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N° 6274

August 2007

Thème NUM

Rapport de recherche

ISSN 0249-6399

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Thème NUM — Systèmes numériques
Projet Smash

Rapport de recherche n° 6274 — August 2007 — 23 pages

Abstract: We performed a traveling wave analysis of a two phase isothermal Euler model to exhibit the inner structure of shock waves in two-phase flows. In the model studied in this work, the dissipative regularizing term is not of viscous type but instead comes from relaxation phenomena toward equilibrium between the phases. This gives an unusual structure to the diffusion tensor where dissipative terms appear only in the mass conservation equations. We show that this implies that the mass fractions are not constant inside the shock although the Rankine-Hugoniot relations give a zero jump of the mass fraction through the discontinuities. We also show that there exists a critical speed for the traveling waves above which no $\mathscr{C}^1$ solutions exist. Neverthless for this case, it is possible to construct traveling solutions involving single phase shocks.

Key-words: Asymptotic analysis, Chapman-Enskog expansion, Rankine-Hugoniot relations, Shock structure, Traveling waves, Compressible two phase flows

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Structure des chocs pour un modéle Euler diphasique isotherme

Résumé : Nous avons réalisé une analyse par ondes progressives d'un modèle Euler à deux phases pour décrire la structure des ondes de chocs dans les écoulements diphasiques. Dans ce modèle, les termes de régularisation dissipative ne sont pas de type visqueux mais proviennent de termes de relaxation vers l'équilibre entre les phases. Cela donne une structure inhabituelle au tenseur de diffusion où des termes dissipatifs apparaissent seulement dans les équations de conservations des masses. Nous montrons que ces termes impliquent que les fractions massiques ne sont pas constantes dans les zones de choc bien que les relations de Rankine-Hugoniot prévoient un saut nul des fractions massiques au travers d'un choc. Nous montrons aussi qu'il existe une vitesse d'onde critique au delà de laquelle il n'existe plus de solution $\mathscr{C}^1$. Cependant, dans ce cas, nous montrons que l'on peut construire des solutions qui font apparaitre des chocs monophasiques.

Mots-clés : Analyse asymptotique, Dévelopements de Chapman-Enskog, Relations de Rankine-Hugoniot, Structure des chocs, Ondes progressives, Ecoulements diphasiques compressibles

Contents

List of Figures

1 Introduction

Hyperbolic systems of Partial Differential Equations of the form

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} = 0 \tag{1}$$

where $U \in I\!R^n$ and $A(U)$ is an $n \times n$ matrix, appear in a large number of domains in science and engineering. If $A(U) = d_U F(U)$ for some flux function $F(U)$, system (1) becomes a system of conservation laws of the form :

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0 \tag{2}$$

A paradigmal example of these systems are the Euler equations of gas dynamics expressing the conservation of mass ρ, momentum ρu and energy ρe where $U = (\rho, \rho u, \rho e)^t$ and $F(U) = (\rho u, \rho u^2 + p, (\rho e + p)u)^t$ with p the pressure.
A characteristic feature of systems (1) or (2) is that their solutions can become discontinuous even for analytic initial data. This leads to the concept of shock solutions of these systems : For system of conservation laws of the form (2) a shock traveling with speed $s \in I\!R$ is a weak solution of (2) of the form

$$U(x,t) = \begin{cases} U_L & \text{for} \quad x < st \\ U_R & \text{for} \quad x > st \end{cases} \tag{3}$$

that satisfies the Rankine-Hugoniot relations :

$$F(U_R) - F(U_L) = s(U_R - U_L) \tag{4}$$

However, this concept of shock waves is too wide : Actually many step functions satisfying the Rankine-Hugoniot conditions (4) can exist simultaneously and thus the initial value problems for (2) may have multiple solutions. A successful way to remedy to this problem is to add some dissipative mechanism to (2) and to rely on a traveling wave analysis [6, 14, 7] of the resulting system. More specifically, the system (2) is changed to an enlarged dissipative system of the form

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = \varepsilon\frac{\partial}{\partial x}(D(U)\frac{\partial U}{\partial x}) \tag{5}$$

where $D(U)$ is a dissipative tensor and ε a small positive parameter. Then one look for traveling waves solutions $U(x-st)$ with $U(-\infty) = U_L$ and $U(+\infty) = U_R$ of (5). If such solutions exists, as $\varepsilon \to 0$, $U((x-st)/\varepsilon)$ converges to the discontinuous step function (3) defining in an unique way the shock solution of (2) compatible with the enlarged system (5).

However, the practical realization of a traveling wave analysis faces many difficulties. The most important one concerns the choice of the dissipative tensor $D(U)$. It is clear from the previous discussion that the dissipative tensor selects a particular shock solutions from the possible solutions of (4). In the case of non-conservative systems i.e systems (1) that cannot be put under the form (2), the situation is even worse : Not only the structure of the shock but also the end states on the two sides of the discontinuity depend heavily on the precise shape of the dissipative tensor. Actually, in non-conservative systems, the regularizing effect of the diffusive tensor precisely dictate the amplitude of the jump relations connecting the two states of the discontinuity. We refer for instance to [2] for examples showing the strong influence of the viscosity tensor on the generalized Rankine-Hugoniot relations in non-conservative systems.

For concrete problems, this question is of great practical importance. It implies that the dissipative tensor cannot be arbitrary and must in some sense encode the right physic of the inner structure of a shock.
At present, for concrete problems, the small number of works that have dealt with the effective construction of traveling waves as a means to define shock solutions have mainly considered *viscous* regularizing tensors $D(U)$ [11, 2]. In these works, the dominant regularizing effect is assumed to come from viscosity and the dissipative tensor contains non-zero terms in the momentum equations only.
For simple models of flows, this choice is certainly reasonable. However, for more complex flow models as the ones encountered in two-phase or multicomponent flows, the dissipative tensor can contains, beside viscous effects, many other dissipative effects (Dufour, Soret, etc). This is the case, for instance if the dissipative tensor

has been constructed by the modeling techniques used in non-equilibrium thermodynamics [8]. For these cases, it is not obvious that viscous regularization has the dominant effect.

In the present work, we would like to present an example of another kind of possible regularizing effects than viscous ones in a simple model of two-phase flows. This effect is based on the existence of relaxation phenomena in two-phase systems that drive the two phases toward mechanical and thermodynamical equilibrium. As a consequence, the dominant effect in the regularizing dissipative tensor is not a viscous one leading to a quite unusual structure of this tensor. We will neverthless show that a traveling wave analysis for this system is possible.

Moreover, from a physical point of view, although the system we will consider is simple, we will see that it reveal interesting and unusual features on the structure of shocks in two phase flows. In particular, we will see that this model reveal that the mass fractions in the structure of a two-phase shock is not a constant. This is unexpected since the Rankine-Hugoniot relations for the partial mass equation (that are in conservative form even for other complex non-equilibrium models) predict a zero jump of the mass fraction.

The sequel of this paper is as follows : In section 2, we will present the model and its mathematical properties. In particular, we will show that this model is endowed with a mathematical entropy compatible with the dissipative tensor. Section 3 is devoted to the derivation of this model. It will show that the specific form of the dissipative tensor results from relaxation phenomena between phases that drives two-phase systems toward equilibrium. In section 4, we will study the existence of traveling waves for our model while section 5 will present some numerical examples.

2 The mathematical model

In this work, we are concerned with a traveling wave analysis for the following system

$$\begin{aligned}
\frac{\partial}{\partial t}(\rho) + \frac{\partial}{\partial x}(\rho u) &= 0 \quad (6.1)\\
\frac{\partial}{\partial t}(\rho Y) + \frac{\partial}{\partial x}(\rho Y u) - \varepsilon \frac{\partial}{\partial x}(\rho Y(1-Y)u_r) &= 0 \quad (6.2)\\
\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho (u)^2 + p) &= 0 \quad (6.3)
\end{aligned}$$

This system describes a two-phase medium composed of two immiscible fluids $k = 1, 2$ where the pressures in the phases 1 and 2 are equal and given by barotropic state laws. To be more specific, ρ here denote the total density of the flow, u its velocity while Y is a mass fraction expressing the relative proportion of the mass of one of the two fluid over the total mass. For definitiveness, we will assume that this mass fraction is relative to the fluid 2 : $Y = Y_2$. The mass fractions Y_k are related to the volume fraction α_k by the relation $\alpha_k \rho_k = \rho Y_2$ where ρ_k is the phase densities. The fact that the two fluid cannot mix is expressed by the saturation constraint $\alpha_1 + \alpha_2 = 1$. Finally, the pressures $p_k, k = 1, 2$ in the two phase are given by barotropic state laws $p_k = p_k(\rho_k)$. The phase densities as well as the pressure p are then found by solving the system of equations expressing the equality of the pressures in the two phases as well as the saturation constraint giving :

$$\begin{cases} p_1(\rho_1) = p_2(\rho_2) \\ \dfrac{(1-Y)}{\rho_1} + \dfrac{(Y)}{\rho_2} = \dfrac{1}{\rho} \end{cases}$$

The partial mass conservation equation (6.2) contains a diffusive term that expresses the fact that the velocities of the two phases are not exactly equal. They differs from the center of mass velocity u by a relative velocity u_r. An asymptotic analysis of the non-equilibrium two-phase model to be performed in section 3 provides the following estimate for this term

$$u_r = (Y - \alpha)\frac{\partial p}{\partial x} \tag{7}$$

The system (6) has been introduced in [9] as a model for isothermal bubbly flows. Numerical approximation for this system was proposed in this work and it was shown that despite its simplicity, this system is able to reproduce two-phase computations usually performed with more complex models.

Here our interest in this model is different and we use it to study the possible regularizing effect in two-phase models of second-order perturbations of the form displayed in equation (6.2). Actually, in contrast to many models considered in studies on viscous shock profiles, the model (6) does not contain a viscous regularization in the momentum equation and one may wonder if the diffusive term in (6.2) is sufficient for a regularizing effect to occur. Example of the non-viscous Navier-Stokes system with thermal diffusion [5] shows actually that shock solutions could develop even in the presence of diffusive terms in the equations. Therefore, the question to know if shock solutions of (6) admits diffusive profiles is of interest given the non-standard form of the dissipative tensor. In the remainder of this section we summarize the mathematical properties of this model.

2.1 Entropy and dissipativity

Let us set $U = (\rho, \rho Y, \rho u)^t$ and write the system (6) under the form :

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = \frac{\partial}{\partial x}(D(U)\frac{\partial U}{\partial x}) \tag{8}$$

with obvious notations. The mathematical properties of system (6) have been studied in [9] and we refer to this work for the proof of the results that we recalled in this section. First since the system (6) is barotropic, we can define the Helmholtz free energy $f_k(\rho_k), k = 1, 2$ of each phase by :

$$f_k'(\rho_k) = p_k(\rho_k)/\rho_k^2$$

With these definitions, we introduce the following function

$$\eta(U) = \rho\frac{u^2}{2} + \rho Y f_2(\rho_2) + \rho(1-Y) f_1(\rho_1) \tag{9}$$

We then have

Proposition 2.1 *$\eta(U)$ is an entropy for the system (6). Moreover, in the two-phase case ($Y \in]0,1[$), the diffusion tensor $D(U)$ is dissipative for this entropy.*

Indeed for smooth solutions, $\eta(U)$ verifies the equation

$$\frac{\partial \eta(U)}{\partial t} + \nabla_U \eta(U) \cdot \frac{\partial F(U)}{\partial x} = \frac{\partial}{\partial x}(\nabla_U \eta(U) \cdot D(U)\frac{\partial U}{\partial x}) - \nabla^2_U \eta(U)\frac{\partial U}{\partial x} \cdot D(U)\frac{\partial U}{\partial x} \tag{10}$$

but it can be shown (see [9]) that

$$\nabla_U \eta(U) \cdot \frac{\partial F(U)}{\partial x} = \frac{\partial (\eta(U)+p)u}{\partial x}$$

while

$$\nabla^2_U \eta(U)\frac{\partial U}{\partial x} \cdot D(U)\frac{\partial U}{\partial x} = (u_r)^2$$

and then proposition 2.1 is proved

2.2 Hyperbolicity

If we now concentrate on the first order convective part of system (6)

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0 \tag{11}$$

we have (see [9] for the proof)

Proposition 2.2 *The system (11) is hyperbolic with a complete set of eigenvectors associated to the eigenvalues*

$$\lambda_1(U) = u - a\,, \quad \lambda_2(U) = u\,, \quad \lambda_3(U) = u + a \tag{12}$$

where a is the speed of sound defined by the expression

$$\frac{1}{\rho a^2} = \sum_k \frac{\alpha_k}{\rho_k a_k^2} \tag{13}$$

Moreover, the characteristic fields associated with the waves $\lambda_1(U) = u - a$ and $\lambda_3(U) = u + a$ are genuinely non linear while the characteristic field associated with the wave $\lambda_2(U) = u$ is linearly degenerate.

3 Derivation of the model

In this section, we show that the model (6) can be derived from a barotropic version of the Baer-Nunziato model [1]. For this we will assume that the pressure and velocity relaxation times tend to zero and consider the first-order Chapman-Enskog expansion of the barotropic Baer-Nunziato model in this limit. Note that in [9], the model (6) was also derived from the classical two-velocity, one pressure two fluid model (see for instance [4, 16, 10]) in the limit of large drag coefficients. This section presents an alternate derivation and therefore is not essential to the remainder of the paper. It can be skipped by readers not interested by the details of the derivation of (6).

3.1 The two-pressure, two velocity barotropic model

The non-equilibrium 1986 Baer-Nunziato model [1] or some of its variations have been recently the subject of several works to model two-phase flows [15, 3, 13] . In this section, we describe an isothermal version of this model. We thus consider a flow composed of two immiscible fluids, α_k denote the volume fractions of each phase ($\alpha_1 + \alpha_2 = 1$), ρ_k their phase densities while $\boldsymbol{u}_k$ are the vector velocities. Finally p_k are the pressures related to the phase densities ρ_k by barotropic equations of state

$$p_k = p_k(\rho_k) \tag{14}$$

that in accordance with standard thermodynamic, verify

$$\frac{\partial p_k}{\partial \rho_k} > 0$$

and we note $a_k(\rho_k) = \sqrt{\partial p_k / \partial \rho_k}$ the speed of sound in the phase k. With these notations, the model that we consider is

$$\begin{aligned}
&\frac{\partial \alpha_1 \rho_1}{\partial t} &+& \operatorname{div}(\alpha_1 \rho_1 \boldsymbol{u}_1) &=& 0 &\quad (15.1)\\
&\frac{\partial \alpha_1 \rho_1 \boldsymbol{u}_1}{\partial t} &+& \operatorname{div}(\alpha_1 \rho_1 \boldsymbol{u}_1 \otimes \boldsymbol{u}_1) + \nabla \alpha_1 p_1 &=& p_{\mathrm{I}} \nabla \alpha_1 + \lambda(\boldsymbol{u}_2 - \boldsymbol{u}_1) &\quad (15.2)\\
&\frac{\partial \alpha_2 \rho_2}{\partial t} &+& \operatorname{div}(\alpha_2 \rho_2 \boldsymbol{u}_2) &=& 0 &\quad (15.3)\\
&\frac{\partial \alpha_2 \rho_2 \boldsymbol{u}_2}{\partial t} &+& \operatorname{div}(\alpha_2 \rho_2 \boldsymbol{u}_2 \otimes \boldsymbol{u}_2) + \nabla \alpha_2 p_2 &=& p_{\mathrm{I}} \nabla \alpha_2 - \lambda(\boldsymbol{u}_2 - \boldsymbol{u}_1) &\quad (15.4)\\
&\frac{\partial \alpha_2}{\partial t} &+& \boldsymbol{u}_{\mathrm{I}} . \nabla \alpha_2 &=& \mu(p_2 - p_1) &\quad (15.5)
\end{aligned}$$

In these equations, p_{I} and $\boldsymbol{u}_{\mathrm{I}}$ stand respectively for the inter-facial pressure and velocity. Let us define $Y_k = (\alpha_k \rho_k)/\rho$ the mass fraction of the phase k where $\rho = \sum_{k=1}^{2} \alpha_k \rho_k$ is the mixture density. $\boldsymbol{u}_{\mathrm{I}}$ is here chosen as the center of mass velocity

$$\boldsymbol{u}_{\mathrm{I}} = \sum_{k=1}^{2} Y_k \boldsymbol{u}_k \tag{16}$$

Considerations on the entropy dissipation of the model (to be given later in this section) show that with this choice of the inter-facial velocity, the inter-facial pressure must have the following value :

$$p_{\mathrm{I}} = \sum_{k=1}^{2} Y_{\bar{k}} p_k \tag{17}$$

where $\bar{k} = (2, 1)$ when $k = (1, 2)$. The model (15) contains relaxation parameters λ and $\mu > 0$ that determine the rates at which the velocities and pressures of the two-phases reach equilibrium. In many situations (bubbly pipe flows, bubble column, etc), the pressure equilibrium time is extremely small while the time necessary to reach velocity equality is much larger. This leads to consider the classical two-velocity, one pressure models ([4, 16, 10]) where equation (15.5) is replaced by the assumption of pressure equilibrium $p_1(\rho_1) = p_2(\rho_2)$. However, this last model fails to be hyperbolic and in the sequel, we prefer to derive (6) directly from (15) instead of assuming a priori pressure equality between the two phases.

3.1.1 Quasi-linear form of the model

In the sequel, the material derivative of a quantity ϕ with respect to the velocities $\boldsymbol{u}_k$ of each phase k as well as the inter-facial velocity $\boldsymbol{u}_{\mathrm{I}}$ will be denoted by :

$$\frac{D_k \phi}{Dt} = \frac{\partial \phi}{\partial t} + \boldsymbol{u}_k . \nabla \phi \quad \text{for} \quad k = 1, 2 \text{ and } k = \mathrm{I} \tag{18}$$

Using this notation, the momentum equations (15.2)-(15.5) and the mass conservation equations (15.1)-(15.4), it is easily seen that the velocities $\boldsymbol{u}_k$ obey the following equations :

$$\alpha_1\rho_1\frac{D_1\boldsymbol{u}_1}{Dt}+\nabla\alpha_1p_1 = p_{\mathrm{I}}\nabla\alpha_1+\lambda(\boldsymbol{u}_2-\boldsymbol{u}_1) \quad (19.1)$$

$$\alpha_2\rho_2\frac{D_2\boldsymbol{u}_2}{Dt}+\nabla\alpha_2p_2 = p_{\mathrm{I}}\nabla\alpha_2-\lambda(\boldsymbol{u}_2-\boldsymbol{u}_1) \quad (19.2)$$

Next, using the volume fraction equation (15.7), we can rewrite the mass conservation equations (15.1)-(15.4) in term of phase densities ρ_k under the form

$$\alpha_1\frac{D_1\rho_1}{Dt}+\alpha_1\rho_1\mathrm{div}\boldsymbol{u}_1 = \rho_1(\boldsymbol{u}_{\mathrm{I}}-\boldsymbol{u}_1).\nabla\alpha_1+\mu\rho_1(p_2-p_1) \quad (20.1)$$

$$\alpha_2\frac{D_2\rho_2}{Dt}+\alpha_2\rho_2\mathrm{div}\boldsymbol{u}_2 = \rho_2(\boldsymbol{u}_{\mathrm{I}}-\boldsymbol{u}_2).\nabla\alpha_2-\mu\rho_2(p_2-p_1) \quad (20.2)$$

Finally, to get the equations for the pressures p_k of each phase, we write that since $p_k=p_k(\rho_k)$, then $\frac{D_kp_k}{Dt}=a_k^2\frac{D_k\rho_k}{Dt}$ and we obtain

$$\alpha_1\frac{D_1p_1}{Dt}+\alpha_1\rho_1a_1^2\mathrm{div}\boldsymbol{u}_1=\rho_1a_1^2(\boldsymbol{u}_{\mathrm{I}}-\boldsymbol{u}_1).\nabla\alpha_1+\mu\rho_1a_1^2(p_2-p_1) \quad (21.1)$$

$$\alpha_2\frac{D_2p_2}{Dt}+\alpha_2\rho_2a_2^2\mathrm{div}\boldsymbol{u}_2=\rho_2a_2^2(\boldsymbol{u}_{\mathrm{I}}-\boldsymbol{u}_2).\nabla\alpha_2-\mu\rho_2a_2^2(p_2-p_1) \quad (21.2)$$

Let us summarize these results. In term of the variables ${}^t(\boldsymbol{u}_k,p_k,\alpha_2)$, the isothermal non-equilibrium model (15) can be written under the quasi-linear form

$$\left\{\begin{array}{lcll}
\alpha_1\rho_1\frac{D_1\boldsymbol{u}_1}{Dt}+\nabla\alpha_1p_1 & = & p_{\mathrm{I}}\nabla\alpha_1+\lambda(\boldsymbol{u}_2-\boldsymbol{u}_1) & (22.1)\\
\alpha_2\rho_2\frac{D_2\boldsymbol{u}_2}{Dt}+\nabla\alpha_2p_2 & = & p_{\mathrm{I}}\nabla\alpha_2-\lambda(\boldsymbol{u}_2-\boldsymbol{u}_1) & (22.2)\\
\alpha_1\frac{D_1p_1}{Dt}+\alpha_1\rho_1a_1^2\mathrm{div}\boldsymbol{u}_1 & = & \rho_1a_1^2(\boldsymbol{u}_{\mathrm{I}}-\boldsymbol{u}_1).\nabla\alpha_1+\mu\rho_1a_1^2(p_2-p_1) & (22.3)\\
\alpha_2\frac{D_2p_2}{Dt}+\alpha_2\rho_2a_2^2\mathrm{div}\boldsymbol{u}_2 & = & \rho_2a_2^2(\boldsymbol{u}_{\mathrm{I}}-\boldsymbol{u}_2).\nabla\alpha_2-\mu\rho_2a_2^2(p_2-p_1) & (22.4)\\
\frac{D_{\mathrm{I}}\alpha_2}{Dt} & = & \mu(p_2-p_1) & (22.5)
\end{array}\right.$$

In the sequel, we will denote $C_k=\rho_ka_k^2$ the bulk modulus.

3.1.2 Existence of an entropy

Proposition 3.1 *Let $\eta(\boldsymbol{U}),G(\boldsymbol{U})$ be defined by*

$$\eta=\alpha_1\rho_1(\frac{\boldsymbol{u}_1^2}{2}+f_1(\rho_1))+\alpha_2\rho_2(\frac{\boldsymbol{u}_2^2}{2}+f_2(\rho_2))$$

$$G(\boldsymbol{U})=\boldsymbol{u}_1(\alpha_1\rho_1(\frac{\boldsymbol{u}_1^2}{2}+f_1(\rho_1))+\alpha_1p_1)+\boldsymbol{u}_2(\alpha_2\rho_2(\frac{\boldsymbol{u}_2^2}{2}+f_2(\rho_2))+\alpha_2p_2)$$

then $\eta(\boldsymbol{U}),G(\boldsymbol{U})$ are a couple entropy-flux that verifies :

$$\frac{\partial\eta}{\partial t}+divG\leq 0$$

3.1.3 Hyperbolicity

Proposition 3.2 *The system (15) is hyperbolic with a complete set of eigenvectors associated to the eigenvalues*

$$\begin{array}{l}\lambda_1(U)=u_1-a_1\\ \lambda_2(U)=u_1+a_1\end{array}, \qquad \lambda_3(U)=u_I \qquad \begin{array}{l}\lambda_4(U)=u_2-a_2\\ \lambda_5(U)=u_2+a_2\end{array} \tag{23}$$

where a_i is the speed of sound in the phase i Moreover, the characteristic fields associated with the waves $\lambda_1(U), \lambda_2(U), \lambda_4(U), \lambda_5(U)$ are genuinely non linear while the characteristic field associated with the wave $\lambda_3(U)=u_I$ is linearly degenerate.

3.2 First-order Chapman-Enskog analysis

In this section, we are interested in situations where the relaxation times are small compared with the others characteristic times of the flow and derive a reduced model for these situations. Thus we set $\lambda=\lambda^0/\varepsilon$ and $\mu=\mu^0/\varepsilon$ where λ^0 and μ^0 are $\mathcal{O}(1)$ and we analyze the case $\varepsilon\to 0$. This analysis is similar to the ones performed in [12] and [9] and uses the Chapman-Enskog expansion technique. For the sake of simplicity, this analysis will be done in 1-D. First, we set $\boldsymbol{U}={}^t(u_1,u_2,p_1,p_2,\alpha_2)$. Using the results of paragraph 3.1.1 we write system (22) in the form

$$\frac{\partial \boldsymbol{U}}{\partial t}+A(\boldsymbol{U})\frac{\partial \boldsymbol{U}}{\partial x}=\frac{R(\boldsymbol{U})}{\varepsilon} \tag{24}$$

where the Jacobian matrix $A(\boldsymbol{U})$ is given by :

$$A(\boldsymbol{U})=\begin{pmatrix} u_1 & 0 & 1/\rho_1 & 0 & (p_1-p_{\mathrm{I}})/(\alpha_1\rho_1)\\ 0 & u_2 & 0 & 1/\rho_2 & (p_2-p_{\mathrm{I}})/(\alpha_2\rho_2)\\ C_1 & 0 & u_1 & 0 & C_1(u_1-u_{\mathrm{I}})/\alpha_1\\ 0 & C_2 & 0 & u_2 & C_2(u_2-u_{\mathrm{I}})/\alpha_2\\ 0 & 0 & 0 & 0 & u_{\mathrm{I}} \end{pmatrix} \tag{25}$$

while the source term $R(\boldsymbol{U})$ has the following expression :

$$R(\boldsymbol{U})=\begin{pmatrix} \lambda^0(u_2-u_1)/(\alpha_1\rho_1)\\ -\lambda^0(u_2-u_1)/(\alpha_2\rho_2)\\ \mu^0 C_1(p_2-p_1)/\alpha_1\\ -\mu^0 C_2(p_2-p_1)/\alpha_2\\ \mu^0(p_2-p_1) \end{pmatrix} \tag{26}$$

The equilibrium set

$$\mathcal{E}=\{\boldsymbol{U}\in I\!R^5; R(\boldsymbol{U})=0\} \tag{27}$$

is a smooth manifold that can be parametrized by the mapping $M:\boldsymbol{u}\in I\!R^5\to\boldsymbol{U}\in\mathcal{E}$ defined by

$$\begin{pmatrix}\boldsymbol{u}_1\\ \boldsymbol{u}_2\\ \boldsymbol{u}_3\end{pmatrix}=\begin{pmatrix}u\\ p\\ \alpha_2\end{pmatrix} \longrightarrow \begin{pmatrix}\boldsymbol{u}_1\\ \boldsymbol{u}_1\\ \boldsymbol{u}_2\\ \boldsymbol{u}_2\\ \boldsymbol{u}_3\end{pmatrix}=\begin{pmatrix}u\\ u\\ p\\ p\\ \alpha_2\end{pmatrix} \tag{28}$$

For any $u=(u,p,\alpha_2)\in I\!R^3$, the linearized source term evaluated on an equilibrium state has the expression

$$R'(M(\boldsymbol{u}))=\begin{pmatrix} -\lambda^0/(\alpha_1\rho_1) & \lambda^0/(\alpha_1\rho_1) & 0 & 0 & 0\\ \lambda^0/(\alpha_2\rho_2) & -\lambda^0/(\alpha_2\rho_2) & 0 & 0 & 0\\ 0 & 0 & -\mu^0 C_1/\alpha_1 & \mu^0 C_1^0/\alpha_1 & 0\\ 0 & 0 & \mu^0 C_2/\alpha_2 & -\mu^0 C_2^0/\alpha_2 & 0\\ 0 & 0 & -\mu^0 & \mu^0 & 0 \end{pmatrix} \tag{29}$$

Looking for solutions of (24) close to the equilibrium manifold $\mathcal{E}$ we introduce the ansatz

$$\boldsymbol{U} = M(\boldsymbol{u}) + \varepsilon \boldsymbol{V} = \begin{pmatrix} u^0 \\ u^0 \\ p^0 \\ p^0 \\ \alpha_2^0 \end{pmatrix} + \varepsilon \begin{pmatrix} u_1^1 \\ u_2^1 \\ p_1^1 \\ p_2^1 \\ \alpha_2^1 \end{pmatrix} \tag{30}$$

and in agreement with the Chapman-Enskog asymptotic technique, we choose the first-order correction $\boldsymbol{V}$ in $\mathrm{Rng}(R'(M(\boldsymbol{u})))$, the first step is thus to characterize the elements of this space. For this we have

Lemma 3.1 *Let $\boldsymbol{V} = {}^t(u_1^1, u_2^1, p_1^1, p_2^1, \alpha_2^1) \in Rng(R'(M(\boldsymbol{u})))$ then it exists $\boldsymbol{w} = {}^t(w_1, w_2) \in I\!R^2$ such that*

$$\left\{ \begin{array}{lllllll} u_1^1 & = & -w_1/Y_1 & (31.1) & p_1^1 & = & -w_2 C_1/\alpha_1 \quad (31.3) \\ u_2^1 & = & w_1/Y_2 & (31.2) & p_2^1 & = & w_2 C_2/\alpha_2 \quad (31.4) \\ & & & & \alpha_2^1 & = & -w_2 \quad (31.5) \end{array} \right.$$

Proof: The expression (29) of the linearized source term shows that a basis of $\mathrm{Rng}(R'(M(\boldsymbol{u})))$ is given by the two vectors

$$I^1 = \begin{pmatrix} -1/Y_1^0 \\ 1/Y_2^0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad ; \quad I^2 = \begin{pmatrix} 0 \\ 0 \\ -C_1^0/\alpha_1^0 \\ C_2^0/\alpha_2^0 \\ -1 \end{pmatrix} \tag{32}$$

Thus for any vector, $\boldsymbol{V} \in \mathrm{Rng}(R'(M(\boldsymbol{u})))$, there exists $\boldsymbol{w} \in I\!R^2$ such that $\boldsymbol{V} = \boldsymbol{w}_1 I^1 + \boldsymbol{w}_2 I^2$ and the relations (31) follow.

It remains now to find the expression of the coordinates $\boldsymbol{w}$ in term of the order 0 variables ${}^t(u^0, p^0, \alpha_2^0)$ and their derivatives. This is done in the following result :

Lemma 3.2 *Let $\boldsymbol{V} = {}^t(u_1^1, u_2^1, p_1^1, p_2^1, \alpha_2^1) \in Rng(R'(M(\boldsymbol{u}))$ be the vector of fluctuations and introduce the notations $u_r = u_2^1 - u_1^1$ and $\Delta p = p_2^1 - p_1^1$ then the following relations hold true*

$$\begin{array}{lllr} \lambda^0 u_r & = & \rho^0 Y_1^0 Y_2^0 (\dfrac{1}{\rho_1^0} - \dfrac{1}{\rho_2^0}) \dfrac{\partial p^0}{\partial x} = (Y_2^0 - \alpha_2^0) \dfrac{\partial p^0}{\partial x} & (33.1) \\ \mu^0 \Delta p & = & \alpha_1^0 \alpha_2^0 \dfrac{C_1^0 - C_2^0}{\alpha_1^0 C_2^0 + \alpha_2^0 C_1^0} \dfrac{\partial u^0}{\partial x} & (33.2) \end{array}$$

Proof: Let us introduce the ansatz $\boldsymbol{U} = M(\boldsymbol{u}) + \varepsilon \boldsymbol{V}$ in system (24), up to terms of order $\mathcal{O}(\varepsilon)$ we obtain

$$\frac{\partial M(\boldsymbol{u})}{\partial t} + A(M(\boldsymbol{u})) \frac{\partial M(\boldsymbol{u})}{\partial x} - R'(M(\boldsymbol{u})).\boldsymbol{V} = \mathcal{O}(\varepsilon) \tag{34}$$

or in developed form

$$\begin{array}{lllr} \dfrac{\partial u^0}{\partial t} + u^0 \dfrac{\partial u^0}{\partial x} + \dfrac{1}{\rho_1^0} \dfrac{\partial p^0}{\partial x} - \dfrac{\lambda^0}{\alpha_1^0 \rho_1^0}(u_2^1 - u_1^1) & = & \mathcal{O}(\varepsilon) & (35.1) \\ \dfrac{\partial u^0}{\partial t} + u^0 \dfrac{\partial u^0}{\partial x} + \dfrac{1}{\rho_2^0} \dfrac{\partial p^0}{\partial x} - \dfrac{\lambda^0}{\alpha_2^0 \rho_2^0}(u_1^1 - u_k^2) & = & \mathcal{O}(\varepsilon) & (35.2) \\ \dfrac{\partial p^0}{\partial t} + u^0 \dfrac{\partial p^0}{\partial x} + C_1^0 \dfrac{\partial u^0}{\partial x} - \dfrac{\mu^0 C_1^0}{\alpha_1^0}(p_2^1 - p_1^1) & = & \mathcal{O}(\varepsilon) & (35.3) \\ \dfrac{\partial p^0}{\partial t} + u^0 \dfrac{\partial p^0}{\partial x} + C_2^0 \dfrac{\partial u^0}{\partial x} - \dfrac{\mu^0 C_2^0}{\alpha_2^0}(p_1^1 - p_2^1) & = & \mathcal{O}(\varepsilon) & (35.4) \\ \dfrac{\partial \alpha_2^0}{\partial t} + u^0 \dfrac{\partial \alpha_2^0}{\partial x} - \mu^0 (p_2^1 - p_1^1) & = & \mathcal{O}(\varepsilon) & (35.5) \end{array} \tag{35}$$

Multiplying (35.2) by $\alpha_2^0\rho_2^0$, (35.3) by $\alpha_1^0\rho_1^0$ and taking the difference, one obtains (33.1). The expression for the difference of the pressure fluctuations can be obtained in the same way. $\square$

Now, combining the last two lemmas, we can characterize completely the first order correction $\boldsymbol{V} \in \mathrm{Rng}(R'(M(\boldsymbol{u})))$

Proposition 3.3 *Up to terms of order $\mathcal{O}(\varepsilon)$ the fluctuations $\boldsymbol{V} \in$ Rng$(R'(M(\boldsymbol{u})))$ are given by*

$$\begin{cases} u_1^1 = -Y_2^0 u_r \quad (36.1) & p_1^1 = -\alpha_2^0 \dfrac{C_1}{C}\Delta p \quad (36.3) \\ u_2^1 = Y_1^0 u_r \quad (36.2) & p_2^1 = \alpha_2^0 \dfrac{C_2}{C}\Delta p \quad (36.4) \\ & \alpha_2^1 = -\dfrac{\alpha_1^0\alpha_2^0}{C}\Delta p \quad (36.5) \end{cases}$$

We can now use these results to construct a reduced model valid up to order ε^2. It appears that for this purpose, the use of the conservative variables ${}^t(\alpha_k^0\rho_k^0, \rho^0 u^0, \rho^0 e^0, \alpha_2^0)$ is more appropriate. Thus considering the equations (15), we expand each variable f into a mean part and a first-order correction

$$f = f^0 + \varepsilon f^1 \tag{37}$$

then use Proposition 3.3 to express the first-order corrections and finally neglect all terms of order smaller than ε.

In the sequel, we introduce the notations : $\rho^0 = \sum_{k=1}^2 \alpha_k^0\rho_k^0$ the mixture density. The details of the computations are now as follows

Proposition 3.4 (Mass conservation equations) : *There exists a non negative scalar χ such that the mass conservation equations can be written :*

$$\frac{\partial}{\partial t}(\rho_k^0 Y_k^0) + \frac{\partial}{\partial x}(\rho_k^0 Y_k^0 u^0) - \varepsilon\frac{\partial}{\partial x}(\chi(\tau_k^0 - \tau_k'^0)\frac{\partial p}{\partial x}) = \mathcal{O}(\varepsilon^2)$$

Proof : First, we need to express the first-order fluctuations of the phase densities ρ_k^1. Since we have $\rho_k = \rho_k(p_k)$ we deduce that

$$\rho_k(p + \epsilon p_k^1) = \rho_k(p) + \frac{\epsilon}{a_k^2} p_k^1$$

then $\rho_k^1 = p_k^1/a_k^2$ and with (36.2) to express the first-order velocities we obtain

$$\frac{\partial}{\partial t}(\alpha_k^0\rho_k^0) + \frac{\partial}{\partial x}(\alpha_k^0\rho_k^0 u^0) = \varepsilon\frac{\partial}{\partial x}\left(\frac{(\rho^0 Y_1^0 Y_2^0)^2}{\lambda^0}(\frac{1}{\rho_k^0} - \frac{1}{\rho_k'^0})\frac{\partial p^0}{\partial x}\right) \tag{38}$$

that are exactly (3.4) with

$$\chi = \frac{(\rho^0 Y_1^0 Y_2^0)^2}{\lambda^0}$$

with the definition $\tau_k = 1/\rho_k$. For future reference, we will denote J_k, the partial mass "diffusive" flux :

$$J_k = \frac{(\rho^0 Y_1^0 Y_2^0)^2}{\lambda^0}(\frac{1}{\rho_k^0} - \frac{1}{\rho_k'^0})\frac{\partial p^0}{\partial x} = \frac{(Y_1^0 Y_2^0)(\alpha_1^0\alpha_2^0)}{\lambda^0}(\rho_k'^0 - \rho_k^0)\frac{\partial p^0}{\partial x} \tag{39}$$

Proposition 3.5 (Mixture momentum conservation equation) : *There exists a non negative scalar coefficient μ such that the momentum conservation equation can be written :*

$$\frac{\partial}{\partial t}(\rho^0 u^0) + \frac{\partial}{\partial x}(\rho^0 (u^0)^2 + p^0) - \varepsilon\frac{\partial}{\partial x}(\mu\frac{\partial u^0}{\partial x}) = 0$$

Proof: First, we remark that the total momentum is up to terms of order $\mathcal{O}(\varepsilon^2)$ equal to its 0-order approximation. This is a direct consequence of (31.1)-(31.2) and (31.3) as we have

$$\sum_{k=1}^{2} \alpha_k \rho_k u_k = \sum_{k=1}^{2} (\alpha_k \rho_k)^0 u^0 + \varepsilon[(\alpha_k \rho_k)^1 u^0 + (\alpha_k \rho_k)^0 u_k^1] + \mathcal{O}(\varepsilon^2)$$

Thus, to obtain the mixture momentum equation, we sum the two momentum equations and obtain

$$\begin{aligned} &\frac{\partial}{\partial t}(\rho^0 u^0) + \frac{\partial}{\partial x}(\rho^0 (u^0)^2 + p^0) + \\ &+\varepsilon \frac{\partial}{\partial x}(\sum_{k=1}^{2} \alpha_k^0 p_k^1) + \varepsilon \frac{\partial}{\partial x}(\sum_{k=1}^{2} \alpha_k^1 p^0) = \mathcal{O}(\varepsilon^2) \end{aligned} \tag{40}$$

Since $\alpha_1^1 + \alpha_2^1 = 0$ we finally obtain :

$$\frac{\partial}{\partial t}(\rho^0 u^0) + \frac{\partial}{\partial x}(\rho^0 (u^0)^2 + p^0) + \varepsilon \frac{\partial}{\partial x}(\sum_{k=1}^{2} \alpha_k^0 p_k^1) = \mathcal{O}(\varepsilon^2) \tag{41}$$

and the result follows from (36.3) with

$$\mu = \frac{(C_2^0 - C_1^0)^2}{\mu_0 (\frac{C_2^0}{\alpha_2^0} + \frac{C_1^0}{\alpha_1^0})^2} = \frac{(\alpha_1^0 \alpha_2^0)^2}{\mu_0} (\frac{1}{C_2^0} - \frac{1}{C_1^0})^2 [\rho^0 (\hat{a}^2)]^2$$

□

To summarize, the first-order Chapman-Enskog expansion of the two-velocity, two-pressure model barotropic model (15) is given by the following set of equations. Observe that this system is formally very close to the compressible Navier-Stokes system:

$$\frac{\partial}{\partial t}(\alpha_1^0 \rho_1^0) + \frac{\partial}{\partial x}(\alpha_1^0 \rho_1^0 u^0) - \varepsilon \frac{\partial}{\partial x} J_1 = 0 \tag{42.1}$$

$$\frac{\partial}{\partial t}(\alpha_2^0 \rho_2^0) + \frac{\partial}{\partial x}(\alpha_2^0 \rho_2^0 u^0) - \varepsilon \frac{\partial}{\partial x} J_2 = 0 \tag{42.2}$$

$$\frac{\partial}{\partial t}(\rho^0 u^0) + \frac{\partial}{\partial x}(\rho^0 (u^0)^2 + p^0) - \varepsilon \frac{\partial}{\partial x}\left(\mu \frac{\partial u^0}{\partial x}\right) = 0 \tag{42.3}$$

where the expression for J_k and μ are given in Propositions 3.4 and 3.5. Comparison between (42) and the model (6) shows that (42) obtained from (15) contains in addition to the dissipative terms present in the model (6) a viscous-like contribution in the momentum equation. This viscous contribution comes from the assumption that pressure relaxation effects are of the same order of magnitude than velocity drag relaxation effects. As pointed out in the begining of this section, in many cases of practical interest, one can consider that the pressure relaxation time is much shorter than the velocity relaxation time and therefore the viscous term in (42.3) is actually much smaller that the drift mass flux and the appropriate reduced model for these situations is (6)[1]. Moreover, for the eos that we will consider in the next section, the viscous contribution in (42) disappears.

4 Traveling wave solutions of the reduced model

In this section, we specialize the discussion to the following equations of state :

$$p_k = \rho_k a_k^2 \quad \text{for} \quad k = 1, 2 \tag{43}$$

[1]However, the opposite can be true for some situations : in plasma physics for instance, the velocities of the different particles, say the ions and the electrons are almost identical while the ionic and electronic pressures can be very different

where a_k, the speed of sound in the phase k is a constant. For definiteness, we will assume with no loss of generality, that $a_2 > a_1$. With these eos, the viscosity-like term in the momentum equation disappears and the system (42) reduces to the model (6):

$$\begin{aligned}
&\frac{\partial}{\partial t}(\rho) + \frac{\partial}{\partial x}(\rho u) &&= 0 \quad (44.1)\\
&\frac{\partial}{\partial t}(\rho Y) + \frac{\partial}{\partial x}(\rho Y u) - \varepsilon\frac{\partial}{\partial x}(\rho Y(1-Y)u_r) &&= 0 \quad (44.2)\\
&\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho (u)^2 + p) &&= 0 \quad (44.3)
\end{aligned}$$

where u_r is given by the following relation :

$$u_r = (Y - \alpha)\frac{\partial p}{\partial x} \tag{45}$$

We want to establish the existence of a certain class of solutions of this system, namely the traveling waves solutions defined as follows

Definition $U(t,x)$ is a traveling wave solution of (44) if

1. There exists a real s and a one-parameter function $\hat{U}(\xi)$ such that $U(t,x) = \hat{U}(x - st)$
2. There exist two state vectors U_L and U_R such that

$$\begin{cases}
\lim_{\xi\to-\infty} \hat{U}(\xi) = U_L & (46.1)\\
\lim_{\xi\to-\infty} \hat{U}'(\xi) = 0 & (46.2)\\
\lim_{\xi\to+\infty} \hat{U}(\xi) = U_R & (46.3)\\
\lim_{\xi\to+\infty} \hat{U}'(\xi) = 0 & (46.4)
\end{cases}$$

If such solutions exist, they are characterized by the differential system of degree 2 :

$$(F(\hat{U}))' - s\hat{U}' = (D(\hat{U})\hat{U}')' \tag{47}$$

In the sequel, we will assume with no loss of generality that $s > 0$. The right state U_R is therefore the unperturbed state (before the passage of the wave) while the left state U_L is the perturbed state (after the passage of the wave). Since the system (6) is in conservative form (47) can be integrated once to yield a first degree differential system. Moreover integrating the system (44) between $\xi = -\infty$ and $\xi = \infty$ yields the Rankine-Hugoniot conditions that the two end states U_L and U_R must satisfy if they are connected by a shock wave. These jump conditions are :

Lemma 4.1

$$\begin{aligned}
\rho_R(u_R - s) &= \rho_L(u_L - s) && (48.1)\\
Y_R\rho_R(u_R - s) &= Y_L\rho_L(u_L - s) && (48.2)\\
u_R\rho_R(u_R - s) + p_R &= u_L\rho_L(u_L - s) + p_L && (48.3)
\end{aligned}$$

Let us define $M = \rho_R(u_R - s) = \rho_L(u_L - s)$ the constant mass flux across the wave. Equation (48.1) establishes that the mass flux M is a constant. Two cases are therefore possible, $M > 0$ or $M < 0$. They are associated respectively with a 1-compression wave or a 3-compression wave. According to Lax shock criterion, a 3-compression wave has to verify

$$\begin{cases}
u- < s\\
u^- - a^- > s > u^+ - a^+
\end{cases}$$

for the two states U^- and U^+ on the two sides of the wave. In this study, we assume that $s > 0$ and thus $M = \rho(u - s) < 0$. The case of a 1-compression wave $M > 0$ can be handled similarly by changing ξ in $-\xi$.

We also note that if $M \neq 0$ then the relation (48.2) implies that the mass fractions are equal on the two sides of the discontinuity.

In the sequel we are interested to show the existence of viscous profile connecting two end states verifying (48), that is we are interested to find a solution U of the profile equation

$$D(\hat{U})\hat{U}' = F(\hat{U}) - F(\hat{U}_R) - s(\hat{U} - \hat{U}_R) \tag{49}$$

such that $\hat{U} \to \hat{U}_L$ for $\xi \to -\infty$. For convenience, we will omit the notation $\hat{}$ in the sequel. The integration between ξ and $+\infty$ of the equations (44.1) and (44.3) gives using the boundary conditions (46)

Lemma 4.2

$$\begin{aligned} \rho(\xi)(u(\xi) - s) &= M = constant && (50.1) \\ M^2(\tau^R - \tau(\xi)) + p^R - p(\xi) &= 0 && (50.2) \end{aligned}$$

where we have used the notation $\tau = 1/\rho$.

The next step in the analysis of the differential system (47) is to reduce it to a first degree autonomous differential equation. This is done in the following result :

Lemma 4.3 *The differential system (47) associated with the right boundary conditions*

$$\begin{cases} \lim_{\xi \to +\infty} U(\xi) = U_R \\ \lim_{\xi \to +\infty} U'(\xi) = 0 \end{cases}$$

can be reduced to the following first order autonomous system of dimension 1 in the variable $z = p - p_R$

$$Y(1-Y)(\alpha - Y)z' = \tau \frac{z(M^2\tau_R - p_R - z)}{M(a_2^2 - a_1^1)} \qquad (51)$$

associated with the following definitions for the specific volume and velocity

$$\begin{cases} \tau = \tau^R - \dfrac{z}{M^2} & (52.1) \\[2ex] u = u^R - \dfrac{z}{M} & (52.2) \end{cases}$$

and where Y and α are functions of z given by the expressions

$$\begin{cases} Y(z) = Y_R + \dfrac{z(M^2\tau_R - p_R - z)}{M^2(a_2^2 - a_1^2)} & (53.1) \\[2ex] \alpha(z) = \dfrac{a_2^2 Y(z)}{\tau(z)(p_R + z)} & (53.2) \end{cases}$$

Proof: We first begin to prove (53.1). The equations of state $p_k = \rho_k a_k^2$ allow to write

$$p\tau = Y_1 a_1^1 + Y_2 a_2^2 = (a_2^2 - a_1^2)Y + a_1^2$$

that can be solved for Y to obtain

$$Y = \frac{p\tau - a_1^2}{(a_2^2 - a_1^2)} = \frac{p(M^2\tau_R + p_R - p) - M^2 a_1^2}{M^2(a_2^2 - a_1^2)}$$

from which (53.1) follows by noting that

$$p_R\tau_R = (a_2^2 - a_1^2)Y_R + a_1^2$$

Next, we consider (53.2). The definition of $\alpha = \alpha_2$ together with the state law gives

$$\alpha = \frac{Y}{\tau\rho_2} = \frac{a_2^2 Y(z)}{p\tau}$$

that is exactly (53.2). Next, the relations (52) comes from (50). Finally, the integration between ξ and $+\infty$ of the mass fraction equation gives

$$M(Y - Y_R) = \rho Y(1-Y)(Y-\alpha)p'$$

that together with (53.1) gives (51).

Let us define z_L by $z_L = p_L - p_R = M^2\tau_R - p_R$. The existence of viscous profiles is thus reduced to the study of the ode

$$Y(1-Y)(\alpha - Y)z' = \tau \frac{z(z_L - z)}{M(a_2^2 - a_1^1)} \tag{54}$$

We begin to note that $z_L > 0$ since by the Lax criterion we have $s - u_R > a_R$ that implies $\tau_R\rho_R^2(s-u_R)^2 > \rho_R a_R^2 = p_R$. The ode (54) can therefore be studied between $z = 0$ and $z = z_L$ that are two equilibrium points. Before studying the stability of these equilibria, we have to check if (54) can become singular. We are thus lead to study the function $Y(z)(1 - Y(z))(\alpha(z) - Y(z))$ for $z \in [0, z_L]$.
We begin to remark that $(\alpha(z) - Y(z)) > 0$ for $Y(1 - Y) > 0$ since from the state law and the definition of α we have

$$\alpha(z) - Y(z) = Y(z)\frac{\rho - \rho_2}{\rho_2} = Y(z)(1 - Y(z))\frac{(a_2^2 - a_1^2)}{(p_R + z)(\tau(z))}$$

Therefore, it just remains to study $Y(z)(1 - Y(z))$ where $Y(z)$ is given by (53.1). Between $z = 0$ and $z = z_L$, $Y(z) > Y_R$ and thus the only possibility for the ode (54) to become singular is that $Y(z) \geq 1$. We are then lead to consider two different situations.

4.1 Weak shocks

Consider the function

$$Y(z) = Y_R + \frac{z(z_L - z)}{M^2(a_2^2 - a_1^2)} \tag{55}$$

This is a second degree polynomial in z whose maximum is reached for $z = z_L/2$. The value of this maximum is

$$Y_{\max} = Y_R + \frac{z_L^2}{4M^2(a_2^2 - a_1^2)} = \frac{(M^2\tau_R - p_R)^2}{4M^2(a_2^2 - a_1^2)}$$

and is a increasing function of M^2. Therefore it exists a critical value of the mass flux M_{crit} such that if $M^2 < M_{crit}^2$ then $Y_{\max} < 1$ and then the ode (54) is never singular. Some algebra gives :

$$M_{crit}^2 = \frac{a^2 + 2(1 - Y_2)(a_1^2 - a_2^2) + 2\sqrt{(1 - Y_2)(a_1^2 - a_2^2)(a^2 + (1 - Y_2)(a_1^2 - a_2^2))}}{\tau_R^2} \tag{56}$$

with $a^2 = \tau_R p_R$. To end the study of this case, we just have to study the stability of the two equilibria $z = 0$ and $z = z_L$. Linearizing (54) in the vicinity of these two points shows easily that $z = 0$ is a stable equilibrium while $z = z_L$ is unstable. We summarize this case in the result :

Proposition 4.1 : Weak shock case *If the mass flux verifies $M^2 < M_{crit}^2$ the ode (54) is never singular and therefore, it exists a unique $\mathscr{C}^1$ solution connecting the two equilibriums $z = 0$ and $z = z_L$ and in consequence a viscous profile connecting the two states U_L and U_R.*

4.2 Strong shocks

We are now interested in the strong shock case, for which

$$M^2 > M_{crit}^2$$

in this case, the equation

$$1 = Y_R + \frac{z(z_L - z)}{M^2(a_2^2 - a_1^2)} \tag{57}$$

admits exactly 2 roots (eventually identical) that we will denote z_L^* and z_R^* with $0 < z_R^* \leq z_L^* < z_L$. For $z = z_R^*$ or $z = z_L^*$, the ode (54) becomes singular and it is not possible to connect $z = 0$ to $z = z_L$ by a $\mathscr{C}^1$ orbit. However, the important point to note is that since $z = 0$ and $z = z_L$ are equilibria corresponding to two states U_R and U_L that satisfy the Rankine-Hugoniot relations (4.1) then the two states U_R^* and U_L^* corresponding to z_R^* and z_L^* also satisfy the same Rankine-Hugoniot relations (4.1) with the same mass flux :

Lemma 4.4 *Let $U^*_{L,R} = {}^t(\tau^*_{L,R}, u^*_{L,R}, Y^*_{L,R})$ be the two states corresponding to the variables $z^*_{L,R}$ given by the relations (52) and $Y^*_{L,R} = 1$ then $U^*_{L,R}$ satisfy the Rankine-Hugoniot relations (4.1) with the same jump velocity s*

Proof: This can be seen by noting that the Rankine-Hugoniot relations results from the integration of (47) : Let ξ^*_L and ξ^*_R be the coordinates where $z = z^*_L$ and $z = z^*_R$ respectively. Integration of (47) between $\xi = -\infty$ and $\xi = \xi^*_L$ and between $\xi = \xi^*_R$ and $\xi = +\infty$ respectively yields :

$$F(U^*_L) - F(U_L) = s(U^*_L - U_L)$$

$$F(U_R) - F(U^*_R) = s(U_R - U^*_R)$$

because in $\xi = \xi^*_{L,R}$ the diffusion tensor disappears : $D(U^*_{L,R}) = 0$ while in $\xi = \pm\infty$ the gradients $d_\xi U$ are null. Adding the previous two equations and using the fact that U_L and U_R verify the Rankine-Hugoniot equations shows that they are also verified by U^*_R and U^*_L □

Now, although the ode (54) becomes singular in $z = z^*_R$ and $z = z^*_L$, this ode can be integrated to connect $z = 0$ to $z = z^*_R$ and $z = z^*_L$ to z_L respectively. Together with the previous lemma, this allows the definition of traveling wave solutions connecting the two end states U_R and U_L :

Proposition 4.2 : Strong shock case *If the mass flux verifies $M^2 > M^2_{crit}$, there is no $\mathscr{C}^1$ viscous profile connecting the states U_L and U_R. The ode (54) is singular in z^*_L and z^*_R and there exist an infinite number of orbits connecting the two equilibria $z = 0$ and $z = z_L$. These orbits are composed of*

- *a $\mathscr{C}^1$ two-phase solution connecting the equilibrium $z = 0$ and $z = z^*_R$ or equivalently U_R and U^*_R*
- *a discontinuous one-phase shock connecting the two states U^*_R and U^*_L*
- *a $\mathscr{C}^1$ two-phase solution connecting $z = z^*_L$ and the equilibrium $z = z_L$ or equivalently U^*_L and U_L*

Note that the width of the one phase region is arbitrary. Therefore the profile equation (49) admits an infinity of solutions that differ only by the width of this region. In particular, this one phase zone can be reduced to a single point.

5 Numerical applications

We illustrate the results of the previous section with some numerical applications. In these examples, the equations of state characterizing the two fluids are

$$p_k = \rho_k a_k^2 \quad \text{with} \quad a_1 = 1000 \quad \text{and} \quad a_2 = 3000 \tag{58}$$

The unperturbed state is given by

$$Y_1 = 0.5 \quad p = 8910 \quad u = 10 \tag{59}$$

and according to the state laws (58), this corresponds to the following values of the densities :

$$\rho_1 = 8.91 \times 10^{-3} \qquad \rho_2 = 9.9 \times 10^{-2} \tag{60}$$

and $\rho = \alpha_1\rho_1 + \alpha_2\rho_2 = 1.634862385321101 \times 10^{-2}$. With these values, the minimum mass flux to have a shock is $M = \rho_R(u_R - s) = 12.0692269235486$ and according to expression (56), the minimum mass flux corresponding to a blow-up of $\mathscr{C}^1$ solutions is $M_{crit} = 27.3763735493943$.

Computation of a $\mathscr{C}^1$ solution : We first compute a $\mathscr{C}^1$ solution corresponding to a mass flux equal to $M = 17$. According to the Rankine-Hugoniot relations (4.1) the state after the passage of the shock is given by

$$Y_1 = 0.5 \quad p = 17677.3288439955 \quad u = -505.725226117383 \tag{61}$$

corresponding to the following values of the densities

$$\rho_1 = 1.767732884399551 \times 10^{-2} \qquad \rho_2 = 0.196414764933283 \tag{62}$$

and $\rho = 3.24354657687991 \times 10^{-2}$. This solution was computed by two different methods. The first method integrates the ode (54) with initial conditions given by a slight perturbation of the pre-shock state (59). The computation is stopped when a steady state is obtained. We will check that this state correspond indeed to the post-shock state. From the numerical point of view, a Runge-Kutta method of order 4 with adaptive time stepping has been used.
The second method is a finite volume method that solves directly the PDE system (6). The numerical method is the one of [9] except that the hyperbolic solver has been changed for a Roe scheme. This was done because Roe scheme has the property of computing exactly the stationary discontinuities. Therefore in this way we minimize the influence of the numerical viscosity of the scheme on the width of the shock profile and expect the results to depend only on the dissipative Darcy-Drift model. These finite volume computations have been done in the coordinate system of the shock, starting from discontinuous initial conditions given by the Rankine-Hugoniot conditions and integrating in time the system (6) until a steady state is reached. For these FV computations, a 1000 node mesh was used.
Figure 1 shows the computed profiles in the shock region. In agreement with the theoretical results a smooth shock profile of non-zero width is obtained and the mass fraction passes through a maximum. One can also note that the two numerical methods give identical results.

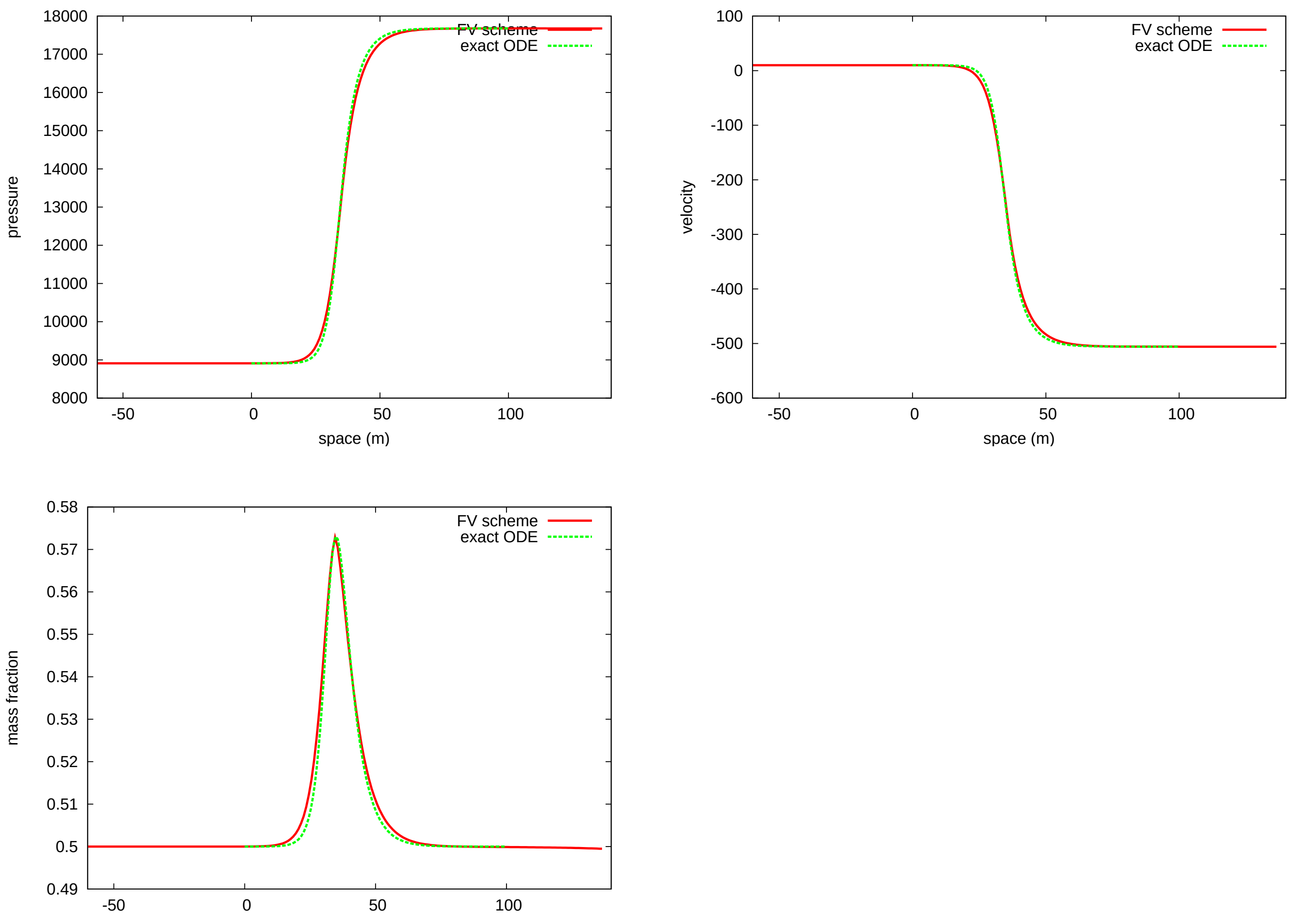

Figure 1: Profiles in the shock for the weak shock case, upper left : pressure, upper right : velocity, lower left : mass fraction

Computation of a discontinuous traveling wave solution : Next, we turn to the computation of a strong shock. The pre-shock state and fluid characteristics are the same than in the previous computations but we use now a value of the mass flux equal to $M = 29 > M_{crit}$. For this case, the Rankine-Hugoniot relations gives the following values for the post-shock state

$$(Y_1)_L = 0.5 \quad p_L = 51441.6386083053 \quad u_L = -1456.60822787260 \tag{63}$$

According to the analysis since $M = 29 > M_{crit}$, no $\mathscr{C}^1$ solutions can exist and the solution will blow-up in two states U_L^* and U_R^* given by the roots of the equation (57). Solving this equation, gives the following values :

$$\begin{aligned} (Y_1)_R^* &= 1 \quad p_R^* = 21834.3523662052 \quad u_R^* = -435.667322972591 \\ (Y_1)_L^* &= 1 \quad p_L^* = 38517.2862421001 \quad u_L^* = -1010.94090490000 \end{aligned} \tag{64}$$

One can check that this is exactly what is reproduced by the numerical methods. Integration of the edo (54) from the two states $z_R = 0$ and z_L shows that the values z_R^* and z_L^* are asymptotically approached for $t \to \pm\infty$. The corresponding states U_L^* and U_R^* satisfy the Rankine-Hugoniot relations with the same mass flux M and thus it is possible to connect them by a single phase shock corresponding to $Y_1 = 1$. This is what is shown in figure 2. In these figures, the width of the single phase zone is arbitrary. In particular, it can be

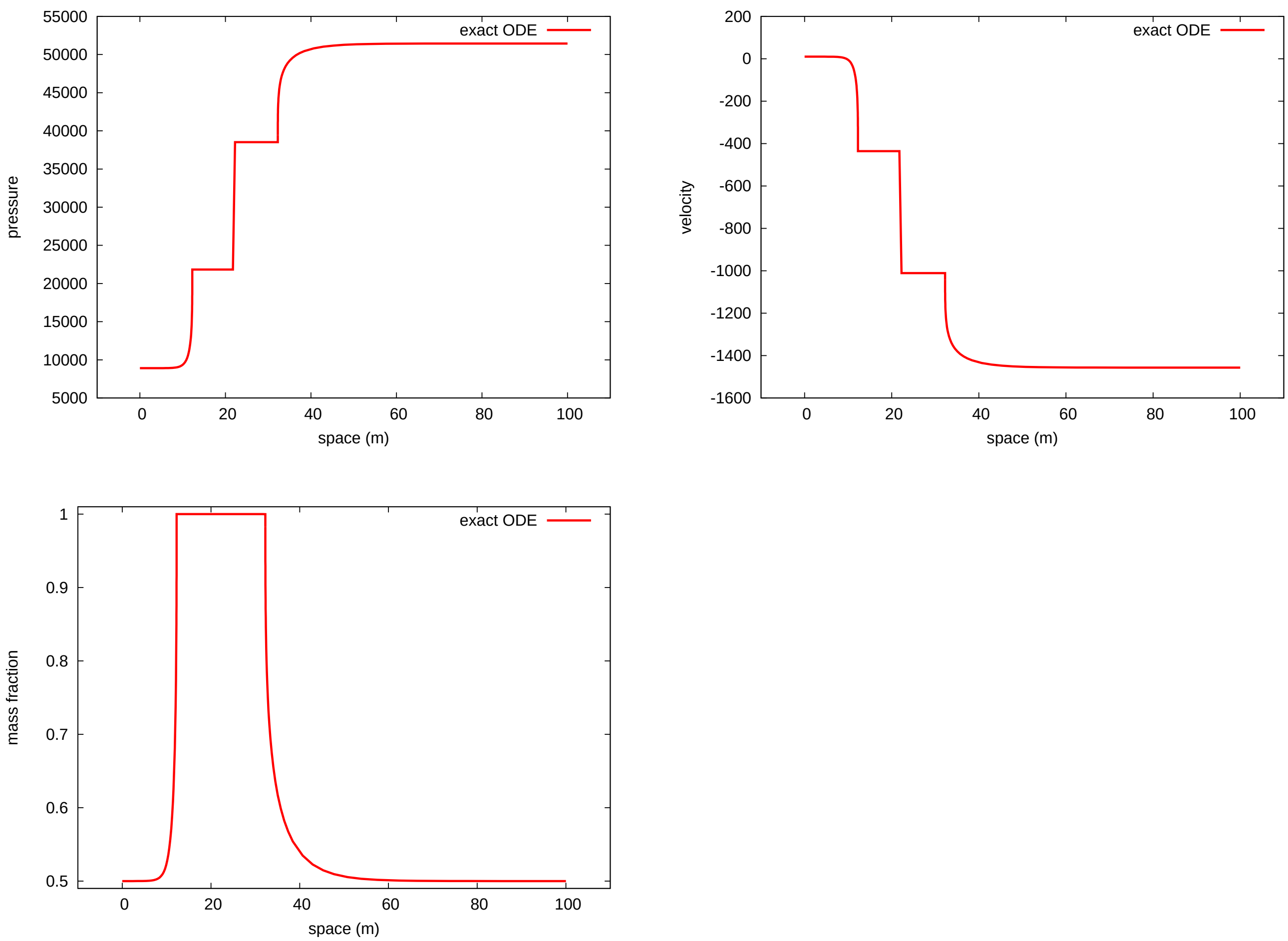

Figure 2: Profiles in the shock for the strong shock case obtained from numerical integration of (54) upper left : pressure, upper right : velocity, lower left : mass fraction

equal to zero and the mono-phase region reduced to a single point. On figure 2, we gave a non-zero width to this region in order to show that the shock profile has infinite derivatives when the states U_L^* and U_R^* are reached.

We have also repeated this computation with the finite volume scheme of [9] to integrate the system (6). The initial condition is given by an inviscid shock between the two states (59) and (63). The steady state obtained is displayed in figure 3 for the mass fraction and compared with the result obtained from the numerical integration of (54). In these computations the width of the mono-phase region is zero and the solution displays an infinite derivative at this point. An extremely large number of grid points is therefore needed to approximate the derivative of the solution at this point. This is what is shown in figure 4 where the number of grid points have been increased in order to capture this region of large derivatives. It is seen that as the number of grid nodes is increased, the maximum value of the mass fraction tends to 1 in perfect agreement with the analytical results of the previous section. Actually, for the 8000 node mesh, the maximum value of the mass fraction was 0.985661816.

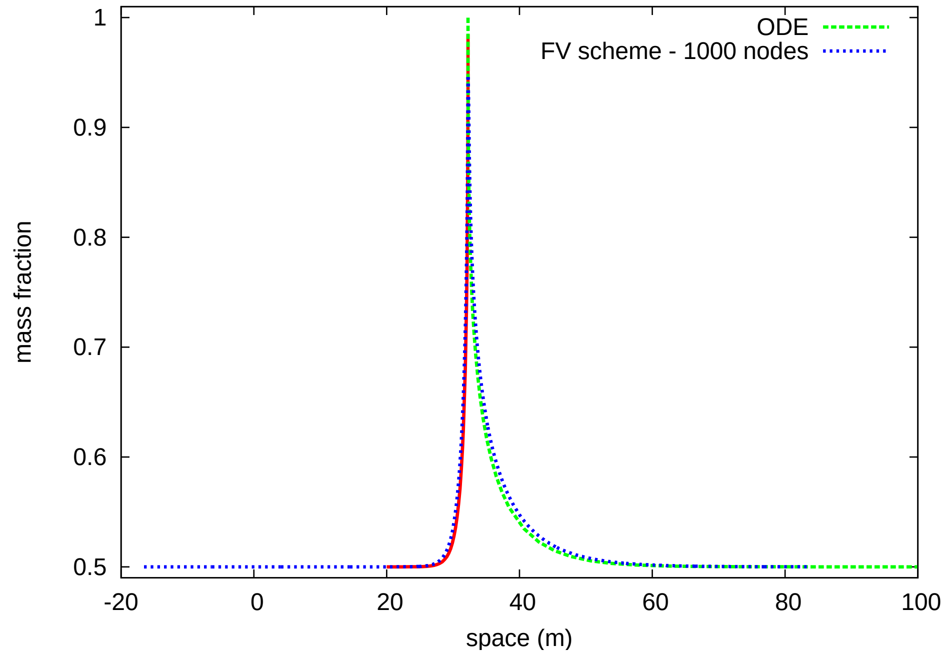

Figure 3: Mass fraction profile in the shock for the strong shock case, FV scheme compared with the numerical integration of (54)

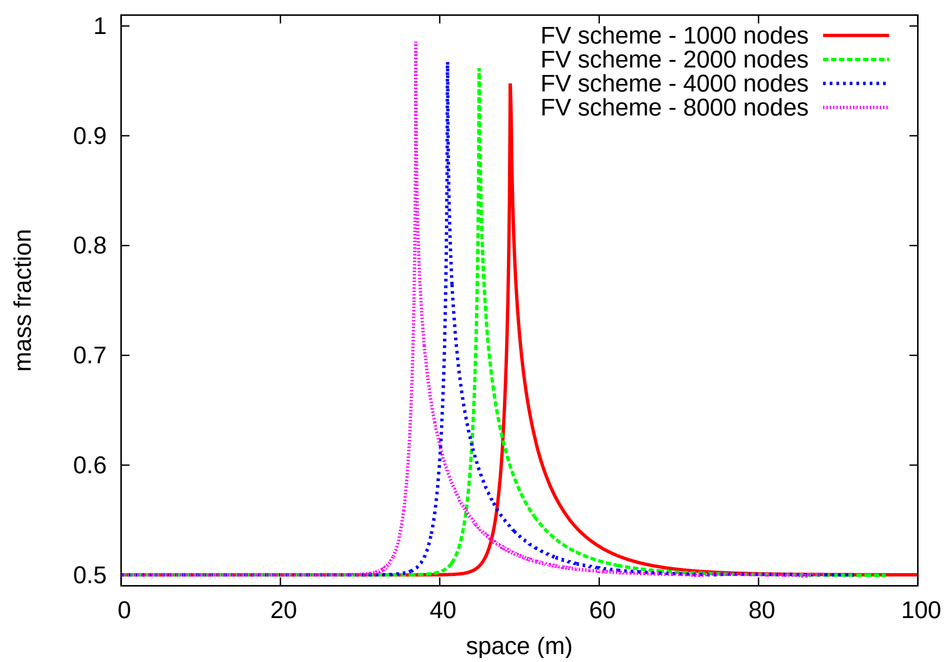

Figure 4: Mass fraction profile in the shock for the strong shock case, using an increasing number of grid points. The various curves are shifted from each other in order to have a better representation of the maximum value of the curve in the shock region.

6 Conclusion

The correct definition of shock solutions in hyperbolic systems is generally an open question. This is particularly true for models in non-conservative form such as in many two-phase flow models. From a mathematical point of view, the definition of shock solutions for these systems necessarily demand some kind of regularization. In principle, traveling waves analysis [6, 14, 7] provide a satisfactory way to describe the inner structure of a shock and consequently should allow a rigorous definition of shock waves.
However, in this case, the choice of the dissipative tensor appears to be crucial as it defines the valid shock solutions. In this work, we have investigated on a simple model of two phase flow, a particular structure of dissipative tensor that differs from the usual viscous tensors constructed from a Navier-Stokes analogy.
Using Chapman-Enskog expansion, we have shown that this type of tensor results from mechanical relaxation toward equilibrium between two phases and that it represents the dominant effect in two phase flows.
Traveling wave analysis for this model has been performed. This analysis has revealed that despite the zero mass fraction jump implied by the Rankine-Hugoniot relations, the mass fraction is not constant in the shock region. Moreover, this analysis has also shown the existence of a critical speed of the waves above which no $\mathscr{C}^1$ solutions exist. However above this critical speed, we have shown the possibility to construct traveling solutions involving single phase shocks.

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Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399