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Olivier Faugeras, François Grimbert, Jean-Jacques Slotine

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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Stability and Synchronization in Neural Fields*

Olivier Faugeras †, François Grimbert ‡, Jean-Jacques Slotine §

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Abstract: Neural fields are an interesting option for modelling macroscopic parts of the cortex involving several populations of neurons, like cortical areas. Two classes of neural field equations are considered: voltage and activity based. The spatio-temporal behaviour of these fields is described by nonlinear integro-differential equations. The integral term, computed over a compact subset of $\mathbb{R}^q$, $q = 1, 2, 3$, involves space and time varying, possibly non-symmetric, intra-cortical connectivity kernels. Contributions from white matter afferents are represented as external input. Sigmoidal nonlinearities arise from the relation between average membrane potentials and instantaneous firing rates. Using methods of functional analysis, we characterize the existence and uniqueness of a solution of these equations for general, homogeneous (i.e. independent of the spatial variable), and locally homogeneous inputs. In all cases we give sufficient conditions on the connectivity functions for the solutions to be absolutely stable, that is to say independent of the initial state of the field. These conditions bear on some compact operators defined from the connectivity kernels, the sigmoids, and the time constants used in describing the temporal shape of the post-synaptic potentials. Numerical experiments are presented to illustrate the theory. An important contribution of our work is the application of the theory of compact operators in a Hilbert space to the problem of neural fields with the effect of providing very simple mathematical answers to the questions asked by neuroscience modellers.

Key-words: neural fields, integro-differential equations, compact operators, Hilbert space, stability, synchronization, neural masses, cortical columns.

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† INRIA/ENS, Odyssée Team, 2004 route des Lucioles, Sophia-Antipolis, France
‡ INRIA/ENS, Odyssée Team, 2004 route des Lucioles, Sophia-Antipolis, France
§ MIT, Nonlinear Systems Laboratory, 77 Massachusetts Avenue, Cambridge, MA, USA

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex (France)
Téléphone : +33 4 92 38 77 77 —Télécopie : +33 4 92 38 77 65

Stabilité et synchronisation dans des champs neuronaux

Résumé : Les champs neuronaux offrent la possibilité de modéliser des parties macroscopiques du cortex faisant intervenir plusieurs populations de neurones, comme les aires corticales. On considère deux classes d'équations de champs neuronaux, basées respectivement sur le potentiel et l'activité. Le comportement spatio-temporel de ces champs est décrit par des équations intégro-différentielles non-linéaires. Le terme intégral, calculé sur une partie compacte de $\mathbb{R}^q$, $q = 1, 2, 3$, fait intervenir des noyaux de connectivité intra-corticale dépendants du temps et de l'espace, et généralement non symétriques. Les contributions provenant de la matière blanche sont prises en compte par un terme d'entrée extérieure. Des non-linéarités sigmoïdales sont introduites par la relation entre les potentiels de membrane moyens et les taux de décharge instantanés. Grâce à des méthodes d'analyse fonctionnelle, on caractérise l'existence et l'unicité d'une solution à ces équations pour des entrées quelconques, homogènes (i.e., indépendantes de la variable d'espace) et localement homogènes. Dans tous ces cas, on donne des conditions suffisantes sur les fonctions de connectivité pour que les solutions soient absolument stables, c'est-à-dire indépendantes de l'état initial du champ. Ces conditions portent sur des opérateurs compacts définis à partir des noyaux de connectivité, des sigmoïdes et des constantes de temps caractéristiques des potentiels post-synaptiques. Des exemples numériques sont présentés pour illustrer la théorie. Une contribution importante de notre travail est l'application de la théorie des opérateurs compacts sur un espace de Hilbert aux problèmes de champs neuronaux, qui donne des réponses mathématiques simples aux questions posées par les modélisateurs en neurosciences.

Mots-clés : champs neuronaux, équations intégro-différentielles, opérateurs compacts, espaces de Hilbert, stabilité, synchronisation, masses neuronales, colonnes corticales.

1 Introduction

We model neural fields as continuous networks of cortical units, and focus on the ability of these units to synchronize. We therefore emphasize is the dynamics and spatio-temporal behaviour.
Cortical units are built from a local description of the dynamics of a number of interacting neuron populations, called *neural masses* [12], where the spatial structure of the connections is neglected. These "vertically" built units can be thought of as *cortical columns* [22, 23, 2]. Probably the most well-known neural mass based column model is that of Jansen and Rit [17] based on the original work of Lopes Da Silva, Van Rotterdam and colleagues [19, 20, 28]. A complete analysis of the bifurcations of this model can be found in [14]. More realistic models can be derived from experimental connectivity studies, such as the one shown in figure 1. This figure, adapted from [15], is based on the work of Alex Thomson and colleagues [27]. It shows the local connectivity graph of six populations of neurons and can be thought of as a model of a column comprising six interacting neural masses.

Such columns are then assembled spatially to form the neural field, which is meant to represent a macroscopic part of the neocortex, e.g. a visual area such as V1. Connections between columns are intra-cortical (gray matter) connections. Connections made via white matter with, e.g., such visual areas as the LGN or V2 are also considered in our models, but are treated as input/output quantities.

There are at least three reasons why we think this is the relevant granularity to do modelling

- Realistic modelling of a macroscopic part of the brain at the scale of the neuron is still difficult for obvious complexity reasons. Starting from mesoscopic building blocks like neural masses, described by the average activity of their neurons, is therefore a reasonable choice.
- While MEG and scalp EEG recordings mostly give a bulk signal of a cortical area, multi-electrode recordings, in vitro experiments on pharmacologically treated brain slices and new imaging techniques like extrinsic optical imaging can provide a spatially detailed description of neural masses dynamics in a macroscopic part of the brain like an area.
- The column/area scales correspond to available local connectivity data. Indeed, these are obtained by averaging on local populations of neurons we can think of as neural masses. Besides, local connectivity is supposed to be spatially invariant within an area.

We now present a general mathematical framework for neural field modelling that agrees with the ideas of using average descriptions of neuronal activity and spatial invariance of the local connectivity across the field.

In section 2 we describe the local and spatial models of neural masses and derive the equations that govern their spatio-temporal variations. In section 3 we analyze the problem

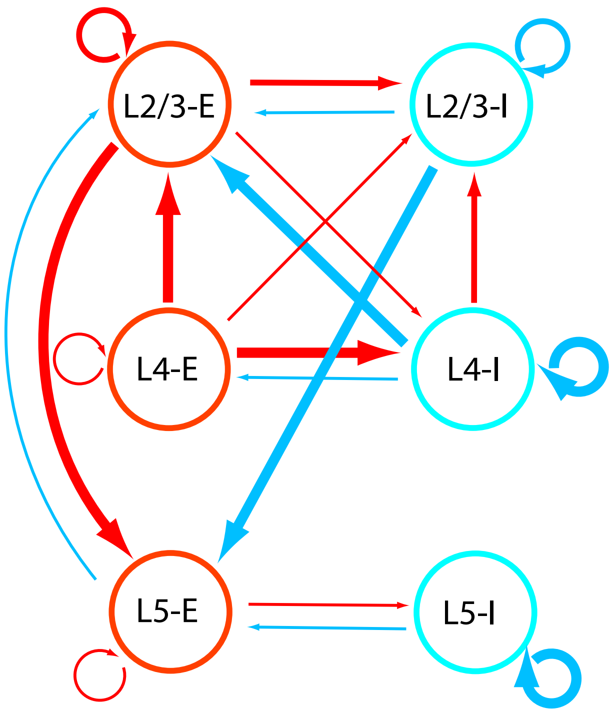

Figure 1: A simplified model of local cortical interactions based on six neuron populations. This local connectivity graph can be seen as a model of a cortical column composed of six interacting neural masses. There are three layers corresponding to cortical layers II/III, IV and V, and two types of neurons (excitatory ones in red and inhibitory ones in blue) in each of these layers. The size of the arrows gives an idea of the strength of the connectivity between populations. This figure is adapted from [15].

of the existence and uniqueness of the smooth general and homogeneous solutions of these equations. In section 4 we study the stability of these solutions with respect to the initial conditions. In section 5 we extend this analysis to the stability of the homogeneous solutions, when they exist. In section 6 we revisit the functional framework of our analysis and extend our results to non-smooth functions with the effect that we can discuss spatially local stability and synchronization of neural masses. In section 7 we present a number of numerical expereriments to illustrate the theory and conclude in section 8.

2 The models

We discuss local and spatial models.

2.1 The local models

We consider n interacting populations of neurons such as those shown in figure 1. The following derivation is built after Ermentrout's review [8]. We consider that each neural population i is described by its average membrane potential $V_i(t)$ or by its average instantaneous firing rate $\nu_i(t)$, the relation between the two quantities being of the form $\nu_i(t) = S_i(V_i(t))$ [13, 6], where S_i is sigmoidal. The functions S_i, $i = 1, \cdots, n$ satisfy the following properties introduced in the

Definition 2.1 *For all* $i = 1, \cdots, n$, S_i *and* S_i' *are positive and bounded. We note* $S_{im} = \sup_x S_i(x)$, $S_m = \max_i S_{im}$, $S'_{im} = \sup_x S'_i(x)$ *and* $DS_m = \max_i S'_{im}$. *Finally, we define* $D\mathbf{S}_m$ *as the diagonal matrix* $diag(S'_{im})$.

Neurons in population j are connected to neurons in population i. A single action potential from neurons in population j is seen as a post-synaptic potential $PSP_{ij}(t-s)$ by neurons in population i, where s is the time of the spike hitting the terminal and t the time after the spike. We neglect the delays due to the distance travelled down the axon by the spikes.

Assuming that the post-synaptic potentials sum linearly, the average membrane potential of population i is

$$V_i(t) = \sum_{j,k} PSP_{ij}(t - t_k)$$

where the sum is taken over the arrival times of the spikes produced by the neurons in population j. The number of spikes arriving between t and $t + dt$ is $\nu_j(t)dt$. Therefore we have

$$V_i(t) = \sum_j \int_{t_0}^{t} PSP_{ij}(t-s)\nu_j(s)\, ds = \sum_j \int_{t_0}^{t} PSP_{ij}(t-s)S_j(V_j(s))\, ds,$$

or, equivalently

$$\nu_i(t) = S_i\left(\sum_j \int_{t_0}^{t} PSP_{ij}(t-s)\nu_j(s)\, ds\right) \tag{1}$$

The PSP_{ij}s can depend on several variables in order to account for adaptation, learning, etc ...
There are two main simplifying assumptions that appear in the literature [8] and yield two different models.

2.1.1 The voltage-based model

The assumption, made in [16], is that the post-synaptic potential has the same shape no matter which presynaptic population caused it, the sign and amplitude may vary though. This leads to the relation

$$PSP_{ij}(t) = w_{ij} PSP_i(t).$$

If $w_{ij} > 0$ the population j excites population i whereas it inhibits it when $w_{ij} < 0$.

Finally, if we assume that $PSP_i(t) = k_i e^{-t/\tau_i} Y(t)$ (where Y is the Heaviside distribution), or equivalently that

$$\tau_i \frac{dPSP_i(t)}{dt} + PSP_i(t) = k_i \tau_i \delta(t), \tag{2}$$

we end up with the following system of ordinary differential equations

$$\frac{dV_i(t)}{dt} + \frac{V_i(t)}{\tau_i} = k_i \sum_j w_{ij} S_j(V_j(t)) + I^i_{\text{ext}}(t), \tag{3}$$

that describes the dynamic behaviour of a cortical column. We have added an external current $I_{\text{ext}}(t)$ to model the non-local connections of population i.

We introduce the $n \times n$ matrices $\mathbf{W}$ such that $W_{ij} = k_i w_{ij}$, and the function $\mathbf{S}$, $\mathbb{R}^n \to \mathbb{R}^n$ such that $\mathbf{S}(\mathbf{x})$ is the vector of coordinates $S_i(x_i)$. We rewrite (3) in vector form and obtain the following system of n ordinary differential equations

$$\dot{\mathbf{V}} = -\mathbf{L}\mathbf{V} + \mathbf{W}\mathbf{S}(\mathbf{V}) + \mathbf{I}_{\text{ext}}, \tag{4}$$

where $\mathbf{L}$ is the diagonal matrix $\mathbf{L} = \text{diag}(1/\tau_{\text{i}})$.

2.1.2 The activity-based model

The assumption is that the shape of a PSP depends only on the nature of the presynaptic cell, that is

$$PSP_{ij}(t) = w_{ij} PSP_j(t).$$

As above we suppose that $PSP_i(t)$ satisfies the differential equation (2) and define the activity to be

$$A_j(t) = \int_{t_0}^{t} PSP_j(t-s) \nu_j(s)\, ds.$$

A similar derivation yields the following set of n ordinary differential equations

$$\frac{dA_i(t)}{dt} + \frac{A_i(t)}{\tau_i} = k_i S_i \left(\sum_j w_{ij} A_j(t) + I^i_{\rm ext}(t) \right), \quad i = 1, \cdots, n.$$

We include the k_is in the sigmoids S_i, set $W_{ij} = w_{ij}$, and rewrite this in vector form

$$\dot{\mathbf{A}} = -\mathbf{L}\mathbf{A} + \mathbf{S}(\mathbf{W}\mathbf{A} + \mathbf{I}_{\rm ext}), \tag{5}$$

2.2 Neural fields models

We now combine these local models to form a continuum of columns, e.g., in the case of a model of a significant part Ω of the cortex. From now on we consider a compact subset Ω of $\mathbb{R}^q$, $q = 1, 2, 3$. This encompasses several cases of interest.

When $q = 1$ we deal with one-dimensional neural fields. Even though this appears to be of limited biological interest, it is one of the most widely studied cases because of its relative mathematical simplicity and because of the insights one can gain of the more realistic situations.

When $q = 2$ we discuss properties of two-dimensional neural fields. This is perhaps more interesting from a biological point of view since Ω can be viewed as a piece of cortex where the third dimension, its thickness, is neglected. This case has received by far less attention than the previous one, probably because of the increased mathematical difficulty.

Finally $q = 3$ allows us to discuss properties of volumes of neural masses, e.g. cortical sheets where their thickness is taken into account [18, 3].

The results that are presented in this paper are independent of q. Nevertheless, we have a good first approximation of a real cortical area with $q = 2$, and cortical depth given by the index $i = 1, \cdots, n$ of the considered cortical population, following the idea of a field composed of columns, or equivalently, of interconnected cortical layers.

We note $\mathbf{V}(\mathbf{r}, t)$ (respectively $\mathbf{A}(\mathbf{r}, t)$) the n-dimensional state vector at the point $\mathbf{r}$ of the continuum and at time t. We introduce the $n \times n$ matrix function $\mathbf{W}(\mathbf{r}, \mathbf{r}', t)$ which describes how the neural mass at point $\mathbf{r}'$ influences that at point $\mathbf{r}$ at time t. More precisely, $W_{ij}(\mathbf{r}, \mathbf{r}', t)$ describes how population j at point $\mathbf{r}'$ influences population i at point $\mathbf{r}$ at time t. We call $\mathbf{W}$ the connectivity matrix function. Equation (4) can now be extended to

$$\mathbf{V}_t(\mathbf{r}, t) = -\mathbf{L}\mathbf{V}(\mathbf{r}, t) + \int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\mathbf{S}(\mathbf{V}(\mathbf{r}', t))\, d\mathbf{r}' + \mathbf{I}_{\rm ext}(\mathbf{r}, t), \tag{6}$$

and equation (5) to

$$\mathbf{A}_t(\mathbf{r}, t) = -\mathbf{L}\mathbf{A}(\mathbf{r}, t) + \mathbf{S}\left(\int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\mathbf{A}(\mathbf{r}', t))\, d\mathbf{r}' + \mathbf{I}_{\rm ext}(\mathbf{r}, t) \right). \tag{7}$$

A special case which will be considered later is when $\mathbf{W}$ is translation invariant, $\mathbf{W}(\mathbf{r}, \mathbf{r}', t) = \mathbf{W}(\mathbf{r} - \mathbf{r}', t)$. We give below sufficient conditions on $\mathbf{W}$ and $\mathbf{I}_{\text{ext}}$ for equations (6) and (7) to be well-defined and study their solutions.

3 Existence and uniqueness of a solution

In this section we deal with the problem of the existence and uniqueness of a solution to (6) and (7) for a given set of initial conditions. Unlike previous authors [10, 4, 21] we consider the case of a neural field with the effect that we have to use the tools of functional analysis to characterize their properties.

We start with the assumption that the state vectors $\mathbf{V}$ and $\mathbf{A}$ are differentiable (respectively continuous) functions of the time (respectively the space) variable. This is certainly reasonable in terms of the temporal variations because we are essentially modeling large populations of neurons and do not expect to be able to represent time transients. It is far less reasonable in terms of the spatial dependency since one should allow neural masses activity to be spatially distributed in a locally non-smooth fashion with areas of homogeneous cortical activity separated by smooth boundaries. A more general assumption is proposed in section 6. But it turns out that most of the groundwork can be done in the setting of continuous functions.

Let $\mathcal{F}$ be the set $\mathbf{C}_n(\Omega)$ of the continuous functions from Ω to $\mathbb{R}^n$. This is a Banach space for the norm $\|\mathbf{V}\|_{n,\infty} = \max_{1 \leq i \leq n} \sup_{\mathbf{r} \in \Omega} |\mathbf{V}_i(\mathbf{r})|$, see appendix A. We denote by J a closed interval of the real line containing 0.

We will several times need the following

Lemma 3.1 *We have the following inequalities for all* $\mathbf{x}$, $\mathbf{y} \in \mathcal{F}$ *and* $\mathbf{r}' \in \Omega$

$$\|\mathbf{S}(\mathbf{x}(\mathbf{r}')) - \mathbf{S}(\mathbf{y}(\mathbf{r}'))\|_\infty \leq DS_m \|\mathbf{x}(\mathbf{r}') - \mathbf{y}(\mathbf{r}')\|_\infty \quad \text{and} \quad \|\mathbf{S}(\mathbf{x}) - \mathbf{S}(\mathbf{y})\|_{n,\infty} \leq DS_m \|\mathbf{x} - \mathbf{y}\|_{n,\infty}.$$

Proof. $\mathbf{S}$ is smooth so we can perform a zeroth-order Taylor expansion with integral remainder, write

$$\mathbf{S}(\mathbf{x}(\mathbf{r}')) - \mathbf{S}(\mathbf{y}(\mathbf{r}')) = \left(\int_0^1 D\mathbf{S}(\mathbf{y}(\mathbf{r}') + \zeta(\mathbf{x}(\mathbf{r}') - \mathbf{y}(\mathbf{r}')))\, d\zeta \right) (\mathbf{x}(\mathbf{r}') - \mathbf{y}(\mathbf{r}')),$$

and, because of lemma A.1 and definition 2.1

$$\|\mathbf{S}(\mathbf{x}(\mathbf{r}')) - \mathbf{S}(\mathbf{y}(\mathbf{r}'))\|_\infty \leq \int_0^1 \|D\mathbf{S}(\mathbf{y}(\mathbf{r}') + \zeta(\mathbf{x}(\mathbf{r}') - \mathbf{y}(\mathbf{r}')))\|_\infty \, d\zeta \, \|\mathbf{x}(\mathbf{r}') - \mathbf{y}(\mathbf{r}')\|_\infty \leq DS_m \|\mathbf{x}(\mathbf{r}') - \mathbf{y}(\mathbf{r}')\|_\infty.$$

This proves the first inequality. The second follows immediately. □

3.1 General solution

A function $\mathbf{V}(t)$ is thought of as a mapping $\mathbf{V} : \mathrm{J} \rightarrow \mathcal{F}$ and equations (6) and (7) are formally recast as an initial value problem:

$$\begin{cases} \mathbf{V}'(t) &= f(t, \mathbf{V}(t)) \\ \mathbf{V}(0) &= \mathbf{V}_0 \end{cases} \tag{8}$$

where $\mathbf{V}_0$ is an element of $\mathcal{F}$ and the function f from $\mathrm{J} \times \mathcal{F}$ is equal to f_v defined by the righthand side of (6):

$$f_v(t, \mathbf{x})(\mathbf{r}) = -\mathbf{L}\mathbf{x}(\mathbf{r}) + \int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\mathbf{S}(\mathbf{x}(\mathbf{r}'))\, d\mathbf{r}' + \mathbf{I}_{\mathrm{ext}}(\mathbf{r}, t) \quad \forall \mathbf{x} \in \mathcal{F}, \tag{9}$$

or to f_a defined by the righthand side of (7):

$$f_a(t, \mathbf{x})(\mathbf{r}) = -\mathbf{L}\mathbf{x}(\mathbf{r}) + \mathbf{S}\left(\int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\mathbf{x}(\mathbf{r}')\, d\mathbf{r}' + \mathbf{I}_{\mathrm{ext}}(\mathbf{r}, t)\right) \quad \forall \mathbf{x} \in \mathcal{F}. \tag{10}$$

We have the

Proposition 3.2 *If the following two hypotheses are satisfied*

1. *The connectivity function* $\mathbf{W}$ *is in* $C(\mathrm{J}; \mathbf{C}_{\mathrm{n}\times\mathrm{n}}(\Omega \times \Omega))$,
2. *The external current* $\mathbf{I}_{\mathrm{ext}}$ *is in* $C(\mathrm{J}; \mathbf{C}_{\mathrm{n}}(\Omega))$,

then the mappings f_v *and* f_a *are from* $\mathrm{J} \times \mathcal{F}$ *to* $\mathcal{F}$*, continuous, and Lipschitz continuous with respect to their second argument, uniformly with respect to the first (*$\mathbf{C}_{n\times n}(\Omega \times \Omega)$ *and* $\mathbf{C}_n(\Omega)$ *are defined in appendix A).*

Proof. Let $t \in \mathrm{J}$ and $\mathbf{x} \in \mathcal{F}$. We introduce the mapping

$$g_v : (t, \mathbf{x}) \rightarrow g_v(t, \mathbf{x}) \quad \text{such that} \quad g_v(t, \mathbf{x})(\mathbf{r}) = \int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\mathbf{S}(\mathbf{x}(\mathbf{r}'))\, d\mathbf{r}'$$

$g_v(t, \mathbf{x})$ is well defined for all $\mathbf{r} \in \Omega$ because, thanks to the first hypothesis, it is the integral of the continuous function $\mathbf{W}(\mathbf{r}, ., t)\mathbf{S}(\mathbf{x}(.))$ on a compact domain. For all $\mathbf{r}' \in \Omega$, $\mathbf{W}(., \mathbf{r}', t)\mathbf{S}(\mathbf{x}(\mathbf{r}'))$ is continuous (first hypothesis again) and we have (lemma A.1)

$$\|\mathbf{W}(\mathbf{r}, \mathbf{r}', t)\mathbf{S}(\mathbf{x}(\mathbf{r}'))\|_\infty \leq \|\mathbf{W}(., ., t)\|_{n\times n,\infty}\|\mathbf{S}(\mathbf{x}(\mathbf{r}'))\|_\infty.$$

Since $\|\mathbf{S}(\mathbf{x}(.))\|_\infty$ is bounded, it is integrable in Ω and we conclude that $g_v(t, \mathbf{x})$ is continuous on Ω. Then it is easy to see that $f_v(t, \mathbf{x})$ is well defined and belongs to $\mathcal{F}$.

Let us prove that f_v is continuous.

$$\begin{aligned} f_v(t,\mathbf{x}) - f_v(s,\mathbf{y}) &= -\mathbf{L}(\mathbf{x}-\mathbf{y}) + \int_\Omega (\mathbf{W}(\cdot,\mathbf{r}',t)\mathbf{S}(\mathbf{x}(\mathbf{r}')) - \mathbf{W}(\cdot,\mathbf{r}',s)\mathbf{S}(\mathbf{y}(\mathbf{r}')))\,d\mathbf{r}' \\ &\qquad + \mathbf{I}_{\text{ext}}(\cdot,t) - \mathbf{I}_{\text{ext}}(\cdot,s) \\ &= -\mathbf{L}(\mathbf{x}-\mathbf{y}) + \int_\Omega (\mathbf{W}(\cdot,\mathbf{r}',t) - \mathbf{W}(\cdot,\mathbf{r}',s))\mathbf{S}(\mathbf{x}(\mathbf{r}'))\,d\mathbf{r}' \\ &\qquad + \int_\Omega \mathbf{W}(\cdot,\mathbf{r}',s)(\mathbf{S}(\mathbf{x}(\mathbf{r}')) - \mathbf{S}(\mathbf{y}(\mathbf{r}'))\,d\mathbf{r}' + \mathbf{I}_{\text{ext}}(\cdot,t) - \mathbf{I}_{\text{ext}}(\cdot,s) \end{aligned}$$

It follows from lemma 3.1 that

$$\begin{aligned} \|f_v(t,\mathbf{x}) - f_v(s,\mathbf{y})\|_{n,\infty} \le \|\mathbf{L}\|_\infty \|\mathbf{x}-\mathbf{y}\|_{n,\infty} + |\Omega|\, S_m \|\mathbf{W}(\cdot,\cdot,t) - \mathbf{W}(\cdot,\cdot,s)\|_{n\times n,\infty} + \\ |\Omega|\, \|\mathbf{W}(\cdot,\cdot,s)\|_{n\times n,\infty} DS_m \|\mathbf{x}-\mathbf{y}\|_{n,\infty} + \|\mathbf{I}_{\text{ext}}(\cdot,t) - \mathbf{I}_{\text{ext}}(\cdot,s)\|_{n,\infty}. \end{aligned}$$

Because of the hypotheses we can choose $|t-s|$ small enough so that $\|\mathbf{W}(\cdot,\cdot,t)-\mathbf{W}(\cdot,\cdot,s)\|_{n\times n,\infty}$ and $\|\mathbf{I}_{\text{ext}}(\cdot,t) - \mathbf{I}_{\text{ext}}(\cdot,s)\|_{n,\infty}$ are arbitrarily small. Similarly, since $\mathbf{W}$ is continuous on the compact interval J, it is bounded there and $\|\mathbf{W}(\cdot,\cdot,s)\|_{n\times n,\infty} \le w > 0$ for all $s \in$ J. This proves the continuity of f_v.

It follows from the previous inequality that

$$\|f_v(t,\mathbf{x}) - f_v(t,\mathbf{y})\|_{n,\infty} \le \|\mathbf{L}\|_\infty \|\mathbf{x}-\mathbf{y}\|_{n,\infty} + |\Omega|\, \|\mathbf{W}(\cdot,\cdot,t)\|_{n\times n,\infty} DS_m \|\mathbf{x}-\mathbf{y}\|_{n,\infty},$$

and because $\|\mathbf{W}(\cdot,\cdot,t)\|_{n\times n,\infty} \le w > 0$ for all ts in J, this proves the Lipschitz continuity of f_v with respect to its second argument, uniformly with respect to the first.
A very similar proof applies to f_a. □
We continue with the proof that there exists a unique solution to the abstract initial value problem (8) in the two cases of interest.

Proposition 3.3 *Subject to the hypotheses of proposition 3.2 for any element* $\mathbf{V}_0$ *(resp.* $\mathbf{A}_0$*) of* $\mathcal{F}$ *there is a unique solution* $\mathbf{V}$ *(resp.* $\mathbf{A}$*), defined on a subinterval of* J *containing 0 and continuously differentiable, of the abstract initial value problem (8) for* $f = f_v$ *(resp.* $f = f_a$*).*

Proof. All conditions of the Picard-Lindelöf theorem on differential equations in Banach spaces [7, 1] are satisfied, hence the proposition. □
This solution, defined on the subinterval J of $\mathbb{R}$ can in fact be extended to the whole real line and we have the

Proposition 3.4 *If the following two hypotheses are satisfied*

1. *The connectivity function* $\mathbf{W}$ *is in* $C(\mathbb{R}; \mathbf{C}_{\text{n}\times\text{n}}(\Omega\times\Omega))$,
2. *The external current* $\mathbf{I}_{\text{ext}}$ *is in* $C(\mathbb{R}; \mathbf{C}_{\text{n}}(\Omega))$,

then for any function $\mathbf{V}_0$ *(resp.* $\mathbf{A}_0$*) in* $\mathcal{F}$ *there is a unique solution* $\mathbf{V}$ *(resp.* $\mathbf{A}$*), defined on* $\mathbb{R}$ *and continuously differentiable, of the abstract initial value problem (8) for* $f = f_v$ *(resp.* $f = f_a$*).*

Proof. In theorem B.1 of appendix B, we prove the existence of a constant $\tau > 0$ such that for any initial condition $(t_0, \mathbf{V}_0) \in \mathbb{R} \times \mathcal{F}$, there is a unique solution defined on the closed interval $[t_0 - \tau, t_0 + \tau]$. We can then cover the real line with such intervals and finally obtain the global existence and uniqueness of the solution of the initial value problem. □

3.2 Homogeneous solution

A homogeneous solution to (6) or (7) is a solution $\mathbf{U}$ that does not depend upon the space variable $\mathbf{r}$, for a given homogeneous input $\mathbf{I}_{\text{ext}}(t)$ and a constant initial condition $\mathbf{U}_0$. If such a solution $\mathbf{U}(t)$ exists, then it satisfies the following equation

$$\mathbf{U}'(t) = -\mathbf{L}\mathbf{U}(t) + \int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\mathbf{S}(\mathbf{U}(t))\, d\mathbf{r}' + \mathbf{I}_{\text{ext}}(t),$$

in the case of (6) and

$$\mathbf{U}'(t) = -\mathbf{L}\mathbf{U}(t) + \mathbf{S}\left(\int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\mathbf{U}(t)\, d\mathbf{r}' + \mathbf{I}_{\text{ext}}(t)\right),$$

in the case of (7). The integral $\int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\mathbf{S}(\mathbf{U}(t))\, d\mathbf{r}'$ is equal to $\left(\int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\, d\mathbf{r}'\right) \mathbf{S}(\mathbf{U}(t))$. The integral $\int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\mathbf{U}(t)\, d\mathbf{r}'$ is equal to $\left(\int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\, d\mathbf{r}'\right) \mathbf{U}(t)$. They must be independent of the position $\mathbf{r}$. Hence a necessary condition for the existence of a homogeneous solution is that

$$\int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\, d\mathbf{r}' = \overline{\mathbf{W}}(t), \tag{11}$$

where the $n \times n$ matrix $\overline{\mathbf{W}}(t)$ does not depend on the spatial coordinate $\mathbf{r}$.

In the special case where $\mathbf{W}(\mathbf{r}, \mathbf{r}', t)$ is translation invariant, $\mathbf{W}(\mathbf{r}, \mathbf{r}', t) \equiv \mathbf{W}(\mathbf{r}-\mathbf{r}', t)$, the condition is not satisfied in general because of the border of Ω. In all cases, the homogeneous solutions satisfy the differential equation

$$\mathbf{U}'(t) = -\mathbf{L}\mathbf{U}(t) + \overline{\mathbf{W}}(t)\mathbf{S}(\mathbf{U}(t)) + \mathbf{I}_{\text{ext}}(t), \tag{12}$$

for (6) and

$$\mathbf{U}'(t) = -\mathbf{L}\mathbf{U}(t) + \mathbf{S}\left(\overline{\mathbf{W}}(t)\mathbf{U}(t)) + \mathbf{I}_{\text{ext}}(t)\right), \tag{13}$$

for (7), with initial condition $\mathbf{U}(0) = \mathbf{U}_0$, a vector of $\mathbb{R}^n$. The following proposition gives a sufficient condition for the existence of a homogeneous solution.

Theorem 3.5 *If the external current* $\mathbf{I}_{\text{ext}}(t)$ *and the connectivity matrix* $\overline{\mathbf{W}}(t)$ *are continuous on some closed interval* J *containing 0, then for all vector* $\mathbf{U}_0$ *of* $\mathbb{R}^n$*, there exists a unique solution* $\mathbf{U}(t)$ *of (12) or (13) defined on a subinterval* J_0 *of* J *containing 0 such that* $\mathbf{U}(0) = \mathbf{U}_0$*.*

Proof. The proof is an application of Cauchy's theorem on differential equations. Consider the mapping $f_{hv} : \mathbb{R}^n \times \mathrm{J} \to \mathbb{R}^n$ defined by

$$f_{hv}(\mathbf{x}, t) = -\mathbf{L}\mathbf{x} + \overline{\mathbf{W}}(t)\mathbf{S}(\mathbf{x}) + \mathbf{I}_{\text{ext}}(t)$$

We have

$$\|f_{hv}(\mathbf{x}, t) - f_{hv}(\mathbf{y}, t)\|_\infty \leq \|\mathbf{L}\|_\infty \|\mathbf{x} - \mathbf{y}\|_\infty + \|\overline{\mathbf{W}}(t)\|_\infty \|\mathbf{S}(\mathbf{x}) - \mathbf{S}(\mathbf{y})\|_\infty$$

It follows from lemma 3.1 that $\|\mathbf{S}(\mathbf{x}) - \mathbf{S}(\mathbf{y})\|_\infty \leq DS_m \|\mathbf{x} - \mathbf{y}\|_\infty$ and, since $\overline{\mathbf{W}}$ is continuous on the compact interval J, it is bounded there by $w > 0$ and

$$\|f_{hv}(\mathbf{x}, t) - f_{hv}(\mathbf{y}, t)\|_\infty \leq (\|\mathbf{L}\|_\infty + wDS_m)\|\mathbf{x} - \mathbf{y}\|_\infty$$

for all $\mathbf{x}$, $\mathbf{y}$ of $\mathbb{R}^n$ and all $t \in \mathrm{J}$. A similar proof applies to (13) and the conclusion of the proposition follows. ☐
As in proposition 3.4, this existence and uniqueness result extends to the whole time real line if $\mathbf{I}$ and $\overline{\mathbf{W}}$ are continuous on $\mathbb{R}$.

This homogeneous solution can be seen as describing a state where the columns of the continuum are synchronized: they receive the same input $\mathbf{I}_{\text{ext}}(t)$ and produce the same output $\mathbf{U}(t)$.

3.3 Some remarks about the case $\Omega = \mathbb{R}^q$

A significant amount of work has been done on equations of the type (6) or (7) in the case of a one-dimensional infinite continuum, $\Omega = \mathbb{R}$, or a two-dimensional infinite continuum, $\Omega = \mathbb{R}^2$. The reader is referred to the review papers by Ermentrout [8] and by Coombes [5] as well as to [26, 11, 25].

Beside the fact that an infinite cortex is unrealistic, the case $\Omega = \mathbb{R}^q$ raises some mathematical questions. Indeed, the choice of the functional space $\mathcal{F}$ is problematic. A natural idea would be to choose $\mathcal{F} = \mathbf{L}^2$, the space of square-integrable functions. If we make this choice we immediately encounter the problem that the homogeneous solutions (constant with respect to the space variable) do not belong to that space. A further difficulty is that $\mathbf{S}(\mathbf{x})$ does not in general belong to $\mathcal{F}$ if $\mathbf{x}$ does. As shown in this article, these difficulties vanish if Ω is compact.

4 Stability of the general solution

We investigate the stability of a solution to (6) and (7) for a given input $\mathbf{I}_{\text{ext}}$. We give sufficient conditions for the solution to be independent of the initial conditions. We first consider the general case before looking at an approximation of the convolution case.

4.1 The general case

We define a number of matrices and linear operators that are useful in the sequel

Definition 4.1 *Let*

$$\mathbf{W}_{cm} = \mathbf{W}D\mathbf{S}_m \quad \mathbf{W}_{mc} = D\mathbf{S}_m\mathbf{W}$$

and

$$\mathbf{W}^L_{cm} = \mathbf{L}^{-1/2}\mathbf{W}_{cm}\mathbf{L}^{-1/2} \quad \mathbf{W}^L_{mc} = \mathbf{L}^{-1/2}\mathbf{W}_{mc}\mathbf{L}^{-1/2}$$

Consider also the linear operators, noted g_m, and h_m defined on $\mathcal{F}$:

$$g_m(\mathbf{x})(\mathbf{r}, t) = \int_\Omega \mathbf{W}_{cm}(\mathbf{r}, \mathbf{r}', t)\mathbf{x}(\mathbf{r}')\, d\mathbf{r}' \quad \forall \mathbf{x} \in \mathcal{F},$$

and

$$h_m(\mathbf{x})(\mathbf{r}, t) = \int_\Omega \mathbf{W}_{mc}(\mathbf{r}, \mathbf{r}', t)\mathbf{x}(\mathbf{r}')\, d\mathbf{r}' \quad \forall \mathbf{x} \in \mathcal{F},$$

as well as g^L_m and h^L_m that are constructed from $\mathbf{W}^L_{cm}$ and $\mathbf{W}^L_{mc}$, respectively.

We start with a lemma.

Lemma 4.2 *With the hypotheses of proposition 3.2, the operators g_m, g^L_m, h_m, and h^L_m are compact operators from $\mathcal{F}$ to $\mathcal{F}$ for each time $t \in$ J.*

Proof. This is a direct application of the theory of Fredholm's integral equations [7]. We prove it for g^L_m.

Because of the hypothesis 1 in proposition 3.2, at each time instant t in J, $\mathbf{W}_{cm}$ is continuous on the compact set $\Omega \times \Omega$, therefore it is uniformly continuous. Hence, for each $\varepsilon > 0$ there exists $\eta(t) > 0$ such that $\|\mathbf{r}_1 - \mathbf{r}_2\| \le \eta(t)$ implies that $\|\mathbf{W}_{cm}(\mathbf{r}_1, \mathbf{r}', t) - \mathbf{W}_{cm}(\mathbf{r}_2, \mathbf{r}', t)\|_\infty \le \varepsilon$ for all $\mathbf{r}' \in \Omega$, and, for all $\mathbf{x} \in \mathcal{F}$

$$\|g_m(\mathbf{x})(\mathbf{r}_1, t) - g_m(\mathbf{x})(\mathbf{r}_2, t)\|_\infty \le \varepsilon|\Omega|\|\mathbf{x}\|_{n,\infty}$$

This shows that the image $g_m(B)$ of any bounded subset B of $\mathcal{F}$ is equicontinuous.

Similarly, if we set $w(t) = \|\mathbf{W}_{cm}(., ., t)\|_{n\times n,\infty}$, we have $\|g_m(\mathbf{x})(\mathbf{r}, t)\|_\infty \le w(t)|\Omega|\|\mathbf{x}\|_{n,\infty}$. This shows that for every $\mathbf{r} \in \Omega$, the set $\{\mathbf{y}(\mathbf{r}), \mathbf{y} \in g_m(B)\}$, is bounded in $\mathbb{R}^n$, hence relatively compact. From the Arzelà-Ascoli theorem, we conclude that the subset $g_m(B)$ of $\mathcal{F}$ is relatively compact for all $t \in$ J. And so the operator is compact.

The same proof applies to g^L_m, h_m, and h^L_m. □

To study the stability of solutions of (6) and (7) it is convenient to use an inner product on $\mathcal{F}$. It turns out that the natural inner-product will pave the ground for the generalization in section 6. We therefore consider the pre-Hilbert space $\mathcal{G}$ defined on $\mathcal{F}$ by the usual inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_\Omega \mathbf{x}(\mathbf{r})^T \mathbf{y}(\mathbf{r})\, d\mathbf{r}$$

We note $\|\mathbf{x}\|_{n,2}$ the corresponding norm to distinguish it from $\|\mathbf{x}\|_{n,\infty}$, see appendix A. It is easy to show that all previously defined operators are also compact operators from $\mathcal{G}$ to $\mathcal{G}$. We have the

Lemma 4.3 *g_m, g_m^L, h_m, and h_m^L are compact operators from $\mathcal{G}$ to $\mathcal{G}$ for each time $t \in$ J.*

Proof. We give the proof for g_m.

The identity mapping $\mathbf{x} \to \mathbf{x}$ from $\mathcal{F}$ to $\mathcal{G}$ is continuous since $\|\mathbf{x}\|_{n,2} \leq \sqrt{n|\Omega|}\,\|\mathbf{x}\|_{n,\infty}$. Consider now g_m as a mapping from $\mathcal{G}$ to $\mathcal{F}$. As in the proof of lemma 4.2, for each $\varepsilon > 0$ there exists $\eta(t) > 0$ such that $\|\mathbf{r}_1 - \mathbf{r}_2\| \leq \eta(t)$ implies $\|\mathbf{W}_{cm}(\mathbf{r}_1, \mathbf{r}', t) - \mathbf{W}_{cm}(\mathbf{r}_2, \mathbf{r}', t)\|_\infty \leq \varepsilon$ for all $\mathbf{r}' \in \Omega$. Therefore the ith coordinate $g_m^i(\mathbf{x})(\mathbf{r}_1, t) - g_m^i(\mathbf{x})(\mathbf{r}_2, t)$ satisfies (Cauchy-Schwarz' inequalities):

$$|g_m^i(\mathbf{x})(\mathbf{r}_1, t) - g_m^i(\mathbf{x})(\mathbf{r}_2, t)| \leq \sum_j \int_\Omega |W_{cm,\,ij}(\mathbf{r}_1, \mathbf{r}', t) - W_{cm,\,ij}(\mathbf{r}_2, \mathbf{r}', t)|\, |x_j(\mathbf{r}')|\, d\mathbf{r}' \leq$$

$$\varepsilon \sum_j \int_\Omega |x_j(\mathbf{r}')|\, d\mathbf{r}' \leq \varepsilon \sqrt{|\Omega|} \sum_j \left(\int_\Omega |x_j(\mathbf{r}')|^2\, d\mathbf{r}' \right)^{1/2} \leq \varepsilon \sqrt{n\,|\Omega|} \|\mathbf{x}\|_{n,2},$$

and the image $g_m(B)$ of any bounded set B of $\mathcal{G}$ is equicontinuous. Similarly, if we set $w(t) = \|\mathbf{W}_{cm}(.,.,t)\|_{n\times n,\infty}$ in $\Omega \times \Omega$, we have $|g_m^i(\mathbf{x})(\mathbf{r}, t)| \leq w(t)\sqrt{n\,|\Omega|}\,\|\mathbf{x}\|_{n,2}$. The same reasoning as in lemma 4.2 shows that the operator $\mathbf{x} \to g_m(\mathbf{x})$ from $\mathcal{G}$ to $\mathcal{F}$ is compact and since the identity from $\mathcal{F}$ to $\mathcal{G}$ is continuous, $\mathbf{x} \to g_m(\mathbf{x})$ is compact from $\mathcal{G}$ to $\mathcal{G}$.

The same proof applies to g_m^L, h_m, and h_m^L. □

We then proceed with the following

Lemma 4.4 *The adjoint of the operator g_m of $\mathcal{G}$ is the operator g_m^* defined by*

$$g_m^*(\mathbf{x})(\mathbf{r}, t) = \int_\Omega \mathbf{W}_{cm}^T(\mathbf{r}', \mathbf{r}, t)\mathbf{x}(\mathbf{r}')\, d\mathbf{r}'$$

It is a compact operator.

Proof. The adjoint, if it exists, is defined by the condition $\langle g_m(\mathbf{x}), \mathbf{y}\rangle = \langle \mathbf{x}, g_m^*(\mathbf{y})\rangle$ for all $\mathbf{x}$, $\mathbf{y}$ in $\mathcal{G}$. We have

$$\langle g_m(\mathbf{x}), \mathbf{y}\rangle = \int_\Omega \mathbf{y}(\mathbf{r})^T \left(\int_\Omega \mathbf{W}_{cm}(\mathbf{r}, \mathbf{r}', t)\mathbf{x}(\mathbf{r}')\, d\mathbf{r}' \right) d\mathbf{r} =$$

$$\int_\Omega \mathbf{x}(\mathbf{r}')^T \left(\int_\Omega \mathbf{W}_{cm}^T(\mathbf{r}, \mathbf{r}', t)\mathbf{y}(\mathbf{r})\, d\mathbf{r} \right) d\mathbf{r}',$$

from which the conclusion follows. Since $\mathcal{G}$ is not a Hilbert space the adjoint of a compact operator is not necessarily compact. But the proof of compactness of g_m in lemma 4.3 extends easily to g_m^*. □

Similar expressions apply to $g_m^{L\,*}$, h_m^* and $h_m^{L\,*}$.

We now state an important result of this section.

Theorem 4.5 *A sufficient condition for the stability of a solution to (6) is*

$$\|g_m^L\|_{\mathcal{G}} < 1$$

where $\|.\|_{\mathcal{G}}$ is the operator norm.

Proof. Let us note $\underline{\mathbf{S}}$ the function $D\mathbf{S}_m^{-1}\mathbf{S}$ and rewrite equation (6) as follows

$$\mathbf{V}_t(\mathbf{r},t) = -\mathbf{L}\mathbf{V}(\mathbf{r},t) + \int_\Omega \mathbf{W}_{cm}(\mathbf{r},\mathbf{r}',t)\,\underline{\mathbf{S}}(\mathbf{V}(\mathbf{r}',t))\,d\mathbf{r}' + \mathbf{I}_{\text{ext}}(\mathbf{r},t).$$

Let $\mathbf{U}$ be its unique solution with initial conditions $\mathbf{U}(0) = \mathbf{U}_0$, an element of $\mathcal{G}$. Let also $\mathbf{V}$ be the unique solution of the same equation with different initial conditions $\mathbf{V}(0) = \mathbf{V}_0$, another element of $\mathcal{G}$. We introduce the new function $\mathbf{X} = \mathbf{V} - \mathbf{U}$ which satisfies

$$\mathbf{X}_t(\mathbf{r},t) = -\mathbf{L}\mathbf{X}(\mathbf{r},t) + \int_\Omega \mathbf{W}_{cm}(\mathbf{r},\mathbf{r}',t)\,\mathbf{H}(\mathbf{X},\mathbf{U})(\mathbf{r}',t)\,d\mathbf{r}' = -\mathbf{L}\mathbf{X}(\mathbf{r},t) + g_m(\mathbf{H}(\mathbf{X},\mathbf{U}))(\mathbf{r},t) \tag{14}$$

where the vector $\mathbf{H}(\mathbf{X},\mathbf{U})$ is given by $\mathbf{H}(\mathbf{X},\mathbf{U})(\mathbf{r},t)) = \underline{\mathbf{S}}(\mathbf{V}(\mathbf{r},t)) - \underline{\mathbf{S}}(\mathbf{U}(\mathbf{r},t)) = \underline{\mathbf{S}}(\mathbf{X}(\mathbf{r},t)+\mathbf{U}(\mathbf{r},t)) - \underline{\mathbf{S}}(\mathbf{U}(\mathbf{r},t))$. Consider now the functional

$$V(\mathbf{X}) = \frac{1}{2}\,\langle\,\mathbf{X},\,\mathbf{X}\,\rangle$$

Its time derivative is $\langle\,\mathbf{X},\,\mathbf{X}_t\,\rangle$. We replace $\mathbf{X}_t$ by its value from (14) in this expression to obtain

$$\frac{dV(\mathbf{X})}{dt} = -\,\langle\,\mathbf{X},\,\mathbf{L}\mathbf{X}\,\rangle + \langle\,\mathbf{X},\,g_m(\mathbf{H}(\mathbf{X},\mathbf{U}))\,\rangle = \\ -\left\langle\,,\mathbf{L}^{1/2}\mathbf{X},\,\mathbf{L}^{1/2}\mathbf{X}\right\rangle + \left\langle\,\mathbf{L}^{1/2}\mathbf{X},\,g_m^L(\mathbf{L}^{1/2}\mathbf{H}(\mathbf{X},\mathbf{U}))\,\right\rangle$$

Let $\mathbf{Y} = \mathbf{L}^{1/2}\mathbf{X}$. Using lemma 3.1, we write

$$\mathbf{H}(\mathbf{X},\mathbf{U}) = \mathcal{D}_m\mathbf{X},$$

where $\mathcal{D}_m$ is a diagonal matrix whose diagonal elements are between 0 and 1. Multiplying both sides with $\mathbf{L}^{1/2}$ we obtain

$$\mathbf{L}^{1/2}\mathbf{H}(\mathbf{X},\mathbf{U}) = \mathcal{D}_m\mathbf{L}^{1/2}\mathbf{X} = \mathcal{D}_m\mathbf{Y},$$

and therefore

$$\left|\left\langle\,\mathbf{Y},\,g_m^L(\mathbf{L}^{1/2}\mathbf{H}(\mathbf{X},\mathbf{U}))\,\right\rangle\right| \leq \|\mathbf{Y}\|_{n,2}\,\left\|g_m^L(\mathbf{L}^{1/2}\mathbf{H}(\mathbf{X},\mathbf{U}))\right\|_{n,2} \leq \\ \|\mathbf{Y}\|_{n,2}\,\|g_m^L\|_{\mathcal{G}}\,\left\|\mathbf{L}^{1/2}\mathbf{H}(\mathbf{X},\mathbf{U})\right\|_{n,2} \leq \|\mathbf{Y}\|_{n,2}^2\,\|g_m^L\|_{\mathcal{G}}$$

and the conclusion follows. □
We now give a sufficient condition for the stability of a solution to (7).

Theorem 4.6 *A sufficient condition for the stability of a solution to (7) is*

$$\|h_m^L\|_{\mathcal{G}} < 1$$

Proof. Let $\mathbf{U}$ be the unique solution of (7) with an external current $\mathbf{I}_{\text{ext}}(\mathbf{r},t)$ and initial conditions $\mathbf{U}(0) = \mathbf{U}_0$. As in the proof of theorem 4.5 we introduce the new function $\mathbf{X} = \mathbf{V} - \mathbf{U}$, where $\mathbf{V}$ is the unique solution of the same equation with different initial conditions. We have

$$\mathbf{X}_t(\mathbf{r},t) = -\mathbf{L}\mathbf{X}(\mathbf{r},t) + \mathbf{S}\left(\int_\Omega \mathbf{W}(\mathbf{r},\mathbf{r}',t)\mathbf{V}(\mathbf{r}',t)\,d\mathbf{r}' + \mathbf{I}_{\text{ext}}(\mathbf{r},t)\right) - \mathbf{S}\left(\int_\Omega \mathbf{W}(\mathbf{r},\mathbf{r}',t)\mathbf{U}(\mathbf{r}',t)\,d\mathbf{r}' + \mathbf{I}_{\text{ext}}(\mathbf{r},t)\right) \quad (15)$$

Using the idea in the proof of lemma 3.1, this equation can be rewritten as

$$\mathbf{X}_t(\mathbf{r},t) = -\mathbf{L}\mathbf{X}(\mathbf{r},t) + \left(\int_0^1 D\mathbf{S}\Big(\int_\Omega \mathbf{W}(\mathbf{r},\mathbf{r}',t)\mathbf{U}(\mathbf{r}'t)\,d\mathbf{r}' + \mathbf{I}_{\text{ext}}(\mathbf{r},t) + \zeta\int_\Omega \mathbf{W}(\mathbf{r},\mathbf{r}',t)\mathbf{X}(\mathbf{r}',t)\,d\mathbf{r}'\Big)\,d\zeta\right)\left(\int_\Omega \mathbf{W}(\mathbf{r},\mathbf{r}',t)\mathbf{X}(\mathbf{r}',t)\,d\mathbf{r}'\right)$$

We use the same functional as in the proof of theorem 4.5

$$V(\mathbf{X}) = \frac{1}{2}\langle\,\mathbf{X},\,\mathbf{X}\,\rangle.$$

Its time derivative is readily obtained with the help of equation (15)

$$\frac{dV(\mathbf{X})}{dt} = -\langle\,\mathbf{X},\,\mathbf{L}\mathbf{X}\,\rangle + \langle\,\mathbf{X},\,\mathcal{D}_m h_m(\mathbf{X})\,\rangle,$$

where $\mathcal{D}_m$ is defined by

$$\mathcal{D}_m(\mathbf{U},\mathbf{X},\mathbf{r},t) = \int_0^1 D\mathbf{S}\Big(\int_\Omega \mathbf{W}(\mathbf{r},\mathbf{r}',t)\mathbf{U}(t)\,d\mathbf{r}' + \mathbf{I}_{\text{ext}}(\mathbf{r},t) + \zeta\int_\Omega \mathbf{W}(\mathbf{r},\mathbf{r}',t)\mathbf{X}(\mathbf{r}',t)\,d\mathbf{r}'\Big)\,D\mathbf{S}_m^{-1}\,d\zeta,$$

a diagonal matrix whose diagonal elements are between 0 and 1. We rewrite $\frac{dV(\mathbf{X})}{dt}$ in a slightly different manner, introducing the operator h_m^L

$$\frac{dV(\mathbf{X})}{dt} = -\Big\langle\,\mathbf{L}^{1/2}\mathbf{X},\,\mathbf{L}^{1/2}\mathbf{X}\,\Big\rangle + \Big\langle\,\mathcal{D}_m\mathbf{L}^{1/2}\mathbf{X},\,h_m^L(\mathbf{L}^{1/2}\mathbf{X})\,\Big\rangle,$$

Letting $\mathbf{Y} = \mathbf{L}^{1/2}\mathbf{X}$, from the Cauchy-Schwarz' inequality and the property of $\mathcal{D}_m$ we obtain

$$\big|\langle\,\mathcal{D}_m\mathbf{Y},\,h_m^L(\mathbf{Y})\,\rangle\big| \le \|\mathcal{D}_m\mathbf{Y}\|_{n,2}\,\|h_m^L(\mathbf{Y})\|_{n,2} \le \|\mathbf{Y}\|_{n,2}\,\|h_m^L(\mathbf{Y})\|_{n,2}$$

And since $\|h_m^L(\mathbf{Y})\|_{n,2} \le \|h_m^L\|_{\mathcal{G}}\,\|\mathbf{Y}\|_{n,2}$, a sufficient condition for $\frac{dV(\mathbf{X})}{dt}$ to be negative is that $\|h_m^L\|_{\mathcal{G}} < 1$. □

Note that $\|g_m^L\|_{\mathcal{G}} = \|g_m^L\|_{\mathbf{L}^2}$ and $\|h_m^L\|_{\mathcal{G}} = \|h_m^L\|_{\mathbf{L}^2}$ by density of $\mathcal{G}$ in $\mathbf{L}^2$ (see section 6). In appendix A, we show how to compute such operator norms.

4.2 The convolution case

In the case where $\mathbf{W}$ is translation invariant we can obtain a slightly easier to exploit sufficient condition for the stability of the solutions than in the theorems 4.5 and 4.6. We first consider the case of a general compact Ω and then the case where Ω is an interval. Translation invariance means that $\mathbf{W}(\mathbf{r}+\mathbf{a},\mathbf{r}'+\mathbf{a},t)=\mathbf{W}(\mathbf{r},\mathbf{r}',t)$ for all $\mathbf{a}$, so we can write $\mathbf{W}(\mathbf{r},\mathbf{r}',t)=\mathbf{W}(\mathbf{r}-\mathbf{r}',t)$. Hence $\mathbf{W}(\mathbf{r},t)$ must be defined for all $\mathbf{r}\in\widehat{\Omega}=\{\mathbf{r}-\mathbf{r}',\text{ with }\mathbf{r},\,\mathbf{r}'\in\Omega\}$ and we suppose it continuous on $\widehat{\Omega}$ for each t. $\widehat{\Omega}$ is a symmetric with respect to the origin 0 compact subset of $\mathbb{R}^q$.

4.2.1 General Ω

We note $\mathbf{1}_A$ the characteristic function of the subset A of $\mathbb{R}^q$ and $\mathbf{M}^*=\overline{\mathbf{M}}^T$ the conjugate transpose of the complex matrix $\mathbf{M}$.

We prove the following

Theorem 4.7 *If the eigenvalues of the Hermitian matrix*

$$\widetilde{\mathbf{W}}^*(\mathbf{f},t)\,\widetilde{\mathbf{W}}(\mathbf{f},t) \tag{16}$$

are strictly less than 1 for all $\mathbf{f}\in\mathbb{R}^q$ *and all* $t\in\mathrm{J}$, *then the system (6) is absolutely stable.* $\widetilde{\mathbf{W}}(\mathbf{f},t)$ *is the Fourier transform with respect to the space variable* $\mathbf{r}$ *of* $\mathbf{1}_{\widehat{\Omega}}(\mathbf{r})\,\mathbf{W}_{cm}^L(\mathbf{r},t)$,

$$\widetilde{\mathbf{W}}(\mathbf{f},t)=\int_{\widehat{\Omega}}\mathbf{W}_{cm}^L(\mathbf{r},t)e^{-2i\pi\mathbf{r}\cdot\mathbf{f}}\,d\mathbf{r}$$

Proof. We prove the theorem for $\widetilde{\mathbf{W}}(\mathbf{f},t)=\int_{\widehat{\Omega}}\mathbf{W}_{cm}^{L\,T}(-\mathbf{r},t)e^{-2i\pi\mathbf{r}\cdot\mathbf{f}}\,d\mathbf{r}$, the Fourier transform of $\mathbf{1}_{\widehat{\Omega}}(\mathbf{r})\,\mathbf{W}_{cm}^{L\,T}(-\mathbf{r},t)$, because we deal with $g_m^{L\,*}$ in the following proof. Then the theorem naturally holds for $\widetilde{\mathbf{W}}(\mathbf{f},t)=\int_{\widehat{\Omega}}\mathbf{W}_{cm}^L(\mathbf{r},t)e^{-2i\pi\mathbf{r}\cdot\mathbf{f}}\,d\mathbf{r}$ since the corresponding families of Hermitian matrices for both definitions of $\widetilde{\mathbf{W}}$ have the same spectral properties. The proof proceeds exactly as that of theorem 4.5. We have

$$\left\langle \mathbf{Y},\,g_m^L(\mathbf{L}^{1/2}\mathbf{H}(\mathbf{X},\mathbf{U}))\right\rangle=\left\langle g_m^{L\,*}(\mathbf{Y}),\,\mathcal{D}_m\mathbf{Y}\right\rangle$$

Therefore the condition

$$\|g_m^{L\,*}(\mathbf{Y})\|_{n,2}^2<\|\mathbf{Y}\|_{n,2}^2$$

is sufficient to ensure the negativity of $dV(\mathbf{X})/dt$.

For all vector $\mathbf{Z}$ in $\mathbb{R}^q$, we have

$$\mathbf{Z}^*\,\widetilde{\mathbf{W}}^*(\mathbf{f},t)\,\widetilde{\mathbf{W}}(\mathbf{f},t)\,\mathbf{Z}<\mathbf{Z}^*\,\mathbf{Z}\quad\forall\mathbf{f},\,\forall t\in\mathrm{J},$$

thanks to the hypothesis of the theorem. Hence

$$\widetilde{\mathbf{Y}}^*(\mathbf{f},t)\,\widetilde{\mathbf{W}}^*(\mathbf{f},t)\,\widetilde{\mathbf{W}}(\mathbf{f},t)\,\widetilde{\mathbf{Y}}(\mathbf{f},t)<\widetilde{\mathbf{Y}}^*(\mathbf{f},t)\,\widetilde{\mathbf{Y}}(\mathbf{f},t)\quad\forall\mathbf{f},\,\forall t\in\mathrm{J},$$

where $\widetilde{\mathbf{Y}}(\mathbf{f},t)$ is the Fourier transform of $\mathbf{1}_\Omega\,\mathbf{Y}$, which is well defined and belongs to $\mathbf{L}^2_n(\mathbb{R}^q)$ (Plancherel's theorem). We integrate the previous inequality in $\mathbb{R}^q$ and obtain

$$\|\widetilde{\mathbf{W}}\,\widetilde{\mathbf{Y}}\|^2_{\mathbb{R}^q,n,2} < \|\widetilde{\mathbf{Y}}\|^2_{\mathbb{R}^q,n,2}.$$

Then, Parseval's theorem gives ($\star$ denotes the spatial convolution)

$$\|\mathbf{1}_{\widehat{\Omega}}\,\mathbf{W}^{L\,T}_{cm}(-.,t)\star\mathbf{1}_\Omega\,\mathbf{Y}(.,t)\|^2_{\mathbb{R}^q,n,2} < \|\mathbf{1}_\Omega\,\mathbf{Y}(.,t)\|^2_{\mathbb{R}^q,n,2},$$

from which $\|g^{L\,*}_m(\mathbf{Y})\|^2_{n,2} < \|\mathbf{Y}\|^2_{n,2}$ follows. □

The case of the activation-based model can be addressed in a similar fashion. We have the

Theorem 4.8 *If the eigenvalues of the Hermitian matrix*

$$\widetilde{\mathbf{W}}^*(\mathbf{f},t)\,\widetilde{\mathbf{W}}(\mathbf{f},t)$$

are strictly less than 1 for all $\mathbf{f}$ *and all* $t\in\mathrm{J}$ *then the system (7) is absolutely stable.* $\widetilde{\mathbf{W}}(\mathbf{f},t)$ *is the Fourier transform of* $\mathbf{W}^L_{mc}(\mathbf{r},t)$ *with respect to the space variable* $\mathbf{r}$.

Proof. The proof follows that of theorem 4.6 and then exploits the relation between convolution and Fourier transform and Parseval's relation by expressing the constraint

$$\|h^L_m(\mathbf{Y})\|^2_{n,2} < \|\mathbf{Y}\|^2_{n,2}$$

in the Fourier domain. □

These two theorems are somewhat unsatisfactory since they replace a condition that must be satisfied over a countable set, the spectrum of a compact operator, as in theorems 4.5 and 4.6, by a condition that must be satisfied over a continuum. Nonetheless one may consider that the computation of the Fourier transforms of the matrixes $\mathbf{W}^L_{cm}$ and $\mathbf{W}^L_{mc}$ are easier than that of the spectra of the operators g^L_m and h^L_m, but see section A.

4.2.2 Ω is an interval

In the case where Ω is an interval, i.e. an interval of $\mathbb{R}$ ($q=1$), a parallelogram ($q=2$), or a parallelepiped ($q=3$), we can state different sufficient conditions. We can always assume that Ω is the q-dimensional interval $[0,1]^q$ by applying an affine change of coordinates. The connectivity matrix $\mathbf{W}$ is defined on $\mathrm{J}\times[-1,1]^{\mathrm{q}}$ and extended to a q-periodic function of periods 2 on $\mathrm{J}\times\mathbb{R}^{\mathrm{q}}$. Similarly, the state vectors $\mathbf{V}$ and $\mathbf{A}$ as well as the external current $\mathbf{I}_{\mathrm{ext}}$ defined on $\mathrm{J}\times[0,1]^{\mathrm{q}}$ are extended to q-periodic functions of the same periods over $\mathrm{J}\times\mathbb{R}^{\mathrm{q}}$ by padding them with zeros in the complement in the interval $[-1,1]^q$ of the interval $[0,1]^q$. $\mathcal{G}$ is now the space $\mathbf{L}^2_n(2)$ of the square integrable q-periodic functions of periods 2.

We define the functions $\psi_{\mathbf{m}}(\mathbf{r})\equiv e^{-\pi i(r_1m_1+\cdots+r_qm_q)}$, for $\mathbf{m}\in\mathbb{Z}^q$ and consider the matrix $\widetilde{\mathbf{W}}(\mathbf{m})$ whose elements are given by

$$\tilde{w}_{ij}(\mathbf{m}) = \int_{[0,2]^q} w_{ij}(\mathbf{r})\psi_{\mathbf{m}}(\mathbf{r})\,d\mathbf{r} \quad 1\le i,\,j\le n.$$

We recall the

Definition 4.9 *The matrix* $\widetilde{\mathbf{W}}(\mathbf{m})$ *is the* $\mathbf{m}$*th element of the Fourier series of the periodic matrix function* $\mathbf{W}(\mathbf{r})$.

The theorems 4.7 and 4.8 can be stated in this framework.

Theorem 4.10 *If the eigenvalues of the Hermitian matrix*

$$\widetilde{\mathbf{W}}^*(\mathbf{m},t)\,\widetilde{\mathbf{W}}(\mathbf{m},t) \tag{17}$$

are strictly less than 1 for all $\mathbf{m} \in \mathbb{Z}^q$ *and all* $t \in \mathrm{J}$, *then the system (6) (resp. (7)) is absolutely stable.* $\widetilde{\mathbf{W}}(\mathbf{m},t)$ *is the* $\mathbf{m}$*th element of the Fourier series of the* q*-periodic matrix function* $\mathbf{W}^L_{cm}(\mathbf{r},t)$ *(resp.* $\mathbf{W}^L_{mc}(\mathbf{r},t)$*) with periods 2 at time* t.

5 Stability of the homogeneous solution

We next investigate the stability of a homogeneous solution to (6) and (7). As in the previous section we distinguish the general and convolution cases.

5.1 The general case

The homogeneous solutions are characterized by the fact that they are spatially constant at each time instant. We consider the subspace $\mathcal{G}_c$ of $\mathcal{G}$ of the constant functions. We have the following

Lemma 5.1 $\mathcal{G}_c$ *is a complete linear subspace of* $\mathcal{G}$. *The orthogonal projection operator* $\mathcal{P}_{\mathcal{G}_c}$ *from* $\mathcal{G}$ *to* $\mathcal{G}_c$ *is defined by*

$$\mathcal{P}_{\mathcal{G}_c}(\mathbf{x}) = \overline{\mathbf{x}} = \frac{1}{|\Omega|}\int_\Omega \mathbf{x}(\mathbf{r})\,d\mathbf{r}$$

The orthogonal complement $\mathcal{G}_c^\perp$ *of* $\mathcal{G}_c$ *is the subset of functions of* $\mathcal{G}$ *that have a zero average. The orthogonal projection*[1] *operator* $\mathcal{P}_{\mathcal{G}_c^\perp}$ *is equal to* $\mathrm{Id} - \mathcal{P}_{\mathcal{G}_c}$. *We also have*

$$\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{M}\mathbf{x} = \mathbf{M}\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{x} \quad \forall\,\mathbf{x} \in \mathcal{G},\, \mathbf{M} \in \mathcal{M}_{n\times n} \tag{18}$$

Proof. The constant functions are clearly in $\mathcal{G}$. Any Cauchy sequence of constants is converging to a constant hence $\mathcal{G}_c$ is closed in the pre-Hilbert space $\mathcal{G}$. Therefore there exists an orthogonal projection operator from $\mathcal{G}$ to $\mathcal{G}_c$ which is linear, continuous, of unit norm, positive and self-adjoint. $\mathcal{P}_{\mathcal{G}_c}(\mathbf{x})$ is the minimum with respect to the constant vector $\mathbf{a}$ of the integral $\int_\Omega \|\mathbf{x}(\mathbf{r}) - \mathbf{a}\|^2\,d\mathbf{r}$. Taking the derivative with respect to $\mathbf{a}$, we obtain the necessary condition

$$\int_\Omega (\mathbf{x}(\mathbf{r}) - \mathbf{a})\,d\mathbf{r} = 0$$

[1] To be accurate, this is the projection on the closure of $\mathcal{G}_c^\perp$ in the closure of $\mathcal{G}$ which is the Hilbert space $\mathbf{L}^2_n(\Omega)$.

and hence $\mathbf{a}_{min} = \overline{\mathbf{x}}$. Conversely, $\mathbf{x} - \mathbf{a}_{min}$ is orthogonal to $\mathcal{G}_c$ since $\int_\Omega (\mathbf{x}(\mathbf{r}) - \mathbf{a}_{min})\mathbf{b}\, d\mathbf{r} = 0$ for all $\mathbf{b} \in \mathcal{G}_c$.

Let $\mathbf{y} \in \mathcal{G}$, $\int_\Omega \mathbf{x}\,\mathbf{y}(\mathbf{r})\, d\mathbf{r} = \mathbf{x} \int_\Omega \mathbf{y}(\mathbf{r})\, d\mathbf{r} = 0$ for all $\mathbf{x} \in \mathcal{G}_c$ if and only if $\mathbf{y} \in \mathcal{G}_c^\perp$.

Finally

$$\mathcal{P}_{\mathcal{G}_c^\perp} \mathbf{M}\mathbf{x} = \mathbf{M}\mathbf{x} - \overline{\mathbf{M}\mathbf{x}} = \mathbf{M}\mathbf{x} - \mathbf{M}\overline{\mathbf{x}} = \mathbf{M}(\mathbf{x} - \overline{\mathbf{x}}) = \mathbf{M}\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{x}$$

□

We are now ready to prove the theorem on the stability of the homogeneous solutions to (6).

Theorem 5.2 *If* $\mathbf{W}$ *satisfies (11), a sufficient condition for the stability of a homogeneous solution to (6) is that the norm* $\|g_m^{L\,*}\|_{\mathcal{G}_c^\perp}$ *of the restriction to* $\mathcal{G}_c^\perp$ *of the compact operator* $g_m^{L\,*}$ *be less than 1 for all* $t \in \mathrm{J}$.

Proof. This proof is inspired by [24]. Note that $\mathcal{G}_c^\perp$ is invariant by g_m^* and hence by $g_m^{L\,*}$. Indeed, from lemma 4.4 and equation (11) we have

$$\overline{g_m^*(\mathbf{x})} = \overline{\mathbf{W}_{cm}^T}(t)\overline{\mathbf{x}} = 0 \quad \forall \mathbf{x} \in \mathcal{G}_c^\perp$$

Let $\mathbf{V}_p$ be the unique solution of (6) with homogeneous input $\mathbf{I}_{\text{ext}}(t)$ with initial conditions $\mathbf{V}_p(0) = \mathbf{V}_{p0}$ and consider the initial value problem

$$\left\{ \begin{array}{lcl} \mathbf{X}'(t) & = & \mathcal{P}_{\mathcal{G}_c^\perp}\left(f_v(t, \mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X} + \mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p)\right) \\ \mathbf{X}(0) & = & \mathbf{X}_0 \end{array} \right. \tag{19}$$

$\mathbf{X} = \mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{V}_p$ is a solution with initial condition $\mathbf{X}_0 = \mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{V}_{p0}$ since $\mathcal{P}^2_{\mathcal{G}_c^\perp} = \mathcal{P}_{\mathcal{G}_c^\perp}$ and $\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{V}_p + \mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p = \mathbf{V}_p$. But $\mathbf{X} = 0$ is also a solution with initial condition $\mathbf{X}_0 = 0$ since $\mathcal{G}_c$ is flow-invariant because of (11), that is $f_v(t, \mathcal{G}_c) \subset \mathcal{G}_c$, and hence $\mathcal{P}_{\mathcal{G}_c^\perp}(f_v(t, \mathcal{G}_c)) = 0$. We therefore look for a sufficient condition for the system (19) to be absolutely stable at $\mathbf{X} = 0$.

We consider again the functional $V(\mathbf{X}) = \frac{1}{2}\langle \mathbf{X}, \mathbf{X} \rangle$ with time derivative $\frac{dV(\mathbf{X})}{dt} = \langle \mathbf{X}, \mathbf{X}_t \rangle$. We substitute $\mathbf{X}_t$ with its value from (19) which can be rewritten as

$$\mathbf{X}_t = \mathcal{P}_{\mathcal{G}_c^\perp}\left(-\mathbf{L}(\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X} + \mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p) + \int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\,\mathbf{S}(\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X}(\mathbf{r}', t) + \mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p(\mathbf{r}', t))\, d\mathbf{r}' \right)$$

Because of lemma 5.1 this yields

$$\mathbf{X}_t = -\mathbf{L}\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X} + \mathcal{P}_{\mathcal{G}_c^\perp}\left(\int_\Omega \mathbf{W}_{cm}(\mathbf{r}, \mathbf{r}', t)\,\underline{\mathbf{S}}(\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X}(\mathbf{r}', t) + \mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p(\mathbf{r}', t))\, d\mathbf{r}' \right)$$

We write

$$\underline{\mathbf{S}}(\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X} + \mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p) = \underline{\mathbf{S}}(\mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p) + \left(\int_0^1 D\underline{\mathbf{S}}(\mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p + \zeta\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X})\, d\zeta \right)\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X},$$

and since $\underline{\mathbf{S}}(\mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p) \in \mathcal{G}_c$, and because of (11)

$$\mathcal{P}_{\mathcal{G}_c^\perp}\left(\int_\Omega \mathbf{W}_{cm}(\mathbf{r},\mathbf{r}',t)\,\underline{\mathbf{S}}(\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X}(\mathbf{r}',t)+\mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p(\mathbf{r}',t))\,d\mathbf{r}'\right) = \mathcal{P}_{\mathcal{G}_c^\perp}\left(\int_\Omega \mathbf{W}_{cm}(\mathbf{r},\mathbf{r}',t)\left(\int_0^1 D\underline{\mathbf{S}}(\mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p(\mathbf{r}',t)+\zeta\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X}(\mathbf{r}',t))\,d\zeta\right)\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X}(\mathbf{r}',t)\,d\mathbf{r}'\right)$$

We use (18) and the fact that $\mathcal{P}_{\mathcal{G}_c^\perp}$ is self-adjoint and idempotent to write

$$\frac{dV(\mathbf{X})}{dt} = -\langle \mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X},\, \mathbf{L}\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X}\rangle + \left\langle \mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X},\, \left(\int_\Omega \mathbf{W}_{cm}(\mathbf{r},\mathbf{r}',t)\left(\int_0^1 D\underline{\mathbf{S}}(\mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p(\mathbf{r}',t)+\zeta\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X}(\mathbf{r}',t))\,d\zeta\right)\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X}(\mathbf{r}',t)\,d\mathbf{r}'\right)\right\rangle$$

Let us denote $\mathcal{D}_v(\mathbf{r}')$ the diagonal matrix $\int_0^1 D\underline{\mathbf{S}}(\mathcal{P}_{\mathcal{G}_c}\mathbf{V}_p(\mathbf{r}',t)+\zeta\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X}(\mathbf{r}',t))\,d\zeta$. Its diagonal elements are between 0 and 1. Letting $\mathbf{Y} = \mathbf{L}^{1/2}\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{X}$ we rewrite the previous equation in operator form

$$\frac{dV(\mathbf{X})}{dt} = -\langle \mathbf{Y},\, \mathbf{Y}\rangle + \left\langle \mathbf{Y},\, g_m^L(\mathcal{D}_v\,\mathbf{Y})\right\rangle$$

By definition of the adjoint

$$\left\langle \mathbf{Y},\, g_m^L(\mathcal{D}_v\,\mathbf{Y})\right\rangle = \left\langle g_m^{L\,*}(\mathbf{Y})\,,\, \mathcal{D}_v\,\mathbf{Y}\right\rangle$$

From the Cauchy-Schwarz' inequality

$$\left|\left\langle g_m^{L\,*}(\mathbf{Y})\,,\, \mathcal{D}_v\,\mathbf{Y}\right\rangle\right| \le \left\|g_m^{L\,*}(\mathbf{Y})\right\|_{n,2}\,\|\mathcal{D}_v\,\mathbf{Y}\|_{n,2} \le \left\|g_m^{L\,*}(\mathbf{Y})\right\|_{n,2}\,\|\mathbf{Y}\|_{n,2}\,,$$

and since

$$\left\|g_m^{L\,*}(\mathbf{Y})\right\|_{n,2} \le \|g_m^{L\,*}\|_{\mathcal{G}_c^\perp}\,\|\mathbf{Y}\|_{n,2}\,,$$

the conclusion follows. □

Note that $\|g_m^{L\,*}\|_{\mathcal{G}_c^\perp} = \|g_m^L\|_{\mathbf{L}_0^2}$ by density of $\mathcal{G}_c^\perp$ in $\mathbf{L}_0^2$, where $\mathbf{L}_0^2$ is the subspace of $\mathbf{L}^2$ of zero mean functions. We show in appendix A how to compute this norm.

We prove a similar theorem in the case of (7).

Theorem 5.3 *If $\mathbf{W}$ satisfies (11), a sufficient condition for the stability of a homogeneous solution to (7) is that the norm $\|h_m^L\|_{\mathcal{G}_c^\perp}$ of the restriction to $\mathcal{G}_c^\perp$ of the compact operator h_m^L be less than 1 for all $t \in$ J.*

Proof. The proof is similar to that of theorem 5.2. We consider $\mathbf{A}_p$ the unique solution to (7) with homogeneous input $\mathbf{I}_{\text{ext}}(t)$, initial conditions $\mathbf{A}_p(0) = \mathbf{A}_{p0}$, and consider the initial value problem

$$\begin{cases} \mathbf{A}'(t) &= \mathcal{P}_{\mathcal{G}_c^\perp}\left(f_a(t, \mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{A} + \mathcal{P}_{\mathcal{G}_c}\mathbf{A}_p)\right) \\ \mathbf{A}(0) &= \mathbf{A}_0 \end{cases} \tag{20}$$

$\mathbf{A} = \mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{A}_p$ is a solution with initial conditions $\mathbf{A}_0 = \mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{A}_{p0}$ since $\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{A}_p + \mathcal{P}_{\mathcal{G}_c}\mathbf{A}_p = \mathbf{A}_p$. But $\mathbf{A} = 0$ is also a solution with initial conditions $\mathbf{A}_0 = 0$ since $\mathcal{G}_c$ is flow-invariant because of (11), that is $f_a(t, \mathcal{G}_c) \subset \mathcal{G}_c$, and hence $\mathcal{P}_{\mathcal{G}_c^\perp}(f_a(t, \mathcal{G}_c)) = 0$. We therefore look for a sufficient condition for the system (19) to be absolutely stable at $\mathbf{A} = 0$.

Consider again the functional $V(\mathbf{A}) = \frac{1}{2}\langle \mathbf{A},\, \mathbf{A}\rangle$ with time derivative $\frac{dV(\mathbf{A})}{dt} = \langle \mathbf{A},\, \mathbf{A}_t \rangle$. We substitute $\mathbf{A}_t$ with its value from (20) which, using (11), can be rewritten as

$$\mathbf{A}_t =$$
$$\mathcal{P}_{\mathcal{G}_c^\perp}\left(-\mathbf{L}(\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{A} + \mathcal{P}_{\mathcal{G}_c}\mathbf{A}_p) + \mathbf{S}\left(\int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\,\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{A}(\mathbf{r}', t)\,d\mathbf{r}' + \overline{\mathbf{W}}(t)\mathcal{P}_{\mathcal{G}_c}\mathbf{A}_p + \mathbf{I}_{\text{ext}}(t)\right)\right)$$

We do a first-order Taylor expansion with integral remainder of the $\mathbf{S}$ term and introduce the operator h_m:

$$\mathbf{S}\left(\int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\,\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{A}(\mathbf{r}', t)\,d\mathbf{r}' + \overline{\mathbf{W}}(t)\mathcal{P}_{\mathcal{G}_c}\mathbf{A}_p + \mathbf{I}_{\text{ext}}(t)\right) = \mathbf{S}\left(\overline{\mathbf{W}}(t)\mathcal{P}_{\mathcal{G}_c}\mathbf{A}_p + \mathbf{I}_{\text{ext}}(t)\right) +$$
$$\left(\int_0^1 D\underline{\mathbf{S}}\Big(\overline{\mathbf{W}}(t)\mathcal{P}_{\mathcal{G}_c}\mathbf{A}_p + \mathbf{I}_{\text{ext}}(t) + \zeta\int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\,\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{A}(\mathbf{r}', t)\,d\mathbf{r}'\Big)d\zeta\right) h_m(\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{A})(\mathbf{r}, t)$$

Let us define

$$\mathcal{D}_a(\mathbf{r}, t) = \int_0^1 D\underline{\mathbf{S}}\Big(\overline{\mathbf{W}}(t)\mathcal{P}_{\mathcal{G}_c}\mathbf{A}_p + \mathbf{I}_{\text{ext}}(t) + \zeta\int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\,\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{A}(\mathbf{r}', t)\,d\mathbf{r}'\Big)d\zeta,$$

a diagonal matrix whose diagonal elements are between 0 and 1. Letting $\mathbf{Y} = \mathbf{L}^{1/2}\mathcal{P}_{\mathcal{G}_c^\perp}\mathbf{A}$ we write

$$\frac{dV(\mathbf{A})}{dt} = -\langle \mathbf{Y},\, \mathbf{Y}\rangle + \big\langle \mathbf{Y},\, \mathcal{D}_a\, h_m^L(\mathbf{Y})\big\rangle$$

and the conclusion follows from the Cauchy-Schwarz' inequality:

$$\left|\big\langle \mathbf{Y},\, \mathcal{D}_a\, h_m^L(\mathbf{Y})\big\rangle\right| \le \|\mathbf{Y}\|_{n,2}\, \left\|\mathcal{D}_a\, h_m^L(\mathbf{Y})\right\|_{n,2} \le \|\mathbf{Y}\|_{n,2}\, \left\|h_m^L(\mathbf{Y})\right\|_{n,2} \le \left\|h_m^L\right\|_{\mathcal{G}_c^\perp}\, \|\mathbf{Y}\|_{n,2}^2$$

□

5.2 The convolution cases

We know from the analysis done in section 3.2 that there are in general no homogeneous solutions to (6) or (7) since condition (11) cannot be satisfied. Indeed the integral $\int_\Omega \mathbf{W}(\mathbf{r} - \mathbf{r}', t)\,d\mathbf{r}'$ is in general a function of $\mathbf{r}$. But propositions 3.3, 3.4, 6.2 and 6.3 show that there exists a solution for a given initial condition. We call these solutions pseudo-homogeneous.

5.2.1 General Ω

There are echoes of theorems 4.7 and 4.8 which are stated in the following

Theorem 5.4 *If* $\mathbf{W}$ *satisfies (11), a sufficient condition for the stability of a pseudo-homogeneous solution to (6) (resp. (7)) is that the eigenvalues of the Hermitian matrix*

$$\widetilde{\mathbf{W}}^*(\mathbf{f},t)\widetilde{\mathbf{W}}(\mathbf{f},t)$$

are strictly less than 1 for all $\mathbf{f} \neq 0$ *and all* $t \in \mathrm{J}$. $\widetilde{\mathbf{W}}(\mathbf{f},t)$ *is the Fourier transform of* $\mathbf{W}^L_{cm}(\mathbf{r},t)$ *(resp.* $\mathbf{W}^L_{mc}(\mathbf{r},t)$*) with respect to the space variable* $\mathbf{r}$.

Proof. The proof is an adaptation of the proofs of theorems 5.2 and 5.3. We exploit again the relation between convolution and Fourier transform, Parseval's relation and the fact that if $\mathbf{x} \in \mathcal{G}_c^\perp$, $\tilde{\mathbf{x}}(0) = 0$. □
The only difference with theorems 4.7 and 4.8 is that there are no constraints on the Fourier transforms of the connectivity matrices at the origin of the spatial frequencies plane. This is due to the fact that we only "look" at the subspace of $\mathcal{G}$ of functions with zero spatial average.

5.2.2 Ω is an interval

There are also echoes of theorem 4.10 in the following

Theorem 5.5 *If* $\mathbf{W}$ *satisfies (11), a sufficient condition for the stability of a pseudo-homogeneous solution to (6) (resp. (7)) is that the eigenvalues of the Hermitian matrices*

$$\widetilde{\mathbf{W}}^*(\mathbf{m},t)\widetilde{\mathbf{W}}(\mathbf{m},t)$$

are strictly less than 1 for all $\mathbf{m} \neq \mathbf{0} \in \mathbb{Z}^q$ *and all* $t \in \mathrm{J}$. $\widetilde{\mathbf{W}}(\mathbf{m},t)$ *is the* $\mathbf{m}$*th element of the Fourier series of the* q*-periodic function* $\mathbf{W}^L_{cm}(\mathbf{r},t)$ *(resp.* $\mathbf{W}^L_{mc}(\mathbf{r},t)$*) with respect to the space variable* $\mathbf{r}$.

6 Extending the theory

We have developed our analysis of (6) and (7) in the Banach space $\mathcal{F}$ of continuous functions of the spatial coordinate $\mathbf{r}$ even though we have used a structure of pre-Hilbert space $\mathcal{G}$ on top of it. But there remains the fact that the solutions that we have been discussing are smooth, i.e., continuous with respect to the space variable. It may be interesting to also consider non-smooth solutions, e.g., piecewise continuous solutions that can be discontinuous along curves of Ω. A natural setting, given the fact that we are interested in having a structure of Hilbert space, is $\mathbf{L}^2_n(\Omega)$, the space of square-integrable functions from Ω to $\mathbb{R}^n$, see appendix A. It is a Hilbert space and $\mathcal{G}$ is a dense subspace.

6.1 Existence and uniqueness of a solution

The theory developed in the previous sections can be readily extended to $\mathbf{L}^2_n(\Omega)$: the analysis of the stability of the general and homogeneous solutions has been done using the pre-Hilbert space structure of $\mathcal{G}$ and all the operators that have been shown to be compact in $\mathcal{G}$ are also compact in its closure $\mathbf{L}^2_n(\Omega)$ [7]. The only point that has to be re-worked is the problem of existence and uniqueness of a solution addressed in propositions 3.2 and 3.3. This allows us to *relax* the rather stringent spatial smoothness hypotheses imposed on the connectivity function $\mathbf{W}$ and the external current $\mathbf{I}_{\text{ext}}$, thereby bringing in more flexibility to the model. We have the following

Proposition 6.1 *If the following two hypotheses are satisfied. At each time instant $t \in \mathrm{J}$*

1. *The mapping $\mathbf{W}$ is in $C(\mathrm{J}; \mathbf{L}^2_{\mathrm{n}\times\mathrm{n}}(\Omega \times \Omega))$.*
2. *The external current $\mathbf{I}$ is in $C(\mathrm{J}; \mathbf{L}^2_{\mathrm{n}}(\Omega))$,*

then the mappings f_v and f_a are from $\mathrm{J} \times \mathbf{L}^2_{\mathrm{n}}(\Omega)$ to $\mathbf{L}^2_n(\Omega)$, continuous, and Lipschitz continuous with respect to their second argument, uniformly with respect to the first.

Proof. Because of the first hypothesis, the fact that $\mathbf{S}(\mathbf{x})$ is in $\mathbf{L}^2_n(\Omega)$ for all $\mathbf{x} \in \mathbf{L}^2_n(\Omega)$, and lemma A.2, f_v is well-defined. Let us prove that it is continuous. As in the proof of proposition 3.2 we write

$$f_v(t, \mathbf{x}) - f_v(s, \mathbf{y}) = -\mathbf{L}(\mathbf{x} - \mathbf{y}) + \int_\Omega (\mathbf{W}(\cdot, \mathbf{r}', t) - \mathbf{W}(\cdot, \mathbf{r}', s))\mathbf{S}(\mathbf{x}(\mathbf{r}'))\, d\mathbf{r}' + \int_\Omega \mathbf{W}(\cdot, \mathbf{r}', s)(\mathbf{S}(\mathbf{x}(\mathbf{r}')) - \mathbf{S}(\mathbf{y}(\mathbf{r}'))\, d\mathbf{r}' + \mathbf{I}_{\text{ext}}(\cdot, t) - \mathbf{I}_{\text{ext}}(\cdot, s),$$

from which we obtain, using lemma A.2

$$\|f_v(t, \mathbf{x}) - f_v(s, \mathbf{y})\|_{n,2} \le \|\mathbf{L}\|_F \|\mathbf{x} - \mathbf{y}\|_{n,2} + \sqrt{n|\Omega|} S_m \|\mathbf{W}(\cdot, \cdot, t) - \mathbf{W}(\cdot, \cdot, s)\|_F + DS_m \|\mathbf{W}(\cdot, \cdot, s)\|_F \|\mathbf{x} - \mathbf{y}\|_{n,2} + \|\mathbf{I}_{\text{ext}}(\cdot, t) - \mathbf{I}_{\text{ext}}(\cdot, s)\|_{n,2},$$

and the continuity follows from the hypotheses. $\| \, \|_F$ is the Frobenius norm, see appendix A. Note that since $\mathbf{W}$ is continuous on the compact interval J, it is bounded and $\|\mathbf{W}(\cdot, \cdot, t)\|_F \le w$ for all $t \in \mathrm{J}$ for some positive constant w. The Lipschitz continuity with respect to the second argument uniformly with respect to the first one follows from the previous inequality by choosing $s = t$.

The proof for f_a is similar. □

From this proposition we deduce the existence and uniqueness of a solution over a subinterval of $\mathbb{R}$:

Proposition 6.2 *Subject to the hypotheses of proposition 6.1 for any element $\mathbf{V}_0$ of $\mathbf{L}^2_n(\Omega)$ there is a unique solution $\mathbf{V}$, defined on a subinterval of J containing 0 and continuously differentiable, of the abstract initial value problem (8) for $f = f_v$ and $f = f_a$ such that $\mathbf{V}(0) = \mathbf{V}_0$.*

Proof.
All conditions of the Picard-Lindelöf theorem on differential equations in Banach spaces (here a Hilbert space) [7, 1] are satisfied, hence the proposition. □
We can also prove that this solution exists for all times, as in proposition 3.4:

Proposition 6.3 *If the following two hypotheses are satisfied*

1. *The connectivity function* $\mathbf{W}$ *is in* $C(\mathbb{R}; \mathbf{L}^2_{\mathrm{n}\times\mathrm{n}}(\Omega \times \Omega))$,
2. *The external current* $\mathbf{I}_{\mathrm{ext}}$ *is in* $C(\mathbb{R}; \mathbf{L}^2_{\mathrm{n}}(\Omega))$,

then for any function $\mathbf{V}_0$ *in* $\mathbf{L}^2_n(\Omega)$ *there is a unique solution* $\mathbf{V}$, *defined on* $\mathbb{R}$ *and continuously differentiable, of the abstract initial value problem (8) for* $f = f_v$ *and* $f = f_a$.

Proof. The proof is similar to the one of proposition 3.4. □

6.2 Locally homogeneous solutions and their stability

An application of the previous extension is the following. Consider a partition of Ω into P subregions Ω_i, $i = 1, \ldots, P$. We assume that the Ω_is are closed, hence compact, subsets of Ω intersecting along piecewise regular curves. These curves form a set of 0 Lebesgue measure of Ω. We consider locally homogeneous input current functions

$$\mathbf{I}_{\mathrm{ext}}(\mathbf{r}, t) = \sum_{k=1}^{P} \mathbf{1}_{\Omega_k}(\mathbf{r}) \mathbf{I}^k_{\mathrm{ext}}(t), \tag{21}$$

where the P functions $\mathbf{I}^k_{\mathrm{ext}}(t)$ are continuous on some closed interval J containing 0. On the border between two adjacent regions the value of $\mathbf{I}_{\mathrm{ext}}(\mathbf{r}, t)$ is undefined. Since this set of borders is of 0 measure, the functions defined by (21) are in $\mathbf{L}^2_n(\Omega)$ at each time instant. We assume that the connectivity matrix $\mathbf{W}$ satisfies the following conditions

$$\int_{\Omega_k} \mathbf{W}(\mathbf{r}, \mathbf{r}', t)\, d\mathbf{r}' = \sum_{i=1}^{P} \mathbf{1}_{\Omega_i}(\mathbf{r}) \mathbf{W}_{ik}(t) \quad k = 1, \cdots, P. \tag{22}$$

These conditions are analogous to (11). A locally homogeneous solution of (6) or (7) can be written

$$\mathbf{V}(\mathbf{r}, t) = \sum_{i=1}^{P} \mathbf{1}_{\Omega_i}(\mathbf{r}) \mathbf{V}_i(t),$$

where the functions $\mathbf{V}_i$ satisfy the following system of ordinary differential equations

$$\mathbf{V}'_i(t) = -\mathbf{L}\mathbf{V}_i(t) + \sum_{k=1}^{P} \mathbf{W}_{ik}(t) \mathbf{S}(\mathbf{V}_k(t)) + \mathbf{I}^i_{\mathrm{ext}}(t), \tag{23}$$

for the voltage-based model and

$$\mathbf{V}_i'(t) = -\mathbf{L}\mathbf{V}_i(t) + \mathbf{S}\left(\sum_{k=1}^{P} \mathbf{W}_{ik}(t)\mathbf{V}_k(t) + \mathbf{I}_{\text{ext}}^i(t)\right), \tag{24}$$

for the activity-based model. The conditions for the existence and uniqueness of a locally homogeneous solution are given in the following theorem, analog to theorem 3.5:

Theorem 6.4 *If the external currents $\mathbf{I}_{\text{ext}}^k(t)$, $k = 1, \cdots, P$ and the connectivity matrixes $\mathbf{W}_{ik}(t)$, $i, k = 1, \cdots, P$ are continuous on some closed interval* J *containing 0, then for all sets of P vectors $\mathbf{U}_0^k, k = 1, \cdots, P$ of $\mathbb{R}^n$, there exists a unique solution $(\mathbf{U}_1(t), \cdots, \mathbf{U}_P(t))$ of (23) or (24) defined on a subinterval* J_0 *of* J *containing 0 such that $\mathbf{U}_k(0) = \mathbf{U}_0^k, k = 1, \cdots, P$.*

Proof. The system (23) can be written in the form

$$\mathcal{V}'(t) = -\mathcal{L}\mathcal{V}(t) + \mathcal{W}(t)\mathcal{S}(\mathcal{V}(t)) + \mathcal{I}_{\text{ext}}(t), \tag{25}$$

where $\mathcal{V}$ is the nP dimensional vector $\begin{pmatrix} \mathbf{V}_1 \\ \vdots \\ \mathbf{V}_P \end{pmatrix}$, $\mathcal{I}_{\text{ext}} = \begin{pmatrix} \mathbf{I}_{\text{ext}}^1 \\ \vdots \\ \mathbf{I}_{\text{ext}}^P \end{pmatrix}$, $\mathcal{S}(\mathcal{X}) = \begin{pmatrix} \mathbf{S}(\mathbf{X}_1) \\ \vdots \\ \mathbf{S}(\mathbf{X}_P) \end{pmatrix}$, $\mathcal{W}$ is the block matrix $(\mathbf{W}_{ik})_{i,k}$ and $\mathcal{L}$ is the block diagonal matrix whose diagonal elements are all equal to $\mathbf{L}$. Then we are dealing with a classical initial value problem of dimension nP and the proof of existence and uniqueness is similar to the one of theorem 3.5. A similar proof can be written in the case of system (24). □

Again, if $\mathcal{I}_{\text{ext}}$ and $\mathcal{W}$ are continuous on $\mathbb{R}$, the existence and uniqueness result extends to the whole time line $\mathbb{R}$.

Having proved existence and uniqueness of a locally homogeneous solution we consider the problem of characterizing its stability. The method is the same as in section 5. We consider the subset, noted $\mathcal{G}_c^P$, of the functions that are constant in the interior $\overset{\circ}{\Omega}_i$ of each region $\Omega_i, i = 1, \cdots, P$ (the interior $\overset{\circ}{A}$ of a subset A is defined as the biggest open subset included in A). We have the following lemma that echoes lemma 5.1

Lemma 6.5 *$\mathcal{G}_c^P$ is a complete linear subspace of $\mathbf{L}_n^2(\Omega)$. The orthogonal projection operator $\mathcal{P}_{\mathcal{G}_c^P}$ from $\mathbf{L}_n^2(\Omega)$ to $\mathcal{G}_c^P$ is defined by*

$$\mathcal{P}_{\mathcal{G}_c^P}(\mathbf{x})(\mathbf{r}) = \overline{\mathbf{x}}^P = \sum_{k=1}^{P} \mathbf{1}_{\Omega_k}(\mathbf{r}) \frac{1}{|\Omega_k|} \int_{\Omega_k} \mathbf{x}(\mathbf{r}')\, d\mathbf{r}'$$

The orthogonal complement $\mathcal{G}_c^{P\perp}$ of $\mathcal{G}_c^P$ is the subset of functions of $\mathbf{L}_n^2(\Omega)$ that have a zero average in each $\Omega_i, i = 1, \cdots, P$. The orthogonal projection operator $\mathcal{P}_{\mathcal{G}_c^{P\perp}}$ is equal to $\mathrm{Id} - \mathcal{P}_{\mathcal{G}_c^P}$. We also have

$$\mathcal{P}_{\mathcal{G}_c^{P\perp}}\mathbf{M}\mathbf{x} = \mathbf{M}\mathcal{P}_{\mathcal{G}_c^{P\perp}}\mathbf{x} \quad \forall \mathbf{x} \in \mathbf{L}_n^2(\Omega),\ \mathbf{M} \in \mathcal{M}_{n\times n} \tag{26}$$

Proof. The proof of this lemma is similar to the one of lemma 5.1. □
We have the following theorem, corresponding to theorems 5.2 and 5.3.

Theorem 6.6 *If* $\mathbf{W}$ *satisfies (22), a sufficient condition for the stability of a locally homogeneous solution to (6) (respectively (7)) is that the norm* $\|g_m^{L*}\|_{\mathcal{G}_c^{P\perp}}$ *(respectively* $\|h_m^L\|_{\mathcal{G}_c^{P\perp}}$*) of the restriction to* $\mathcal{G}_c^{P\perp}$ *of the compact operator* g_m^{L*} *(respectively* h_m^L*) be less than 1 for all* $t \in \mathrm{J}$.

Proof. The proof strictly follows the lines of the ones of theorems 5.2 and 5.3. □
This is true for every partition of Ω therefore we have the following

Proposition 6.7 *If the operator* g_m^{L*} *(respectively* h_m^L*) satisfies the condition of theorem 5.2 (respectively of theorem 5.3), then for every partition of* Ω*, corresponding locally homogeneous current, and* $\mathbf{W}$ *satisfying (22), the locally homogeneous solution of (6) (respectively (7)) is stable.*

Proof. It is clear that $\mathcal{G}_c \subset \mathcal{G}_c^P$, therefore $\mathcal{G}_c^{P\perp} \subset \mathcal{G}_c^{\perp}$ and $\|g_m^{L*}\|_{\mathcal{G}_c^{P\perp}} \leq \|g_m^{L*}\|_{\mathcal{G}_c^{\perp}}$ (respectively $\|h_m^L\|_{\mathcal{G}_c^{P\perp}} \leq \|h_m^L\|_{\mathcal{G}_c^{\perp}}$). □
Note that even if condition (22) is not satisfied by $\mathbf{W}$, i.e. if we do not guarantee the existence of a locally homogeneous solution for a given partition of Ω, we still have the result that, given that the operator g_m^{L*} (respectively h_m^L) satisfies the condition of theorem 5.2 (respectively of theorem 5.3), the "pseudo" locally homogeneous solutions, corresponding to a locally homogeneous input current, are stable. A numerical example is given below, see figure 13.

7 Numerical examples

We consider two ($n = 2$) one-dimensional ($q = 1$) populations of neurons, population 1 being excitatory and population 2 inhibitory. The set Ω is simply the closed interval $[0, 1]$. We note x the spatial variable and f the spatial frequency variable. We consider Gaussian functions, noted $G_{ij}(x), i, j = 1, 2$, from which we define the connectivity functions. Hence we have $G_{ij} = \mathcal{G}(0, \sigma_{ij})$. We consider three cases. In the first case we assume that the connectivity matrix is translation invariant (see sections 4.2 and 5.2). In the second case we relax this assumption and study the stability of the homogeneous solutions. The third case, finally, covers the case of the locally homogeneous solutions and their stability. In this section we have $S_1(x) = S_2(x) = 1/(1 + e^{-x})$.

7.1 The convolution case

We define $W_{ij}(x, x') = \alpha_{ij}\, G_{ij}(x - x')$, where the α_{ij}s are positive weights. The connectivity functions and their Fourier transforms are then given by

$$W_{ij}(x) = \frac{\alpha_{ij}}{\sqrt{2\pi\sigma_{ij}^2}} e^{-\frac{x^2}{2\sigma_{ij}^2}} \quad \widetilde{W}_{ij}(f) = \alpha_{ij} e^{-2\pi^2 f^2 \sigma_{ij}^2}$$

The matrices $\mathbf{W}(x)$ and $\widetilde{\mathbf{W}}(f)$ can be written[2]

$$\mathbf{W}(x) = \begin{bmatrix} \frac{\alpha_{11}}{\sqrt{2\pi\sigma_{11}^2}} e^{-\frac{x^2}{2\sigma_{11}^2}} & -\frac{\alpha_{12}}{\sqrt{2\pi\sigma_{12}^2}} e^{-\frac{x^2}{2\sigma_{12}^2}} \\ \frac{\alpha_{21}}{\sqrt{2\pi\sigma_{21}^2}} e^{-\frac{x^2}{2\sigma_{21}^2}} & -\frac{\alpha_{22}}{\sqrt{2\pi\sigma_{22}^2}} e^{-\frac{x^2}{2\sigma_{22}^2}} \end{bmatrix}$$

$$\widetilde{\mathbf{W}}(f) = \begin{bmatrix} \alpha_{11} e^{-2\pi^2 f^2 \sigma_{11}^2} & -\alpha_{12} e^{-2\pi^2 f^2 \sigma_{12}^2} \\ \alpha_{21} e^{-2\pi^2 f^2 \sigma_{21}^2} & -\alpha_{22} e^{-2\pi^2 f^2 \sigma_{22}^2} \end{bmatrix}$$

Therefore we have

$$\widetilde{\mathbf{W}}_{cm}^L(f) = \begin{bmatrix} \alpha_{11} S'_{1m} \tau_1 e^{-2\pi^2 f^2 \sigma_{11}^2} & -\alpha_{12} S'_{2m} \sqrt{\tau_1 \tau_2} e^{-2\pi^2 f^2 \sigma_{12}^2} \\ \alpha_{21} S'_{1m} \sqrt{\tau_1 \tau_2} e^{-2\pi^2 f^2 \sigma_{21}^2} & -\alpha_{22} S'_{2m} \tau_2 e^{-2\pi^2 f^2 \sigma_{22}^2} \end{bmatrix},$$

and

$$\overline{\widetilde{\mathbf{W}}}_{cm}^{L\,T}(f) \widetilde{\mathbf{W}}_{cm}^L(f) \overset{def}{=} \mathbf{X}_{cm}^L(f) = \begin{bmatrix} A & C \\ C & B \end{bmatrix},$$

where

$$A = S_1'^2 \tau_1 \left(\alpha_{11}^2 \tau_1 e^{-4\pi^2 \sigma_{11}^2 f^2} + \alpha_{21}^2 \tau_2 e^{-4\pi^2 \sigma_{21}^2 f^2} \right)$$

$$B = S_2'^2 \tau_2 \left(\alpha_{22}^2 \tau_2 e^{-4\pi^2 \sigma_{22}^2 f^2} + \alpha_{12}^2 \tau_1 e^{-4\pi^2 \sigma_{12}^2 f^2} \right),$$

and

$$C = -S_1' S_2' \sqrt{\tau_1 \tau_2} \left(\alpha_{21} \alpha_{22} \tau_2 e^{-2\pi^2 (\sigma_{21}^2 + \sigma_{22}^2) f^2} + \alpha_{12} \alpha_{11} \tau_1 e^{-2\pi^2 (\sigma_{12}^2 + \sigma_{11}^2) f^2} \right)$$

By construction the eigenvalues of this matrix are positive, the largest one, λ_{max}, being given by

$$\lambda_{max} = \frac{1}{2} \left(A + B + \sqrt{(A-B)^2 + 4C^2} \right)$$

Introducing the parameters $A_1 = (\tau_1 S_1' \alpha_{11})^2$, $A_2 = (\tau_2 S_2' \alpha_{22})^2$, $r = \tau_1/\tau_2$, $x_1 = \alpha_{21}/\alpha_{11}$, $x_2 = \alpha_{12}/\alpha_{22}$ we can rewrite A, B and C as follows

$$A = A_1 \left(e^{-4\pi^2 \sigma_{11}^2 f^2} + \frac{x_1^2}{r} e^{-4\pi^2 \sigma_{21}^2 f^2} \right) \quad B = A_2 \left(e^{-4\pi^2 \sigma_{22}^2 f^2} + r x_2^2 e^{-4\pi^2 \sigma_{12}^2 f^2} \right),$$

[2]We ignore for simplicity the convolution with the Fourier transform of $\mathbf{1}_{[-1,1]}(x)$.

and

$$C = -\sqrt{A_1 A_2}\left(\frac{x_1}{\sqrt{r}} e^{-2\pi^2(\sigma_{21}^2 + \sigma_{22}^2)f^2} + x_2\sqrt{r} e^{-2\pi^2(\sigma_{12}^2 + \sigma_{11}^2)f^2}\right)$$

The necessary and sufficient condition that the eigenvalues are less than 1 for all f is therefore

$$c(f) \stackrel{def}{=} 2 - A - B - \sqrt{(A-B)^2 + 4C^2} > 0 \quad \forall f \tag{27}$$

It depends on the spatial frequency f and the nine parameters A_1, A_2, x_1, x_2, r, and σ, the 2×2 matrix σ_{ij}, $i, j = 1, 2$.

We have solved equation (6) on $\Omega = [0, 1]$. We have sampled the interval with 100 points corresponding to 100 neural masses. The input $\mathbf{I}_{\text{ext}}$ is equal to $[W_1(t), W_2(t)]^T$, where where the $W_i(t)$s, $i = 1, 2$ are realizations of independent Brownian/Wiener processes shown in figure 2. We know that the solution is not homogeneous for the reasons exposed above.

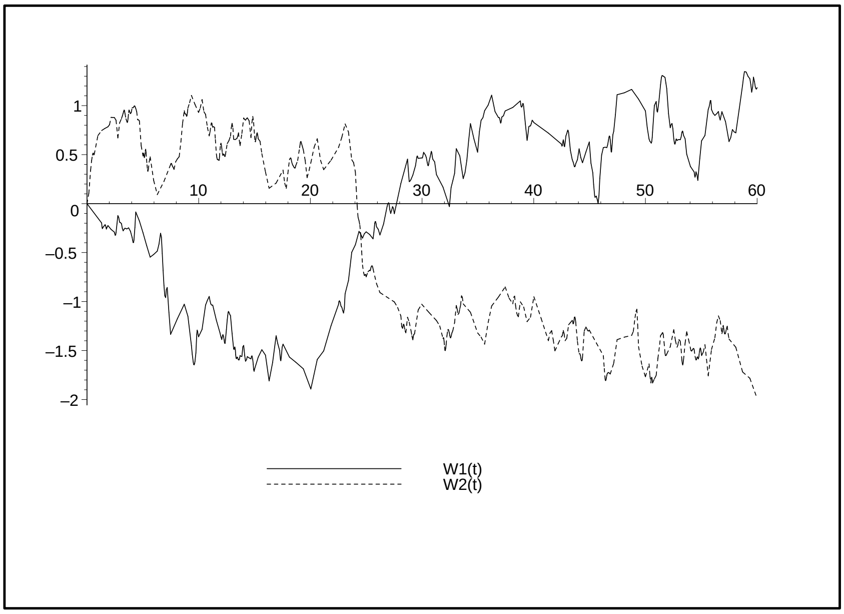

Figure 2: The two coordinates of the input $\mathbf{I}_{\text{ext}}(t)$ are realizations of independent Wiener processes.

This is illustrated in figure 3. The initial conditions are homogeneous and equal to either $(0, 0)$ or $(1, -1)$ for all neural masses state vectors $\mathbf{V}$. The figure 4 shows the stability of the first coordinate of the solution for one of the hundred neural masses: solutions become identical independently of the initial conditions. The bottom of the figure shows the plot of the function c defined in (27) as a function of the spatial frequency f. The constraint is

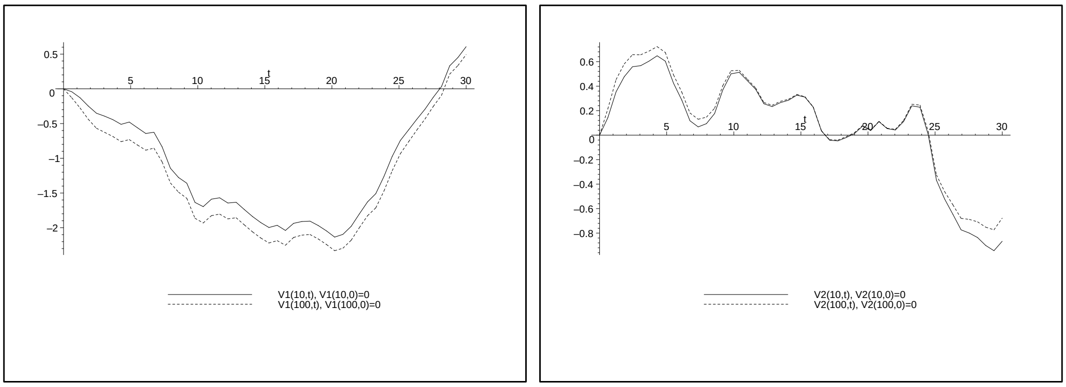

Figure 3: An illustration of the fact that when the connectivity matrix is translation invariant there does not exist in general a homogeneous solution: the neural masses do not synchronize. The lefthand graph shows the first coordinate of the solution when the input and the initial conditions are homogeneous. The righthand graph shows the same for the second coordinate.

satisfied for all frequencies. The parameters are

$$\boldsymbol{\alpha} = \begin{bmatrix} 2 & 1.414 \\ 1.414 & 2 \end{bmatrix} \quad \boldsymbol{\sigma} = \begin{bmatrix} 1 & 0.1 \\ 0.1 & 1 \end{bmatrix} \quad S'_{1m} = S'_{2m} = 0.25 \quad \tau_1 = \tau_2 = 1$$

We show in figure 5 a case where the stability is lost. The values of the parameters are

$$\boldsymbol{\alpha} = \begin{bmatrix} 565.7 & 565.7 \\ 565.7 & 565.7 \end{bmatrix} \quad \boldsymbol{\sigma} = \begin{bmatrix} 0.01 & 0.01 \\ 0.1 & 0.1 \end{bmatrix} \quad S'_{1m} = S'_{2m} = 0.25 \quad \tau_1 = \tau_2 = 1$$

The bottom of the figure shows the plot of the function c defined in (27) as a function of the spatial frequency f. The constraint is not satisfied at low spatial frequencies.

7.2 Homogeneous solutions

In the previous case the translation invariance of the connectivity matrix forbid the existence of homogeneous solutions. We can obtain a connectivity matrix satisfying condition (11) by defining

$$W_{ij}(x, x') = \alpha\alpha_{ij} \frac{G_{ij}(x - x')}{\int_0^1 G_{ij}(x - y)\, dy} \quad i, j = 1, 2,$$

where α and the α_{ij}s are connectivity weights. These functions are well defined since the denominator is never equal to 0 and the resulting connectivity matrix is in $\mathbf{L}^2_{2\times 2}([0,1]\times[0,1])$. It is shown in figure 6. The values of the parameters are given in (28). Proposition 6.3 guarantees the existence and uniqueness of a homogeneous solution for an initial condition in $\mathbf{L}^2_2(\Omega)$. According to theorem 5.2, a sufficient condition for this solution to be stable is that $\|g_m^{L*}\|_{\mathcal{G}_c^\perp} < 1$.

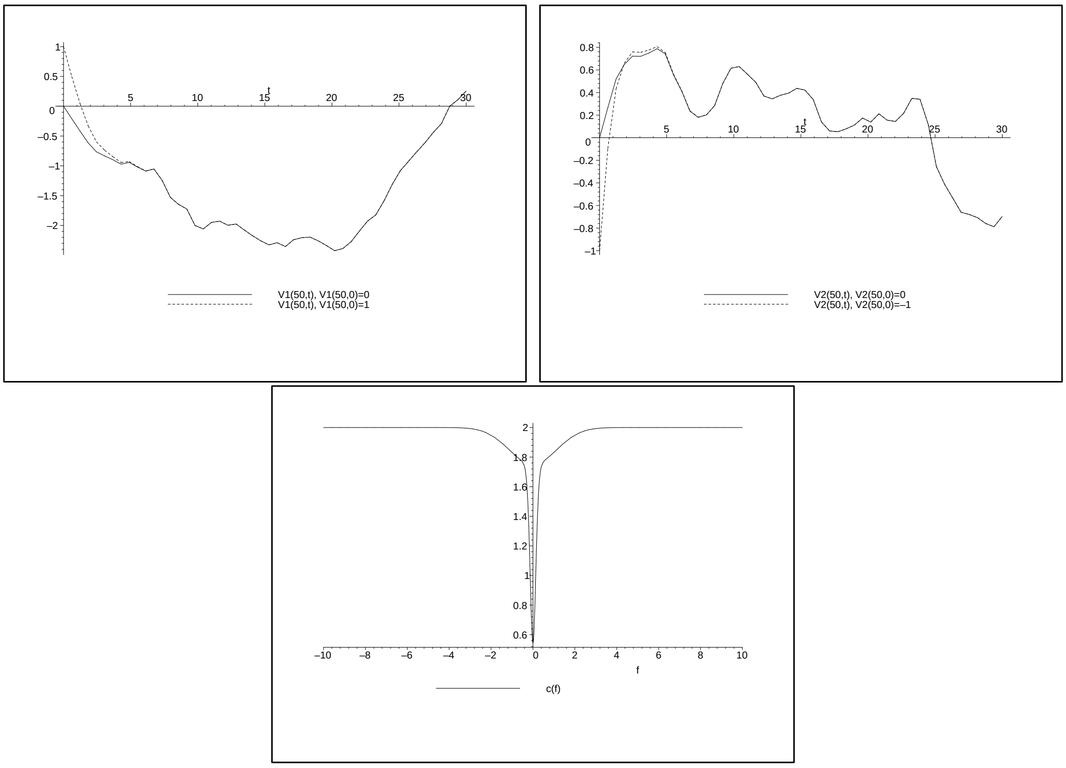

Figure 4: An illustration of the stability of the two coordinates of the solution with respect to the choice of the initial condition. Results are shown for the neural mass 50. The function $c(f)$ defined in (27) is shown at the bottom. It is positive for all spatial frequencies f.

We have solved equation (6) in $\Omega = [0, 1]$. We have sampled the interval with 100 points corresponding to 100 columns and solved the resulting system of 2x100 ordinary differential equations using Maple. The parameters are

$$\boldsymbol{\alpha} = \begin{bmatrix} 5.20 & 5.20 \\ 2.09 & 2.09 \end{bmatrix} \quad \boldsymbol{\sigma} = \begin{bmatrix} 0.1 & 0.1 \\ 1 & 1 \end{bmatrix} \quad \tau_1 = \tau_2 = 1 \quad \alpha = 1/20 \tag{28}$$

For these values we have $\|g_m^{L*}\|_{\mathcal{G}_c^\perp} \simeq 0.01$. All operator norms have been computed using the method described in appendix A. The initial conditions are drawn randomly and independently from the uniform distribution on $[-2, 2]$. The input $\mathbf{I}_{\text{ext}}(t)$ is equal to $[W_1(t), W_2(t)]^T$, where the $W_i(t)$s, $i = 1, 2$ are realizations of independent Brownian/Wiener processes shown in figure 2.

We show in figure 7 the synchronization of four (numbers 10, 26, 63 and 90) of the hundred neural masses. If we increase the value of α, the sufficient condition will eventually

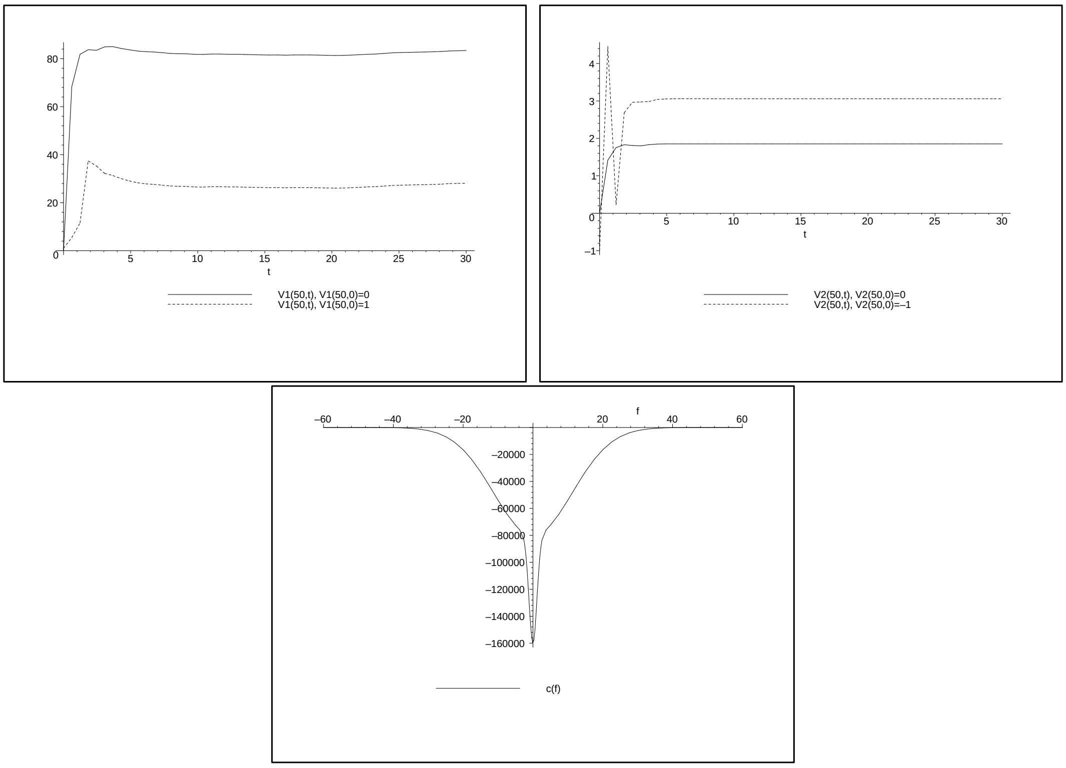

Figure 5: An illustration of the lack of stability of the two coordinates of the solution with respect to the choice of the initial condition. Results are shown for the neural mass 50. The function $c(f)$ defined in (27) is shown at the bottom. It is negative for a range of spatial frequencies f.

not be satisfied and we may be able to witness lack of synchronization. This is shown in figure 8 for $\alpha = 15$ corresponding to an operator norm $\|g_m^{L\,*}\|_{\mathcal{G}_c^\perp} \simeq 2.62$.

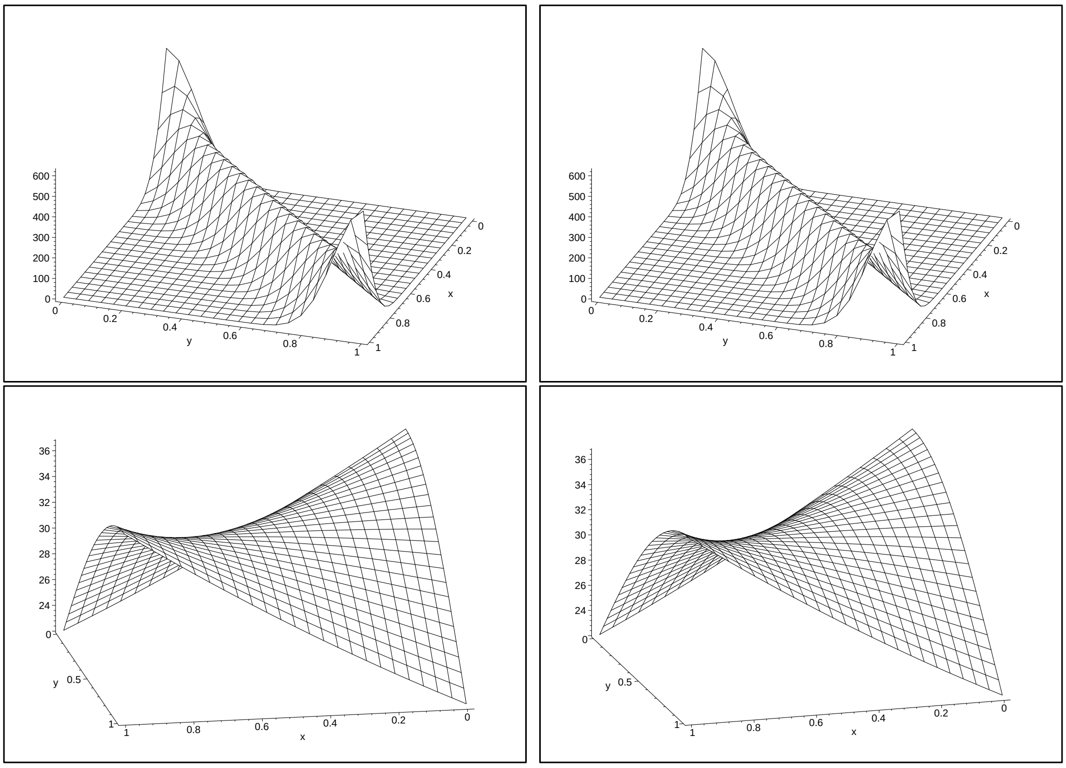

Figure 6: The four elements of the matrix $\mathbf{W}(x, x')$ in the homogeneous case.

7.3 Locally homogeneous solutions

We partition $\Omega = [0, 1]$ into $\Omega_1 = [0, 1/2[$ and $\Omega_2 = [1/2, 1]$. We can obtain a connectivity matrix satisfying condition (22) by defining

$$W_{ij}(x, x') = \begin{cases} \alpha\alpha_{ij}(x, x') \dfrac{G_{ij}(x - x')}{\displaystyle\int_0^{1/2} G_{ij}(x - y)\, dy}, \; x' \in \Omega_1 \\ \alpha\alpha_{ij}(x, x') \dfrac{G_{ij}(x - x')}{\displaystyle\int_{1/2}^{1} G_{ij}(x - y)\, dy}, \; x' \in \Omega_2 \end{cases},$$

with $\alpha_{ij}(x, x') = \alpha_{ij}^{kl}, \; x \in \Omega_k, \; x' \in \Omega_l, \; k, l = 1, 2.$
The resulting connectivity matrix is in $\mathbf{L}^2_{2\times 2}([0, 1] \times [0, 1])$. It is shown in figure 9.
The input $\mathbf{I}_{\text{ext}}(t)$ is equal to $[W_1(t), W_2(t)]^T$ in Ω_1 and to $[W_3(t), W_4(t)]^T$ in Ω_2, where the $W_i(t)$s, $i = 1, \cdots, 4$ are realizations of independent Brownian/Wiener processes shown in

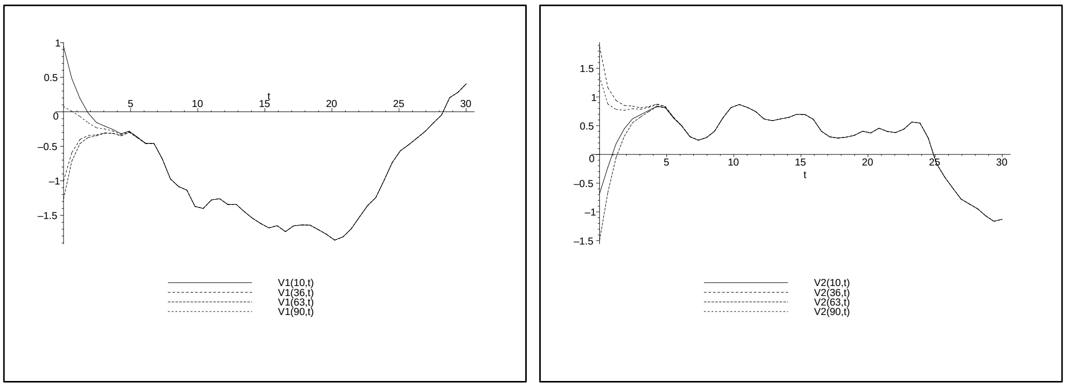

Figure 7: The synchronization of four of the hundred neural masses. The input is shown in figure 2. The first components are shown on the left, the second on the right.

figure 10. According to proposition 6.3 there exists a unique solution to (6) for a given initial condition in $\mathbf{L}_2^2(\Omega)$.

The parameters are

$$\boldsymbol{\alpha}^{11} = \begin{bmatrix} 5.21 & 0.23 \\ 0.23 & 5.21 \end{bmatrix} \quad \boldsymbol{\alpha}^{12} = \begin{bmatrix} 4.98 & 0.34 \\ 0.34 & 4.98 \end{bmatrix}$$
$$\boldsymbol{\alpha}^{21} = \begin{bmatrix} 4.75 & 0.45 \\ 0.45 & 4.75 \end{bmatrix} \quad \boldsymbol{\alpha}^{22} = \begin{bmatrix} 5.39 & 0.13 \\ 0.13 & 5.39 \end{bmatrix}$$
$$\boldsymbol{\sigma} = \begin{bmatrix} 0.05 & 0.075 \\ 0.1 & 0.03 \end{bmatrix} \quad \tau_1 = \tau_2 = 1 \quad \alpha = 1$$

For these values we have $\|g_m^{L\,*}\|_{\mathcal{G}_c^{P\perp}} \simeq 0.23$. The initial conditions are drawn randomly and independently from the uniform distribution on $[-10, 10]$ and $[-2, 2]$ for Ω_1 and on $[-20, 20]$ and $[-2, 2]$ for Ω_2.

We show in figure 11 the synchronization of two neural masses (numbers 10 and 26) in Ω_1 and two neural masses (numbers 63 and 90) in Ω_2 of the hundred neural masses. If we increase the value of α, the sufficient condition will eventually not be satisfied and we may be able to witness lack of synchronization. This is shown in figure 12 for $\alpha = 10$ corresponding to an operator norm $\|g_m^{L\,*}\|_{\mathcal{G}_c^{P\perp}} \simeq 2.3$.

As mentioned at the end of section 6.2, even if the connectivity function satisfies condition (11) but not condition (22) and the operator $g_m^{L\,*}$ satifies the condition of theorem 5.2 but not that of theorem 6.6 the existence of locally homogeneous solutions is not guaranteed but their stability is because of proposition 6.7. As shown in figure 13 these solutions can be very close to being locally homogeneous. This is potentially very interesting from the application viewpoint since one may say that if the system admits homogeneous solutions and if they are stable it can have locally homogeneous solutions without "knowing" the partition, and they are stable.

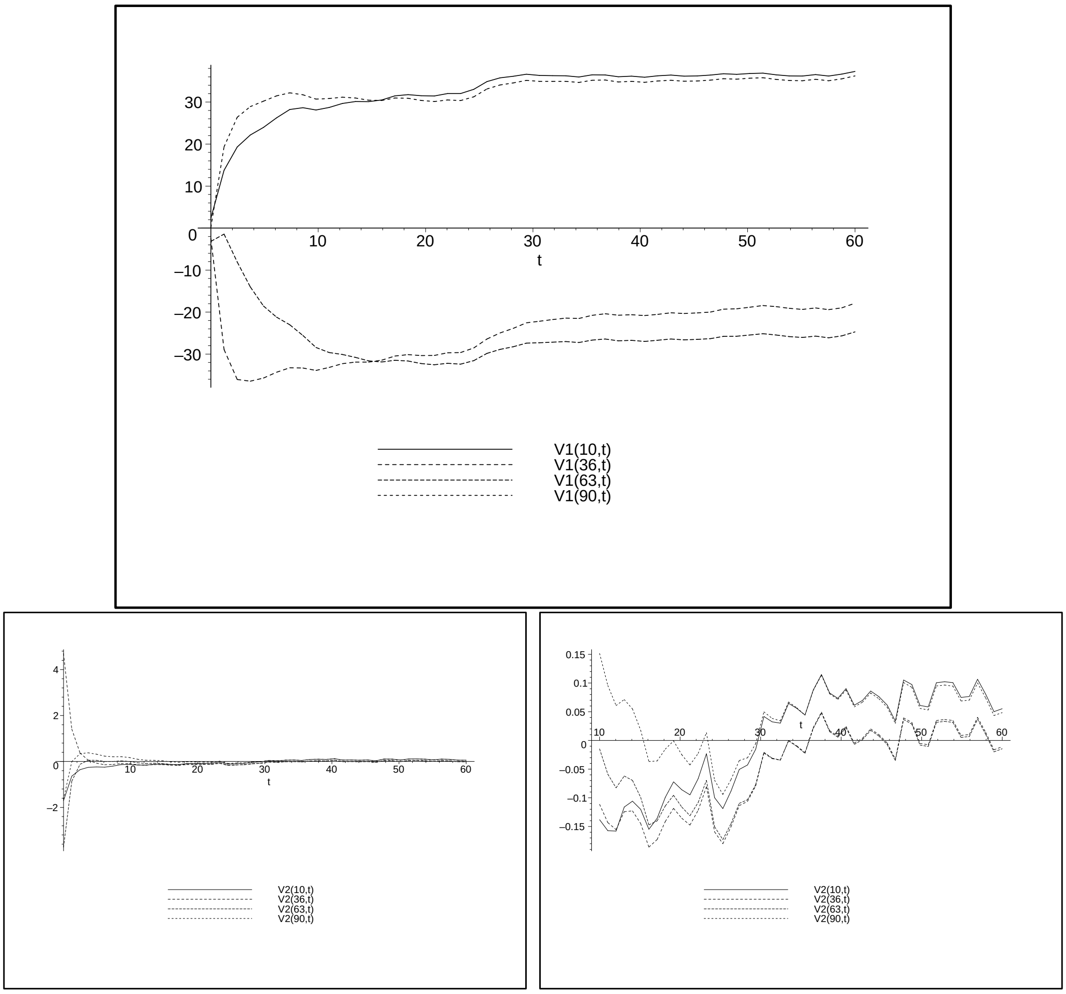

Figure 8: The lack of synchronization of four of the hundred neural masses when the sufficient condition of theorem 5.2 is not satisfied. The input is the same as in the previous example. The first components are shown on the first row, the second on the second. The left figure shows the complete graph for $0 \leq t \leq 60s$, the right figure is a zoom on $10 \leq t \leq 60s$.

8 Conclusion

We have studied the existence, uniqueness, and stability with respect to the initial conditions of a solution to two examples of nonlinear integro-differential equations that describe the spatio-temporal activity of sets of neural masses. These equations involve space and time

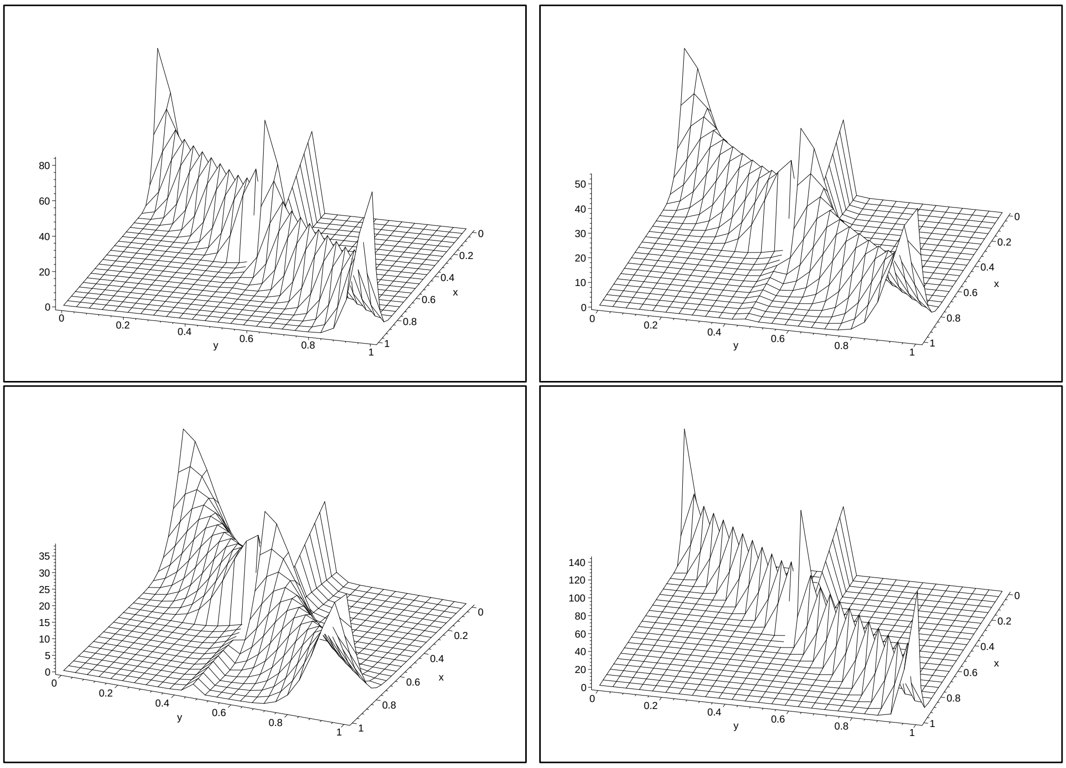

Figure 9: The four elements of the matrix $\mathbf{W}(x, x')$ in the locally homogeneous case.

varying, possibly non-symmetric, intra-cortical connectivity kernels. Contributions from white matter afferents are represented by external inputs. Sigmoidal nonlinearities arise from the relation between average membrane potentials and instantaneous firing rates. The intra-cortical connectivity functions have been shown to naturally define compact operators of the functional space of interest. Using methods of functional analysis, we have characterized the existence, uniqueness, and stability to the initial conditions of a solution of these equations for general, homogeneous (i.e. independent of the spatial variable), and locally homogeneous inputs. In all cases we have provided sufficient conditions for the solutions to be absolutely stable, that is to say independent of the initial state of the field. These conditions involve the connectivity functions, the maximum slopes of the sigmoids, as well as the time constants used to described the time variation of the postsynaptic potentials. We think that an important contribution of our work is the application of the theory of compact operators in a Hilbert space to the problem of neural fields with the effect of providing very simple mathematical answers to the questions asked by modellers in neuroscience. This application may not be limited to the specific topic of this paper.

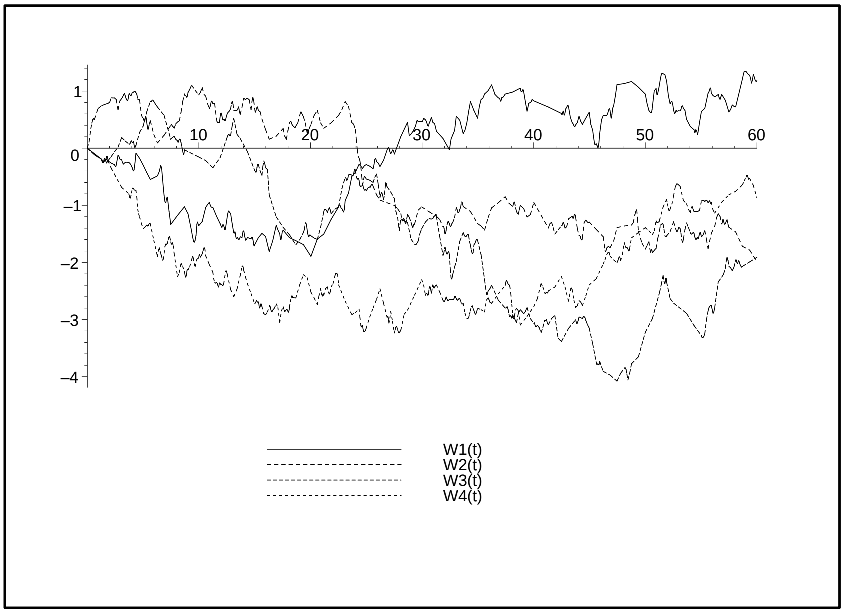

Figure 10: The two coordinates of the input $\mathbf{I}_{\text{ext}}(t)$ in Ω_1 and Ω_2 are realizations of four independent Wiener processes (W_1 and W_2 are identical to those shown in figure 2).

A Notations and background material

A.1 Matrix norms and spaces of functions

We note $\mathcal{M}_{n\times n}$ the set of $n \times n$ real matrices. We consider the matrix norm,

$$\|\mathbf{M}\|_\infty = \max_i \sum_j |M_{ij}|$$

We note $\mathbf{C}_{n\times n}(\Omega)$ the set of continuous functions from Ω to $\mathcal{M}_{n\times n}$ with the infinity norm. This is a Banach space for the norm induced by the infinity norm on $\mathcal{M}_{n\times n}$. Let $\mathbf{M}$ be an element of $\mathbf{C}_{n\times n}(\Omega)$, we note and define $\|\mathbf{M}\|_{n\times n,\infty}$ as

$$\|\mathbf{M}\|_{n\times n,\infty} = \sup_{\mathbf{r}\in\Omega} \max_i \sum_j |M_{ij}(\mathbf{r})| = \max_i \sup_{\mathbf{r}\in\Omega} \sum_j |M_{ij}(\mathbf{r})|$$

We also note $\mathbf{C}_n(\Omega)$ the set of continuous functions from Ω to $\mathbb{R}^n$ with the infinity norm. This is also a Banach space for the norm induced by the infinity norm of $\mathbb{R}^n$. Let $\mathbf{x}$ be an element of $\mathbf{C}_n(\Omega)$, we note and define $\|\mathbf{x}\|_{n,\infty}$ as

$$\|\mathbf{x}\|_{n,\infty} = \sup_{\mathbf{r}\in\Omega} \max_i |x_i(\mathbf{r})| = \max_i \sup_{\mathbf{r}\in\Omega} |x_i(\mathbf{r})|$$

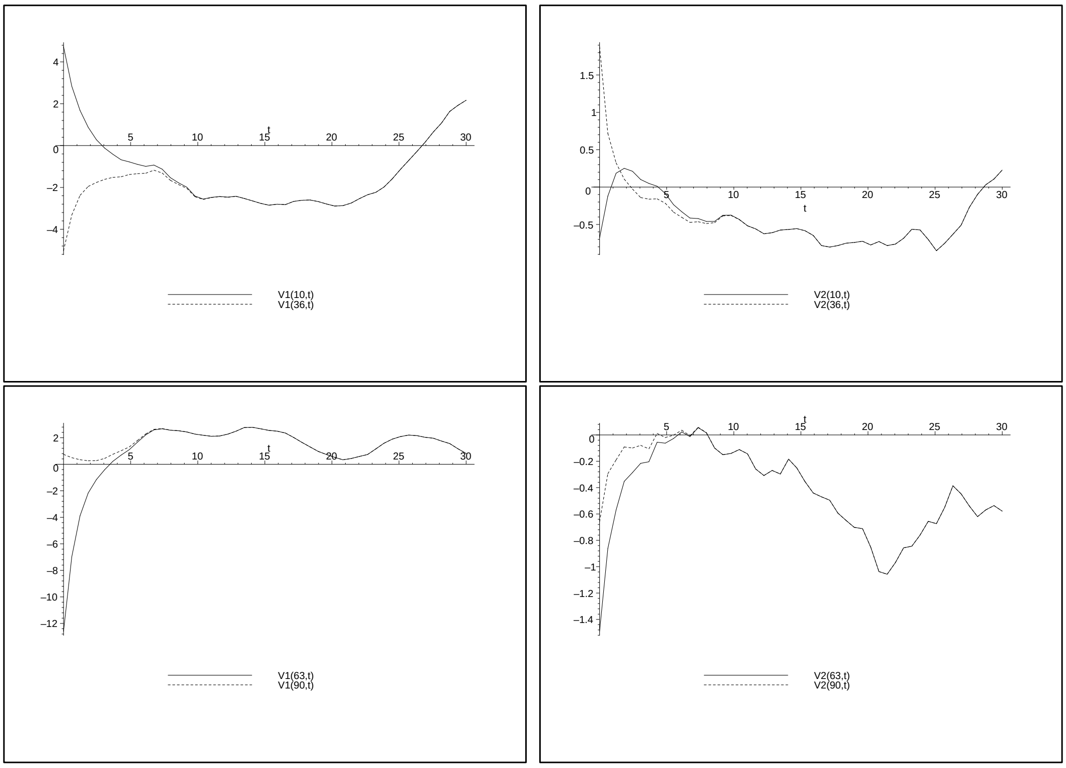

Figure 11: The synchronization of two neural masses in Ω_1 and two neural masses in Ω_2. The input is shown in figure 10. The first components are shown on the left, the second on the right.

We can similarly define the norm $\|.\|_{n\times n,\infty}$ (resp. $\|.\|_{n,\infty}$) for the space $\mathbf{C}_{n\times n}(\Omega\times\Omega)$ (resp. $\mathbf{C}_n(\Omega\times\Omega)$).
We have the following

Lemma A.1 *Given* $\mathbf{x}\in\mathbf{C}_n(\Omega)$ *and* $\mathbf{M}\in\mathbf{C}_{n\times n}(\Omega)$ *we have*

$$\|\mathbf{M}\,\mathbf{x}\|_{n,\infty} \le \|\mathbf{M}\|_{n\times n,\infty}\,\|\mathbf{x}\|_{n,\infty}$$

More precisely, we have for all $\mathbf{r}\in\Omega$

$$\|\mathbf{M}(\mathbf{r})\,\mathbf{x}(\mathbf{r})\|_\infty \le \|\mathbf{M}(\mathbf{r})\|_\infty\|\mathbf{x}(\mathbf{r})\|_\infty$$

The same results hold for $\Omega\times\Omega$ *instead of* Ω.

Proof. Let $\mathbf{y}=\mathbf{M}\,\mathbf{x}$, we have

$$y_i(\mathbf{r}) = \sum_j M_{ij}(\mathbf{r})x_j(\mathbf{r})$$

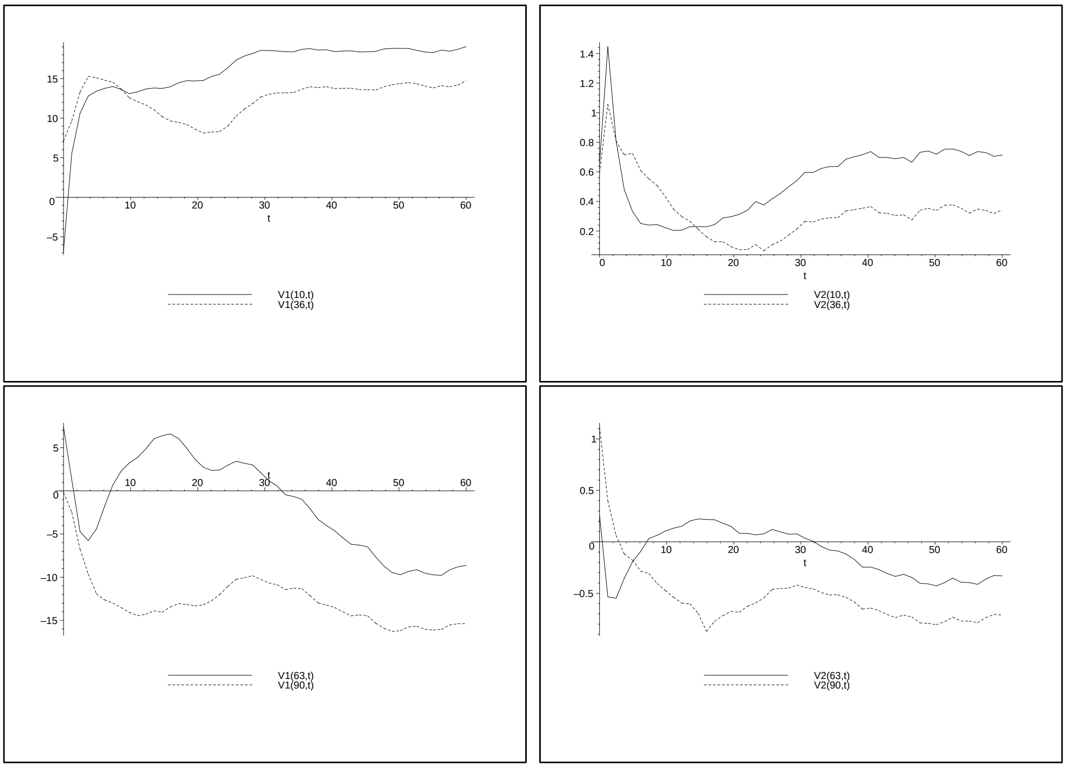

Figure 12: The lack of synchronization of two neural masses in Ω_1 and two neural masses in Ω_2. The input is shown in figure 10. The first components are shown on the left, the second on the right.

and therefore

$$|y_i(\mathbf{r})| \leq \sum_j |M_{ij}(\mathbf{r})| \, |x_j(\mathbf{r})| \leq \sum_j |M_{ij}(\mathbf{r})| \, \|\mathbf{x}(\mathbf{r})\|_\infty,$$

so, taking the $\max_i$

$$\|\mathbf{y}(\mathbf{r})\|_\infty \leq \|\mathbf{M}(\mathbf{r})\|_\infty \, \|\mathbf{x}(\mathbf{r})\|_\infty$$

from which the first statement easily comes. □
We also consider the Frobenius norm on $\mathcal{M}_{n\times n}$

$$\|\mathbf{M}\|_F = \sqrt{\sum_{i,j=1}^{n} M_{ij}^2},$$

and consider the space $\mathbf{L}^2_{n\times n}(\Omega\times\Omega)$ of the functions from $\Omega\times\Omega$ to $\mathcal{M}_{n\times n}$ whose Frobenius norm is in $L^2(\Omega\times\Omega)$. If $\mathbf{W} \in \mathbf{L}^2_{n\times n}(\Omega\times\Omega)$ we note $\|\mathbf{W}\|_F^2 = \int_{\Omega\times\Omega} \|\mathbf{W}(\mathbf{r},\mathbf{r}')\|_F^2 \, d\mathbf{r}\, d\mathbf{r}'$.

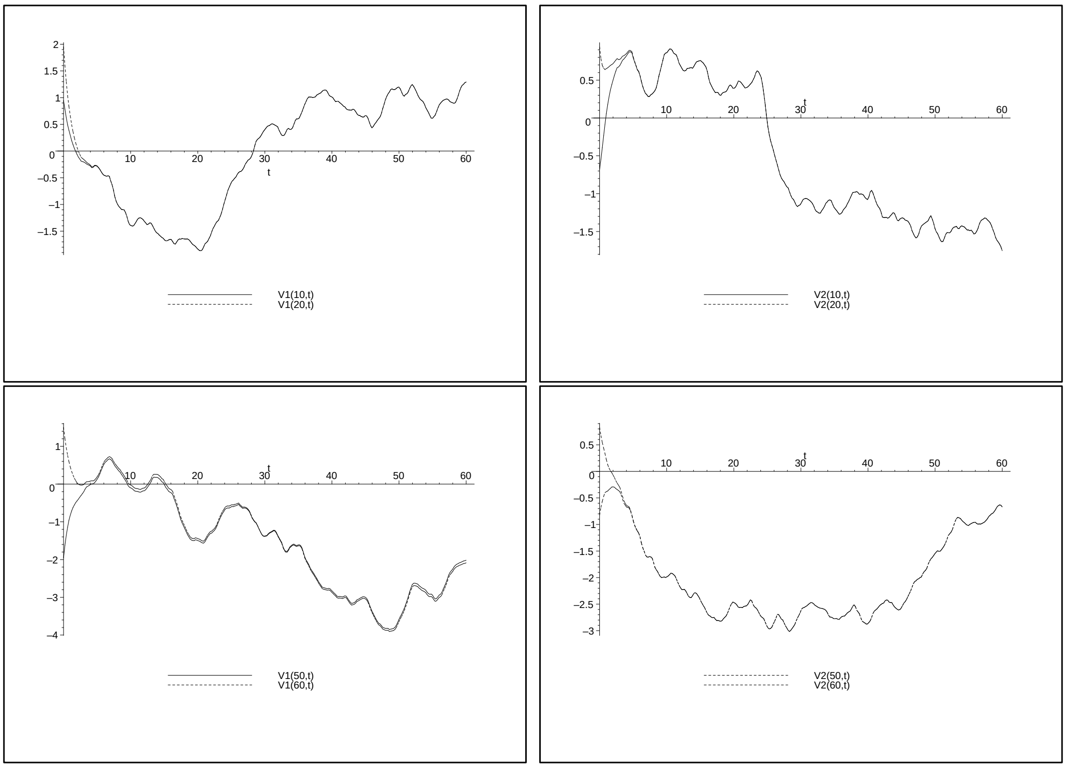

Figure 13: The connectivity function satisfies condition (11) but not condition (22) and the operator $g_m^L{}^*$ satifies the condition of theorem 5.2, not that of theorem 6.6. The input is locally homogeneous, as in figure 11. The solution is stable, because of theorem 5.2 and almost locally homogeneous.The first components are shown on the left, the second on the right.

Note that this implies that each element w_{ij}, i, $j = 1, \cdots, n$ is in $L^2(\Omega \times \Omega)$. We note $\mathbf{L}^2_n(\Omega)$ the set of square-integrable mappings from Ω to $\mathbb{R}^n$ and $\|\mathbf{x}\|_{n,2} = (\sum_j \|x_j\|_2^2)^{1/2}$ the corresponding norm. We have the following

Lemma A.2 *Given* $\mathbf{x} \in \mathbf{L}^2_n(\Omega)$ *and* $\mathbf{W} \in \mathbf{L}^2_{n\times n}(\Omega\times\Omega)$*, we define* $\mathbf{y}(\mathbf{r}) = \int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}')\mathbf{x}(\mathbf{r}')\, d\mathbf{r}'$*. This integral is well defined for almost all* $\mathbf{r}$*,* $\mathbf{y}$ *is in* $\mathbf{L}^2_n(\Omega)$ *and we have*

$$\|\mathbf{y}\|_{n,2} \leq \|\mathbf{W}\|_F \, \|\mathbf{x}\|_{n,2}.$$

Proof. Since each w_{ij} is in $L^2(\Omega \times \Omega)$, $w_{ij}(\mathbf{r}, .)$ is in $L^2(\Omega)$ for almost all $\mathbf{r}$, thanks to Fubini's theorem. So $w_{ij}(\mathbf{r}, .)x_j(.)$ is integrable for almost all $\mathbf{r}$ from what we deduce that $\mathbf{y}$ is well-defined for almost all $\mathbf{r}$. Next we have

$$|y_i(\mathbf{r})| \leq \sum_j \left| \int_\Omega w_{ij}(\mathbf{r}, \mathbf{r}')\, x_j(\mathbf{r}')\, d\mathbf{r}' \right|$$

and (Cauchy-Schwarz):

$$|y_i(\mathbf{r})| \leq \sum_j \left(\int_\Omega w_{ij}^2(\mathbf{r}, \mathbf{r}')\, d\mathbf{r}' \right)^{1/2} \|x_j\|_2,$$

from where it follows that (Cauchy-Schwarz again, discrete version):

$$|y_i(\mathbf{r})| \leq \left(\sum_j \|x_j\|_2^2 \right)^{1/2} \left(\sum_j \int_\Omega w_{ij}^2(\mathbf{r}, \mathbf{r}')\, d\mathbf{r}' \right)^{1/2} = \|\mathbf{x}\|_{n,2} \left(\sum_j \int_\Omega w_{ij}^2(\mathbf{r}, \mathbf{r}')\, d\mathbf{r}' \right)^{1/2},$$

from what it follows that $\mathbf{y}$ is in $\mathbf{L}^2_n(\Omega)$ (thanks again to Fubini's theorem) and

$$\|\mathbf{y}\|_{n,2}^2 \leq \|\mathbf{x}\|_{n,2}^2 \sum_{i,j} \int_{\Omega\times\Omega} w_{ij}^2(\mathbf{r}, \mathbf{r}')\, d\mathbf{r}'\, d\mathbf{r} = \|\mathbf{x}\|_{n,2}^2 \, \|\mathbf{W}\|_F^2.$$

□

A.2 Banach space-valued functions

A useful viewpoint that is used in this article is to consider the state vector of the neural field as a mapping from a closed time interval J containing the origin 0 into one of the spaces discussed in the previous section. We note $C(\mathrm{J}; \mathbf{C}_\mathrm{n}(\Omega))$ the set of continuous mappings from J to the Banach space $\mathbf{C}_n(\Omega)$ and $C(\mathrm{J}; \mathbf{L}^2_\mathrm{n}(\Omega))$ the set of continuous mappings from J to the Hilbert (hence Banach) space $\mathbf{L}^2_n(\Omega)$, see, e.g., [9].

A.3 Computation of operator norms

We give a method to compute the norms $\|g\|_{\mathcal{G}}$ and $\|g^*\|_{\mathcal{G}_c^\perp}$ for an operator g of the form

$$g(\mathbf{x})(\mathbf{r}) = \int_\Omega \mathbf{W}(\mathbf{r}, \mathbf{r}')\, \mathbf{x}(\mathbf{r}')\, d\mathbf{r}'.$$

Since $\mathcal{G}$ (respectively $\mathcal{G}_c^\perp$) is dense in the Hilbert space $\mathbf{L}^2(\Omega)$ (respectively $\mathbf{L}_0^2(\Omega)$, the subspace of $\mathbf{L}^2(\Omega)$ of functions with zero mean), we have $\|g\|_{\mathcal{G}}=\|g\|_{\mathbf{L}^2}$ and $\|g^*\|_{\mathcal{G}_c^\perp} = \|g^*\|_{\mathbf{L}_0^2}$.
We consider the compact self-adjoint operators

$$G = g^* g : \mathbf{L}^2 \to \mathbf{L}^2$$

and

$$G_c^\perp = g^* \mathcal{P} g : \mathbf{L}_0^2 \to \mathbf{L}_0^2,$$

where $\mathcal{P}$ is the orthogonal projection on $\mathbf{L}_0^2$. We compute the norms of the two self-adjoint positive operators G and $G_c^\perp$, and use the relations

$$\|G\|_{\mathbf{L}^2} = \|g\|_{\mathbf{L}^2}^2,$$

and

$$\|G_c^\perp\|_{\mathbf{L}_0^2} = \|g^*\mathcal{P}^*\mathcal{P}g\|_{\mathbf{L}_0^2} = \|g^*\mathcal{P}^*\|_{\mathbf{L}_0^2}^2 = \|g^*\|_{\mathbf{L}_0^2}^2.$$

Let T be a compact self-adjoint positive operator on a Hilbert space $\mathcal{H}$. Its largest eigenvalue is $\lambda = \|T\|_{\mathcal{H}}$. Let $x \in \mathcal{H}$. If $x \notin \mathrm{Ker}(\lambda \mathrm{Id} - T)^\perp$, then, according to, e.g., [7],

$$\lim_{n\to\infty} \|T^n x\|_{\mathcal{H}} / \|T^{n-1}x\|_{\mathcal{H}} = \lambda.$$

This method can be applied to g_m^L and h_m^L, and generalized to the computation of the $\|.\|_{\mathcal{G}_c^{P\,\perp}}$ norm.

B Global existence of solutions

In this appendix, we complete the proof of proposition (3.4) by computing the constant $\tau > 0$ such that for any initial condition $(t_0, \mathbf{V}_0) \in \mathbb{R} \times \mathcal{F}$, the existence and uniqueness of the solution $\mathbf{V}$ is guaranteed on the closed interval $[t_0 - \tau, t_0 + \tau]$.
We refer to [1] and exploit the

Theorem B.1 *Let $\mathcal{F}$ be a Banach space and $c > 0$. We consider the initial value problem:*

$$\begin{cases} \mathbf{V}'(t) &= f(t, \mathbf{V}(t)) \\ \mathbf{V}(t_0) &= \mathbf{V}_0 \end{cases}$$

for $|t - t_0| < c$ where $\mathbf{V}_0$ is an element of $\mathcal{F}$ and $f : [t_0 - c, t_0 + c] \times \mathcal{F} \to \mathcal{F}$ is continuous. Let $b > 0$. We define the set $Q_{b,c} \equiv \{(t, \mathbf{X}) \in \mathbb{R} \times \mathcal{F},\ |t - t_0| \le c \quad \text{and} \quad \|\mathbf{X} - \mathbf{V}_0\| \le b\}$.

Assume the function $f : Q_{b,c} \to \mathcal{F}$ is continuous and uniformly Lipschitz continuous with respect to its second argument, ie

$$\|f(t,\mathbf{X}) - f(t,\mathbf{Y})\| \leq K_{b,c}\|\mathbf{X}-\mathbf{Y}\|,$$

where $K_{b,c}$ is a constant independent of t.
Let $M_{b,c} = \sup_{Q_{b,c}} \|f(t,\mathbf{X})\|$ and $\tau_{b,c} = \min\{b/M_{b,c}, c\}$.
Then the initial value problem has a unique continuously differentiable solution $\mathbf{V}(.)$ defined on the interval $[t_0 - \tau_{b,c}, t_0 + \tau_{b,c}]$.

In our case, $f = f_v$ and all the hypotheses of the theorem hold, thanks to proposition 3.2 and the hypotheses of proposition 3.4, with

$$K_{b,c} = \|\mathbf{L}\|_\infty + |\Omega|\, DS_m \sup_{|t-t_0|\leq c} \|\mathbf{W}(\cdot,\cdot,t)\|_{n\times n,\infty},$$

where the sup is well defined (continuous function on a compact domain).
We have

$$M_{b,c} \leq \|\mathbf{L}\|_\infty\,(\|\mathbf{V}_0\|_{n,\infty} + b) + |\Omega|\, S_m\, W + I,$$

where $W = \sup_{|t-t_0|\leq c} \|\mathbf{W}(\cdot,\cdot,t)\|_{n\times n,\infty}$ and $I = \sup_{|t-t_0|\leq c} \|\mathbf{I}_{\text{ext}}(\cdot,t)\|_{n,\infty}$.
So

$$b/M_{b,c} \geq \frac{1}{\|\mathbf{L}\|_\infty + \frac{\|\mathbf{L}\|_\infty\,\|\mathbf{V}_0\|_{n,\infty} + |\Omega|\, S_m\, W + I}{b}}.$$

Hence, for $c \geq \frac{1}{2\|\mathbf{L}\|_\infty}$ and b big enough, we have $\tau_{b,c} \geq \frac{1}{2\|\mathbf{L}\|_\infty}$ and we can set $\tau = \frac{1}{2\|\mathbf{L}\|_\infty}$.

A similar proof applies to the case $f = f_a$ and the one of proposition 6.3.

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Contents

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Unité de recherche INRIA Futurs : Parc Club Orsay Université - ZAC des Vignes
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