Batch Processor Sharing with Hyper-Exponential Service Time

Natalia Osipova

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Batch Processor Sharing with Hyper-Exponential Service Time

Natalia Osipova

N° 6180

April 2007

Thème COM

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--6180--FR+ENG

Batch Processor Sharing with Hyper-Exponential Service Time

Natalia Osipova * †

Thème COM — Systèmes communicants
Projet MAESTRO

Rapport de recherche n° 6180 — April 2007 — 21 pages

Abstract: We study Processor-Sharing queueing model with the hyper-exponential service time distribution and Poisson batch arrival process. One of the main goals to study Batch Processor-Sharing (BPS) is the possibility of its application to size-based scheduling, which is used in differentiation between short and long flows in the Internet. In the case of hyper-exponential service time distribution we find an analytical expression for the expected conditional response time for the BPS queue. We show that the expected conditional response time is a concave function of the service time. We apply the obtained results to the Two Level Processor-Sharing (TLPS) model with the hyper-exponential service time distribution and find an expression of the expected response time for the TLPS model. The TLPS scheduling discipline can be applied to size-based differentiation in TCP/IP networks and Web server request handling.

Key-words: Batch Processor Sharing, Two Level Processor sharing, Hyper-Exponential distribution, Laplace transform, Cauchy matrices.

* PhD student at INRIA Sophia Antipolis, France, e-mail: Natalia.Osipova@sophia.inria.fr

† The work was supported by France Telecom R&D Grant “Modélisation et Gestion du Trafic Réseaux Internet” no. 42937433.

La File d'Attente Avec Service à Temps Partagé Avec Arrivées en Rafale

Résumé : Nous étudions la file d'attente avec service à temps partagé avec arrivées en rafale, BPS ("Batch Processor Sharing"), lorsque les temps de service ont une distribution hyper-exponentielle et que le processus d'arrivée des rafales est Poisson. Une des motivations de cette étude est que la file BPS apparaît naturellement lorsque l'on étudie les procédures d'ordonnancement favorisant les connexions courtes sur un réseau internet. Dans le cas d'une distribution hyper-exponentielle des temps de service nous trouvons une expression analytique pour le temps de réponse moyen conditionné sur le temps de service pour une file d'attente BPS. Nous montrons que ce temps de réponse moyen conditionné est une fonction concave du temps de service. Nous appliquons les résultats obtenus à la file d'attente munie d'une politique à Temps Partagées avec Deux-Niveaux "Two Level Processor Sharing" (TLPS) avec distribution hyper-exponentielle des temps de service. Nous trouvons l'expression du temps de réponse moyen pour la file TLPS. La discipline de service TLPS peut être utilisée pour ordonner l'accès aux ressources en fonction de la taille dans un réseau TCP/IP ou sur un serveur Web.

Mots-clés : La file d'attente avec service à temps partagé avec arrivées en rafale, la file d'attente munie d'une politique à Temps Partagées avec Deux-Niveaux, réseau TCP/IP, distribution hyper-exponentielle, Laplace transforment, Matrices de Cauchy.

1 Introduction

The Processor-Sharing (PS) queueing systems are now often used to model communication and computer systems. The PS systems were first introduced by Kleinrock (see [6] and references therein). Under the PS policy each job receives an equal share of the processor.

PS with batch arrivals (BPS) is not yet characterized fully. Kleinrock *et al.* [5] first studied BPS. They found that the derivative of the expected response time satisfies the integral equation and found the analytical solution in the case when the job size (service time) distribution function has the form $F(x) = 1 - p(x)e^{-\mu x}$ where $p(x)$ is a polynomial.

Bansal [4], using Kleinrock's integral equation, obtained the solution for the Laplace transform of the expected conditional service time as a solution of the system of linear equations, when the job size distribution is a hyper-exponential distribution. Also he considers distributions with a rational Laplace transform. Rege and Sengupta [12] obtained the expression for the response time in condition upon the number of customers in the system. Feng and Mishra [8] provided bounds for the expected conditional response time, the bounds depend on the second moment of the service time distribution. Avrachenkov *et al.* [2] proved existence and uniqueness of the solution of the Kleinrock's integral equation and provided asymptotic analysis and bounds on the expected conditional response time.

We study the BPS model with the hyper-exponential service time distribution. For this distribution we provide the solution of the Kleinrock's integral equation according to the derivative of the expected conditional response time. We prove the concavity of the expected conditional sojourn time function for the BPS model with the hyper-exponential job size distribution function. We note that the concavity of the expected conditional sojourn time for the BPS with the hyper-exponential job size distribution was proven by another method in [10].

One of the main goals to study BPS is the possibility of its application to age-based scheduling and the possibility to take into account the burstiness of the arrival process. Bursty arrivals often occur in such modern systems as web server. Age-based scheduling is used in differentiation of short and long flows in the Internet. A quite general set of age-based scheduling mechanisms was introduced by Kleinrock and termed as Multy-Level PS (MLPS).

In MLPS jobs are classified into different classes depending on their attained amount of service. Jobs within the same class are served according to FCFS (First Come First Serve), PS (Processor Sharing) or FB (Foreground Background) policy. The classes themselves are served according to the FB policy, so that the priority is given to the jobs with small sizes.

We study the particular case of MLPS, Two-Level PS (TLPS) scheduling mechanism. It is based on the differentiation of jobs according to some threshold and gives priority to jobs with small sizes. The TLPS scheduling mechanism could be used to model such applications as size based differentiation in TCP/IP networks and Web server request differentiation.

It was shown in [1] that when a job size distribution has a decreasing hazard rate, then with the selection of the threshold the expected sojourn time of the TLPS system could be reduced in comparison to standard PS system.

The distribution of file sizes in the Internet has a decreasing hazard rate and often modelled with heavy-tailed distributions. In [9] it is shown that heavy-tailed distributions such as Pareto distribution could be approximated with the hyper-exponential distributions with a significant number of phases.

Therefore, we study the TLPS model with the hyper-exponential service time distribution. We apply the results of the BPS queueing model to the TLPS model and find an expression of expected conditional sojourn time for the TLPS model.

The paper is organized as follows. In Section 2 the BPS scheduling mechanism is considered in the case when the job service time distribution is the hyper-exponential distribution. For this case we provide the solution of the Kleinrock's integral equation. In Section 3 the results obtained for the BPS model are applied to the TLPS model, where the job size distribution is also hyper-exponential. An analytical expression of the expected sojourn time is found. We put some technical proofs in the Appendix.

2 The Analysis of the Batch Arrival Processor Sharing Queue

2.1 Main definitions

Let us consider an M/G/1 system with batch arrivals and Processor-Sharing (PS) queueing discipline. The batches arrive according to a Poisson process with arrival rate λ. Let $\overline{n} > 0$ be the average size of a batch. Let $b > 0$ be the average number of jobs that arrive with (and in addition to) the tagged job.

Let $B(x)$ be the required job size (service time) distribution and $\overline{B}(x) = 1 - B(x)$ be its complementary distribution function.

The load is given by $\rho = \lambda \overline{n} m$, with $m = \int_0^\infty x dB(x)$. We consider that the system is stable, $\rho < 1$.

Let $\alpha(x)$ be the expected conditional response time for a job with service time x and $\alpha'(x)$ be its derivative. Kleinrock showed in [6] that $\alpha'(x)$ satisfies the following integro-differential equation

$$\alpha'(x) = \lambda \overline{n} \int_0^\infty \alpha'(y) \overline{B}(x+y) dy + \lambda \overline{n} \int_0^x \alpha'(y) \overline{B}(x-y) dy + b\overline{B}(x) + 1. \tag{1}$$

We are interested in the case when the job size distribution is a hyper-exponential function

$$B(x) = 1 - \sum_{i=1}^{N} p_i \, e^{-\mu_i \, x}, \ \sum_i p_i = 1, \ p_i > 0, \ \mu_i > 0, \quad i = 1, ..., N, \quad 1 < N \leq \infty. \tag{2}$$

By $\sum_i$ and $\prod_i$ we mean $\sum_{i=1}^N$ and $\prod_{i=1}^N$. By $\sum_{i \neq j}$ or $\prod_{i \neq j}$ we mean $\sum_{i=1,\ldots,N, \, i \neq j}$ and $\prod_{i=1,\ldots,N, \, i \neq j}$. By $\forall i$ we mean $i = 1, ..., N$.

Without loss of generality, we can assume that

$$0 < \mu_N < \mu_{N-1} < ... < \mu_2 < \mu_1 < \infty. \tag{3}$$

2.2 The expected conditional sojourn time for the BPS model

Let us first prove additional Lemma.

Lemma 1. *The zeros b_i of the rational function*

$$\Psi(s) =^{def} 1 - \lambda \overline{n} \sum_i \frac{p_i}{s + \mu_i} = \frac{\prod_i (s + b_i)}{\prod_i (s + \mu_i)} \tag{4}$$

are all real, distinct, positive and satisfy the following inequalities:

$$0 < b_N < \mu_N, \ \ \mu_{i+1} < b_i < \mu_i, \quad i = 1, ..., N-1. \tag{5}$$

Proof. Let us study the following equation

$$\Psi(s) = 0.$$

Following the approach of [7] we get that it has N_1 roots $-b_i$, $i = 1, ..., N_1$, where N_1 is the number of distinct elements within μ_i. As μ_i, $i = 1, ..., N$ are all different, $\mu_i \neq \mu_j$, $i \neq j$, then $N_1 = N$ and there are N different roots $-b_i$. All $-b_i$ are real, distinct, negative, satisfy the following inequalities: $0 > -b_N > -\mu_N$, $-\mu_{i+1} > -b_i > -\mu_i$, $\quad i = 1, ..., N-1$. With this we prove the statement of Lemma 1. So, then it is possible to present $\Psi(s)$ in the following form:

$$\Psi(s) = \frac{\prod_i (s + b_i)}{\prod_i (s + \mu_i)},$$

and from here we have (4). □

Before presenting our main result let us prove an auxiliary Lemma.

Lemma 2. *The solution of the following system of linear equations:*

$$\sum_j \frac{x_j}{\mu_q^2 - b_j^2} = 1, \quad q = 1, ..., N, \tag{6}$$

is given by

$$x_k = \frac{\prod\limits_{q=1,...,N} (\mu_q^2 - b_k^2)}{\prod\limits_{q \neq k} (b_q^2 - b_k^2)}, \quad k = 1, ..., N. \tag{7}$$

Proof. The proof is given in the appendix. □

Corollary 3. *The solution of equation (6) is positive. Namely,* $x_k > 0$ *for* $k = 1, ..., N$.

Proof. As (3) and (5), then the statement of the Corollary holds. □

Now we can prove our main result.

Theorem 4. *The expected conditional response time for BPS queue with hyper-exponential job size distribution function as in (2) is given by:*

$$\alpha(x) = c_0 x - \sum_k \frac{c_k}{b_k} e^{-b_k x} + \sum_k \frac{c_k}{b_k}, \quad \alpha(0) = 0, \tag{8}$$

where

$$c_0 = \frac{1}{1-\rho},$$

$$c_k = \frac{b}{2\lambda\overline{n}} \left(\frac{\prod_q (\mu_q^2 - b_k^2)}{b_k \prod_{q \neq k} (b_q^2 - b_k^2)} \right),$$

and where b_k are the solutions of the following equation:

$$1 - \lambda\overline{n} \sum_i \frac{p_i}{s + \mu_i} = 0,$$

and are all positive, distinct, real, satisfy the following inequalities

$$0 < b_N < \mu_N, \ \ \mu_{i+1} < b_i < \mu_i, \quad i = 1, ..., N-1.$$

Proof. We can rewrite integral equation (1) in the following way:

$$\alpha'(x) = \lambda\overline{n} \int_0^\infty \alpha'(y) \sum_i p_i e^{-\mu_i(x+y)} dy + \lambda\overline{n} \int_0^x \alpha'(y)\overline{B}(x-y)dy + b\overline{B}(x) + 1,$$

$$\alpha'(x) = \lambda\overline{n} \sum_i p_i e^{-\mu_i x} \int_0^\infty \alpha'(y) e^{-\mu_i y} dy + \lambda\overline{n} \int_0^x \alpha'(y)\overline{B}(x-y)dy + b\overline{B}(x) + 1.$$

We note that in the latter equation $\int_0^\infty \alpha'(y)e^{-\mu_i y}dy, \quad i = 1, ..., N$ are the Laplace transforms of $\alpha'(y)$ evaluated at $\mu_i, \ i = 1, ..., N$. Denote

$$L_i = \int_0^\infty \alpha'(y)e^{-\mu_i y}dy, \quad i = 1, ..., N.$$

Then, we have

$$\alpha'(x) = \lambda\overline{n} \sum_i p_i L_i e^{-\mu_i x} + \lambda\overline{n} \int_0^x \alpha'(y)\overline{B}(x-y)dy + b\overline{B}(x) + 1.$$

Now taking the Laplace transform of the above equation and using the convolution property, we get the following equation. Here we denote $L_{\alpha'}(s)$ the Laplace transform of $\alpha'(x)$.

$$L_{\alpha'}(s) = \lambda\overline{n} \sum_i \frac{p_i L_i}{s+\mu_i} + \lambda\overline{n} \sum_i \frac{p_i L_{\alpha'}(s)}{s+\mu_i} + b \sum_i \frac{p_i}{s+\mu_i} + \frac{1}{s},$$

$$\Longrightarrow \qquad L_{\alpha'}(s) \left(1 - \lambda\overline{n} \sum_i \frac{p_i}{s+\mu_i} \right) = \lambda\overline{n} \sum_i \frac{p_i L_i}{s+\mu_i} + b \sum_i \frac{p_i}{s+\mu_i} + \frac{1}{s}.$$

Let us note that $L_{\alpha'}(\mu_i) = L_i, \quad i = 1, ..., N$. According to Lemma 1 and (4), the following equation holds:

$$L_{\alpha'}(s)\frac{\prod_i(s+b_i)}{\prod_i(s+\mu_i)} = \lambda\overline{n}\sum_i \frac{p_i L_i}{s+\mu_i} + b\sum_i \frac{p_i}{s+\mu_i} + \frac{1}{s}, \tag{9}$$

$$\Longrightarrow \qquad L_{\alpha'}(s) = \lambda\overline{n}\sum_i p_i L_i \frac{\prod_{k\neq i}(s+\mu_k)}{\prod_i(s+b_i)} + b\sum_i p_i \frac{\prod_{k\neq i}(s+\mu_k)}{\prod_i(s+b_i)} + \frac{1}{s}\frac{\prod_i(s+\mu_i)}{\prod_i(s+b_i)}. \tag{10}$$

From here we see that there exist such c_k that:

$$L_{\alpha'}(s) = \frac{c_0}{s} + \sum_k \frac{c_k}{s+b_k}. \tag{11}$$

Then, taking the inversion of the Laplace transform, we have the expression for $\alpha'(x)$:

$$\alpha'(x) = c_0 + \sum_k c_k e^{-b_k x}.$$

and as $\alpha(0) = 0$, then for $\alpha(x)$ we have

$$\alpha(x) = c_0 x - \sum_k \frac{c_k}{b_k} e^{-b_k x} + \sum_k \frac{c_k}{b_k}, \quad \alpha(0) = 0.$$

Now let us find c_k. To find c_0 let us multiply both parts of the equation (11) by s and find the value at the point $s = 0$. Then, we have that

$$c_0 = L_{\alpha'}(s)s|_{s=0} = \frac{\prod_i \mu_i}{\prod_i b_i}. \tag{12}$$

From (4) we have the following:

$$\frac{\prod_i b_i}{\prod_i \mu_i} = \Psi(s)|_{s=0} = 1 - \lambda\overline{n}\sum_i \frac{p_i}{\mu_i} = 1 - \lambda\overline{n}m = 1 - \rho.$$

So, then

$$c_0 = \frac{1}{1-\rho}.$$

Let us find the other coefficients c_i. We denote:

$$L^*_{\alpha'}(s) = \sum_i \frac{c_i}{s+b_i},$$

$$\Longrightarrow \qquad L_{\alpha'}(s) = \frac{c_0}{s} + \sum_i \frac{c_i}{s+b_i} = \frac{c_0}{s} + L^*_{\alpha'}(s).$$

According to the system (9), (10) we have the following:

$$L_j^* = \sum_i \frac{c_i}{\mu_j + b_i}, \quad j = 1, ..., N,$$

$$(9) \Longrightarrow \quad L_{\alpha'}^*(s) \frac{\prod_i (s+b_i)}{\prod_i (s+\mu_i)} = \lambda\overline{n} \sum_i \frac{p_i L_i^*}{s+\mu_i} + b \sum_i \frac{p_i}{s+\mu_i},$$

$$\Longrightarrow \quad L_{\alpha'}^*(s) \frac{\prod_i (s+b_i)}{\prod_i (s+\mu_i)} (s+\mu_q)|_{s=-\mu_q} = \lambda\overline{n} \sum_i \frac{p_i L_i^*}{s+\mu_i} (s+\mu_q)|_{s=-\mu_q} +$$

$$+ b \sum_i \frac{p_i}{s+\mu_i} (s+\mu_q)|_{s=-\mu_q}, \quad q = 1, ..., N,$$

$$\Longrightarrow \quad \sum_j \frac{c_j}{b_j - \mu_q} \frac{\prod_i (b_i - \mu_q)}{\prod_{i \neq q} (\mu_i - \mu_q)} = \lambda\overline{n} p_q L_q^* + b p_q, \quad q = 1, ..., N,$$

$$\Longrightarrow \quad \sum_j \frac{c_j}{b_j - \mu_q} \frac{\prod_i (b_i - \mu_q)}{\prod_{i \neq q} (\mu_i - \mu_q)} = \lambda\overline{n} p_q \sum_j \frac{c_j}{b_j + \mu_q} + b p_q, \quad q = 1, ..., N.$$

Let us notice that from (4) we have the following:

$$\frac{\prod_i (b_i - \mu_q)}{\prod_{i \neq q} (\mu_i - \mu_q)} = \frac{\prod_i (s+b_i)}{\prod_i (s+\mu_i)} (s+\mu_q)|_{s=-\mu_q} = \Psi(s)(s+\mu_q)|_{s=-\mu_q} =$$

$$= \left(1 - \lambda\overline{n} \sum_i \frac{p_i}{s+\mu_i}\right) (s+\mu_q)|_{s=-\mu_q} = -\lambda\overline{n}\, p_q, \quad q = 1, ..., N,$$

then

$$\Longrightarrow \quad \sum_j \frac{c_j}{b_j - \mu_q} (-\lambda\overline{n}\, p_q) = \lambda\overline{n}\, p_q \sum_j \frac{c_j}{b_j + \mu_q} + b p_q, \quad q = 1, ..., N,$$

$$\Longrightarrow \quad \sum_j \frac{c_j}{\mu_q - b_j} - \sum_j \frac{c_j}{\mu_q + b_j} = \frac{b}{\lambda\overline{n}}, \quad q = 1, ..., N,$$

$$\Longrightarrow \quad \sum_j \frac{c_j b_j}{\mu_q^2 - b_j^2} = \frac{b}{2\lambda\overline{n}}, \quad q = 1, ..., N.$$

So, c_j are solutions of the following linear system:

$$\sum_j \frac{c_j b_j}{\mu_q^2 - b_j^2} = \frac{b}{2\lambda\overline{n}}, \quad q = 1, ..., N. \tag{13}$$

If we denote

$$x_k = \frac{c_k b_k}{\frac{b}{2\lambda\overline{n}}}, \quad k = 1, ..., N,$$

the system (13) will take the form (6) and by Lemma 2 for c_j the final result is as follows:

$$c_k = \frac{b}{2\lambda\overline{n}}\left(\frac{x_k}{b_k}\right) = \frac{b}{2\lambda\overline{n}}\left(\frac{\prod_q (\mu_q^2 - b_k^2)}{b_k \prod_{q\neq k} (b_q^2 - b_k^2)}\right), \quad k = 1, ..., N.$$

This completes the proof of Theorem 4. □

Corollary 5. *The expected conditional sojourn time function in the BPS system with hyper-exponential job size distribution as in (2) is a strictly concave function with respect to job sizes.*

Proof. The function

$$\alpha(x) = c_0 x - \sum_k \frac{c_k}{b_k} e^{-b_k x} + \sum_k \frac{c_k}{b_k}, \quad \alpha(0) = 0$$

is a strictly concave function if $\alpha''(x) < 0$.

$$\alpha''(x) = -\sum_k c_k b_k e^{-b_k x} < 0$$

as $c_k > 0,\ b_k > 0,\ k = 1, ..., N$, which follows from $b > 0, \overline{n} > 0$, Corollary 3 and Lemma 1. □

Corollary 6. *The expected sojourn time in the BPS system with hyper-exponential job size distribution as in (2) is given by*

$$\overline{T}^{BPS} = \frac{m}{1-\rho} + \sum_{i,j} \frac{p_i c_j}{\mu_i + b_j}. \tag{14}$$

Proof. As the expected sojourn time $\overline{T}^{BPS}$ is given by

$$\overline{T}^{BPS} = \int_0^\infty \alpha'(x)\overline{B}(x)dx,$$

then using (8) we receive the statement of the Corollary. □

3 The Analysis of the Two Level Processor Sharing Model

3.1 Main definitions

We study the Two Level Processor Sharing (TLPS) scheduling discipline with the hyper-exponential job size distribution. The model description is as follows.

Jobs arrive to the system according to a Poisson process with rate λ. Let θ be a given threshold. The jobs in the system that attained a service less than θ are assigned to the high priority queue. If in addition there are jobs with attained service greater than θ, such a job is separated into two parts. The first part of size θ is assigned to the high priority queue and the second part of size $x-\theta$ waits in the lower priority queue. The low priority queue is served when the high priority queue is empty. Both queues are served according to the Processor Sharing (PS) discipline.

Let us denote the job size distribution by $F(x)$. By $\overline{F}(x) = 1 - F(x)$ we denote the complementary distribution function. We consider the case, when $F(x)$ is a hyper-exponential distribution function, namely

$$F(x) = 1 - \sum_{i=1}^{N} \widetilde{p}_i e^{-\mu_i x}, \quad \sum_i \widetilde{p}_i = 1,\ \widetilde{p}_i > 0,\ \mu_i > 0, \quad i = 1, ..., N, \quad 1 < N \leq \infty. \quad (15)$$

The mean job size is given by $m = \int_0^\infty x dF(x)$ and the system load is $\rho = \lambda m$. We assume that the system is stable ($\rho < 1$) and is in steady state.

3.2 The expected conditional sojourn time for the TLPS model

Let us denote by $\overline{T}^{TLPS}(x)$ the expected conditional sojourn time in the TLPS system for a job of size x and by $\overline{T}(\theta)$ the expected sojourn time of the system.

According to [6] the expected conditional sojourn time of the system is given by:

$$\overline{T}^{TLPS}(x) = \begin{cases} \dfrac{x}{1-\rho_\theta}, & x \in [0, \theta], \\ \dfrac{\overline{W}(\theta) + \theta + \alpha(x-\theta)}{1-\rho_\theta}, & x \in (\theta, \infty), \end{cases}$$

where ρ_θ is the utilization factor for the truncated distribution $\rho_\theta = \lambda \overline{X_\theta^1}$, the n-th moment of the distribution truncated at θ is

$$\overline{X_\theta^n} = \int_0^\theta n y^{n-1} \overline{F}(y) dy,$$

$(\overline{W}(\theta)+\theta)/(1-\rho_\theta)$ expresses the time needed to reach the low priority queue. This time consists of the time $\theta/(1-\rho_\theta)$ spent in the high priority queue, where the flow is served up to the threshold θ, plus the time $\overline{W}(\theta)/(1-\rho_\theta)$ which is spent waiting for the high priority queue to empty. Here $\overline{W}(\theta) = \frac{\lambda \overline{X_\theta^2}}{2(1-\rho_\theta)}$. The remaining term $\alpha(x-\theta)/(1-\rho_\theta)$ is the time spent in the low priority queue.

To find $\alpha(x)$ we use the interpretation of the lower priority queue as a PS system with batch arrivals.

Let us denote

$$\overline{F_\theta^i} = \widetilde{p}_i e^{-\mu_i \theta}, \quad i = 1, ..., N,$$

and prove the following Theorem:

Theorem 7. *In TLPS priority queue with the hyper-exponential job size distribution as in (15):*

$$\alpha(x) = c_0(\theta)x - \sum_k \frac{c_k(\theta)}{b_k(\theta)} e^{-b_k(\theta)x} + \sum_k \frac{c_k(\theta)}{b_k(\theta)}, \quad \alpha(0) = 0,$$

where

$$c_0(\theta) = \frac{1-\rho_\theta}{1-\rho},$$

$$c_k(\theta) = \frac{b}{2\lambda\overline{n}} \left(\frac{\prod\limits_{q=1,\dots,N} (\mu_q^2 - b_k^2(\theta))}{b_k(\theta) \prod\limits_{q\neq k} (b_q^2(\theta) - b_k^2(\theta))} \right), \quad k = 1, ..., N,$$

and $b_i(\theta)$ are roots of

$$1 - \frac{\lambda}{1-\rho_\theta} \sum_i \frac{\overline{F_\theta^i}}{s+\mu_i} = 0,$$

and satisfy the following inequalities:

$$0 < b_N(\theta) < \mu_N, \ \ \mu_{i+1} < b_i(\theta) < \mu_i, \quad i = 1, ..., N-1.$$

Proof. As was shown in [6], $\alpha'(x) = d\alpha/dx$ is the solution of the integral equation we looked at in Section 2

$$\alpha'(x) = \lambda\overline{n} \int_0^\infty \alpha'(y)\overline{B}(x+y)dy + \lambda\overline{n} \int_0^x \alpha'(y)\overline{B}(x-y)dy + b\overline{B}(x) + 1.$$

Here the average batch size is given by $\overline{n} = \frac{1-F(\theta)}{1-\rho_\theta}$ and the average number of jobs that arrive to the low priority queue in addition to the tagged job is given by $b = \frac{2\lambda\,(1-F(\theta))\left(\overline{W}(\theta)+\theta\right)}{(1-\rho_\theta)}$. The complementary truncated distribution is $\overline{B}(x) = \frac{1-F(x+\theta)}{1-F(\theta)}$, then:

$$\overline{B}(x) = \frac{\overline{F}(x+\theta)}{\overline{F}(\theta)} = \frac{1}{\overline{F}(\theta)}\sum_i \widetilde{p}_i e^{-\mu_i(x+\theta)} = \sum_i \frac{\widetilde{p}_i e^{-\mu_i \theta}}{\overline{F}(\theta)} e^{-\mu_i x} = \sum_i p_i e^{-\mu_i x},$$

$$p_i = \frac{1}{\overline{F}(\theta)}\widetilde{p}_i e^{-\mu_i\theta} = \frac{1}{\overline{F}(\theta)}\overline{F^i_\theta}, \quad i = 1, ..., N.$$

We apply the results of Theorem 4 for the TLPS model. To calculate c_0 we use (12) and (4) and get the following

$$\frac{1}{c_0(\theta)} = \frac{\prod_i b_i(\theta)}{\prod_i \mu_i} = \Psi(s)|_{s=0} = 1 - \lambda\overline{n}\sum_i \frac{p_i}{\mu_i} = 1 - \frac{\lambda}{1-\rho_\theta}\sum_i \frac{\overline{F^i_\theta}}{\mu_i} = 1 - \frac{\lambda(m - \overline{X^1_\theta})}{1-\rho_\theta} = \frac{1-\rho}{1-\rho_\theta}.$$

So, then

$$c_0(\theta) = \frac{1-\rho_\theta}{1-\rho}.$$

For $c_k(\theta), b_k(\theta)$ we use the results of Theorem 4 and get the statement of Theorem 7. □

Corollary 8. *The coefficients* $c_k(\theta) > 0, \quad k = 1, ..., N$ *if* $\theta > 0$.

Proof. As (3), $0 < b_N(\theta) < \mu_N, \ \ \mu_{i+1} < b_i(\theta) < \mu_i, \quad i = 1, ..., N-1$ and $b > 0, \overline{n} > 0$ when $\theta > 0$, then the statement of Corollary holds. □

Corollary 9. *For the TLPS queue with hyper-exponential job size distribution as in (15) the function* $\alpha(x)$ *is a strictly concave function with respect to job sizes with positive values of* θ.

Proof. The function

$$\alpha(x) = c_0(\theta)x - \sum_k \frac{c_k(\theta)}{b_k(\theta)} e^{-b_k(\theta)x} + \sum_k \frac{c_k(\theta)}{b_k(\theta)}, \quad \alpha(0) = 0$$

is a strictly concave function if $\alpha''(x) < 0$.

$$\alpha''(x) = -\sum_k c_k(\theta) b_k(\theta) e^{-b_k(\theta)x} < 0$$

as $c_k(\theta) > 0, b_k(\theta) > 0, \ \ k = 1, ..., N$, if $\theta > 0$ as it follows from Corollary 8 and Theorem 7. □

Corollary 10. *The expected conditional sojourn time for the TLPS queue with hyper-exponential job size distribution as in (15) is not a concave function with respect to job sizes.*

Proof. The function $\overline{T}^{TLPS}(x)$ is a concave function, when

$$\begin{aligned}
&\overline{T}^{TLPS}{}'(x)|_{x=\theta-0} \geq \overline{T}^{TLPS}{}'(x)|_{x=\theta+0}\\
&\frac{1}{1-\rho_\theta}|_{x=\theta-0} \geq \frac{\alpha'(x-\theta)}{1-\rho_\theta}|_{x=\theta+0}\\
&1 \geq \alpha'(0+).
\end{aligned}$$

As shown in [2],

$$\begin{aligned}
\alpha(x) &\geq \frac{x}{1-\rho},\\
\alpha'(x) &\geq \frac{1}{1-\rho} > 1.
\end{aligned}$$

Then, it follows that $\overline{T}^{TLPS}(x)$ is not a concave function. $\square$

3.3 The expected sojourn time for the TLPS model

Theorem 11. *The expected sojourn time in the TLPS system with hyper-exponential distribution function (15) is given by the following equation:*

$$\overline{T}(\theta) = \frac{\overline{X^1_\theta} + \overline{W}(\theta)\overline{F}(\theta)}{1-\rho_\theta} + \frac{(m - \overline{X^1_\theta})}{1-\rho} + \frac{(\overline{W}(\theta)+\theta)}{1-\rho_\theta}\sum_{i,j}\frac{\overline{F^i_\theta}}{b_j(\theta)(\mu_i + b_j(\theta))}\frac{\prod_q (\mu_q^2 - b_j^2(\theta))}{\prod_{q\neq j}(b_q^2(\theta) - b_j^2(\theta))},$$

where $b_i(\theta)$ are defined as in Theorem 7.

Proof. According to [6] and Theorem 7 we have the following:

$$\begin{aligned}
&\overline{T}(\theta) = \frac{\overline{X^1_\theta} + \overline{W}(\theta)\overline{F}(\theta)}{1-\rho_\theta} + \frac{1}{1-\rho_\theta}\overline{T}^{BPS}(\theta),\\
&\overline{T}^{BPS}(\theta) = \int_\theta^\infty \alpha(x-\theta)dF(x) = \int_0^\infty \alpha'(x)\overline{F}(x+\theta)dx,\\
(Theorem\ 7) \Longrightarrow \quad &\overline{T}^{BPS}(\theta) = c_0(\theta)(m - \overline{X^1_\theta}) + \sum_{i,j}\frac{\overline{F^i_\theta}c_j(\theta)}{\mu_i + b_j(\theta)},
\end{aligned}$$

then

$$\overline{T}(\theta) = \frac{\overline{X_\theta^1} + \overline{W}(\theta)\overline{F}(\theta)}{1-\rho_\theta} + \frac{(m - \overline{X_\theta^1})}{1-\rho} + \frac{(\overline{W}(\theta)+\theta)}{1-\rho_\theta} \sum_{i,j} \frac{\overline{F_\theta^i}}{b_j(\theta)(\mu_i + b_j(\theta))} \frac{\prod_q (\mu_q^2 - b_j^2(\theta))}{\prod_{q \neq j} (b_q^2(\theta) - b_j^2(\theta))}.$$

□

4 Conclusion

We study the BPS queueing model, when the job size distribution is hyper-exponential, and we find an analytical expression of the expected conditional response time and for the expected sojourn time. We show that the function of the expected conditional sojourn time in the BPS system with hyper-exponential job size distribution is a concave function with respect to job sizes. We apply the results obtained for the BPS model to the TLPS scheduling mechanism with the hyper-exponential job size distribution and we find the expressions of the expected conditional response time and expected response time for the TLPS model.

Acknowledgment

I would like to thank K. Avrachenkov and P. Brown for fruitful discussions and suggestions.

5 Appendix

Lemma 2. The solution of the following system of linear equations

$$\sum_j \frac{x_j}{\mu_q^2 - b_j^2} = 1, \quad q = 1, ..., N, \tag{16}$$

is given by:

$$x_k = \frac{\prod\limits_{q=1,\dots,N} (\mu_q^2 - b_k^2)}{\prod\limits_{q \neq k} (b_q^2 - b_k^2)}. \tag{17}$$

Proof. Let

$$x = [x_1, x_2, ..., x_N]^T,$$
$$\underline{1} = [1, 1, ..., 1]^T|_{1\times N}$$

be two vectors of size N,

$$D = \begin{pmatrix} \frac{1}{\mu_1^2 - b_1^2} & \frac{1}{\mu_1^2 - b_2^2} & \cdots & \frac{1}{\mu_1^2 - b_N^2} \\ \frac{1}{\mu_2^2 - b_1^2} & \frac{1}{\mu_2^2 - b_2^2} & \cdots & \frac{1}{\mu_2^2 - b_N^2} \\ \cdots\cdots & & & \\ \frac{1}{\mu_N^2 - b_1^2} & \frac{1}{\mu_N^2 - b_2^2} & \cdots & \frac{1}{\mu_N^2 - b_N^2} \end{pmatrix}_{n \times N}$$

be the matrix of size $N \times N$. Then equation (16) could be rewritten as

$$Dx = \underline{1}.$$

Applying the Cramer formulas [11] we obtain:

$$x_k = \frac{\det D_k}{\det D} = \frac{\det [D_{[1]}, ...D_{[k-1]}, \underline{1}, D_{[k+1]}, ...D_{[N]}]}{\det D}, \quad k = 1, ..., N, \tag{18}$$
$$D_k = [D_{[1]}, ...D_{[k-1]}, \underline{1}, D_{[k+1]}, ...D_{[N]}].$$

Since D is a Cauchy matrix, its determinant is known [11]:

$$\det D = \frac{\prod\limits_{1 \le j < k \le N} (\mu_j^2 - \mu_k^2)(b_k^2 - b_j^2)}{\prod\limits_{j,k=1,\dots,N} (\mu_j^2 - b_k^2)} . \tag{19}$$

As $0 < b_N < \mu_N < ... < \mu_{i+1} < b_i < \mu_i < ... < b_1 < \mu_1$, then $\det D > 0$ and we can use Cramer formulas to calculate x_k. Let us find $\det D_k = \det\left[D_{[1]}, ...D_{[k-1]}, \underline{1}, D_{[k+1]}, ...D_{[N]}\right]$.

$$D_k = \begin{pmatrix} \frac{1}{\mu_1^2-b_1^2} & \cdots & \frac{1}{\mu_1^2-b_{k-1}^2} & 1 & \frac{1}{\mu_1^2-b_{k+1}^2} & \cdots & \frac{1}{\mu_1^2-b_N^2} \\ \frac{1}{\mu_2^2-b_1^2} & \cdots & \frac{1}{\mu_2^2-b_{k-1}^2} & 1 & \frac{1}{\mu_2^2-b_{k+1}^2} & \cdots & \frac{1}{\mu_2^2-b_N^2} \\ ..... \\ \frac{1}{\mu_N^2-b_1^2} & \cdots & \frac{1}{\mu_N^2-b_{k-1}^2} & 1 & \frac{1}{\mu_N^2-b_{k+1}^2} & \cdots & \frac{1}{\mu_N^2-b_N^2} \end{pmatrix}_{N\times N}$$

$$\det D_k = \begin{vmatrix} \frac{1}{\mu_1^2-b_1^2} & \cdots & \frac{1}{\mu_1^2-b_{k-1}^2} & 1 & \frac{1}{\mu_1^2-b_{k+1}^2} & \cdots & \frac{1}{\mu_1^2-b_N^2} \\ \frac{1}{\mu_2^2-b_1^2} & \cdots & \frac{1}{\mu_2^2-b_{k-1}^2} & 1 & \frac{1}{\mu_2^2-b_{k+1}^2} & \cdots & \frac{1}{\mu_2^2-b_N^2} \\ ..... \\ \frac{1}{\mu_N^2-b_1^2} & \cdots & \frac{1}{\mu_N^2-b_{k-1}^2} & 1 & \frac{1}{\mu_N^2-b_{k+1}^2} & \cdots & \frac{1}{\mu_N^2-b_N^2} \end{vmatrix}_{N\times N}$$

$$= (-1)^{k-1} \begin{vmatrix} 1 & \frac{1}{\mu_1^2-b_1^2} & \cdots & \frac{1}{\mu_1^2-b_{k-1}^2} & \frac{1}{\mu_1^2-b_{k+1}^2} & \cdots & \frac{1}{\mu_1^2-b_N^2} \\ 1 & \frac{1}{\mu_2^2-b_1^2} & \cdots & \frac{1}{\mu_2^2-b_{k-1}^2} & \frac{1}{\mu_2^2-b_{k+1}^2} & \cdots & \frac{1}{\mu_2^2-b_N^2} \\ ..... \\ 1 & \frac{1}{\mu_N^2-b_1^2} & \cdots & \frac{1}{\mu_N^2-b_{k-1}^2} & \frac{1}{\mu_N^2-b_{k+1}^2} & \cdots & \frac{1}{\mu_N^2-b_N^2} \end{vmatrix}_{N\times N}$$

To simplify the following computations let us introduce the following notations:

$$\beta_i = -b_{i-1}^2, \quad i = 2, ..., k,$$
$$\beta_i = -b_i^2, \quad i = k+1, ..., N.$$

Then, we have

$$\det D_k = (-1)^{k-1} \begin{vmatrix} 1 & \frac{1}{\mu_1^2+\beta_2} & \cdots & \frac{1}{\mu_1^2+\beta_k} & \frac{1}{\mu_1^2+\beta_{k+1}} & \cdots & \frac{1}{\mu_1^2+\beta_N} \\ 1 & \frac{1}{\mu_2^2+\beta_2} & \cdots & \frac{1}{\mu_2^2+\beta_k} & \frac{1}{\mu_2^2+\beta_{k+1}} & \cdots & \frac{1}{\mu_2^2+\beta_N} \\ ..... \\ 1 & \frac{1}{\mu_N^2+\beta_2^2} & \cdots & \frac{1}{\mu_N^2+\beta_k} & \frac{1}{\mu_N^2+\beta_{k+1}} & \cdots & \frac{1}{\mu_N^2+\beta_N} \end{vmatrix}_{N\times N}$$

$$\det D_k = (-1)^{k-1} \begin{vmatrix} 1 & \frac{1}{\mu_1^2+\beta_2} & \cdots & \frac{1}{\mu_1^2+\beta_k} & \cdots & \frac{1}{\mu_1^2+\beta_N} \\ 0 & \frac{\mu_1^2-\mu_2^2}{(\mu_2^2+\beta_2)(\mu_1^2+\beta_2)} & \cdots & \frac{\mu_1^2-\mu_2^2}{(\mu_2^2+\beta_k)(\mu_1^2+\beta_k)} & \cdots & \frac{\mu_1^2-\mu_2^2}{(\mu_2^2+\beta_N)(\mu_1^2+\beta_N)} \\ ..... \\ 0 & \frac{\mu_1^2-\mu_N^2}{(\mu_N^2+\beta_2)(\mu_1^2+\beta_2)} & \cdots & \frac{\mu_1^2-\mu_N^2}{(\mu_N^2+\beta_k)(\mu_1^2+\beta_k)} & \cdots & \frac{\mu_1^2-\mu_N^2}{(\mu_N^2+\beta_N)(\mu_1^2+\beta_N)} \end{vmatrix}_{N\times N}$$

$$= (-1)^{k-1} \frac{(\mu_1^2-\mu_2^2)...(\mu_1^2-\mu_N^2)}{(\mu_1^2+\beta_2)...(\mu_1^2+\beta_N)} \begin{vmatrix} \frac{1}{\mu_2^2+\beta_2} & \cdots & \frac{1}{\mu_2^2+\beta_k} & \frac{1}{\mu_2^2+\beta_{k+1}} & \cdots & \frac{1}{\mu_2^2+\beta_N} \\ ..... \\ \frac{1}{\mu_N^2+\beta_2^2} & \cdots & \frac{1}{\mu_N^2+\beta_k} & \frac{1}{\mu_N^2+\beta_{k+1}} & \cdots & \frac{1}{\mu_N^2+\beta_N} \end{vmatrix}_{(N-1)\times(N-1)}$$

So, as the above matrix under the sign of determinant is a Cauchy matrix of size $N-1$, the following equation holds:

$$\det D_k = (-1)^{k-1} \frac{(\mu_1^2 - \mu_2^2)...(\mu_1^2 - \mu_N^2)}{(\mu_1^2 + \beta_2)...(\mu_1^2 + \beta_N)} \frac{\prod\limits_{2 \le j < q \le N} (\mu_j^2 - \mu_q^2)(\beta_j - \beta_q)}{\prod\limits_{j,q=2,...,N} (\mu_j^2 + \beta_q)} .$$

Let us recall that $\beta_i = -b_{i-1}^2 \quad i = 2, k, \quad \beta_i = -b_i^2 \quad i = k+1, ..., N$, then

$$\begin{aligned}
\det D_k &= (-1)^{k-1} \frac{\prod\limits_{q=2,...,N} (\mu_1^2 - \mu_q^2)}{\prod\limits_{q=2,...,N} (\mu_1^2 + \beta_q)} \frac{\prod\limits_{2 \le j < q \le N} (\mu_j^2 - \mu_q^2)(\beta_j - \beta_q)}{\prod\limits_{j,q=2,...,N} (\mu_j^2 + \beta_q)} \\
&= (-1)^{k-1} \frac{\prod\limits_{1 \le j < q \le N} (\mu_j^2 - \mu_q^2) \prod\limits_{2 \le j < q \le N} (\beta_j - \beta_q)}{\prod\limits_{j=1,...,N, q=2,...,N} (\mu_j^2 + \beta_q)} \\
&= (-1)^{k-1} \frac{\prod\limits_{1 \le j < q \le N} (\mu_j^2 - \mu_q^2) \prod\limits_{1 \le j < q \le N,\, j,q \ne k} (b_q^2 - b_j^2)}{\prod\limits_{j,q=1,...,N, q \ne k} (\mu_j^2 - b_q^2)} \\
&= (-1)^{k-1} \frac{\prod\limits_{1 \le j < q \le N} (\mu_j^2 - \mu_q^2) \prod\limits_{1 \le j < q \le N} (b_q^2 - b_j^2)}{\prod\limits_{j,q=1,...,N, q \ne k} (\mu_j^2 - b_q^2)(-1)^{k-1} \prod\limits_{q=1,...,N,\, q \ne k} (b_q^2 - b_k^2)} \\
&= \frac{\prod\limits_{1 \le j < q \le N} (\mu_j^2 - \mu_q^2)(b_q^2 - b_j^2) \prod\limits_{j=1,...,N} (\mu_j^2 - b_k^2)}{\prod\limits_{j,q=1,...,N} (\mu_j^2 - b_q^2) \prod\limits_{q=1,...,N,\, q \ne k} (b_q^2 - b_k^2)}.
\end{aligned}$$

Then, from (18) and (19), we have the following expressions for x_k:

$$x_k = \frac{\prod\limits_{q=1,...,N} (\mu_q^2 - b_k^2)}{\prod\limits_{q=1,...,N,\, q \ne k} (b_q^2 - b_k^2)},$$

what proves Lemma 2. □

References

[1] S. Aalto and U. Ayesta, "Mean delay analysis of multilevel processor sharing disciplines", In Proceedings of IEEE INFOCOM 2006.

[2] K. Avrachenkov, U. Ayesta, P. Brown, "Batch Arrival Processor-Sharing with Application to Multi-Level Processor-Sharing Scheduling", Queueing Systems 50, pp.459-480, 2005.

[3] K. Avrachenkov, P. Brown, N. Osipova, "Optimal choice of Threshold in Two Level Processor Sharing", preprint. Also available as INRIA Research Report RR-6215, http://hal.inria.fr/inria-00153966/

[4] N. Bansal. "Analysis of the M/G/1 Processor-Sharing Queue with Bulk Arrivals", Operation Research Letters, 31(5), September 2003, pp.401-405.

[5] Kleinrock, L., R. R. Muntz, and E. Rodemich, "The Processor-Sharing Queueing Model for Time-Shared Systems with Bulk Arrivals," Networks Journal,1,1-13 (1971).

[6] L. Kleinrock, *Queueing systems, vol. 2*, John Wiley and Sons, 1976.

[7] G. Fayolle, I. Mitrani, and R. Iasnogorodski, "Sharing a processor among many job classes", Journal of the ACM, 27(3):519-532, 1980.

[8] H. Feng and V. Misra, "Asymptotic bounds for $M^X/G/1$ processor sharing queues", Technical report CUCS-00-04, Columbia University, jul.2003

[9] A. Feldmann, W. Whitt, "Fitting mixtures of exponentials to long-tail distributions to analyze network performance models", *Performance Evaluation*, v.31, 1998, pp.245-258.

[10] J. Kim and B. Kim, "Concavity of the conditional mean sojourn time in the M/G/1 processor-sharing queue with batch arrivals", preprint.

[11] A.G. Kurosh, "Higher algebra", MIR, 1972.

[12] K.M. Rege and B. Sengupta, "The M/G/1 Processor-Sharing Queue with Bulk Arrivals". In Proceedings of Modelling and Evaluation of ATM Networks 1993, pp.417-432.

Contents

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Unité de recherche INRIA Futurs : Parc Club Orsay Université - ZAC des Vignes
4, rue Jacques Monod - 91893 ORSAY Cedex (France)
Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier (France)
Unité de recherche INRIA Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399