# Deep Inference for Hybrid Logic

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**Abstract.** This paper describes work in progress on using deep inference for designing a deductive system for hybrid logic. We will see a cut-free system and prove its soundness and completeness. An immediate observation about the system is that there is no need for additional rewrite rules as in Blackburn's tableaux, nor substitution rules as in Seligman's sequent system.

## 1 Introduction

The point of hybrid logics is to internalize constructs of the meta level into the syntax of the object level. This idea has been employed in the case of modal logics whose semantics is usually given in terms of Kripke-frames. While the ordinary modalities $\Box$ and $\Diamond$ do only have access to points in the frame which are reachable from the current point, the hybrid language has full access to every single point in the frame.[1] This leads to an increased expressivity (e.g., we can now speak about irreflexive reachability relations) without loss in complexity (satisfiability remains PSPACE-complete) [Bla00].

Such an enrichment of the language imposes certain challenges to the deductive system. For example the sequent calculus system proposed by Seligman in [Sel97] needs substitution rules which act globally on the sequent, the tableau system by Tzakova [Tza99] (see also [BB06]) needs to use prefixes, and the tableau system introduced by Blackburn [Bla00] needs additional rewrite rules which have a different behaviour than usual tableau rules.

The actual reason for the necessity of these alien constructs in the deductive systems is that *the meta language of the deductive formalism* (here sequent calculus and tableaux) *is different from the meta language of the logic* (here hybrid modal logic). Whenever there is such a discrepancy between the two meta languages, one has to expect difficulties in designing a concrete deductive system for the logic in question. The bigger the discrepancies, the bigger the difficulties. Another well-known example of such a situation is the modal logic $\mathsf{S5}$, for which there is no cut-free sequent system, unless one resorts to constructs like hypersequents, higher arity sequents, displayed sequents, or the usage of a hybrid language (see [Sto04] for a survey).

However, recently a new deductive formalism, called *the calculus of structures*, has been introduced, which has no "built-in meta language" because it collapses object and meta level. This collapse is achieved by the consequent use of *deep inference*: the inference rules do not work on the root connective of the formula in question, but can do arbitrary rewriting deep inside the formulas. This simple idea has been successful for various logics imposing problems on the sequent calculus, e.g., non-commutative logics [Gug07,DG04] and various modal logics [SS05], including $\mathsf{S5}$ [Sto04].

Given this success together with the first sentence of this introduction, one should expect that the calculus of structures provides the right formalism for dealing with hybrid logics.

[1] Strictly speaking, only the named points in a model are accessible. But since a formula is a finite object, it can directly speak about only a finite number of points in the model, and each of them can be given a name.

The purpose of this work is to investigate to what extend these expectations can be fulfilled. We are going to carry out the exercise of producing a cut-free deductive system for hybrid logic employing deep inference.

## 2 Formulas and Inference Rules

The syntax that we use here is a hybrid between the one used by Blackburn in [Bla00] and the one usually used for deep inference systems (e.g., [BT01,GS01]). We start from two sets of primitives, the set $\mathscr{V} = \{a, b, c, \ldots\}$ of *propositional variables*, and the set $\mathscr{N} = \{s, r, u, \ldots\}$ of *nominals*. The elements of the set $\mathscr{A} = \mathscr{V} \cup \mathscr{N}$ are called *atoms*. The set $\mathscr{F}$ of formulas is generated by the grammar:

$$\mathscr{F} ::= \mathscr{A} \mid \bar{\mathscr{A}} \mid \mathbf{f} \mid \mathbf{t} \mid [\mathscr{F}, \mathscr{F}] \mid (\mathscr{F} \wedge \mathscr{F}) \mid \Diamond\mathscr{F} \mid \Box\mathscr{F} \mid \langle \mathscr{N} : \mathscr{F} \rangle$$

The elements of the set $\mathscr{F}$ are denoted by capital Latin letters ($A$, $B$, $C$, ...). The formula $[A, B]$ denotes the *disjunction* of $A$ and $B$, and the formula $(A \wedge B)$ denotes the *conjunction* of $A$ and $B$. The constants $\mathbf{f}$ and $\mathbf{t}$ stand for *falsum* and *truth*, respectively. The $\Box$ and $\Diamond$ are the usual modalities. The difference to usual modal logics lies in the formulas of the shape $\langle s \colon A \rangle$, where the left subformula has to be a nominal. Informally speaking, the meaning is that "$A$ is true in state $s$".

Note that the negation $\overline{(-)}$ is defined *a priory* only on atoms, but via the usual De Morgan equations we can define negation for all formulas:

$$\begin{array}{lll} \bar{\mathbf{f}} = \mathbf{t} & \overline{\Diamond A} = \Box \bar{A} & \overline{[A, B]} = (\bar{B} \wedge \bar{A}) \\ \bar{\mathbf{t}} = \mathbf{f} & \overline{\Box A} = \Diamond \bar{A} & \overline{(A \wedge B)} = [\bar{B}, \bar{A}] \\ \bar{\bar{a}} = a & \bar{\bar{s}} = s & \overline{\langle s \colon A \rangle} = \langle s \colon \bar{A} \rangle \end{array}$$

It follows immediately that $\bar{\bar{A}} = A$ for all formulas $A$. An implication $A \rightarrow B$ is encoded via negation and disjunction as $[\bar{A}, B]$.

We are now ready to see the inference rules. Figure 1 shows the rules of *system* $\mathsf{BH}{\downarrow}$. The letters $\mathsf{B}$ and $\mathsf{H}$ stand for "Basic Hybrid logic", and the $\downarrow$ indicates that we have here the so called *down fragment*, which represents the cut-free version of the system.[2] We can obtain the full system by adding to each rule its up-version, which is obtained by negating and exchanging premise and conclusion of the rule [Brü03,Str03]. The resulting system $\mathsf{BH}{\downarrow\uparrow}$ is shown in Figure 2. The inference rules in Figures 1 and 2 are (almost) all of the shape

$$\rho \frac{S\{A\}}{S\{B\}} \quad .$$

They should be read as usual term rewriting rules $A \rightarrow B$ that can be applied anywhere inside a formula context $S\{\ \}$. Only the rules $\mathsf{v}{\downarrow}$ and $\mathsf{v}{\uparrow}$ are of different shape. They can be applied only in contexts of a special shape. They also have the side condition that the nominal $v$ may not appear anywhere else in the formula. A *derivation* $\Delta$, denoted by

$$\begin{array}{c} P \\ \mathsf{S} \Big\| \Delta \\ Q \end{array}$$

is a rewriting path using the inference rules in $\mathsf{S}$ starting with $P$ and ending with $Q$. The formula $P$ is called the *premise* of $\Delta$, and $Q$ is the *conclusion* of $\Delta$. A *proof* of a formula

[2] This $\downarrow$ should not be confused with the binding operator $\downarrow x$.

$$\mathsf{ai}\!\downarrow \frac{S\{\mathbf{t}\}}{S\{[\bar{a}, a]\}} \qquad \mathbf{t}\!\downarrow \frac{S\{A\}}{S\{(A \wedge \mathbf{t})\}} \qquad \mathbf{f}\!\downarrow \frac{S\{A\}}{S\{[A, \mathbf{f}]\}}$$

$$\mathsf{s}\, \frac{S\{(A \wedge [B, C])\}}{S\{[(A \wedge B), C]\}} \qquad \sigma\!\downarrow \frac{S\{[B, A]\}}{S\{[A, B]\}} \qquad \alpha\!\downarrow \frac{S\{[A, [B, C]]\}}{S\{[[A, B], C]\}}$$

$$\mathsf{w}\!\downarrow \frac{S\{\mathbf{f}\}}{S\{A\}} \qquad \mathsf{c}\!\downarrow \frac{S\{[A, A]\}}{S\{A\}}$$

$$\mathsf{e}^{\Box}\!\downarrow \frac{S\{\mathbf{t}\}}{S\{\Box\mathbf{t}\}} \qquad \mathsf{k}^{\Box}\!\downarrow \frac{S\{\Box[A, B]\}}{S\{[\Box A, \Diamond B]\}}$$

$$\mathsf{e}^{:}\!\downarrow \frac{S\{\mathbf{t}\}}{S\{\langle s\colon \mathbf{t}\rangle\}} \qquad \mathsf{k}^{:}\!\downarrow \frac{S\{\langle s\colon [A, B]\rangle\}}{S\{[\langle s\colon A\rangle, \langle s\colon B\rangle]\}}$$

$$\mathsf{n}\!\downarrow \frac{S\{\langle s\colon A\rangle\}}{S\{[\bar{s}, A]\}} \qquad \mathsf{n}^{\Box}\!\downarrow \frac{S\{\langle s\colon A\rangle\}}{S\{\Box\langle s\colon A\rangle\}} \qquad \mathsf{n}^{:}\!\downarrow \frac{S\{\langle s\colon A\rangle\}}{S\{\langle r\colon \langle s\colon A\rangle\rangle\}}$$

$$\mathsf{r}\!\downarrow \frac{S\{\mathbf{t}\}}{S\{\langle s\colon s\rangle\}} \qquad \sigma_{\mathsf{n}}\!\downarrow \frac{S\{\langle r\colon \bar{s}\rangle\}}{S\{\langle s\colon \bar{r}\rangle\}} \qquad \mathsf{b}\!\downarrow \frac{S\{\langle s\colon \Box\bar{u}\rangle\}}{S\{[\langle s\colon \Box\bar{r}\rangle, \langle r\colon \bar{u}\rangle]\}}$$

$$\mathsf{v}\!\downarrow \frac{(C \wedge [[\langle s\colon \Box\bar{v}\rangle, \langle v\colon A\rangle], B] \wedge D)}{(C \wedge [\langle s\colon \Box A\rangle, B] \wedge D)} \quad \text{$v$ does not appear in $A$, $B$, $C$, nor $D$}$$

**Fig. 1.** System $\mathsf{BH}\!\downarrow$

$Q$ is a derivation with premise $\mathbf{t}$ and conclusion $Q$, and a *refutation* of a formula $P$ is a derivation with premise $P$ and conclusion $\mathbf{f}$. By the up-down duality of the rules in $\mathsf{BH}\!\downarrow\uparrow$, every refutation of a formula $P$ in $\mathsf{BH}\!\uparrow$ corresponds to a proof of $\bar{P}$ in $\mathsf{BH}\!\downarrow$, and vice versa. Figure 3 shows an example of a proof in system $\mathsf{BH}\!\downarrow$. Its conclusion is the formula

$$[s\colon \Box[\bar{r}, \bar{A}], s\colon \Box[\bar{r}, \bar{B}], s\colon \Diamond(A \wedge B)] \tag{1}$$

where $A$ and $B$ can be arbitrary formulas. The formula (1) might be more familiar to the reader acquainted with hybrid logic, when it written as implication

$$s\colon \Diamond(r \wedge A), s\colon \Diamond(r \wedge B) \rightarrow s\colon \Diamond(A \wedge B) \tag{2}$$

where the comma on the left has to be read as conjunction. Informally speaking, the formula (2) says that if for a state $s$ there are a reachable state in which $r$ and $A$ hold and a reachable state in which $r$ and $B$ hold, then there is a reachable state in which $A$ and $B$ hold. This formula is valid in hybrid logic because for each nominal $r$ there is exactly one state in which $r$ holds. The proof in Figure 3 is the result of translating the sequent proof given in [Bla00, Section 8] into $\mathsf{BH}\!\downarrow$.

In order to ease readability, we used in the proof in Figure 3 the following syntactic conventions:

- Sometimes we omit the context parentheses for formulas $s\colon A$.
- We omit instances of the rule $\alpha\!\downarrow$, and omit nested $[\ldots]$ brackets. Sometimes we also leave the rule $\sigma\!\downarrow$ implicit.

$$
\mathsf{ai}{\downarrow}\,\frac{S\{\mathbf{t}\}}{S\{[\bar{a},a]\}} \qquad \mathsf{ai}{\uparrow}\,\frac{S\{(a\wedge\bar{a})\}}{S\{\mathbf{f}\}}
$$

$$
\mathsf{s}\,\frac{S\{(A\wedge[B,C])\}}{S\{[(A\wedge B),C]\}}
$$

$$
\sigma{\downarrow}\,\frac{S\{[B,A]\}}{S\{[A,B]\}} \qquad \alpha{\downarrow}\,\frac{S\{[A,[B,C]]\}}{S\{[[A,B],C]\}} \qquad \alpha{\uparrow}\,\frac{S\{(A\wedge(B\wedge C))\}}{S\{((A\wedge B)\wedge C)\}} \qquad \sigma{\uparrow}\,\frac{S\{(A\wedge B)\}}{S\{(B\wedge A)\}}
$$

$$
\mathbf{t}{\downarrow}\,\frac{S\{A\}}{S\{(A\wedge\mathbf{t})\}} \qquad \mathbf{f}{\downarrow}\,\frac{S\{A\}}{S\{[A,\mathbf{f}]\}} \qquad \mathbf{f}{\uparrow}\,\frac{S\{(\mathbf{t}\wedge A)\}}{S\{A\}} \qquad \mathbf{t}{\uparrow}\,\frac{S\{[\mathbf{f},A]\}}{S\{A\}}
$$

$$
\mathsf{w}{\downarrow}\,\frac{S\{\mathbf{f}\}}{S\{A\}} \qquad \mathsf{c}{\downarrow}\,\frac{S\{[A,A]\}}{S\{A\}} \qquad \mathsf{c}{\uparrow}\,\frac{S\{A\}}{S\{(A\wedge A)\}} \qquad \mathsf{w}{\uparrow}\,\frac{S\{A\}}{S\{\mathbf{t}\}}
$$

$$
\mathsf{e}^{\Box}{\downarrow}\,\frac{S\{\mathbf{t}\}}{S\{\Box\mathbf{t}\}} \qquad \mathsf{k}^{\Box}{\downarrow}\,\frac{S\{\Box[A,B]\}}{S\{[\Box A,\Diamond B]\}} \qquad \mathsf{k}^{\Box}{\uparrow}\,\frac{S\{(\Box A\wedge\Diamond B)\}}{S\{\Diamond(A\wedge B)\}} \qquad \mathsf{e}^{\Box}{\uparrow}\,\frac{S\{\Diamond\mathbf{f}\}}{S\{\mathbf{f}\}}
$$

$$
\mathsf{e}^{:}{\downarrow}\,\frac{S\{\mathbf{t}\}}{S\{\langle s\colon\mathbf{t}\rangle\}} \qquad \mathsf{k}^{:}{\downarrow}\,\frac{S\{\langle s\colon[A,B]\rangle\}}{S\{[\langle s\colon A\rangle,\langle s\colon B\rangle]\}} \qquad \mathsf{k}^{:}{\uparrow}\,\frac{S\{(\langle s\colon A\rangle\wedge\langle s\colon B\rangle)\}}{S\{\langle s\colon(A\wedge B)\rangle\}} \qquad \mathsf{e}^{:}{\uparrow}\,\frac{S\{\langle s\colon\mathbf{f}\rangle\}}{S\{\mathbf{f}\}}
$$

$$
\mathsf{n}{\downarrow}\,\frac{S\{\langle s\colon A\rangle\}}{S\{[\bar{s},A]\}} \qquad \mathsf{n}{\uparrow}\,\frac{S\{(s\wedge A)\}}{S\{\langle s\colon A\rangle\}}
$$

$$
\mathsf{n}^{\Box}{\downarrow}\,\frac{S\{\langle s\colon A\rangle\}}{S\{\Box\langle s\colon A\rangle\}} \qquad \mathsf{n}^{:}{\downarrow}\,\frac{S\{\langle s\colon A\rangle\}}{S\{\langle r\colon\langle s\colon A\rangle\rangle\}} \qquad \mathsf{n}^{:}{\uparrow}\,\frac{S\{\langle r\colon\langle s\colon A\rangle\rangle\}}{S\{\langle s\colon A\rangle\}} \qquad \mathsf{n}^{\Box}{\uparrow}\,\frac{S\{\Diamond\langle s\colon A\rangle\}}{S\{\langle s\colon A\rangle\}}
$$

$$
\mathsf{r}{\downarrow}\,\frac{S\{\mathbf{t}\}}{S\{\langle s\colon s\rangle\}} \qquad \sigma_{\mathsf{n}}{\downarrow}\,\frac{S\{\langle r\colon\bar{s}\rangle\}}{S\{\langle s\colon\bar{r}\rangle\}} \qquad \sigma_{\mathsf{n}}{\uparrow}\,\frac{S\{\langle r\colon s\rangle\}}{S\{\langle s\colon r\rangle\}} \qquad \mathsf{r}{\uparrow}\,\frac{S\{\langle s\colon\bar{s}\rangle\}}{S\{\mathbf{f}\}}
$$

$$
\mathsf{b}{\downarrow}\,\frac{S\{\langle s\colon\Box\bar{u}\rangle\}}{S\{[\langle s\colon\Box\bar{r}\rangle,\langle r\colon\bar{u}\rangle]\}} \qquad \mathsf{b}{\uparrow}\,\frac{S\{(\langle s\colon\Diamond r\rangle\wedge\langle r\colon u\rangle)\}}{S\{\langle s\colon\Diamond u\rangle\}}
$$

$$
\mathsf{v}{\downarrow}\,\frac{(C\wedge[[\langle s\colon\Box\bar{v}\rangle,\langle v\colon A\rangle],B]\wedge D)}{(C\wedge[\langle s\colon\Box A\rangle,B]\wedge D)} \qquad \mathsf{v}{\uparrow}\,\frac{[C,(\langle s\colon\Diamond A\rangle\wedge B),D]}{[C,((\langle s\colon\Diamond v\rangle\wedge\langle v\colon A\rangle)\wedge B),D]}
$$

in the rules $\mathsf{v}{\downarrow}$ and $\mathsf{v}{\uparrow}$, the nominal $v$ must not occur in any of $A$, $B$, $C$, or $D$

**Fig. 2.** System $\mathsf{BH}{\downarrow}{\uparrow}$

– Sometimes we apply two rules at once to save space, e.g., $\mathsf{c}{\downarrow};\mathsf{c}{\downarrow}$ means that there are two applications of $\mathsf{c}{\downarrow}$.
– We mark the redex of each rule application *when read from bottom to top* with a gray background.

We also use the rules

$$
\mathsf{i}{\downarrow}\,\frac{S\{\mathbf{t}\}}{S\{[\bar{A},A]\}} \qquad \text{and} \qquad \mathsf{i}{\uparrow}\,\frac{S\{(A\wedge\bar{A})\}}{S\{\mathbf{f}\}} \tag{3}
$$

which are the non-atomic versions of $\mathsf{ai}{\downarrow}$ and $\mathsf{ai}{\uparrow}$. One can easily show by induction on the structure of $A$, that $\mathsf{i}{\downarrow}$ is derivable in $\mathsf{BH}{\downarrow}$, and dually, $\mathsf{i}{\uparrow}$ is derivable in $\mathsf{BH}{\uparrow}$. For the

$$
\begin{array}{rl}
 & \mathbf{t} \\
\mathsf{e}^{:}\!\downarrow & \overline{r\colon \mathbf{t}} \\
\mathsf{i}\!\downarrow & \overline{r\colon [\bar{A}, A]} \\
\mathbf{t}\!\downarrow & \overline{r\colon [\bar{A}, (A \wedge \mathbf{t})]} \\
\mathsf{i}\!\downarrow & \overline{r\colon [\bar{A}, (A \wedge [\bar{B}, B])]} \\
\mathsf{s} & \overline{r\colon [\bar{A}, \bar{B}, (A \wedge B)]} \\
\mathsf{k}^{:}\!\downarrow & \overline{[r\colon \bar{A}, r\colon [\bar{B}, (A \wedge B)]]} \\
\mathsf{k}^{:}\!\downarrow & \overline{[r\colon \bar{A}, r\colon \bar{B}, r\colon (A \wedge B)]} \\
\mathsf{n}^{:}\!\downarrow & \overline{[r\colon \bar{A}, r\colon \bar{B}, s\colon \langle r\colon (A \wedge B)\rangle]} \\
\mathsf{n}^{\Box}\!\downarrow & \overline{[r\colon \bar{A}, r\colon \bar{B}, s\colon \Box\langle r\colon (A \wedge B)\rangle]} \\
\mathsf{n}\!\downarrow & \overline{[r\colon \bar{A}, r\colon \bar{B}, s\colon \Box[\bar{r}, (A \wedge B)]]} \\
\mathsf{k}^{\Box}\!\downarrow & \overline{[r\colon \bar{A}, r\colon \bar{B}, s\colon [\Box\bar{r}, \Diamond(A \wedge B)]]} \\
\mathsf{k}^{:}\!\downarrow & \overline{[r\colon \bar{A}, r\colon \bar{B}, s\colon \Box\bar{r}, s\colon \Diamond(A \wedge B)]} \\
\sigma\!\downarrow & \overline{[s\colon \Box\bar{r}, r\colon \bar{A}, r\colon \bar{B}, s\colon \Diamond(A \wedge B)]} \\
\mathbf{f}\!\downarrow & \overline{[s\colon \Box\bar{r}, r\colon \bar{A}, \mathbf{f}, r\colon \bar{B}, s\colon \Diamond(A \wedge B)]} \\
\mathsf{w}\!\downarrow & \overline{[s\colon \Box\bar{r}, r\colon \bar{A}, s\colon \Box\bar{v}, v\colon \bar{r}, r\colon \bar{B}, s\colon \Diamond(A \wedge B)]} \\
\mathsf{n}^{:}\!\downarrow ; \mathsf{n}^{:}\!\downarrow & \overline{[s\colon \Box\bar{r}, u\colon r\colon \bar{A}, s\colon \Box\bar{v}, v\colon \bar{r}, v\colon r\colon \bar{B}, s\colon \Diamond(A \wedge B)]} \\
\mathsf{n}\!\downarrow ; \mathsf{n}\!\downarrow & \overline{[s\colon \Box\bar{r}, u\colon [\bar{r}, \bar{A}], s\colon \Box\bar{v}, v\colon \bar{r}, v\colon [\bar{r}, \bar{B}], s\colon \Diamond(A \wedge B)]} \\
\mathsf{b}\!\downarrow & \overline{[s\colon \Box\bar{u}, u\colon \bar{r}, u\colon [\bar{r}, \bar{A}], s\colon \Box\bar{v}, v\colon \bar{r}, v\colon [\bar{r}, \bar{B}], s\colon \Diamond(A \wedge B)]} \\
\mathbf{f}\!\downarrow ; \mathbf{f}\!\downarrow & \overline{[s\colon \Box\bar{u}, u\colon [\bar{r}, \mathbf{f}], u\colon [\bar{r}, \bar{A}], s\colon \Box\bar{v}, v\colon [\bar{r}, \mathbf{f}], v\colon [\bar{r}, \bar{B}], s\colon \Diamond(A \wedge B)]} \\
\mathsf{w}\!\downarrow ; \mathsf{w}\!\downarrow & \overline{[s\colon \Box\bar{u}, u\colon [\bar{r}, \bar{A}], u\colon [\bar{r}, \bar{A}], s\colon \Box\bar{v}, v\colon [\bar{r}, \bar{B}], v\colon [\bar{r}, \bar{B}], s\colon \Diamond(A \wedge B)]} \\
\mathsf{c}\!\downarrow ; \mathsf{c}\!\downarrow & \overline{[s\colon \Box\bar{u}, u\colon [\bar{r}, \bar{A}], s\colon \Box\bar{v}, v\colon [\bar{r}, \bar{B}], s\colon \Diamond(A \wedge B)]} \\
\mathsf{v}\!\downarrow & \overline{[s\colon \Box\bar{u}, u\colon [\bar{r}, \bar{A}], s\colon \Box[\bar{r}, \bar{B}], s\colon \Diamond(A \wedge B)]} \\
\mathsf{v}\!\downarrow & \overline{[s\colon \Box[\bar{r}, \bar{A}], s\colon \Box[\bar{r}, \bar{B}], s\colon \Diamond(A \wedge B)]}
\end{array}
$$

**Fig. 3.** Example of a proof in $\mathsf{BH}\!\downarrow$

convenience of the reader, we show the inductive cases for $\mathsf{i}\!\uparrow$:

$$
\mathbf{f}\!\uparrow \frac{(\mathbf{t} \wedge \mathbf{f})}{\mathbf{f}}
\qquad
\begin{array}{rl}
 & ((A \wedge B) \wedge [\bar{B}, \bar{A}]) \\
\alpha\!\uparrow & \overline{(A \wedge (B \wedge [\bar{B}, \bar{A}]))} \\
\mathsf{s} & \overline{(A \wedge [(B \wedge \bar{B}), \bar{A}])} \\
\mathsf{i}\!\uparrow & \overline{(A \wedge [\mathbf{f}, \bar{A}])} \\
\mathbf{t}\!\uparrow & \overline{(A \wedge \bar{A})} \\
\mathsf{i}\!\uparrow & \overline{\mathbf{f}}
\end{array}
\qquad
\begin{array}{rl}
 & (\Box A \wedge \Diamond\bar{A}) \\
\mathsf{k}^{\Box}\!\uparrow & \overline{\Diamond(A \wedge \bar{A})} \\
\mathsf{i}\!\uparrow & \overline{\Diamond\mathbf{f}} \\
\mathsf{e}^{\Box}\!\uparrow & \overline{\mathbf{f}}
\end{array}
\qquad
\begin{array}{rl}
 & (s\colon A \wedge s\colon \bar{A}) \\
\mathsf{k}^{:}\!\uparrow & \overline{s\colon (A \wedge \bar{A})} \\
\mathsf{i}\!\uparrow & \overline{s\colon \mathbf{f}} \\
\mathsf{e}^{:}\!\uparrow & \overline{\mathbf{f}}
\end{array}
$$

## 3 Soundness and Completeness

We assume the reader to be familiar with the standard Kripke semantics for hybrid logic and abstain from repeating the definition here, since we will not need it anyway. For showing soundness and completeness of system $\mathsf{BH}$ with respect to the Kripke semantics, we will refer to Blackburn's tableau system, for which soundness and completeness has been shown in [Bla00]. More precisely, we show that a formula $P$ has a finite closed tableau if and only if there is a refutation for $P$ in $\mathsf{BH}\!\uparrow \cup \{\mathsf{k}^{:}\!\downarrow\}$. For this, observe that any tableau $\tau$ can be written as a formula $F(\tau)$ of the shape

$$[(A_{11} \wedge A_{12} \wedge \cdots \wedge A_{1m_1}), (A_{21} \wedge A_{22} \wedge \cdots \wedge A_{2m_2}), \ldots, (A_{n1} \wedge A_{n2} \wedge \cdots \wedge A_{nm_n})] \qquad (4)$$

with a subformula $(A_{i1} \wedge A_{i2} \wedge \cdots \wedge A_{im_i})$ for each branch in the tableau where $A_{i1}, \ldots, A_{im_i}$ are all formulas occurring in the branch.

**3.1 Theorem (Soundness)** *If there is a proof*

$$\mathsf{BH}{\downarrow}{\uparrow} \left\| \begin{array}{c} \mathbf{t} \\ \Delta \\ Q \end{array} \right.$$

*then the formula $Q$ is valid.*

**Proof:** For the rules of the shape

$$\rho \frac{S\{A\}}{S\{B\}} \tag{5}$$

it suffices to show that $A \rightarrow B$ is a valid implication. We leave this as an exercise to the reader. By induction on $S\{\ \}$ we can then show that $S\{A\} \rightarrow S\{B\}$ is a valid implication. Then by induction on the length of $\Delta$, we get validity of $Q$. It remains to show that also the rules $\mathsf{v}{\downarrow}$ and $\mathsf{v}{\uparrow}$, which do not follow the pattern in (5), are sound in a weak sense. (Note that $\mathsf{v}{\downarrow}$ and $\mathsf{v}{\uparrow}$ are not sound in the strong sense that premise implies conclusion, as it is the case with all other rules.) However, note that the rule $\mathsf{v}{\uparrow}$ is precisely Blackburn's tableau rule for $\Diamond$. Hence, soundness follows immediately. Then the soundness of $\mathsf{v}{\downarrow}$ follows by duality. ☐

**3.2 Theorem (Completeness)** *If a formula $Q$ is valid, then there is a proof*

$$\mathsf{BH}{\downarrow} \cup \{\mathsf{k}{\cdot}{\uparrow}\} \left\| \begin{array}{c} \mathbf{t} \\ \Delta \\ Q \end{array} \right. .$$

**Proof:** First, we are going to show that if there is a closed tableau $\tau$ for a formula $P$, then there is a refutation in $\mathsf{BH}{\uparrow}$ for $P$, that has the following shape:

$$\begin{array}{c} P \\ \mathsf{BH}{\uparrow} \cup \{\mathsf{k}{\cdot}{\downarrow}\} \Big\| \Delta_1 \\ F(\tau) \\ \mathsf{BH}{\uparrow} \Big\| \Delta_2 \\ \mathbf{f} \end{array}$$

where $F(\tau)$ is the formula associated to the tableau $\tau$, as shown in (4). Since $\tau$ is closed, the derivation $\Delta_2$ can be obtained by first applying $\mathsf{w}{\uparrow}$ and $\mathbf{f}{\uparrow}$ to to transform $F(\tau)$ into a formula

$$[(A_1 \wedge \bar{A}_1), (A_2 \wedge \bar{A}_2), \ldots, (A_n \wedge \bar{A}_n)]$$

which is easily refuted by applying $\mathsf{i}{\uparrow}$ and $\mathbf{t}{\uparrow}$. So, let us now concentrate on $\Delta_1$. We proceed by induction on the size of $\tau$ and make a case analysis on the tableau rules, as presented in [Bla00]. The rules involving negation are vacuous because we have pushed negation to the atoms.

– The tableau rule $[\wedge]$ is simulated by

$$\mathbf{f}{\uparrow};\mathbf{f}{\uparrow}\,\frac{\mathsf{w}{\uparrow};\mathsf{w}{\uparrow}\,\frac{\mathsf{c}{\uparrow};\mathsf{c}{\uparrow}\,\frac{[C_1, (\langle s\colon (A \wedge B)\rangle \wedge C_2), C_3]}{[C_1, (\langle s\colon (A \wedge B)\rangle \wedge \langle s\colon (A \wedge B)\rangle \wedge \langle s\colon (A \wedge B)\rangle \wedge C_2), C_3]}}{[C_1, (\langle s\colon (A \wedge \mathbf{t})\rangle \wedge \langle s\colon (\mathbf{t} \wedge B)\rangle \wedge \langle s\colon (A \wedge B)\rangle \wedge C_2), C_3]}}{[C_1, (\langle s\colon A\rangle \wedge \langle s\colon B\rangle \wedge \langle s\colon (A \wedge B)\rangle \wedge C_2), C_3]}$$

– The tableau rule $[\vee]$ is simulated by

$$
\begin{array}{l}
\mathsf{c}\!\uparrow;\mathsf{c}\!\uparrow \dfrac{[C_1,(\langle s\colon[A,B]\rangle\wedge C_2),C_3]}{[C_1,(\langle s\colon[A,B]\rangle\wedge\langle s\colon[A,B]\rangle\wedge C_2\wedge\langle s\colon[A,B]\rangle\wedge C_2),C_3]} \\
\mathsf{k}^{:}\!\downarrow \dfrac{}{[C_1,([s\colon A,s\colon B]\wedge\langle s\colon[A,B]\rangle\wedge C_2\wedge\langle s\colon[A,B]\rangle\wedge C_2),C_3]} \\
\mathsf{s}\dfrac{}{[C_1,([\langle s\colon A\rangle,(\langle s\colon B\rangle\wedge\langle s\colon[A,B]\rangle\wedge C_2)]\wedge\langle s\colon[A,B]\rangle\wedge C_2),C_3]} \\
\mathsf{s}\dfrac{}{[C_1,(\langle s\colon A\rangle\wedge\langle s\colon[A,B]\rangle\wedge C_2),(\langle s\colon B\rangle\wedge\langle s\colon[A,B]\rangle\wedge C_2),C_3]}
\end{array}
$$

– The tableau rule $[:]$ is simulated by

$$
\begin{array}{l}
\mathsf{c}\!\uparrow \dfrac{[C_1,(\langle s\colon\langle r\colon A\rangle\rangle\wedge C_2),C_3]}{[C_1,(\langle s\colon\langle r\colon A\rangle\rangle\wedge\langle s\colon\langle r\colon A\rangle\rangle\wedge C_2),C_3]} \\
\mathsf{n}^{:}\!\uparrow \dfrac{}{[C_1,(\langle r\colon A\rangle\wedge\langle s\colon\langle r\colon A\rangle\rangle\wedge C_2),C_3]}
\end{array}
$$

– The tableau rule $[\Diamond]$ is simulated by

$$
\begin{array}{l}
\mathsf{c}\!\uparrow \dfrac{[C_1,(\langle s\colon\Diamond A\rangle\wedge C_2),C_3]}{[C_1,(\langle s\colon\Diamond A\rangle\wedge\langle s\colon\Diamond A\rangle\wedge C_2),C_3]} \\
\mathsf{v}\!\uparrow \dfrac{}{[C_1,(\langle s\colon\Diamond v\rangle\wedge\langle v\colon A\rangle\wedge\langle s\colon\Diamond A\rangle\wedge C_2),C_3]}
\end{array}
$$

– The tableau rule $[\Box]$ is simulated by

$$
\begin{array}{l}
\mathsf{c}\!\uparrow \dfrac{[C_1,(\langle s\colon\Box A\rangle\wedge\langle s\colon\Diamond r\rangle\wedge C_2),C_3]}{[C_1,(\langle s\colon\Box A\rangle\wedge\langle s\colon\Diamond r\rangle\wedge\langle s\colon\Box A\rangle\wedge\langle s\colon\Diamond r\rangle\wedge C_2),C_3]} \\
\mathsf{k}^{:}\!\uparrow \dfrac{}{[C_1,(\langle s\colon(\Box A\wedge\Diamond r)\rangle\wedge\langle s\colon\Box A\rangle\wedge\langle s\colon\Diamond r\rangle\wedge C_2),C_3]} \\
\mathsf{k}^{\Box}\!\uparrow \dfrac{}{[C_1,(\langle s\colon\Diamond(A\wedge r)\rangle\wedge\langle s\colon\Box A\rangle\wedge\langle s\colon\Diamond r\rangle\wedge C_2),C_3]} \\
\sigma\!\uparrow;\mathsf{n}\!\uparrow \dfrac{}{[C_1,(\langle s\colon\Diamond\langle r\colon A\rangle\rangle\wedge\langle s\colon\Box A\rangle\wedge\langle s\colon\Diamond r\rangle\wedge C_2),C_3]} \\
\mathsf{n}^{\Box}\!\uparrow \dfrac{}{[C_1,(\langle s\colon\langle r\colon A\rangle\rangle\wedge\langle s\colon\Box A\rangle\wedge\langle s\colon\Diamond r\rangle\wedge C_2),C_3]} \\
\mathsf{n}^{:}\!\uparrow \dfrac{}{[C_1,(\langle r\colon A\rangle\wedge\langle s\colon\Box A\rangle\wedge\langle s\colon\Diamond r\rangle\wedge C_2),C_3]}
\end{array}
$$

– For simulating the tableau rule [Ref], we have to observe that introducing a formula $\langle s\colon s\rangle$ does only make sense if that branch is eventually closed by the pair $\langle s\colon s\rangle$ and $\langle s\colon\bar{s}\rangle$. In our simulation this would be mimicked by an instance of $\mathsf{i}\!\uparrow$:

$$
\mathsf{i}\!\uparrow \frac{S\{(\langle s\colon s\rangle\wedge\langle s\colon\bar{s}\rangle)\}}{S\{\mathbf{f}\}}
$$

Due to deep inference, we can take a shortcut by skipping the introduction of $\langle s\colon s\rangle$ and replacing the instance of $\mathsf{i}\!\uparrow$ by

$$
\mathsf{r}\!\uparrow \frac{S\{\langle s\colon\bar{s}\rangle\}}{S\{\mathbf{f}\}}
$$

– The tableau rule [Sym] is simulated by

$$
\begin{array}{l}
\mathsf{c}\!\uparrow \dfrac{[C_1,(\langle s\colon r\rangle\wedge C_2),C_3]}{[C_1,(\langle s\colon r\rangle\wedge\langle s\colon r\rangle\wedge C_2),C_3]} \\
\sigma_{\mathsf{n}}\!\uparrow \dfrac{}{[C_1,(\langle r\colon s\rangle\wedge\langle s\colon r\rangle\wedge C_2),C_3]}
\end{array}
$$

– The tableau rule [Nom] is simulated by

$$
\mathsf{c}\!\uparrow \frac{\mathsf{n}^{\cdot}\!\uparrow \frac{\mathsf{n}\!\uparrow \frac{\mathsf{k}^{\cdot}\!\uparrow \frac{\sigma_{\mathsf{n}}\!\uparrow \frac{[C_1, (\langle s\colon r\rangle \wedge \langle r\colon A\rangle \wedge C_2), C_3]}{[C_1, (\langle s\colon r\rangle \wedge \langle r\colon A\rangle \wedge \langle s\colon r\rangle \wedge \langle r\colon A\rangle \wedge C_2), C_3]}}{[C_1, (\langle r\colon s\rangle \wedge \langle r\colon A\rangle \wedge \langle s\colon r\rangle \wedge \langle r\colon A\rangle \wedge C_2), C_3]}}{[C_1, (\langle r\colon (s \wedge A)\rangle \wedge \langle s\colon r\rangle \wedge \langle r\colon A\rangle \wedge C_2), C_3]}}{[C_1, (\langle r\colon \langle s\colon A\rangle\rangle \wedge \langle s\colon r\rangle \wedge \langle r\colon A\rangle \wedge C_2), C_3]}}{[C_1, (\langle s\colon A\rangle \wedge \langle s\colon r\rangle \wedge \langle r\colon A\rangle \wedge C_2), C_3]}
$$

– Finally, the tableau rule [Bridge] is simulated by

$$
\mathsf{c}\!\uparrow \frac{[C_1, (\langle s\colon \Diamond r\rangle \wedge \langle r\colon u\rangle \wedge C_2), C_3]}{\mathsf{b}\!\uparrow \frac{[C_1, (\langle s\colon \Diamond r\rangle \wedge \langle r\colon u\rangle \wedge \langle s\colon \Diamond r\rangle \wedge \langle r\colon u\rangle \wedge C_2), C_3]}{[C_1, (\langle s\colon \Diamond u\rangle \wedge \langle s\colon \Diamond r\rangle \wedge \langle r\colon u\rangle \wedge C_2), C_3]}}
$$

Now we can complete our proof as follows: For a valid formula $Q$ we have by Blackburn's completeness result a closed tableau for $\langle s\colon \bar{Q}\rangle$ where $s$ is a nominal not appearing in $Q$. By our simulation we get a refutation $\Delta$ in $\mathsf{BH}\!\uparrow \cup \{\mathsf{k}^{\cdot}\!\downarrow\}$ of $\langle s\colon \bar{Q}\rangle$. Since $s$ does not appear in $Q$, this refutation $\Delta$ remains correct, if we remove $s$ everywhere in $\Delta$. It can only happen that some rule instances become vacuous, for example,

$$
\mathsf{k}^{\cdot}\!\uparrow \frac{S\{(\langle s\colon A\rangle \wedge \langle s\colon B\rangle)\}}{S\{\langle s\colon (A \wedge B)\rangle\}} \quad \text{becomes} \quad \mathsf{k}^{\cdot}\!\uparrow \frac{S\{(A \wedge B)\}}{S\{(A \wedge B)\}} \quad ,
$$

which we can remove. This yields a refutation of $\bar{Q}$ in $\mathsf{BH}\!\uparrow \cup \{\mathsf{k}^{\cdot}\!\downarrow\}$. By dualizing it, we get a proof of $Q$ in $\mathsf{BH}\!\downarrow \cup \{\mathsf{k}^{\cdot}\!\uparrow\}$. ☐

**3.3 Remark** We used here Blackburn's tableau system for showing completeness. However, we could equally well have used Seligman's sequent system, which is in spirit closer to the system of this paper (see e.g. [Brü03,Str03] for translation between sequent calculus and calculus of structures).[3] We have chosen here Blackburn's tableau because his completeness proof is easy accessible and his system is small (and hence our proof is short).

## 4 Discussion

The system BH proposed in this short note has two serious design flaws, which indicate that the last word on deep inference for hybrid logic is not yet spoken. Let us briefly discuss them:

1. It is rather annoying that we have a completeness proof only for $\mathsf{BH}\!\downarrow \cup \{\mathsf{k}^{\cdot}\!\uparrow\}$ instead of pure $\mathsf{BH}\!\downarrow$. This means we do not have the strong cut elimination result usually associated to a deep inference system, namely, that the whole up-fragment is admissible. However, in a weak sense $\mathsf{BH}\!\downarrow \cup \{\mathsf{k}^{\cdot}\!\uparrow\}$ can still be considered cut-free. Furthermore, I conjecture that $\mathsf{BH}\!\downarrow$ without $\mathsf{k}^{\cdot}\!\uparrow$ is already complete, and that the need for $\mathsf{k}^{\cdot}\!\uparrow$ in this paper is caused only by the rather naive method of proving completeness. (Note that the proof in Figure 3 does not need $\mathsf{k}^{\cdot}\!\uparrow$, although the naive translation from the sequent calculus would introduce it.)

[3] It is in fact a common property of deep inference deductive systems that they can p-simulate most other deductive systems, e.g., Frege-Hilbert systems, sequent calculus, natural deduction, resolution, and tableaux. See also [BG07].

$$
\begin{array}{rc}
 & \mathbf{t} \\ \hline
\mathsf{e}^{:}\!\downarrow & r\colon \mathbf{t} \\ \hline
\mathsf{i}\!\downarrow & r\colon [\bar{A}, A] \\ \hline
\mathbf{t}\!\downarrow & r\colon [\bar{A}, (A \wedge \mathbf{t})] \\ \hline
\mathsf{i}\!\downarrow & r\colon [\bar{A}, (A \wedge [\bar{B}, B])] \\ \hline
\mathsf{s} & r\colon [\bar{A}, \bar{B}, (A \wedge B)] \\ \hline
\mathsf{k}^{:}\!\downarrow & [r\colon \bar{A}, r\colon [\bar{B}, (A \wedge B)]] \\ \hline
\mathsf{k}^{:}\!\downarrow & [r\colon \bar{A}, r\colon \bar{B}, r\colon (A \wedge B)] \\ \hline
\mathsf{n}^{:}\!\downarrow & [r\colon \bar{A}, r\colon \bar{B}, s\colon \langle r\colon (A \wedge B)\rangle] \\ \hline
\mathsf{n}^{\Box}\!\downarrow & [r\colon \bar{A}, r\colon \bar{B}, s\colon \Box\langle r\colon (A \wedge B)\rangle] \\ \hline
\mathsf{n}\!\downarrow & [r\colon \bar{A}, r\colon \bar{B}, s\colon \Box[\bar{r}, (A \wedge B)]] \\ \hline
\mathsf{k}^{\Box}\!\downarrow & [r\colon \bar{A}, r\colon \bar{B}, s\colon [\Box\bar{r}, \Diamond(A \wedge B)]] \\ \hline
\mathsf{k}^{:}\!\downarrow & [r\colon \bar{A}, r\colon \bar{B}, s\colon \Box\bar{r}, s\colon \Diamond(A \wedge B)] \\ \hline
\sigma\!\downarrow & [s\colon \Box\bar{r}, r\colon \bar{A}, r\colon \bar{B}, s\colon \Diamond(A \wedge B)] \\ \hline
\mathsf{n}^{:}\!\downarrow;\mathsf{n}^{:}\!\downarrow & [s\colon \Box\bar{r}, s\colon \langle r\colon \bar{A}\rangle, s\colon \langle r\colon \bar{B}\rangle, s\colon \Diamond(A \wedge B)] \\ \hline
\mathsf{n}^{\Box}\!\downarrow;\mathsf{n}^{\Box}\!\downarrow & [s\colon \Box\bar{r}, s\colon \Box\langle r\colon \bar{A}\rangle, s\colon \Box\langle r\colon \bar{B}\rangle, s\colon \Diamond(A \wedge B)] \\ \hline
\mathsf{n}\!\downarrow;\mathsf{n}\!\downarrow & [s\colon \Box\bar{r}, s\colon \Box[\bar{r}, \bar{A}], s\colon \Box[\bar{r}, \bar{B}], s\colon \Diamond(A \wedge B)] \\ \hline
\mathbf{f}\!\downarrow & [s\colon \Box[\bar{r}, \mathbf{f}], s\colon \Box[\bar{r}, \bar{A}], s\colon \Box[\bar{r}, \bar{B}], s\colon \Diamond(A \wedge B)] \\ \hline
\mathsf{w}\!\downarrow & [s\colon \Box[\bar{r}, \bar{A}], s\colon \Box[\bar{r}, \bar{A}], s\colon \Box[\bar{r}, \bar{B}], s\colon \Diamond(A \wedge B)] \\ \hline
\mathsf{c}\!\downarrow & [s\colon \Box[\bar{r}, \bar{A}], s\colon \Box[\bar{r}, \bar{B}], s\colon \Diamond(A \wedge B)]
\end{array}
$$

**Fig. 4.** A proof of (1) in $\mathsf{BH}\!\downarrow$ without using $\mathsf{v}\!\downarrow$

2. The more serious flaw lies in the presence of the rules $\mathsf{v}\!\downarrow$ and $\mathsf{v}\!\uparrow$. They are clearly not of the "deep inference kind". And since they do not incorporate proper implications, *we do not have* the strong result

$$
\mathsf{BH}\!\downarrow\uparrow \left\| \begin{array}{c} P \\ \\ Q \end{array} \right. \qquad \text{iff} \qquad \mathsf{BH}\!\downarrow \left\| \begin{array}{c} \mathbf{t} \\ \\ [\bar{P}, Q] \end{array} \right. \qquad \text{iff} \qquad \begin{array}{l} \text{The formula } P \to Q \\ \text{is a valid implication} \\ \text{of the logic.} \end{array} \tag{6}
$$

which would state at the same time soundness, completeness, cut elimination, and the deduction theorem. It is therefore an important problem for future research to find a proper deep inference replacement for $\mathsf{v}\!\downarrow$ and $\mathsf{v}\!\uparrow$, or to show that these rules are not needed for completeness. That this might very well be possible shows the example in Figure 4, which proves the same formula as the proof in Figure 3, but without using $\mathsf{v}\!\downarrow$. It seems that the rules $\mathsf{v}\!\downarrow$ and $\mathsf{v}\!\uparrow$ are artifacts of sequent calculus and tableaux, and are present in this paper only because of the naive completeness proof.[4]

This leads to the following conjecture. Let $\mathsf{BH}'\!\downarrow = \mathsf{BH}\!\downarrow \setminus \mathsf{v}\!\downarrow$.

**4.1 Conjecture** *The system* $\mathsf{BH}'\!\downarrow$ *is complete for basic hybrid logic.*

At the moment, I see two possible ways of proving it. Either, we repeat Blackburn's construction via Hintikka sets for $\mathsf{BH}'\!\downarrow$, or, we resort to a syntactic cut elimination proof

[4] On the other hand, one should note the rules $\mathsf{v}\!\downarrow$ and $\mathsf{v}\!\uparrow$ could be read as the quantification that takes place in the interpretation of the modalities in the Kripke-semantics. Since there are well-behaved deep inference rules for the quantifiers [Brü03], one can at least expect a proper deep inference version of $\mathsf{v}\!\downarrow$ and $\mathsf{v}\!\uparrow$ without side condition.

as done in [Brü03,Str03,Gug07]. As a corollary we would then get the statement in (6) for $\mathsf{BH}'\downarrow$.

Furthermore, it would in principle be possible to use $\mathsf{BH}'\downarrow$ for proving decidability: the only rules in $\mathsf{BH}\downarrow$ that increase the size of the formula (while going up in the derivation) are the rules $\mathsf{v}\downarrow$ and $\mathsf{c}\downarrow$; and contraction can be put under control by incorporating it in the other inference rules, as it is usually done in the sequent calculus.

## 5 Conclusion

We have seen a rough outline of a deep inference system for basic hybrid logic. For proving completeness, we used the up-fragment to simulate tableaux. We could also have used the down-fragment to simulate sequent calculus (and would have encountered the same problems as already mentioned in the previous section).

Although we discussed here only basic hybrid logic, it should be clear that the system $\mathsf{BH}$ can straightforwardly be extended

- by adding inference rules to restrict the logic to certain frame classes, for example

$$\mathsf{4}\downarrow \frac{S\{\Diamond\Diamond s\}}{S\{\Diamond s\}} \qquad \text{and} \qquad \mathsf{4}\uparrow \frac{S\{\Box s\}}{S\{\Box\Box s\}}$$

  for transitive frames (see [SS05] for details), and
- by adding the binder $\downarrow x$ for labels.

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