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Implicitization of rational ruled surfaces with μ-bases

Marc Dohm

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Abstract

Chen, Sederberg, and Zheng introduced the notion of a μ-basis for a rational ruled surface in Chen et al. (2001) and showed that its resultant is the implicit equation of the surface, if the parametrization is generically injective. We generalize this result to the case of an arbitrary parametrization. We also give a new proof for the corresponding theorem in the curve case and treat the reparametrization problem for curves and ruled surfaces.

Introduction

Implicitization is a fundamental problem in Computer Aided Geometric Design and there are numerous applications related to it, e.g. the computation of the intersection of two ruled surfaces, see Fioravanti et al. (2005). The method of μ-bases (also known as "moving lines" or "moving surfaces") constitutes an efficient solution to the implicitization problem. Introduced in 1998 by Cox, Sederberg, and Chen for rational curves in Cox et al. (1998), it was generalized to ruled surfaces in Chen et al. (2001) and Chen and Wang (2003b). Whereas the curve case is very well understood and we know that the resultant of a μ-basis is the implicit equation to the power d, where d is the degree of the parametrization, this result is still to be shown in its full generality (i.e. for arbitrary d) for ruled surfaces. We fill this gap by giving a proof, which relies on a geometric idea that reduces the ruled surface case to the curve case. From a computional point of view, μ-bases are in general more efficient than other resultant-based methods such as the ones introduced in Busé and Chardin (2005) or in Khetan (2003), since they are well adapted to the geometry of ruled surfaces.

1 μ-bases of rational planar curves

As we will need them later on, we will start with some known results about the μ-basis of a rational parametric planar curve $\mathcal{C}$ over an algebraically closed field

$\mathbb{K}$, i.e. one given by a parametrization map

$$\begin{array}{rccc} \Phi_{\mathcal{C}}: & \mathbb{P}^1 & \dashrightarrow & \mathbb{P}^2 \\ & (s:\bar{s}) & \mapsto & (f_0(s,\bar{s}):f_1(s,\bar{s}):f_2(s,\bar{s})) \end{array}$$

where each $f_i \in \mathbb{K}[s,\bar{s}] =: R$ is homogeneous of degree $n > 0$ and $g := gcd(f_0,f_1,f_2)$ is of degree less than n. The first syzygy module of f_0,f_1,f_2 is defined as

$$Syz(f_0,f_1,f_2) = \{P \in R[x,y,z] \mid deg(P) \leq 1, P(f_0,f_1,f_2) = 0\} \subseteq R[x,y,z]$$

Then we have the following well-known result.

Theorem 1 *There exists an isomorphism of graded R-modules*

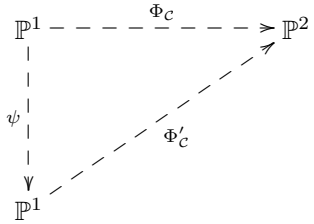
$$Syz(f_0,f_1,f_2) \cong R(-\mu_1) \oplus R(-\mu_2)$$

where $\mu_i \in \mathbb{N}$, $\mu_1 \leq \mu_2$ and

$$\mu_1 + \mu_2 = n - deg(g) = deg(\Phi_{\mathcal{C}}) \cdot deg(\mathcal{C}) =: d$$

A basis (p,q) of $Syz(f_0,f_1,f_2)$ with minimal degrees $deg(p) = \mu_1$ and $deg(q) = \mu_2$ in s and $\bar{s}$ is called a μ-basis of the parametrization $\Phi_{\mathcal{C}}$. One interesting feature of μ-bases is that the resultant of its elements is a power of the implicit equation of $\mathcal{C}$, as was proved in (Cox et al., 1998, Sect. 4, Th. 1). We propose an alternative proof which relies on the idea that we can reduce the problem to the generically injective case by performing a reparametrization, i.e. by using the following

Lemma 2 *There exists $\psi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ parametrized by two coprime homogeneous polynomials h_0 and h_1 of degree $deg(\Phi_{\mathcal{C}})$ and a parametrization Φ' of $\mathcal{C}$ defined by homogeneous polynomials $f'_0(s,\bar{s})$, $f'_1(s,\bar{s})$ and $f'_2(s,\bar{s})$ such that the following diagram commutes:*



It follows that $\Phi'_{\mathcal{C}}$ is a proper (i.e. generically injective) parametrization of $\mathcal{C}$, i.e. $deg(\Phi'_{\mathcal{C}}) = 1$. Moreover, if $gcd(f_0,f_2) = gcd(f_1,f_2) = 1$, we can choose $\Phi'_{\mathcal{C}}$ such that $f_i = f'_i(h_0,h_1)$ for $i \in \{0,1,2\}$.

Proof: First, we treat the case $gcd(f_0, f_2) = gcd(f_1, f_2) = 1$. Then we can dehomogenize $\frac{f_0}{f_2}$ and $\frac{f_1}{f_2}$ by setting $\bar{s} = 1$ without changing the degree as rational functions and decompose them by means of Lüroth's Theorem (van der Waerden, 1970, Section 5.4)) in the following way

$$\frac{f_0}{f_2} = \frac{f_0'}{f_2'} \circ \frac{h_0}{h_1} \qquad\qquad \frac{f_1}{f_2} = \frac{f_1'}{f_2'} \circ \frac{h_0}{h_1}$$

with $gcd(h_0, h_1) = gcd(f_0', f_2') = gcd(f_1', \tilde{f_2'}) = 1$ and $deg(h_0) = deg(h_1) = deg(\Phi_{\mathcal{C}})$ after having rehomogenized them with respect to $\bar{s}$. By multiplying the fractions with a suitable power of h_1 we can consider the f_i' as bivariate homogeneous polynomials

$$\frac{f_0}{f_2} = \frac{f_0'(h_0,h_1)}{f_2'(h_0,h_1)} \qquad\qquad \frac{f_1}{f_2} = \frac{f_1'(h_0,h_1)}{f_2'(h_0,h_1)}$$

Then the numerators and denominators are all coprime, which for the right hand sides follows from (Zippel, 1991, Prop. 6) and we deduce the term-by-term equalities $f_i = f_i'(h_0, h_1)$ for $i \in \{0, 1, 2\}$.
In the general case, we divide the polynomials of the parametrization by their greatest common divisor and perform a generic coordinate change in order to pass to another parametrization of $\mathcal{C}$ which fulfills $gcd(f_0, f_2) = gcd(f_1, f_2) = 1$ and whose polynomial decomposition completes the commutative diagram of rational maps. □

We are now ready to proceed to the main theorem of this section, for which we give a new proof that establishes a link between the μ-basis of $\Phi_{\mathcal{C}}$ and a μ-basis of a proper reparametrization of the curve.

Theorem 3 *Let (p, q) be a μ-basis of the parametrization $\Phi_{\mathcal{C}} : \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$. Then*

$$Res(p, q) = F_{\mathcal{C}}^{deg(\Phi_{\mathcal{C}})}$$

where $F_{\mathcal{C}}$ is an implicit equation of the curve $\mathcal{C}$ defined by $\Phi_{\mathcal{C}}$ and $Res(p, q) \in \mathbb{K}[x, y, z]$ is the homogeneous resultant with respect to the indeterminates s and $\bar{s}$.

Proof: First of all, we may assume that $gcd(f_0, f_2) = gcd(f_1, f_2) = 1$ (if necessary, we divide by $gcd(f_0, f_1, f_2)$ and perform a generic coordinate change, both of which leave the claimed formula invariant). So by Lemma 2 there exist $f_0', f_1', f_2' \in R$ and homogeneous, coprime $h_0, h_1 \in R$ of degree $deg(\Phi_{\mathcal{C}})$, such that

$$\begin{aligned} f_0 &= f_0'(h_0, h_1) \\ f_1 &= f_1'(h_0, h_1) \\ f_2 &= f_2'(h_0, h_1) \end{aligned}$$

Let (p', q') be a μ-basis of the proper reparametrization $\Phi_{\mathcal{C}}'$ of $\mathcal{C}$ defined by the f_i'. Then $p'(h_0, h_1)$ and $q'(h_0, h_1)$ are linearly independent syzygies (i.e. we

substitute h_0 for s and h_1 for $\bar{s}$). It is easy to see that they form a μ-basis by verifying the degree property and if $\mu_1 < \mu_2$, they are related to our original μ-basis (p, q) by

$$\begin{aligned} p'(h_0, h_1) &= \lambda p \\ q'(h_0, h_1) &= ap + q \end{aligned}$$

for some constant $\lambda \neq 0$ and a homogeneous $a \in R$ of degree $deg(q) - deg(p)$. (If $\mu_1 = \mu_2$, we have $p' \circ h = \alpha_1 p + \alpha_2 q$ and $q' \circ h = \beta_1 p + \beta_2 q$ for some constants α_i and β_i (see (Chen and Wang, 2003a, Th. 2)), which leads to computations that are analogous to the ones that follow.)

Now we can apply elementary properties of resultants to calculate

$$\begin{aligned} Res(p, q) &= c \cdot Res(\lambda p, ap + q) && (c \in \mathbb{K}^*) \\ &= c \cdot Res(p'(h_0, h_1), q'(h_0, h_1)) \\ &= c \cdot Res(h_0, h_1)^{deg(p')deg(q')} \cdot Res(p', q')^{deg(h_0)} \\ &= c' \cdot Res(p', q')^{deg(\Phi_C)} && (c' \in \mathbb{K}^*) \qquad (1) \end{aligned}$$

where the second to last equation is a well-known base change formula for resultants, which is proved in (Jouanolou, 1991, 5.12), and the last equation is true because $deg(h_0) = deg(\Phi_C)$ and $Res(h_1, h_2)$ is a constant (since the h_i do not depend on x, y, z) and non-zero (because $gcd(h_1, h_2) = 1$).
So by (1) we have reduced the theorem to the special case, where the parametrization has degree 1, and it remains to show:

$$\begin{array}{ll} 1) & Res(p', q') \neq 0 \\ 2) & F_C \mid Res(p', q') \\ 3) & deg_{x,y,z}(Res(p', q')) \leq deg(C) \end{array}$$

ad 1): Suppose $p = G \cdot H$ were reducible into non-constant $G, H \in R[x, y, z]$, then one of the two, say G, would be independent of x, y, z, because p is linear in those variables and H would define a syzygy with lower degree than p which contradicts the definition of a μ-basis. So p is irreducible and $Res(p, q) = 0$ would mean that q is a multiple of p, which is impossible, for p and q are linearly independent. Hence $Res(p, q) \neq 0$ and by (1) also $Res(p', q') \neq 0$.

ad 2): By construction p and q vanish for all points in $Im(\Phi_C)$. So for any $X = (x_1 : x_2 : x_3) \in Im(\Phi_C)$ we have that $p(X) = q(X) = 0$ which rests true after setting $\bar{s} = 1$, so the two univariate polynomials have a common zero and therefore $Res(p(X), q(X)) = (Res(p, q))(X) = 0$. Again, by (1) we have also $(Res(p', q'))(X) = 0$ and it follows that the implicit equation F_C divides $Res(p', q')$.

ad 3): All the coefficients of p and q are of degree ≤ 1 in x, y, z, so we can give an upper bound for the degree of the resultant in x, y, z:

$$deg_{x,y,z}(Res(p, q)) \leq deg(p) + deg(q) = d = deg(\Phi_C)deg(C)$$

Once again we look at (1) to deduce that

$$deg_{x,y,z}(Res(p', q')) \leq deg(\mathcal{C})$$

Finally, 1), 2), and 3) together imply that $Res(p', q')$ is the implicit equation of $\mathcal{C}$, which concludes the proof. □

2 Implicitization of rational ruled surfaces with μ-bases

Chen, Sederberg, and Zheng introduced the notion of a μ-basis for rational ruled surfaces in Chen et al. (2001), and it was further developed in Chen and Wang (2003b). However, they worked with the restrictive assumption that the parametrization is generically injective. In this section, we will give a proof for the ruled surface version of Theorem 3 in its general form and explain to what extent the ruled surface case can be reduced to the curve case.
First of all, a rational ruled surface $\mathcal{S}$ is given by a rational map

$$\begin{array}{rccc} \Phi_{\mathcal{S}}: & \mathbb{P}^1 \times \mathbb{P}^1 & \dashrightarrow & \mathbb{P}^3 \\ & ((s:\bar{s}),(t:\bar{t})) & \mapsto & (f_0(s,\bar{s},t,\bar{t}) : \ldots : f_3(s,\bar{s},t,\bar{t})) \end{array}$$

where the $f_i \in \mathbb{K}[s, \bar{s}, t, \bar{t}]$ are bihomogeneous of degree $(n, 1)$, by which we mean that they are homogeneous of degree $n + 1$ and that $deg_{s,\bar{s}}(f_i) = n$ and $deg_{t,\bar{t}}(f_i) = 1$ for all $i = 0, \ldots, 3$. We assume that $gcd(f_0, \ldots, f_3) = 1$ and that we can rewrite

$$f_i = \bar{t}\bar{s}^{n_1 - n_0} f_{i0} + t f_{i1} \tag{2}$$

where $f_{i0}, f_{i1} \in \mathbb{K}[s, \bar{s}]$ and $n_0 := max(deg_s(f_{i0}))$ and $n_1 := max(deg_s(f_{i1}))$, and where we have assumed that $n_1 \geq n_0$ (otherwise we may reparametrize (2) by exchanging t and $\bar{t}$) and $n_1 = n$ (otherwise, we may divide the f_i by a suitable power of $\bar{s}$). Finally, we need to make the assumption that $(f_{00}, \ldots, f_{30})$ and $(f_{01}, \ldots, f_{31})$ are R-linearly independent to exclude the degenerate case where $\Phi_{\mathcal{S}}$ does not parametrize a surface.
Let us fix some notation first: The R-module of syzygies on $f_0, \ldots, f_3$ depending only on s and $\bar{s}$ is defined as

$$Syz_R(f_0, \ldots, f_3) = \{P \in R[x, y, z, w] \mid deg(P) = 1, P(f_0, f_1, f_2, f_3) = 0\}$$

Then the structure of this module is well known; see Chen et al. (2001) for a proof of the following

Theorem 4 *There exists an isomorphism of graded R-modules*

$$Syz_R(f_0, \ldots, f_3) \cong R(-\mu_1) \oplus R(-\mu_2)$$

where $\mu_i \in \mathbb{N}$, $\mu_1 \leq \mu_2$ and $\mu_1 + \mu_2 = deg(\Phi_{\mathcal{S}}) \cdot deg(\mathcal{S})$.

A basis (q_1, q_2) of $Syz_R(f_0, f_1, f_2, f_3)$ where q_1 and q_2 are homogeneous of minimal degrees $deg(q_1) = \mu_1$ and $deg(q_2) = \mu_2$ in s and $\bar{s}$ is called a μ-basis of the parametrization $\Phi_{\mathcal{S}}$. As we can see, the syzygy module of the surface $\mathcal{S}$ resembles the one of a curve, which leads to the question if there is some curve with the same syzygy module that can be defined by means of the surface parametrization. The answer to this question is positive and according to an idea due to Busé, Elkadi, and Galligo, we define the curve $\mathcal{C}$ associated to $\mathcal{S}$ by

$$\begin{array}{rccc} \Phi_{\mathcal{C}}: & \mathbb{P}^1 & \dashrightarrow & \mathbb{P}^2 \\ & (s:\bar{s}) & \mapsto & (p_{03}(s,\bar{s}) : p_{13}(s,\bar{s}) : p_{23}(s,\bar{s})) \end{array}$$

where $p_{ij} := f_{i0}f_{j1} - f_{i1}f_{j0} \in R$ are the Plücker coordinates, which are homogeneous of degree $n_1 + n_0$. Let us denote $g := gcd(p_{03}, p_{13}, p_{23})$.
The geometric idea behind this definition is that for almost all parameter values $(s:\bar{s}) \in \mathbb{P}^1$ the image of $\Phi((s:\bar{s}), -)$ is a line $L_{(s:\bar{s})}$ in $\mathbb{P}^3$, hence the surface $\mathcal{S}$ can be viewed as the closure of the union of these lines. The curve defined by all the Plücker coordinates

$$\begin{array}{rccc} \Psi: & \mathbb{P}^1 & \dashrightarrow & \mathbb{P}^5 \\ & (s:\bar{s}) & \mapsto & (p_{ij})_{i,j\in\{0,\ldots,3\},\ i<j} \end{array}$$

is contained in a quadric parametrizing the lines in $\mathbb{P}^3$, more precisely there is a one-to-one correspondance between the points $\Psi((s:\bar{s}))$ on the Plücker curve and the lines $L_{(s:\bar{s})}$ on the ruled surface $\mathcal{S}$, which will allow us to carry over the results about curves to the ruled surface case. However, it is more convenient to work with the curve $\Phi_{\mathcal{C}}$, which is a projection of Ψ to $\mathbb{P}^2$. As we will see, we need to make sure that this projection does not add any base points, which is the statement of the following lemma.

Lemma 5 *If* $gcd(f_{30}, f_{31}) = 1$ *then*

$$gcd(p_{03}, p_{13}, p_{23}) = gcd(p_{03}, p_{13}, p_{23}, p_{01}, p_{02}, p_{12})$$

Proof: Let us suppose $q = gcd(p_{03}, p_{13}, p_{23}) \neq 1$; the case $q = 1$ is trivial. We need to show that q divides the other Plücker coordinates as well. Euclidian division of the f_{ij} by q yields

$$f_{ij} = q \cdot \tilde{f}_{ij} + a_{ij}$$

We have the congruences

$$p_{ij} \equiv f_{i0}f_{j1} - f_{i1}f_{j0} \equiv a_{i0}a_{j1} - a_{i1}a_{j0} \qquad (mod\, q)$$

The other cases being analogous, we only show $p_{12} \equiv 0 \pmod{q}$, i.e. that $a_{10}a_{21} - a_{11}a_{20}$ is divisible by q. We can write $a_{10}a_{31} - a_{11}a_{30} = qr_1$ and $a_{20}a_{31} - a_{21}a_{30} = qr_2$, since p_{13} and p_{23} are divisible by q. As $gcd(f_{30}, f_{31}) = 1$ it follows that they are not both divisible by q, so we may assume $a_{30} \neq 0$. Hence we obtain

$$a_{21}a_{30} = a_{20}a_{31} - qr_2 \qquad \textit{and} \qquad a_{11}a_{30} = a_{10}a_{31} - qr_1$$

from which we deduce

$$a_{30}(a_{10}a_{21} - a_{11}a_{20}) = a_{10}(a_{20}a_{31} - qr_2) - a_{20}(a_{10}a_{31} - qr_1) = q \cdot (r_1 - r_2)$$

and as a_{30} is non-zero and prime to q, we conclude that $a_{10}a_{21} - a_{11}a_{20}$ is divisible by q. □

Later on, we will see in another context why the condition $gcd(f_{30}, f_{31}) = 1$ is necessary. We should note that it is non-restrictive, since it can always be achieved by a generic coordinate change. Next, we state a useful degree formula, which we will use to study the relationship between a ruled surface and its associated curve in more detail.

Proposition 6 (Degree Formula) *With the same notation as before the equality*

$$deg(\mathcal{S})deg(\Phi_{\mathcal{S}}) = n_1 + n_0 - deg(g)$$

holds.

Proof: This formula is an adaptation of the general result

$$deg(\mathcal{S})deg(\Phi_{\mathcal{S}}) = 2n - \sum_{p \in V(f_0,\ldots,f_3)} m_p$$

(see (Fulton, 1984, Prop. 4.4) for a proof, m_p is the multiplicity of p). Our formula follows by counting the base points $\sum_{p \in V(I)} m_p = deg(g) + (n_1 - n_0)$, where $n_1 - n_0$ is the trivial multiplicity of the base point $(\infty, 0) := ((1:0),(0:1))$ and where the other base points (including additional multiplicities of $(\infty, 0)$) can be identified with the roots of g by elementary calculations. □

Next, we proceed to relate $Syz_R(f_0, \ldots, f_3)$ to the syzygy module of the associated curve, given as

$$Syz(p_{03}, p_{13}, p_{23}) = \{P \in R[x, y, z] \mid deg(P) = 1, P(p_{03}, p_{13}, p_{23}) = 0\}$$

Proposition 7 *If $gcd(f_{30}, f_{31}) = 1$, then there exists a canonical isomorphism of graded R-modules*

$$Syz_R(f_0, \ldots, f_3) \cong Syz(p_{03}, p_{13}, p_{23})$$

and $deg(\Phi_{\mathcal{S}}) \cdot deg(\mathcal{S}) = deg(\Phi_{\mathcal{C}}) \cdot deg(\mathcal{C})$.

Proof: As a direct consequence of Theorem 1 and the degree formula, we obtain

$$deg(\Phi_{\mathcal{C}}) \cdot deg(\mathcal{C}) = n_1 + n_0 - deg(g) = deg(\Phi_{\mathcal{S}}) \cdot deg(\mathcal{S})$$

and it remains to construct an isomorphism of degree zero between the syzygy modules. Let $h_0x + h_1y + h_2z + h_3w \in Syz_R(f_0, \ldots, f_3)$. As it does not depend on t and $\bar{t}$, we can deduce from (2) that

$$\begin{aligned} h_0f_{00} + h_1f_{10} + h_2f_{20} + h_3f_{30} &= 0 \\ h_0f_{01} + h_1f_{11} + h_2f_{21} + h_3f_{31} &= 0 \end{aligned}$$

By multiplying the first equation by f_{31} and the second one by f_{30} and by substracting the second from the first we get

$$h_0p_{03} + h_1p_{13} + h_2p_{23} = 0 \tag{3}$$

which is a syzygy on the p_{i3}. Hence, by setting $w = 0$ we obtain a well-defined morphism

$$\begin{aligned} \varphi : \quad Syz_R(f_0, \ldots, f_3) &\to Syz(p_{03}, p_{13}, p_{23}) \\ h_0x + h_1y + h_2z + h_3w &\mapsto h_0x + h_1y + h_2z \end{aligned}$$

which has obviously degree zero.
Now φ is injective, because if $h_0 = h_1 = h_2 = 0$ for a syzygy on the f_i, then $h_3 = 0$ as well. To see why it is also surjective, let $h_0x + h_1y + h_2z \in Syz(p_{03}, p_{13}, p_{23})$ and by rewriting (3) we have

$$(h_0f_{00} + h_1f_{10} + h_2f_{20})f_{31} = (h_0f_{01} + h_1f_{11} + h_2f_{21})f_{30}$$

The assumption that f_{30} and f_{31} are coprime implies that there is a polynomial $h \in K[s, \bar{s}]$ such that $hf_{30} = h_0f_{00} + h_1f_{10} + h_2f_{20}$ and by substituting this in the above equation also $hf_{31} = h_0f_{01} + h_1f_{11} + h_2f_{21}$. These two relations show that $h_0x + h_1y + h_2z - hw \in Syz_R(f_0, \ldots, f_m)$, hence φ is surjective and the proof is complete. □

Corollary 1 *If we perform a generic coordinate change beforehand, we also have $deg(\mathcal{S}) = deg(\mathcal{C})$ and $deg(\Phi_{\mathcal{S}}) = deg(\Phi_{\mathcal{C}})$ in the situation of the preceding Proposition 7.*

Proof: As we have seen in the proof of the proposition, the associated curve is obtained by intersecting the surface with the plane $w = 0$ and the isomorphism of the syzygy modules is induced by the projection map. If this plane is generic, the theorem of Bézout ensures that this intersection preserves the degree. □

An important remark is that the isomorphism described above takes μ-bases to μ-bases, as it is of degree 0 and hence preserves the minimality of the degrees. This leads to an efficient method for the computation of the μ-basis of the surface: One computes the μ-basis of the associated curve (e.g. with the algorithm presented in Chen and Wang (2003a)) and takes its image under the above isomorphism. Of course, the actual computations should be done after dehomogenizing, i.e. setting $\bar{s} = 1$.

ALGORITHM (μ-basis of a ruled surface)

INPUT: $f_{0i}, f_{1i}, f_{2i}, f_{3i} \in \mathbb{K}[s]$ for $i = 0, 1$

1.) Calculate p_{03},p_{13}, and p_{23}.

2.) Calculate a μ-basis $(\tilde{q}_1, \tilde{q}_2) = (q_{10}x + q_{11}y + q_{12}z, q_{20}x + q_{21}y + q_{22}z)$ of the curve defined by p_{03},p_{13}, and p_{23} with an algorithm for planar curves.

3.) Compute $q_{j3} = -\frac{q_{j0}f_{00}+q_{j1}f_{10}+q_{j2}f_{20}}{f_{30}}$ for $j = 1, 2$.

4.) Set $q_1 = q_{10}x + q_{11}y + q_{12}z + q_{13}w$ and $q_2 = q_{20}x + q_{21}y + q_{22}z + q_{23}w$.

OUTPUT: A μ-basis (q_1, q_2) of the parametrization

One can see Theorem 4 as a corollary of Theorem 1 and Proposition 7. Let us also note that Theorem 1 and Theorem 4 can easily be generalized to higher dimension and the proofs are completely analogous to the ones given here. For example, the μ-basis of a curve in $\mathbb{P}^m$ consists of $m-1$ syzygies whose degrees in s and $\bar{s}$ sum up to d. We are now ready to show our main result.

Theorem 8 *Let (q_1, q_2) be a μ-basis of the parametrization $\Phi_{\mathcal{S}} : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$. Then*

$$Res(q_1, q_2) = F_{\mathcal{S}}^{deg(\Phi_{\mathcal{S}})}$$

where $F_{\mathcal{S}}$ is an implicit equation of the ruled surface $\mathcal{S}$ and where the resultant is taken with respect to s and $\bar{s}$.

Proof: First, we can ensure that the hypotheses of Proposition 7 are fulfilled by performing a generic linear coordinate change in $\mathbb{P}^1 \times \mathbb{P}^1$, which leaves both the implicit equation and the resultant unchanged (up to a constant). We will show that $Res(q_1, q_2)$ is the power of an irreducible polynomial, i.e. that it defines an irreducible hypersurface in $\mathbb{P}^3$. Let us consider the incidence variety $\mathcal{W} := \{((s_0 : \bar{s_0}), (x_0 : y_0 : z_0 : w_0)) \in \mathbb{P}^1 \times \mathbb{P}^3 \mid q_i(s_0, \bar{s_0}, x_0, y_0, z_0, w_0) = 0 \}$ then we have the following diagram

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\pi_2} & \mathbb{P}^3 \\ \downarrow{\scriptstyle \pi_1} & & \\ \mathbb{P}^1 & & \end{array}$$

where π_1 and π_2 are the canonical projections. $\mathcal{W}$ is a vector bundle over $\mathbb{P}^1$,

as the q_i are linear in x, y, z, and w, and for any parameter $(s_0 : \bar{s_0})$ the fiber is a $\mathbb{K}$-vector space of codimension 2 (because $q_1(s_0, \bar{s_0})$ and $q_2(s_0, \bar{s_0})$ are linearly independent, as was proved in (Chen and Wang, 2003b, Sect. 2, Prop. 3)). As $\mathbb{P}^1$ is irreducible, it follows that $\mathcal{W}$ is irreducible too (see (Shafarevich, 1977, Ch.6, Th.8)), hence so is $Im(\pi_2)$. (If $Im(\pi_2) = A \cup B$ for two closed sets A and B, $\mathcal{W} = \pi_2^{-1}(A) \cup \pi_2^{-1}(B)$, which implies $\mathcal{W} = \pi_2^{-1}(A)$ or $\mathcal{W} = \pi_2^{-1}(B)$, since $\mathcal{W}$ is irreducible and, consequently, $Im(\pi_2) = A$ or $Im(\pi_2) = B$). Now the points of $Im(\pi_2)$ are exactly those for which the q_i have a common zero in s and $\bar{s}$, so by definition of the resultant they are the zeros of $Res(q_1, q_2)$. In other words, we have shown that $V(Res(q_1, q_2)) = Im(\pi_2)$ is irreducible, so $Res(q_1, q_2)$ is the power of an irreducible polynomial.
By definition, the syzygies of $\Phi_{\mathcal{S}}$ vanish on all of $Im(\Phi_{\mathcal{S}})$ and hence on all of $\mathcal{S}$, so $F_{\mathcal{S}} \,|\, Res(q_1, q_2)$. This implies that $Res(q_1, q_2)$ is a power of $F_{\mathcal{S}}$ and it remains to verify that it has the correct degree $deg(\Phi_{\mathcal{S}}) \cdot deg(\mathcal{S})$.
In the proof of Theorem 7, we have seen the isomorphism of R-modules

$$\begin{array}{rccc} \varphi : & Syz_R(f_0, f_1, f_2, f_3) & \rightarrow & Syz(p_{03}, p_{13}, p_{23}) \\ & h_0 x + h_1 y + h_2 z + h_3 w & \mapsto & h_0 x + h_1 y + h_2 z \end{array}$$

between the syzygies of the parametrization $\Phi_{\mathcal{S}}$ and of the parametrization $\Phi_{\mathcal{C}}$ of its associated curve $\mathcal{C}$. We note that φ extends to the morphism of R-algebras defined by

$$\begin{array}{rcl} R[x, y, z, w] & \rightarrow & R[x, y, z] \\ x & \mapsto & x \\ y & \mapsto & y \\ z & \mapsto & z \\ w & \mapsto & 0 \end{array}$$

which we will also call φ by abuse of notation.
As remarked earlier on, φ takes μ-bases to μ-bases, so $(\varphi(q_1), \varphi(q_2))$ is a μ-basis of $\Phi_{\mathcal{C}}$. Applying Theorem 3 yields

$$\begin{array}{rcl} F_{\mathcal{C}}^{deg(\Phi_{\mathcal{C}})} & = & Res(\varphi(q_1), \varphi(q_2)) \\ & = & \varphi(Res(q_1, q_2)) \end{array}$$

where the last equality is true, because φ is the specialisation $w = 0$ and as such commutes with the resultant.
Finally, we have $deg(\varphi(Res(q_1, q_2))) = deg(Res(q_1, q_2))$, as $Res(q_1, q_2)$ is homogeneous, which shows that

$$deg(Res(q_1, q_2)) = deg(\Phi_{\mathcal{C}}) \cdot deg(\mathcal{C}) = deg(\Phi_{\mathcal{S}}) \cdot deg(\mathcal{S})$$

so $Res(q_1, q_2)$ has indeed the correct degree, which concludes the proof. □

Example

Let us consider the ruled surface $\mathcal{S}$ defined by the polynomials

$$\begin{array}{rcl} f_0 & = & t(-s^6 - 2s^2\bar{s}^4) \\ f_1 & = & \bar{t}\bar{s}^6 + t(-s^2\bar{s}^4 + \bar{s}^6) \\ f_2 & = & \bar{t}\bar{s}^6 + t(-s^6 + \bar{s}^6) \\ f_3 & = & \bar{t}s^2\bar{s}^4 + t(s^2\bar{s}^4 - \bar{s}^6) \end{array}$$

Then its associated curve $\mathcal{C}$ is parametrized by the Plücker coordinates

$$p_{03} = s^8 + 2s^4\bar{s}^4 \qquad p_{13} = s^4\bar{s}^4 - \bar{s}^8 \qquad p_{23} = s^8 - \bar{s}^8$$

and we have $deg(\Phi_{\mathcal{C}}) \cdot deg(\mathcal{C}) = deg(\Phi_{\mathcal{S}}) \cdot deg(\mathcal{S}) = 8$ which follows from the degree formulæ. By computing a μ-basis for $\Phi_{\mathcal{C}}$ and applying the isomorphism φ^{-1} we can calculate the following μ-basis for $\Phi_{\mathcal{S}}$

$$\begin{array}{rcl} q_1 & = & (s^4 + \bar{s}^4)y - \bar{s}^4 z - s^2\bar{s}^2 w \\ q_2 & = & (-s^4 + \bar{s}^4)x - \bar{s}^4 y + (s^4 + \bar{s}^4)z - s^2\bar{s}^2 w \end{array}$$

and we obtain

$$\begin{aligned} Res(q_1, q_2) = (&4x^2y^2 - 4xy^3 + y^4 - 4x^2yz + 2xy^2z + x^2z^2 + 4xyz^2 - 2y^2z^2 - 2xz^3 \\ &+ z^4 - x^2w^2 + xyw^2 + 2y^2w^2 - xzw^2 - 4yzw^2 + 2z^2w^2)^2 \end{aligned}$$

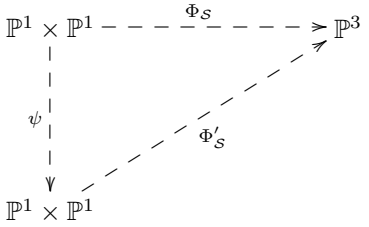
which is indeed the square of an implicit equation $F_{\mathcal{S}}$ of $\mathcal{S}$.

We have seen and used the equality $deg(\Phi_{\mathcal{C}}) \cdot deg(\mathcal{C}) = deg(\Phi_{\mathcal{S}}) \cdot deg(\mathcal{S})$ between the surface $\mathcal{S}$ and its associated curve $\mathcal{C}$. It is natural to ask whether $deg(\mathcal{C}) = deg(\mathcal{S})$ also holds. However, this is not true in our example: we have $deg(\mathcal{C}) = 2$, but $deg(\mathcal{S}) = 4$. According, to the corollary to Proposition 7, we would have had to perform a generic coordinate change in order to ensure the equality of the degrees.

Let us compare the μ-basis method to some others. In our example, $F_{\mathcal{S}}^2$ is obtained as a determinant of a 8×8-matrix, the Sylvester matrix of q_1 and q_2. After dehomogenizing our surface and homogenizing back to $\mathbb{P}^2$ we can use approximation complexes to implicitize, as in Busé and Chardin (2005), and we obtain $F_{\mathcal{S}}^2$ as the quotient of a 28×28-determinant by a 12×12-determinant and an additional term that arises because we add base points when passing from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$, which is by far not as efficient.
Another possibility is to use the classical formula $F_{\mathcal{S}}^2(w = 1) = Res(f_0 - xf_3, f_1 - yf_3, f_2 - zf_3)$ combined with an efficient method to calculate the resultant such as Khetan (2003). $F_{\mathcal{S}}^2$ is obtained as the determinant of 10×10-matrix, which is larger than our Sylvester matrix and whose entries are themselves determinants of smaller matrices.

3 Remark on the reparametrization problem for ruled surfaces

In the proof of Theorem 3 about the implicit equation of a planar curve, we reduced the general case to the proper case by reparametrizing the curve. For rational surfaces there also exists a proper reparametrization by the theorem of Castelnuovo, i.e. there exists a commutative diagram



where $\psi = (\sigma, \tau)$ is of degree $deg(\mathcal{S})$ and $\Phi'_{\mathcal{S}}$ is a proper reparametrization of $\mathcal{S}$. As far as we know, this problem is yet to be solved algorithmically. However, Pérez-Díaz (2006) gives a criterion for the existence of a reparametrization of a rational surface such that $\sigma = \sigma(s, \bar{s})$ depends only on s and $\bar{s}$ and $\tau = \tau(t, \bar{t})$ depends only on t and $\bar{t}$ and proposes an algorithm for its computation if it exists. If we restrict our attention to ruled surfaces we can also treat the case where $\tau = (\bar{t}\alpha + t\beta, \bar{t}\gamma + t\delta)$ with $\alpha, \beta, \gamma, \delta \in \mathbb{K}[s, \bar{s}]$ such that $\alpha\delta - \beta\gamma \neq 0$ by using the associated curve. So let us suppose that there exists a reparametrization such that we can write

$$f_i = \bar{t}(\alpha f'_{i0}(\sigma) + \gamma f'_{i1}(\sigma)) + t(\beta f'_{i0}(\sigma) + \delta f'_{i1}(\sigma)) \tag{4}$$

for $i = 0, \ldots, 3$, where the f'_{ij} define a proper parametrization $\Phi'_{\mathcal{S}}$ of $\mathcal{S}$. We can deduce that $deg(\psi) = deg(\sigma) = deg(\Phi_{\mathcal{S}})$, because τ is a homography with respect to t. We have the following identity

$$p_i = \begin{vmatrix} f_{i0} & f_{i1} \\ f_{30} & f_{31} \end{vmatrix} = \begin{vmatrix} \alpha f'_{i0}(\sigma) + \gamma f'_{i1}(\sigma) & \beta f'_{i0}(\sigma) + \delta f'_{i1}(\sigma) \\ \alpha f'_{30}(\sigma) + \gamma f'_{31}(\sigma) & \beta f'_{30}(\sigma) + \delta f'_{31}(\sigma) \end{vmatrix} = (\alpha\delta - \beta\gamma) p'_i(\sigma)$$

from which we conclude that σ yields a proper reparametrization of the associated curve in the generic case $deg(\Phi_{\mathcal{S}}) = deg(\Phi_{\mathcal{C}})$. On the other hand, any $\lambda(s, \bar{s})$ defining a proper reparametrization $p_i = p''_i(\lambda)$ of $\mathcal{C}$ differs from σ only by a homography, so we can assume $\lambda = \sigma$, which provides us with a (naive) method for calculating the reparametrization: We compute σ with a reparametrization algorithm for curves such as in Pérez-Díaz (2006) and consider (4) as a linear system of equations by comparing the coefficients of the left hand side and the right hand side, where we leave the coefficients of $\alpha, \beta, \gamma, \delta$ and the f'_{ij} undetermined. Then any solution of this system defines a proper reparametrization of the ruled surface. However, the systems are generally too large and further research is needed to develop an efficient algorithmic solution to the reparametrization problem.

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