}$ and $\mathbf{t}$. There are four cases:

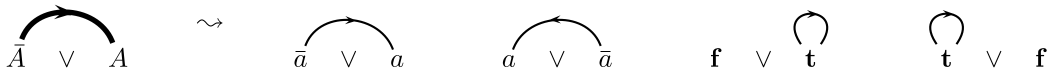


(51)

It is an easy exercise to check that all the equations demanded by the definitions do indeed hold. We show here only the case of the contracibility axiom (45):



As above, the bold links represent bundles of normal links:



(52)

The transposition $(\star)$ of proof nets is obtained by simply drawing the net in the transposed way, as it has been done with (46) and (48). That this has the desired properties should be clear from inspecting (51) and (52). □

9.3 Theorem *The category* $\mathbf{SNet}(\mathscr{A})$ *is graphical.*

Proof: Trivial (cf. [LS05a]). □

Note that in $\mathbf{SNet}(\mathscr{A})$ the sum $f + g \colon A \to B$ of two proofs $f, g \colon A \to B$ is obtained by taking the (set theoretical) union of the two corresponding linkings. Hence, $\mathbf{SNet}(\mathscr{A})$ is idempotent, and we have that $f \leq g$ iff the linking for f is a subset of the linking for g.

9.4 Theorem *The category* $\mathbf{SNet}(\mathscr{A})$ *is an* LK*-category.*

Proof: Let $f, g \colon A \to B$ be two maps in $\mathbf{SNet}(\mathscr{A})$. Let f be given by the proof net $P \rhd \bar{A}, B$ and g be given by $Q \rhd \bar{A}, B$. Then, by what has been said above, we define $f \preccurlyeq g$ iff $Q \subseteq P$. After Theorem 9.2 it only remains to show that equation (LK-Δ) holds for all f. But this follows immediately from the definition of composition of proof nets. □

It should be clear, that the category $\mathbf{SNet}(\mathscr{A})$ is quite a degenerated model for proofs of Boolean propositional logic. The size of a proof net is at most quadradic in the size of the sequent. This means in particular, that the information how often a certain link is used in a proof is not present in the proof net. For this reason we will now allow more than on link between a pair of complementary atoms.[15] But as shown in [LS05b], doing this naively means losing confluence of cut elimination via path composition. I.e., we do not get a category with associative arrow composition. A possible solution has been suggested in [Str05]:

[15] In terms of [LS05b] this means stepping from the Boolean semiring of weights to the natural numbers semiring of weights.

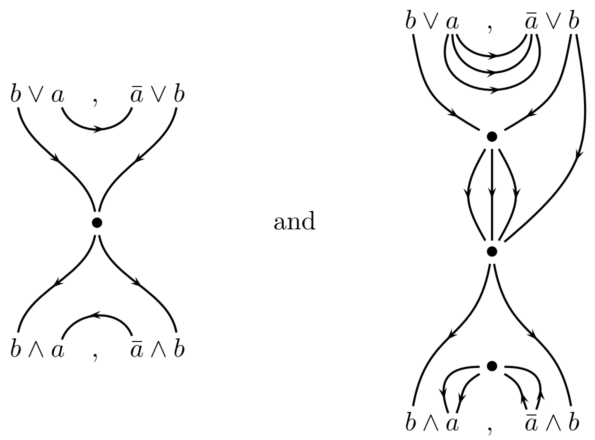
9.5 Definition An *extended prenet* consists of a sequent Γ, a finite set K of *anchors*, an *anchor labelling* $\ell\colon K \to \mathscr{A}$, and a linking (which is now no longer a binary relation, but a binary function to the naturals) $P\colon (\mathscr{L}(\Gamma) \cup K) \times (\mathscr{L}(\Gamma) \cup K) \to \mathbb{N}$, such that

(i) for every $i \in \mathscr{L}(\Gamma)$ with $\ell(i) = \mathbf{t}$ we have $P(i,i) = 1$,
(ii) for every $k \in K$ we have

$$\sum_{i \in \mathscr{L}(\Gamma) \cup K} P(i,k) \geq 2 \qquad \text{and} \qquad \sum_{j \in \mathscr{L}(\Gamma) \cup K} P(k,j) \geq 2$$

(iii) if $P(i,j) \geq 1$, then one of the following cases must hold:
- $i = j$ and $\ell(i) = \mathbf{t}$ and $P(i,i) = 1$, or
- $i \neq j$ and $i \in \mathscr{L}(\Gamma)$ and $j \in \mathscr{L}(\Gamma) \cup K$ and $\ell(i) = \bar{a}$ and $\ell(j) = a$ for some $a \in \mathscr{A}$, or
- $i \neq j$ and $i \in K$ and $j \in \mathscr{L}(\Gamma) \cup K$ and $\ell(i) = \ell(j) = a$ for some $a \in \mathscr{A}$.

As before, every $\mathbf{t}$ has to be linked to itself. That we allow only one and not many such links is due to Lemma 5.11 (which is a consequence of having proper units, cf. [Str05]). But contrary to what we had before, we do now allow not only many links between two atoms but also "non-direct" links visiting anchors on their way. But each anchor can only serve links between atoms of the same name. Furthermore, an anchor must have at least two incoming and at least two outgoing links. Here are two examples of extended prenets:



(53)

The labellings of the anchors are not shown because they are clear from the linkings. Observe that while in a simple proof net the number of links is at most quadratic in the size of the sequent, in an extended net things can get arbitrarily large, as the second example in (53) shows.

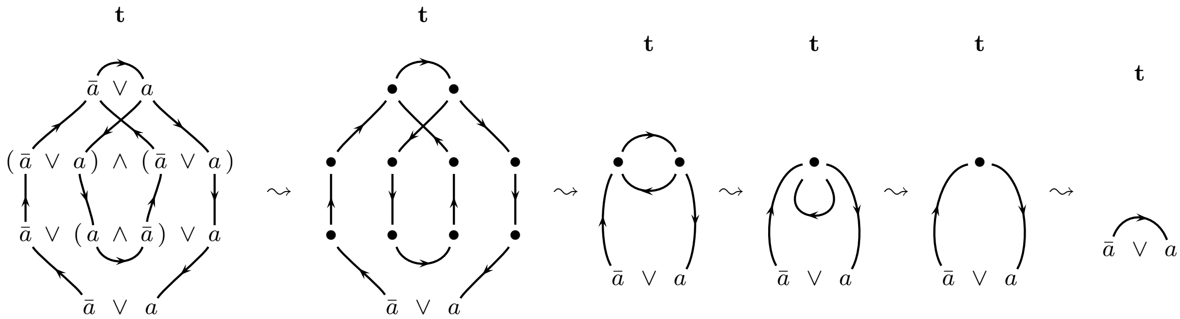
The correctness of extended nets is defined similar as for simple nets. But now the condition is not that every pruning must contain a link, but it must contain a complete path from one leaf to another (in a conjunctive pruning the anchors remain and behave as big disjunctions). A correct extended prenet is an *extended proof net*. The examples in (53) are extended proof nets.

9.6 Cut elimination for extended nets The composition of two extended proof nets is again defined via cut elimination, which can again be understood as path composition. But this time we have to be careful to treat the anchors correctly if we want a well-defined and associative composition. To be formally precise, we define it in two steps:

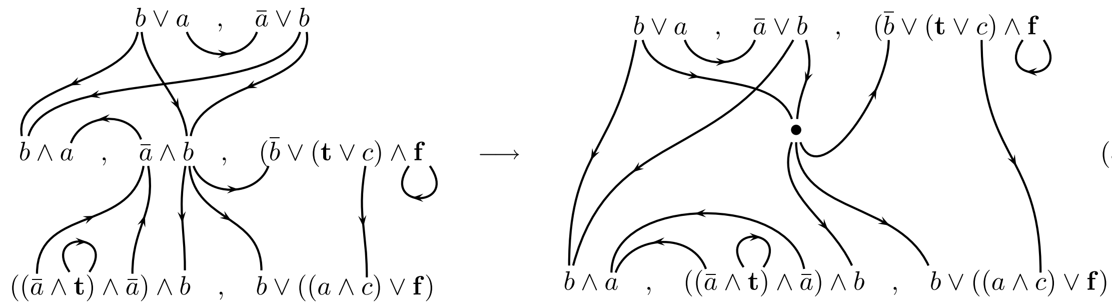
1.) Replace every leaf of the cut formula by an anchor, while the links remain unchanged.
2.) Remove all anchors that have no right to exist (i.e., that have less than two incoming or outgoing links). This is done by repeatedly performing the following reduction steps until no further reduction is possible:
   - Remove every anchor without any outcoming links, and remove all links coming into it.
   - Remove every anchor without any incoming links, and remove all links coming out of it.
   - If there is an anchor k with only one link coming out, i.e., there is exactly one $i \in \mathscr{L}(\Gamma) \cup K$ with $P(k,i) = 1$, and $P(k,j) = 0$ for all $j \in \mathscr{L}(\Gamma) \cup K$ with $j \neq i$, then

- for every $l \in \mathscr{L}(\Gamma) \cup K$ increase the value of $P(l, i)$ by $P(l, k)$, i.e, all links going into k are redirected to i, and
    - remove k.
  - If there is an anchor k with only one link going in, i.e., there is exactly one $i \in \mathscr{L}(\Gamma) \cup K$ with $P(i, k) = 1$, and $P(j, k) = 0$ for all $j \in \mathscr{L}(\Gamma) \cup K$ with $j \neq i$, then
    - for every $l \in \mathscr{L}(\Gamma) \cup K$ increase the value of $P(i, l)$ by $P(k, l)$, i.e, all links coming out of k are replaced by links coming out of i, and
    - remove k.
  - Set $P(k, k) = 0$ for all $k \in K$.
  - Set $P(i, i) = 1$ for all i with $\ell(i) = \mathbf{t}$.

This sounds more complicated than it actually is. As first example, we show here the case of the contractibility axiom (applied to a single atom):

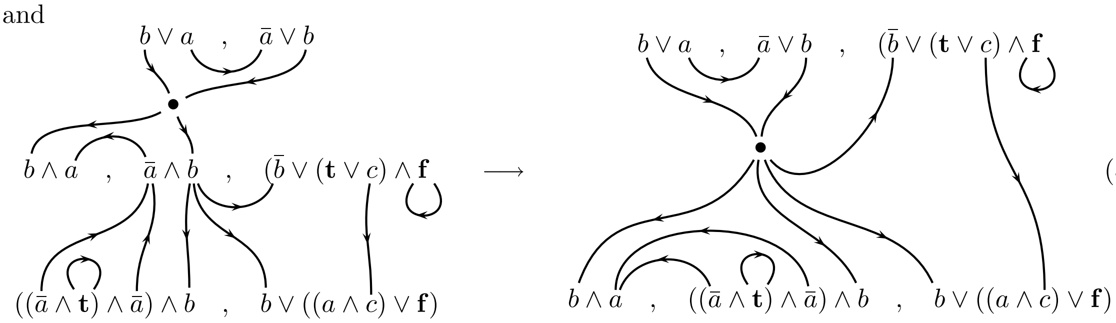


Here are two other examples (compare with (50)):



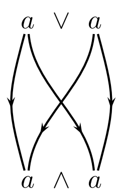
(54)

and

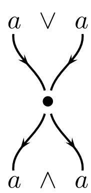


(55)

The whole point of the construction of the extended proof nets is, that we again get a category, which we denote by $\mathbf{ENet}(\mathscr{A})$.[16] This category is again a $\mathsf{B4}'$-category and it is graphical. But it is *not* smooth because the following nets are obviously not equal:



and



(56)

[16] The restrictions made to the anchors in Definition 9.5 are chosen such that we indeed get a well-defined and associative composition.

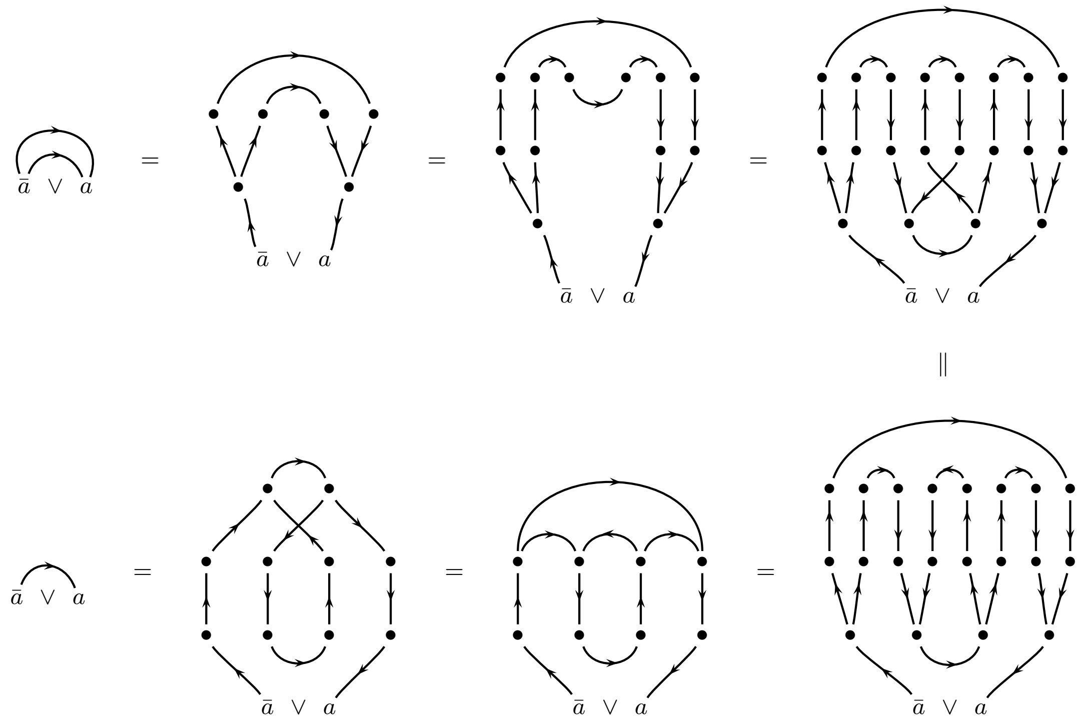


Fig. 1. The idea of the proof of Theorem 8.6

which means that diagram (43) does not commute. It is also easy to see that $\mathbf{ENet}(\mathscr{A})$ is not idempotent, and is therefore not an LK-category. However, we have:

9.7 Theorem *The category* $\mathbf{ENet}(\mathscr{A})$ *is a* $\mathsf{B4}'$*-category that is weakly smooth and contractible.*

Proof: Again, the maps $\hat{\alpha}$, $\hat{\sigma}$ $\hat{\varrho}$, $\hat{\lambda}$, s, m, Π, and Δ are given by the obvious nets. It is an easy exercise to check that the equations demanded by the definitions do still hold. $\square$

9.8 Theorem *The category* $\mathbf{ENet}(\mathscr{A})$ *is graphical.*

Proof: Trivial. $\square$

In Figure 1 we use the notation of the extended proof nets to illustrate the idea behind the proof of Theorem 8.6. The middle equation in the second line is (43), i.e, the identity of the two nets in (56). The left-most equation in the second line and the right-most equation in the first line are both the contracibility equation (45). Everyting else in Figure 1 is rather trivial from the viewpoint of proof nets. However, since we do not have a "coherence theorem", Figure 1 cannot tell us whether the equations are really consequences of the axioms. For this, the proper proof in Section 8 is necessary.

10 More thoughts on order enrichment

Although $\mathbf{ENet}(\mathscr{A})$ is not an LK-category, we can enrich it with a partial order which is induced by cut elimination according to the ideas of [FP04c,FP04a]. This means that that $f \preccurlyeq g$ if g is obtained from f via cut elimination in some formal system (not necessarily LK or another sequent system).

In category theoretical terms, this is achieved by keeping properties (i) and (iii) in Definion 6.1, but by dropping (LK-Π) and (LK-Δ). It should be clear that there is a wide range of possibilities of providing such a partial order. As example we will sketch here the idea which has been proposed in [Str05].

Let $f: A \to B$ be a map in $\mathbf{ENet}(\mathscr{A})$, i.e., an extended proof net, and let $k \in K_f$ be an anchor in f, and let P_f be the linking of f. Then we can remove k according to the cut elimination for simple proof nets (as defined in [LS05b]). Let K' be $K_f \setminus \{k\}$ and for all $i, j \in \mathscr{L}(\Gamma) \cup K'$ let $P'(i,j) = P_f(i,j) + P_f(i,k) \cdot P_f(k,j)$. Now define $g: A \to B$ to be the result of applying the second step of 9.6 to P', K'.

For example the right net in (56) is the result of eliminating the anchor of the left net in (56).

Note that this anchor elimination process is not confluent, i.e., in a net with many anchors, the result of eliminating all of them depends on the order in which they are eliminated. This has been shown in [LS05b], but morally it is a consequence of Theorem 8.6.

There is also a close relationship to cut elimination in the calculus of structures. There is work in progress to nail down the precise relation between the anchor elimination for proof nets defined above and the splitting technique [Gug02b] for elimination the cuts in system SKS [Brü03].

Let us finish this paper by proposing yet another way of enriching $\mathbf{ENet}(\mathscr{A})$ with a partial order: Since map in that category are just directed graphs, we can define $f \preccurlyeq g$ if g is a minor of f in the graph theoretical sense. We have to leave it as problem for future work to investigate the proof theoretical implications of this.

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