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Abstract: We discuss the problem of expansion of optimal control, state and costate when a logarithmic penalty is applied to constraints. We show that, in a simple case, that the variation of (a regular) junction point, and of the optimal control, state and costate is of order $\varepsilon \log \varepsilon$, where ε is the penalty parameter.

Key-words: Optimal control, logarithmic penalty, control constraints, sensitivity.

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Développement de la commande optimale avec pénalisation logarithmique : exemples et problèmes ouverts

Résumé : Nous discutons le problème du développement de la commande optimale, et de l'état et état adjoint associés, quand on applique la pénalisation logarithmique aux contraintes. Nous montrons dans un cas simple que la variation d'un point de jonction des contraintes, comme de la commande optimale et de l'état et état adjoint associés est d'ordre $\varepsilon \log \varepsilon$, où ε est le paramètre de pénalisation logarithmique.

Mots-clés : Commande optimale, pénalisation logarithmique, contraintes sur la commande, sensibilité.

1 Introduction

This short note is intended to discuss some questions related to the perturbation analysis of optimal control problems when the logarithmic penalty is used for inequality constraints. In order to fix ideas, let us consider an optimal control problem of the form

$$\operatorname*{Min}_u \int_0^T \ell(y(t),u(t))\mathrm{d}t + \Phi(y(T)); \left\{ \begin{array}{rcl} \dot{y}(t) & = & f(y(t),u(t)), \text{ a.e. } t \in [0,T]; \\ 0 & \leq & g(y(t),u(t)), \text{ a.e. } t \in [0,T]; \\ y(0) & = & y^0, \end{array} \right. \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^r$, $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, and $\Phi : \mathbb{R}^n \to \mathbb{R}$ are assumed to be smooth functions. The logarithmic penalty consists in introducing a parameter $\varepsilon > 0$ together with the penalizer running cost

$$\ell_\varepsilon(y,u) := \left\{ \begin{array}{ll} \ell(y,u) - \varepsilon \sum_{i=1}^r \log[g_i(y,u)] & \text{if } g(y,u) > 0, \\ +\infty, & \text{otherwise.} \end{array} \right. \tag{2}$$

Noticing that $\ell_\varepsilon(y,u) \downarrow \ell(y,u)$ as $\varepsilon \downarrow 0$ for every pair (y,u) such that $g(y,u) > 0$, the idea behind the method is to solve the penalized optimal control problem

$$\operatorname*{Min}_u \int_0^T \ell_\varepsilon(y(t),u(t))\mathrm{d}t + \Phi(y(T)); \left\{ \begin{array}{rcl} \dot{y}(t) & = & f(y(t),u(t)), \text{ a.e. } t \in [0,T]; \\ y(0) & = & y^0, \end{array} \right. \tag{3}$$

for a small $\varepsilon > 0$, and take this perturbed solution as an approximation of the optimal control for the original problem (1). This is an *interior-point method* in the sense that all the feasible points of (3) satisfy that $\log[g_i(y(t),u(t))]$ is Lebesgue integrable and hence $g_i(y(t),u(t)) > 0$ a.e. in $[0,T]$ for all $i = 1,\ldots,r$. By continuously letting ε tend to 0, this procedure is expected to generate an *optimal path*, that is, a parameterized family $\varepsilon \mapsto u_\varepsilon(\cdot)$ of approximate solutions that asymptotically converge toward an exact solution of (1).

Similar interior-point methods are widely used with great success in linear and nonlinear programming. Within a finite-dimensional framework, the trajectories traced by the optimal paths have been intensively investigated. Their analytic and geometric properties have been exploited to devise *path-following* algorithms that have efficient performances in practice. Polynomial bounds on the computational complexity of these algorithms have been established. For detailed expositions of the theory and recent developments see, for instance, [11, 14, 4] and the references therein.

For optimal control problems the interior-point approach seems to be quite effective since the associated linear algebra may take advantage of the band structure of the resulting matrices. Optimization algorithms based on interior-point algorithms have been applied to optimal control problems by Wright [13], Vicente [12], Leibfritz and Sachs [7], Jockenhövel, Biegler and Wächter [6].

Although there is a vast literature on the mathematical foundations of interior-point methods for finite-dimensional mathematical programming, the situation is quite different for optimal control problems. The only reference we know is [2], where the authors treat the logarithmic penalty for the simpler case of bound constraints of the type $a_i \leq u(t) \leq b_i$. In fact, under such constraints and for strongly convex linear quadratic problems, they prove convergence of optimal control and states (and also costates) in L^2 and L^∞, respectively. They show that similar results hold for a larger class of problems where the Hamiltonian function is convex with respect to the control variable. Also, they establish that controls remain uniformly at a distance of at least $c\varepsilon$ from its bounds, where $c > 0$ is uniform for all small enough ε.

Convergence results as those obtained in [2] are important because they explain the type of convergence one should expect from the penalty method. Nevertheless, for the development and study of new and efficient algorithms, the analysis must be supplemented with a more complete investigation of the asymptotic behavior of optimal paths. For instance, general results on the rate of convergence would be key tools for the construction of good update rules on the penalty parameter, and should play a central role in any complexity analysis of the associated path-following algorithms. To our best knowledge, these are open problems even for relatively simple subclasses of constrained optimal control problems.

In this note, we illustrate through a very simple example the computation of a "first-order" asymptotic expansion of the optimal paths in terms of the penalty parameter.

2 A general restoration theorem

In the example that we will study in the next section, we will deal with an implicit equation that is not differentiable with respect to the perturbation parameter. As a consequence, it is not possible to apply the standard Implicit Function Theorem. Instead, we will use the following Theorem 1, and, more precisely, the Corollary 2. These are variants of the standard surjective mapping theorem of Graves (see [5], and also [3]). Although for our purposes it suffices to have finite-dimensional versions of these results, we state and prove them for abstract Banach spaces to stress the fact that they are valid within such a general framework, and because they may be of independent interest.

Theorem 1. (Restoration theorem) *Let X and Y be Banach spaces, E a metric space and $F : U \subset X \times E \to Y$ a continuous mapping on an open set U. Let $(\hat{x}, \varepsilon_0) \in U$ be such that $F(\hat{x}, \varepsilon_0) = 0$. Assume that there exists a surjective linear continuous mapping $A : X \to Y$ and a function $c : \mathbb{R}_+ \to \mathbb{R}_+$ with $c(\beta) \downarrow 0$ when $\beta \downarrow 0$ such that, if $x \in \overline{B}(\hat{x}, \beta)$, $x' \in \overline{B}(\hat{x}, \beta)$ and $\varepsilon \in B(\varepsilon_0, \beta)$, then*

$$\|F(x', \varepsilon) - F(x, \varepsilon) - A(x' - x)\| \leq c(\beta)\|x' - x\|. \tag{4}$$

Then there exists $\eta > 0$ such that, for all (x, ε) close enough to $(\hat{x}, \varepsilon_0)$, there exists $\bar{x}$ such that $F(\bar{x}, \varepsilon) = 0$ and

$$\|\bar{x} - x\| \leq \eta\|F(x, \varepsilon)\|. \tag{5}$$

Remark. Condition (4) means that $A = D_x F(\hat{x}, \varepsilon_0)$ and that strict differentiability in variable x holds, uniformly in ε.

Proof. By the open mapping theorem, there exists a bounded right inverse of A, which we denote by B, i.e. a (possibly nonlinear) mapping $B : Y \to X$ such that $ABy = y$ for all $y \in Y$, and

$$\|B\| := \sup\{\|By\|/\|y\| \mid y \in Y,\ y \neq 0\} \tag{6}$$

is finite. Fix $\beta > 0$ such that

$$L_\beta := c(\beta)\|B\| < 1. \tag{7}$$

Let $\rho_0 > 0$ and take $x \in B(\hat{x}, \rho_0)$, $\varepsilon \in \overline{B}(\varepsilon_0, \rho_0)$. By taking $\rho_0 > 0$ small enough, as F is continuous, we may assume that

$$\rho_0 + (1 - L_\beta)^{-1}\|B[F(x, \varepsilon)]\| \leq \beta. \tag{8}$$

Let $\{x_n\}$, $n \in \mathbb{N}$, be the sequence defined by $x_0 = x$ and the (modified Newton like) step

$$x_{n+1} = x_n - BF(x_n, \varepsilon). \tag{9}$$

Then

$$\|x_{n+1} - x_n\| = \|B[F(x_n, \varepsilon)]\| \leq \|B\|\,\|F(x_n, \varepsilon)\|. \tag{10}$$

Relation (9) implies

$$F(x_n, \varepsilon) + A(x_{n+1} - x_n) = 0. \tag{11}$$

We show by induction that $\{x_n\}$ remains in $\overline{B}(\hat{x}, \beta)$. By (8), this is true if $n = 0$. For $n = 1$, we have with (10) and (8)

$$\|x_1 - \hat{x}\| \leq \|x_1 - x_0\| + \|x_0 - \hat{x}\| \leq \|B[F(x_0, \varepsilon)]\| + \rho_0 \leq \beta.$$

Then if $x_i \in \overline{B}(\hat{x}, \beta)$, for all $1 \leq i \leq n$, (4) and (11) imply

$$\|F(x_n, \varepsilon)\| \leq c(\beta)\|x_n - x_{n-1}\|. \tag{12}$$

Combining with (10), we get

$$\|x_{n+1} - x_n\| \leq L_\beta\|x_n - x_{n-1}\| \leq \cdots \leq (L_\beta)^n\|x_1 - x_0\|, \tag{13}$$

and hence, with (8),

$$\|x_{n+1} - x_0\| \leq (1 - L_\beta)^{-1}\|x_1 - x_0\| \leq (1 - L_\beta)^{-1}\|B[F(x_0, \varepsilon)]\| \leq \beta - \rho_0.$$

Since $\|x_0 - \hat{x}\| < \rho_0$, we deduce that $x_{n+1} \in B(\hat{x}, \beta)$, and hence, the sequence $\{x_n\}$ remains in $B(\hat{x}, \beta)$. With (12) and (13), we obtain that x_n converges to some $\bar{x}$ such that $F(\bar{x}, \varepsilon) = 0$ and $\|\bar{x} - x_0\| \leq (1 - L_\beta)^{-1}\|B\|\|F(x_0, \varepsilon)\|$, which proves (5) with constant η given by

$$\eta = (1 - L_\beta)^{-1}\|B\|. \tag{14}$$

■

Remark. Note that, in the above proof, we have obtained the estimate (14) for the constant η in (5), where B is a bounded right inverse of A and $L_\beta = c(\beta)\|B\|$. Also, the hypothesis that (x, ε) is "close enough" is satisfied whenever $x \in B(\hat{x}, \rho_0)$ and $\varepsilon \in \overline{B}(\varepsilon_0, \rho_0)$ where ρ_0 is such that (8) holds.

Corollary 2. *Let the hypotheses of Theorem 1 hold, and denote by B a bounded right inverse of A. Then for ε close to ε_0, $F(\cdot, \varepsilon)$ has, in a neighborhood of $\hat{x}$, a zero denoted by x_ε such that*

$$x_\varepsilon = \hat{x} - BF(\hat{x}, \varepsilon) + r(\varepsilon), \tag{15}$$

where the remainder $r(\varepsilon)$ satisfies

$$\|r(\varepsilon)\| \leq c(\beta)(1 - c(\beta)\|B\|)^{-1}\|B\|^2\|F(\hat{x}, \varepsilon)\|. \tag{16}$$

for some $\beta > 0$ small enough.

Proof. Let $\hat{x}(\varepsilon) := \hat{x} - BF(\hat{x}, \varepsilon)$. We have that $F(\hat{x}, \varepsilon) + A(\hat{x}(\varepsilon) - \hat{x}) = 0$ and $\|\hat{x}(\varepsilon) - \hat{x}\| \leq \|B\|\|F(\hat{x}, \varepsilon)\|$. In view of (4), $\|F(\hat{x}(\varepsilon), \varepsilon)\| \leq c(\beta)\|B\|\|F(\hat{x}, \varepsilon)\|$. We conclude with the theorem. ■

Remark. According to (16), for the applications of Corollary 2, better estimates on $c(\beta)$ yield sharper bounds on the remainder in (15). The best constant $c(\beta)$ is given by

$$c(\beta) := \sup_{x,x'} \frac{\|F(x', \varepsilon) - F(x, \varepsilon) - A(x' - x)\|}{\|x' - x\|}, \tag{17}$$

where the supremum is taken over all $x \neq x'$, both in $\overline{B}(\hat{x}, \beta)$.

3 Model example

Consider the following elementary scalar optimal control problem

$$\begin{cases} \text{Min } \frac{1}{2} \displaystyle\int_0^1 (u(t) - 2t)^2 \mathrm{d}t + \frac{1}{2} y(1)^2, \\ \dot{y}(t) = u(t); \quad u(t) \geq 0, \ t \in [0, 1]; \quad y(0) = \frac{3}{4}. \end{cases} \tag{18}$$

According to Prontryaguin's minimum principle, the associated first-order optimality condition is given by the following system:

$$\begin{cases} \dot{y}(t) = H_p(u(t), y(t), p(t), t), \ t \in [0, 1], \ y(0) = \frac{3}{4}, \\ \dot{p}(t) = -H_y(u(t), y(t), p(t), t), \ t \in [0, 1], \ p(1) = y(1), \\ u(t) \in \operatorname*{Argmin}_{v \geq 0} H(v, y(t), p(t), t), \ t \in [0, 1]. \end{cases} \tag{19}$$

By introducing a nonpositive Lagrange multiplier, the last condition in (19) may be equivalently written as

$$\left\{\begin{array}{l} H_u(u(t),y(t),p(t),t)+\lambda(t)=0, \\ \\ u(t)\geq 0,\ \ \lambda(t)\leq 0,\ \ \lambda(t)u(t)=0. \end{array}\right. \tag{20}$$

In the specific case of problem (18) the corresponding Hamiltonian function is given by

$$H(u,y,p,t)=\tfrac{1}{2}(u-2t)^2+pu.$$

Thus for the adjoint state $p(t)$ we get

$$-\dot{p}(t)=0,\ t\in[0,1];\quad p(1)=y(1)=\tfrac{3}{4}+\int_0^1 u(t)\mathrm{d}t. \tag{21}$$

Setting $p_0:=p(1)=p(t)$, $t\in[0,1]$, we obtain that the optimal control is given by

$$u_0(t)=\max\{2t-p_0,0\}=[2t-p_0]_+,\ t\in[0,1], \tag{22}$$

where p_0 solves the equation

$$p_0=y(1)=\tfrac{3}{4}+\int_0^1[2t-p_0]_+\mathrm{d}t, \tag{23}$$

whose unique solution is given by

$$p_0=1.$$

Therefore, the optimal control is

$$u_0(t)=[2t-1]_+=\left\{\begin{array}{lll} 0 & \text{if} & t\leq\frac{1}{2}, \\ 2t-1 & \text{if} & t\geq\frac{1}{2}. \end{array}\right. \tag{24}$$

The constraint on the control has only one junction point at time $\tau_0=p_0/2=1/2$. This junction is regular, in the sense that $\dot{u}_0(t)$ has a nonzero limit when $t\downarrow 1/2$. Notice that one has the corresponding multiplier

$$\lambda_0(t)=-[1-2t]_+=\left\{\begin{array}{lll} 2t-1 & \text{if} & t\leq\frac{1}{2}, \\ 0 & \text{if} & t\geq\frac{1}{2}. \end{array}\right.$$

which satisfies strict complementary, that is, $\lambda_0(t)<0$ when $u_0(t)=0$, except for the junction time $\tau_0=1/2$.

For this example the formulation of the logarithmically penalized problem is, for given $\varepsilon>0$,

$$\left\{\begin{array}{l} \mathrm{Min}\ \frac{1}{2}\displaystyle\int_0^1\left[(u(t)-2t)^2-\varepsilon\log(u(t))\right]\mathrm{d}t+\frac{1}{2}y(1)^2, \\ \dot{y}(t)=u(t),\ t\in[0,1];\quad y(0)=\frac{3}{4}, \end{array}\right.$$

where it is implicitly required that $u(t) > 0$ for a.e. $t \in [0,1]$. The corresponding adjoint state satisfies again (21), and we denote by p_ε its constant value. According to the minimum principle, the optimal control, which we denote by $u_\varepsilon(t)$, satisfies

$$u_\varepsilon(t) > 0; \quad u_\varepsilon(t) - 2t + p_\varepsilon - \varepsilon/u_\varepsilon(t) = 0. \tag{25}$$

Notice that by analogy with (20), we can introduce an “artificial" multiplier variable by setting

$$\lambda_\varepsilon(t) = -\varepsilon/u_\varepsilon(t), \tag{26}$$

which is negative and satisfies the following perturbed complementary condition:

$$\lambda_\varepsilon(t)u_\varepsilon(t) = -\varepsilon.$$

The unique solution of (25) is given by

$$u_\varepsilon(t) = \phi_\varepsilon(2t - p_\varepsilon), \tag{27}$$

for

$$\phi_\varepsilon(x) := \tfrac{1}{2}(x + \sqrt{x^2 + 4\varepsilon}) = \begin{cases} \sqrt{\varepsilon} & \text{if } x = 0, \\ [x]_+ + \varepsilon/|x| + O(\varepsilon^2) & \text{if } x \neq 0. \end{cases} \tag{28}$$

Since $\phi_\varepsilon(-x)\phi_\varepsilon(x) = \varepsilon$, we deduce that

$$\lambda_\varepsilon(t) = -\varepsilon/u_\varepsilon(t) = -\phi_\varepsilon(p_\varepsilon - 2t).$$

It remains to compute the value p_ε of the adjoint state. Using (21) and (27), we get the following equation for p_ε:

$$p_\varepsilon = \tfrac{3}{4} + \int_0^1 \phi_\varepsilon(2t - p_\varepsilon)\mathrm{d}t = \tfrac{3}{4} + \tfrac{1}{2}\left[\Phi_\varepsilon(2 - p_\varepsilon) - \Phi_\varepsilon(-p_\varepsilon)\right], \tag{29}$$

where Φ_ε is a primitive of ϕ_ε, namely

$$\Phi_\varepsilon(x) = \Phi_{1,\varepsilon}(x) + \Phi_{2,\varepsilon}(x). \tag{30}$$

where

$$\Phi_{1,\varepsilon}(x) = \tfrac{1}{4}x^2 + \tfrac{1}{4}x\sqrt{x^2 + 4\varepsilon} \tag{31}$$

and

$$\Phi_{2,\varepsilon}(x) = \varepsilon \log\left(x + \sqrt{x^2 + 4\varepsilon}\right). \tag{32}$$

In order to obtain a closed form for $u_\varepsilon(t)$ we should find p_ε by solving (29). Unfortunately, this transcendent equation is apparently difficult to solve directly. Instead, we will compute an asymptotic expansion of p_ε for ε close to 0. To this end, we define

$$F(x,\varepsilon) = \begin{cases} x - \tfrac{3}{4} - \tfrac{1}{2}\Phi_\varepsilon(2 - x) + \tfrac{1}{2}\Phi_\varepsilon(-x) & \text{if } \varepsilon > 0, \\ x - \tfrac{3}{4} - \tfrac{1}{4}[2 - x]_+^2 + \tfrac{1}{4}[-x]_+^2 & \text{if } \varepsilon = 0, \end{cases} \tag{33}$$

which turns out to be jointly continuous and smooth with respect to x on a neighborhood of $(p_0, 0) = (1, 0)$. We have that for every $\varepsilon \geq 0$, p_ε satisfies

$$F(p_\varepsilon, \varepsilon) = 0. \tag{34}$$

Observe that we cannot apply to (34) the classical Implicit Function Theorem because $(x, \varepsilon) \to F(x, \varepsilon)$ is not continuously differentiable at $(p_0, 0) = (1, 0)$. In fact, although $x \to F(x, \varepsilon)$ is indeed smooth around $p_0 = 1$ and moreover

$$D_x F(p_0, 0) = \tfrac{3}{2}, \tag{35}$$

on the other hand, the partial derivative $D_\varepsilon F(p_0, 0)$ is not well defined because we have that

$$\frac{1}{\varepsilon}[F(p_0, \varepsilon) - F(p_0, 0)] = \frac{1}{\varepsilon} F(p_0, \varepsilon) = \tfrac{1}{2} \log \varepsilon - \tfrac{1}{2} + O(\varepsilon). \tag{36}$$

For overcoming this technical difficulty, we will apply Corollary 2 of Section 2 to (29), or equivalently, to (34). First, by virtue of the decomposition (30)-(32) of Φ_ε, we write $F(x, \varepsilon) = F_1(x, \varepsilon) + F_2(x, \varepsilon)$ where

$$F_1(x, \varepsilon) = \begin{cases} x - \frac{3}{4} - \frac{1}{2}\Phi_{1,\varepsilon}(2 - x) + \frac{1}{2}\Phi_{1,\varepsilon}(-x) & \text{if } \varepsilon > 0, \\ x - \frac{3}{4} - \frac{1}{4}[2 - x]_+^2 + \frac{1}{4}[-x]_+^2 & \text{if } \varepsilon = 0, \end{cases} \tag{37}$$

and

$$F_2(x, \varepsilon) = \begin{cases} -\frac{1}{2}\Phi_{2,\varepsilon}(2 - x) + \frac{1}{2}\Phi_{2,\varepsilon}(-x) & \text{if } \varepsilon > 0, \\ 0 & \text{if } \varepsilon = 0. \end{cases} \tag{38}$$

Both functions are continuous and smooth with respect to x on a neighborhood of $(p_0, \varepsilon_0) = (1, 0)$. Setting $A = D_x F(1, 0)$ and $A_i = D_x F_i(1, 0)$, $i = 1, 2$, we have

$$A_2 = 0, \quad A = A_1 = \tfrac{3}{2}. \tag{39}$$

Since F_1 is smooth (for what follows it suffices F_1 of class C^2) on a neighborhood of $(1, 0)$, we have that (4) holds for F_1 with $c_1(\beta) = c_1' \beta$ for some appropriate constant c_1'. For F_2, using (17) and (39), the best constant is

$$c_2(\beta) := \sup_{x,x'} \frac{|F_2(x', \varepsilon) - F_2(x, \varepsilon)|}{|x' - x|}, \tag{40}$$

where the supremum is over $x \neq x'$, both in the interval $[1 - \beta, 1 + \beta]$. But for positive y and z we have

$$\begin{aligned} \Phi_{2,\varepsilon}(-y) - \Phi_{2,\varepsilon}(-z) \quad &:= \quad \varepsilon \log \frac{-y + \sqrt{y^2 + 4\varepsilon}}{-z + \sqrt{z^2 + 4\varepsilon}} \\ &= \quad \varepsilon \log \frac{z + \sqrt{z^2 + 4\varepsilon}}{y + \sqrt{y^2 + 4\varepsilon}} = O(\varepsilon |y - z|). \end{aligned} \tag{41}$$

Therefore we may take $c_2(\beta) = c_2'\varepsilon$ and finally, since ε is chosen so that $\varepsilon \leq \beta$, we obtain that (4) holds for F with

$$c(\beta) = (c_1' + c_2')\beta. \tag{42}$$

Therefore Theorem 1 and Corollary 2 apply. We may choose $\rho_0 = \varepsilon$, and since $|F(p_0, \varepsilon)| = O(\varepsilon \log \varepsilon)$, (8) implies $\beta = O(\varepsilon \log \varepsilon)$. Thus, using (36) we obtain the following result:

Proposition 3. *The costate for the logarithmically penalized problem satisfies*

$$\begin{aligned} p_\varepsilon &= p_0 - A^{-1}F(p_0, \varepsilon) + O\left((\varepsilon \log \varepsilon)^2\right) \\ &= 1 - \tfrac{1}{3}\varepsilon \log \varepsilon + \tfrac{1}{3}\varepsilon + O(\varepsilon^2) + O\left((\varepsilon \log \varepsilon)^2\right), \\ &= 1 - \tfrac{1}{3}\varepsilon \log \varepsilon + \tfrac{1}{3}\varepsilon + O((\varepsilon \log \varepsilon)^2) \end{aligned} \tag{43}$$

This is to be compared with the standard case (of a smooth perturbation) where the expansion would be of the type $p_0 + d\varepsilon + O(\varepsilon^2)$ for an appropriate constant d.

Recall that, by virtue of (27) and (28), we have for $\varepsilon > 0$

$$u_\varepsilon(t) = \phi_\varepsilon(2t - p_\varepsilon) = \begin{cases} \sqrt{\varepsilon} & \text{if } t = \tfrac{1}{2}p_\varepsilon, \\ [2t - p_\varepsilon]_+ + \varepsilon/|2t - p_\varepsilon| + O(\varepsilon^2) & \text{otherwise,} \end{cases} \tag{44}$$

while

$$u_0(t) = [2t - 1]_+.$$

Setting

$$t_\varepsilon = \tfrac{1}{2}p_\varepsilon = \tfrac{1}{2} - \tfrac{1}{6}\varepsilon \log \varepsilon + O(\varepsilon),$$

then for all $\varepsilon > 0$ small enough to satisfy $t_\varepsilon > t_0 = \frac{1}{2}$, we get

$$u_\varepsilon(t_\varepsilon) - u_0(t_\varepsilon) = \sqrt{\varepsilon} - (2t_\varepsilon - 1) = \sqrt{\varepsilon} + \tfrac{1}{3}\varepsilon \log \varepsilon + O(\varepsilon) = O(\sqrt{\varepsilon}).$$

Since for $t \neq t_\varepsilon$ we have $|u_\varepsilon(t) - u_0(t)| \leq O(\varepsilon \log \varepsilon)$ uniformly with respect to t, we deduce that

$$\|u_\varepsilon - u_0\|_{L^\infty} = O(\sqrt{\varepsilon}).$$

On the other hand, it is not difficult to verify that

$$\|u_\varepsilon - u_0\|_{L^1} = O(\varepsilon \log \varepsilon).$$

In fact,

$$\|u_\varepsilon - u_0\|_{L^1} \geq \int_0^{\frac{1}{2}} \phi_\varepsilon(2t - p_\varepsilon)dt = \tfrac{1}{2}[\Phi_\varepsilon(1 - p_\varepsilon) - \Phi_\varepsilon(-p_\varepsilon)] = -\tfrac{1}{2}\varepsilon \log \varepsilon + O(\varepsilon).$$

On the other hand

$$
\begin{aligned}
\|u_\varepsilon - u_0\|_{L^1} &\leq \int_0^1 [\phi_\varepsilon(2t-p_\varepsilon) - [2t-p_\varepsilon]_+]dt + \int_0^1 [[2t-1]_+ - [2t-p_\varepsilon]_+]dt \\
&= \tfrac{1}{2}[\Phi_\varepsilon(2-p_\varepsilon) - \Phi_\varepsilon(-p_\varepsilon)] - \tfrac{1}{4}(2-p_\varepsilon)^2 + \tfrac{1}{4} - \tfrac{1}{4}(2-p_\varepsilon)^2 \\
&= \tfrac{1}{2}[\Phi_\varepsilon(2-p_\varepsilon) - \Phi_\varepsilon(-p_\varepsilon)] - \tfrac{1}{4} - \tfrac{1}{3}\varepsilon\log\varepsilon + o(\varepsilon\log\varepsilon).
\end{aligned}
$$

By (29), we have that $\frac{1}{2}[\Phi_\varepsilon(2-p_\varepsilon) - \Phi_\varepsilon(-p_\varepsilon)] = p_\varepsilon - \frac{3}{4} = \frac{1}{4} - \frac{1}{3}\varepsilon\log\varepsilon + O(\varepsilon)$. Hence

$$
\|u_\varepsilon - u_0\|_{L^1} \leq -\tfrac{2}{3}\varepsilon\log\varepsilon + o(\varepsilon\log\varepsilon).
$$

Similar computations and estimates hold for $\lambda_\varepsilon(\cdot)$.

4 Towards a possible generalization

Motivated by the previous example, let us consider the more general linear-quadratic problem

$$
\begin{cases}
\min\limits_u \dfrac{1}{2}\|y(T)\|^2 + \dfrac{1}{2}\displaystyle\int_0^T \|u(t)-r(t)\|^2 dt \\
\begin{cases} \dot{y}(t) = Ay(t) + Bu(t), \ t \in [0,T], \\ y(0) = y^0, \end{cases} \\
u \in L^2(0,T;U),
\end{cases}
\tag{45}
$$

where $r : [0,T] \to \mathbb{R}^m$ is some given reference control, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and

$$
U := \mathbb{R}^m_+ = \{u \in \mathbb{R}^m \mid u \geq 0\}.
$$

Here, the Hamiltonian function is given by

$$
H(u,y,p,t) = \frac{1}{2}\|u - r(t)\|^2 + p^\top(Ay + Bu),
$$

and the associated Pontryaguin maximum principle is

$$
\begin{cases}
\dot{y}(t) = H_p(u(t),y(t),p(t),t) = Ay(t) + Bu(t), \ t \in [0,T], \\
\dot{p}(t) = -H_y(u(t),y(t),p(t),t) = -A^\top p(t), \ t \in [0,T], \\
y(0) = y^0, \ p(T) = y(T), \\
u(t) \in \operatorname*{Argmin}_{v\in\mathbb{R}^m_+} H(v,y(t),p(t),t), \ t \in [0,T],
\end{cases}
$$

Therefore, the costate and optimal control are respectively given by

$$p(t) = e^{-tA^\top} p(0) = e^{(T-t)A^\top} p(T)$$

and

$$u_0(t) = [r(t) - B^\top p(t)]_+ = [r(t) - B^\top e^{(T-t)A^\top} p(T)]_+,$$

where $p(T)$ is implicitly characterized by the equation $p(T) = y(T)$. On the other hand, the introduction of the logarithmic penalty yields the optimal control

$$u_\varepsilon(t) = \phi_\varepsilon(r(t) - B^\top p_\varepsilon(t)) = \phi_\varepsilon(r(t) - B^\top e^{(T-t)A^\top} p_\varepsilon(T)),$$

where $\phi_\varepsilon(x)$ is given by (28) and is applied here componentwise. As before, it remains to find $p_\varepsilon(T)$ by solving the equation

$$\begin{aligned} p_\varepsilon(T) &= y_\varepsilon(T) && (46)\\ &= e^{TA}y^0 + \int_0^T e^{(T-s)A} B u_\varepsilon(s) ds && (47)\\ &= e^{TA}y^0 + \int_0^T e^{(T-s)A} B \phi_\varepsilon(r(s) - B^\top e^{(T-s)A^\top} p_\varepsilon(T)) ds. && (48) \end{aligned}$$

Unfortunately, it is not clear how to deal with the latter in order to be able to apply the general restoration theorem as in the previous example.

5 Concluding remarks

For linear or quadratic programming programs it is possible to compute an asymptotic expansion the the central path, see Monteiro and Tsuchiya [10], Mizuno, Jarre and Stoer [8, 9], as well [1, Chapter 18]. This expansion allows a precise analysis of the asymptotic behavior of the path-following algorithms. In particular, the above references discuss correction terms in order to obtain superlinear convergence in the absence of strict complementarity. It would be desirable to derive similar properties in the context of optimal control problems.

The present paper is a first step in that direction. Obviously the example is very simple. Still it happens that the proof is not so simple. An interesting challenge is to extend this kind of results to general convex linear quadratic problems.

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