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BOUNDARY CONDITIONS FOR THE 2D LINEARIZED PEs OF THE OCEAN IN THE ABSENCE OF VISCOSITY

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Abstract

The linearized Primitive Equations with vanishing viscosity are considered. Some new boundary conditions (of transparent type) are introduced in the context of a modal expansion of the solution which consist of an infinite sequence of integral equations. Applying the linear semi-group theory, existence and uniqueness of solutions is established. The case with nonhomogeneous boundary values, encountered in numerical simulations in limited domains, is also discussed.

Introduction

The Primitive Equations of the ocean and the atmosphere are fundamental equations of geophysical fluid mechanics ([14],[24],[20]). In the presence of viscosity, it has been shown, in various contexts, that these equations are well-posed (see e.g. [9],[10], and the review article [23]).The viscosity appearing in [9] is the usual second order dissipation term. Other viscosity terms have also been considered as in the so-called δ-PEs proposed with different motivations in [22] and [20]. It has been shown in [15] and [22] that the mild vertical viscosity appearing in the δ-PEs is sufficient to guarantee well-posedness.
It is generally accepted that the viscosity terms do not affect numerical simulations (predictions) in a limited domain, over a period of a few days, and these viscosities are generally not used, see [25].

Now, for the PEs without viscosity, and to the best of our knowledge, no result of well-posedness has ever been proven, since the negative result of Oliger and Sundström [12] showing that these equations are ill-posed for any set of local boundary conditions (see also the analysis in [22]).
Whereas the analysis of the PEs with viscosity bears some similarity with that of the incompressible Navier Stokes equations (see [9, 10, 23]), it is noteworthy that the result of [12] shows that the

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PEs without viscosity are definitely different from the Euler equations of fluid dynamics, and it is expected that totally different boundary conditions of nonlocal type will be required.
In this article the full 2D-PEs, without viscosity, and linearized around a stratified state with constant velocity are considered. The proposed boundary conditions are of a totally new type ; they consist of nonlocal boundary conditions, defined mode by mode. The well-posedness of the corresponding linearized PEs is established using the linear semi-group theory. Although the use of the Hille-Yosida theorem in this context is classical, the verification of its hypotheses is not straightforward.
Results concerning the linearized 3D-PEs will appear elsewhere. The additional difficulty in dimension three is that the verification of the hypotheses of the Hille-Yosida theorem necessitates the solution of partial differential equations, whereas in space dimension two it involves the resolution of ordinary differential equations.
A few words are in order about the nonlinear case which is our ultimate goal. Concerning well-posedness, we are faced with boundary value problems for nonlinear hyperbolic systems of equations in a limited domain, a subject not yet extensively studied (see however the important results of [11, 4]). We believe and intend to prove that the appropriate boundary conditions for the nonlinear PEs correspond in general to those of the corresponding linearized equations. In any case the study of the well-posedness of the linear primitive equations is a necessary and important step for the problem of well-posedness of the nonlinear PEs, and this fully justifies the attention devoted here to the linearized PEs.

This article is organised as follows. In Section 1 we recall the PEs, describe the equations linearized around the stratified flow, and perform the normal modes expansion, which evidences as in [12, 22] the can not be well-posed for any problem when supplemented with a set of local boundary conditions.

In Section 2 we introduce the boundary conditions which are distinct for the first set of modes (called subcritical modes) and the remaining ones (called supercritical modes). The proposed boundary conditions are furthermore of nonreflective (transparent) type (see e.g. [3, 6]), making them appropriate for computations . This initial boundary value problem is then set as an abstract linear evolution equation in a suitable Hilbert space (Section 2.1). The result of existence and uniqueness of the solution is stated in Section 2.2, and Section 2.3 is devoted to the proof of the hypotheses of the Hille-Yosida theorem. To conclude, we study in Section 2.4 the case, actually encountered in numerical simulations, of nonhomogeneous boundary conditions.

1. Ill-posedness of the classical PEs

The Primitive Equations of the Ocean read :

$$\frac{\partial \tilde{\mathbf{v}}}{\partial t} + (\tilde{\mathbf{v}}.\nabla)\,\tilde{\mathbf{v}} + \tilde{w}\,\frac{\partial \tilde{\mathbf{v}}}{\partial z} + f\,k \times \tilde{\mathbf{v}} + \nabla \tilde{p} = F, \tag{1.1}$$

$$\frac{\partial \tilde{p}}{\partial z} = -\tilde{\rho}\,g, \tag{1.2}$$

$$\nabla\,\tilde{\mathbf{v}} + \frac{\partial \tilde{w}}{\partial z} = 0, \tag{1.3}$$

$$\frac{\partial \tilde{T}}{\partial t} + (\tilde{\mathbf{v}}.\nabla)\,\tilde{T} + \tilde{w}\,\frac{\partial \tilde{T}}{\partial z} = Q_T, \tag{1.4}$$

$$\tilde{\rho} = \rho_0\,(1 - \alpha\,(\tilde{T} - T_0)). \tag{1.5}$$

In these equations $\tilde{\mathbf{v}} = (\tilde{u}, \tilde{v})$ is the horizontal velocity, $\tilde{w}$ the vertical velocity, $\tilde{p}$ the pressure, $\tilde{\rho}$ the density, and $\tilde{T}$ the temperature; g is the gravitational acceleration, and f the Coriolis parameter. The horizontal gradient is denoted by ∇. Equation (1.5) is the equation of state of the fluid, ρ_0 and T_0 are constant reference values of $\tilde{\rho}$ and $\tilde{T}$, and $\alpha > 0$; this equation of state is linear.

Equation (1.2) is the so-called hydrostatic equation. The other equations correspond to the Boussinesq approximation (see e.g. [14] and [20] for more details).

1.1. **Reference flow and stratification.**

We now consider a reference stratified flow with constant velocity $\overline{\mathbf{v}}_0 = (\overline{U_0}, 0) = \overline{U_0}\,e_x$, and density, temperature and pressure of the form $\rho_0 + \overline{\rho}$, $T_0 + \overline{T}$, $p_0 + \overline{p}$ with $d\overline{\rho}/dz$ constant and thus

$$\begin{aligned}
\overline{T}(z) &= \frac{N^2}{\alpha\,g}\,z, && (1.6)\\
\overline{\rho}(z) &= -\rho_0\,\alpha\,\overline{T}(z) = -\frac{\rho_0\,N^2}{g}\,z, && (1.7)\\
\frac{d\overline{T}}{dz}(z) &= \frac{N^2}{\alpha\,g}, && (1.8)\\
\frac{d\overline{\rho}}{dz}(z) &= -\frac{\rho_0}{g}\,N^2, && (1.9)\\
\frac{d\overline{p}}{dz}(z) &= -(\rho_0 + \overline{\rho})\,g. && (1.10)
\end{aligned}$$

Here N is the buoyancy frequency, assumed to be constant.

We then decompose the unknown functions $\tilde{\mathbf{v}}, \tilde{\rho}, \tilde{T}, \tilde{p}$ in the following way:

$$\left\{\begin{array}{lllllll}
\tilde{\mathbf{v}} & = & \overline{U_0}\,e_x & + & \mathbf{v}(x,y,z,t), & & \\
\tilde{\rho} & = & \rho_0 & + & \overline{\rho}(z) & + & \rho(x,y,z,t), \\
\tilde{T} & = & T_0 & + & \overline{T}(z) & + & T(x,y,z,t), \\
\tilde{p} & = & p_0 & + & \overline{p}(z) & + & p(x,y,z,t).
\end{array}\right. \tag{1.11}$$

Equations (1.2), (1.4) and (1.5) become

$$\begin{aligned}
&\frac{\partial p}{\partial z} = -\rho\,g, && (1.12)\\
&\rho = -\rho_0\,\alpha\,T, && (1.13)\\
&\frac{\partial T}{\partial t} + (\tilde{\mathbf{v}}.\nabla)\,T + w\,\frac{\partial T}{\partial z} + \frac{N^2}{\alpha\,g}\,w = F_T. && (1.14)
\end{aligned}$$

Restricting now to a 2D problem, we assume that all variables in (1.11) are independent of y and we infer from (1.1)-(1.5) and (1.12),(1.13) the following equations for u, v, w, $\phi = p/\rho_0$ and

$\psi = \phi_z = \alpha\, g\, T$:

$$\frac{\partial u}{\partial t} + \overline{U_0}\,\frac{\partial u}{\partial x} + u\,\frac{\partial u}{\partial x} + w\,\frac{\partial u}{\partial z} - f\,v + \frac{\partial \phi}{\partial x} = F_u, \tag{1.15}$$

$$\frac{\partial v}{\partial t} + \overline{U_0}\,\frac{\partial v}{\partial x} + u\,\frac{\partial v}{\partial x} + w\,\frac{\partial v}{\partial z} + f\,u = F_v - f\,\overline{U_0}, \tag{1.16}$$

$$\frac{\partial \psi}{\partial t} + \overline{U_0}\,\frac{\partial \psi}{\partial x} + u\,\frac{\partial \psi}{\partial x} + w\,\frac{\partial \psi}{\partial z} + N^2\,w = F_\psi, \tag{1.17}$$

$$\frac{\partial \phi}{\partial z} = -\frac{\rho}{\rho_0}\,g = \psi, \tag{1.18}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{1.19}$$

From equations (1.18) and (1.19)we find:

$$\frac{\partial \phi}{\partial x}(x,z) = \phi_s'(x) - \int_z^0 \frac{\partial \psi}{\partial x}(x,z')\,dz', \tag{1.20}$$

$$w(x,z) = \int_z^0 \frac{\partial u}{\partial x}(x,z')\,dz', \tag{1.21}$$

where $\phi_s(x,t) = \phi(x, z = 0, t)$ is the surface pressure, divided by ρ_0, and ϕ_s' its derivative with respect to x.

The PEs (1.15)-(1.19), linearized around the stratified flow $\overline{\mathbf{v}}_0 = \overline{U_0}\,e_x, \overline{\rho}, \overline{T}, \overline{p}$, read:

$$\frac{\partial u}{\partial t} + \overline{U_0}\,\frac{\partial u}{\partial x} - f\,v + \frac{\partial \phi}{\partial x} = F_u, \tag{1.22}$$

$$\frac{\partial v}{\partial t} + \overline{U_0}\,\frac{\partial v}{\partial x} + f\,u = F_v - f\,\overline{U_0}, \tag{1.23}$$

$$\frac{\partial \psi}{\partial t} + \overline{U_0}\,\frac{\partial \psi}{\partial x} + N^2\,w = F_\psi, \tag{1.24}$$

$$\frac{\partial \phi}{\partial z} = -\frac{\rho}{\rho_0}\,g = \psi, \tag{1.25}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{1.26}$$

We will consider the flow in the 2D domain $\mathcal{M} = (0, L_1) \times (-L_3, 0)$. Naturally, we supplement equations (1.22)-(1.26) with the following top and bottom boundary conditions (just imposed by kinematics):

$$w(x, z = -L_3, t) = w(x, z = 0, t) = 0, \;\; \forall x \in (0, L_1), t > 0. \tag{1.27}$$

The aim of this article is to consider some lateral boundary conditions at $x = 0$ and $x = L_1$ that are both physically reasonable and computationnally satisfying[1] , and that lead to the well-posedness of the problem (1.22)-(1.26).

1.2. **Normal modes.**

[1]Assuming that we are willing to pay the price of a nonlocal (mode by mode) boundary condition, for increased accuracy; see [17] for an alternate solution and a discussion on this issue.

We consider a normal mode decomposition of the solution of the following form (see [22] for the details and the justifications):

$$(u, v, \phi) = \sum_{n \geq 0} \mathcal{U}_n(z) \, (\hat{u}_n, \hat{v}_n, \hat{\phi}_n) \, (x, t), \tag{1.28}$$

$$(w, \psi) = \sum_{n \geq 1} \mathcal{W}_n(z) \, (\hat{w}_n, \hat{\psi}_n) \, (x, t). \tag{1.29}$$

As explained in [22], for every $n \geq 1$ the functions $\mathcal{U}_n$ and $\mathcal{W}_n$ are solutions of the following eigenvalue problem:

$$\begin{aligned}
\Big(\frac{\mathcal{U}_n}{N^2}\Big)_{zz} + \lambda_n^2 \, \mathcal{U}_n &= 0, && (1.30)\\
(\mathcal{W}_n)_{zz} + \lambda_n^2 \, N^2 \, \mathcal{W}_n &= 0, && (1.31)\\
\mathcal{U}_n' &= N^2 \, \mathcal{W}_n \, c_{1,n}, && (1.32)\\
\mathcal{U}_n &= c_{2,n} \, \mathcal{W}_n', && (1.33)\\
\lambda_n^2 = \frac{1}{g \, H_n} &= -\frac{c_{1,n}}{c_{2,n}}. && (1.34)
\end{aligned}$$

where $c_{1,n}, c_{2,n}$ are appropriate constants and the λ_n the eigenvalues of these two-point boundary value problems.

By (1.27) we should have:

$$\left\{ \begin{array}{rclcl} \mathcal{W}_n(0) & = & \mathcal{W}_n(-L_3) & = & 0, \\ \mathcal{U}_n'(0) & = & \mathcal{U}_n'(-L_3) & = & 0. \end{array} \right. \tag{1.35}$$

In a standard manner, we infer from (1.31) and the boundary conditions (1.35), that $\mathcal{W}_n(z) = C_n \, sin(\lambda_n \, N \, z)$ with

$$\lambda_n = \frac{n \, \pi}{N \, L_3} \text{ and thus } H_n = \frac{N^2 \, L_3^2}{g \, n^2 \, \pi^2}. \tag{1.36}$$

One of the constants $c_{1,n}$, $c_{2,n}$ has not yet been imposed; we choose it by orthonormalization of $\mathcal{W}_n$, that is we set $\|\mathcal{W}_n\|_{L^2(-L_3,0)} = 1$, and we find

$$C_n = \sqrt{\frac{2}{L_3}}, \tag{1.37}$$

so that C_n is in fact independent of n. The discussion above refers to the modes $n \geq 1$. For $n = 0$, $\lambda_0 = 0$, so that $\mathcal{W}_0$ vanishes identically, whereas $\mathcal{U}_0$ is constant. Finally we find:

$$\mathcal{U}_0(z) = \frac{1}{\sqrt{L_3}}, \tag{1.38}$$

and for $n \geq 1$:

$$\left\{ \begin{array}{rclcl} \mathcal{U}_n(z) & = & \sqrt{\frac{2}{L_3}} \, cos(\lambda_n \, N \, z) & = & \sqrt{\frac{2}{L_3}} \, cos(\frac{n \, \pi \, z}{L_3}), \\ \mathcal{W}_n(z) & = & \sqrt{\frac{2}{L_3}} \, sin(\lambda_n \, N \, z) & = & \sqrt{\frac{2}{L_3}} \, sin(\frac{n \, \pi \, z}{L_3}). \end{array} \right. \tag{1.39}$$

We notice that $\forall n \geq 1, m \geq 0$, we have, as usual:

$$\left\{\begin{array}{rcl}\displaystyle\int_{-L_3}^{0} \mathcal{U}_n(z)\mathcal{U}_m(z)\,dz &=& \delta_{n,m},\\ \displaystyle\int_{-L_3}^{0} \mathcal{U}_n(z)\,\mathcal{W}_m(z)\,dz &=& 0,\\ \mathcal{U}_n'(z) &=& -N\,\lambda_n\,\mathcal{W}_n(z),\\ \mathcal{W}_n'(z) &=& N\,\lambda_n\,\mathcal{U}_n(z).\end{array}\right. \tag{1.40}$$

Remark 1.1. If we look for a solution "more general" than (1.28), (1.29), that is $u = \sum_{n\geq 1} \mathcal{U}_n \hat{u}_n$, $v = \sum_{n\geq 1} \mathcal{V}_n \hat{v}_n$, $\phi = \sum_{n\geq 1} \Phi_n \hat{\phi}_n$, then (1.22) and (1.23) imply that $\mathcal{U}_n, \mathcal{V}_n, \Phi_n$ are proportional, hence they can be taken equal.

1.3. The modal equations for $(\hat{u}_n, \hat{v}_n, \hat{w}_n, \hat{\psi}_n, \hat{\phi}_n)$.

From now on and when no confusion can occur, we drop the hats and write $(u_n, v_n, w_n, \psi_n, \phi_n)$ instead of $(\hat{u}_n, \hat{v}_n, \hat{w}_n, \hat{\psi}_n, \hat{\phi}_n)$. The constant mode in z (n=0) is different (simpler), and we postpone its study to Section 2.5 below. For every $n \geq 1$, since $\psi(x,z,t) = \phi_z(x,z,t)$ we have:

$$\psi_n(x,t) = -N\,\lambda_n\,\phi_n(x,t). \tag{1.41}$$

We now introduce the expansion (1.28)-(1.29) into equations (1.22)-(1.26). We multiply (1.22), (1.23) and (1.26) by $\mathcal{U}_n$, (1.24) and (1.25) by $\mathcal{W}_n$ and integrate on $(-L_3, 0)$, and we find:

$$\left\{\begin{array}{lllllllll}\dfrac{\partial u_n}{\partial t} &+& \overline{U_0}\,\dfrac{\partial u_n}{\partial x} &-& f\,v_n &+& \dfrac{\partial \phi_n}{\partial x} &=& F_{u,n},\\ \dfrac{\partial v_n}{\partial t} &+& \overline{U_0}\,\dfrac{\partial v_n}{\partial x} &+& f\,u_n & & &=& F_{v,n},\\ \dfrac{\partial \psi_n}{\partial t} &+& \overline{U_0}\,\dfrac{\partial \psi_n}{\partial x} &+& N^2\,w_n & & &=& F_{\psi,n},\\ \phi_n &=& -\dfrac{1}{N\,\lambda_n}\psi_n, & & & & & & \\ w_n &=& -\dfrac{1}{N\,\lambda_n}\,\dfrac{\partial u_n}{\partial x}. & & & & & & \end{array}\right. \tag{1.42}$$

Taking into account the last two equations of (1.42) the first three become:

$$\left\{\begin{array}{lllllllll}\dfrac{\partial u_n}{\partial t} &+& \overline{U_0}\,\dfrac{\partial u_n}{\partial x} &-& f\,v_n &-& \dfrac{1}{N\,\lambda_n}\,\dfrac{\partial \psi_n}{\partial x} &=& F_{u,n},\\ \dfrac{\partial v_n}{\partial t} &+& \overline{U_0}\,\dfrac{\partial v_n}{\partial x} &+& f\,u_n & & &=& F_{v,n},\\ \dfrac{\partial \psi_n}{\partial t} &+& \overline{U_0}\,\dfrac{\partial \psi_n}{\partial x} &-& \dfrac{N}{\lambda_n}\,\dfrac{\partial u_n}{\partial x} & & &=& F_{\psi,n}.\end{array}\right. \tag{1.43}$$

Let us now introduce the lateral boundary conditions which, for each $n \geq 1$, will supplement this system.

1.4. Boundary conditions at $x = 0$ and $x = L_1$.

Looking at (1.43), we find that the characteristic values of this first order system are $\overline{U_0} - 1/\lambda_n$, $\overline{U_0}$ and $\overline{U_0} + 1/\lambda_n$; they are the eigenvalues of the matrix:

$$A_n = \begin{pmatrix} \overline{U_0} & 0 & -\dfrac{1}{N\,\lambda_n} \\ 0 & \overline{U_0} & 0 \\ -\dfrac{N}{\lambda_n} & 0 & \overline{U_0} \end{pmatrix}.$$

Since $\overline{U_0} > 0$, $\lambda_n > 0$, we always have at least two positive eigenvalues. But $\overline{U_0} - 1/\lambda_n$ can either be positive or negative. We say that the corresponding mode is supercritical in the first case and subcritical in the second case, it appears then that the subcritical modes require two boundary values on the left of the domain ($x = 0$) and one boundary value on the right ($x = L_1$), whereas the supercritical modes require three boundary values at $x = 0$. Based on this remark, Oliger and Sundström concluded in [12] that the boundary value problem associated with (1.42)-(1.43) is ill-posed for any set of local boundary conditions (see also [22]). Instead different boundary conditions for the two types of modes must be provided and one of our aims in this article is to show the well-posedness of the system consisting of (1.22)-(1.26) supplemented with an appropriate set of boundary conditions.

Since $\lambda_n = n\,\pi/N\,L_3 \longrightarrow \infty$ as $n \longrightarrow \infty$, there is only a finite number of subcritical modes, let us say n_c:

Definition 1.1. *We denote by n_c the number of subcritical modes, defined by:*

$$\frac{n_c\,\pi}{N\,L_3} = \lambda_{n_c} \le \frac{1}{\overline{U_0}} < \lambda_{n_c+1} = \frac{(n_c+1)\,\pi}{N\,L_3}.$$

In physical applications most of the modes are supercritical, but the few subcritical modes carrying most of the energy are particularly important.

The boundary conditions for the subcritical modes were discussed in [18, 17], they are recalled below. The boundary conditions for the supercritical modes are less problematic, we now present them. For $n > n_c$, a set of natural boundary conditions for system (1.43) is:

$$\left\{\begin{aligned} u_n(0,t) &= 0,\\ v_n(0,t) &= 0,\\ \psi_n(0,t) &= 0. \end{aligned}\right. \tag{1.44}$$

In (1.44) and (1.46) we chose, for simplicity, homogeneous boundary conditions, but we discuss in Section 2.4 below the case of nonzero boundary values.

For $1 \le n \le n_c$, $\overline{U_0} - 1/\lambda_n < 0$, and the corresponding eigenvector is $\eta_n = u_n + \psi_n/N$. The eigenvectors related to $\overline{U_0}$ and $\overline{U_0} + 1/\lambda_n$ are respectively v_n and $\xi_n = u_n - \psi_n/N$. Thanks to (1.41), we have, for $n \ge 1$, $(\xi_n, \eta_n) = (u_n + \lambda_n\,\phi_n, u_n - \lambda_n\,\phi_n)$.

Using the variables ξ_n, v_n, η_n we rewrite (1.43) as follows:

$$\left\{\begin{array}{lclcl} \dfrac{\partial \xi_n}{\partial t} + (\overline{U_0} + \dfrac{1}{\lambda_n})\dfrac{\partial \xi_n}{\partial x} & - & f\,v_n & = & F_{\xi,n},\\ \dfrac{\partial v_n}{\partial t} + \overline{U_0}\,\dfrac{\partial v_n}{\partial x} & + & \frac{1}{2}\,f\,(\xi_n + \eta_n) & = & F_{v,n},\\ \dfrac{\partial \eta_n}{\partial t} + (\overline{U_0} - \dfrac{1}{\lambda_n})\dfrac{\partial \eta_n}{\partial x} & & & = & F_{\eta,n}. \end{array}\right. \tag{1.45}$$

Hence, for these subcritical modes ($n \le n_c$), a set of natural and nonreflective boundary conditions is the following

$$\left\{\begin{aligned} \xi_n(0,t) &= 0,\\ v_n(0,t) &= 0,\\ \eta_n(L_1,t) &= 0. \end{aligned}\right. \tag{1.46}$$

In Section 2 of this article, we will prove the well-posedness of the linear Primitive Equations (1.22)-(1.26) (equivalent mode by mode to (1.42)) with the modal boundary conditions (1.44) and (1.46).

2. Well-posedness of the linear primitive equations with modal boundary conditions

We aim to implement (1.44) and (1.46), and we first set the functional framework appropriate to these boundary conditions.

2.1. **Preliminary settings.**

We aim to write the initial value problem under consideration as a functional evolution in an appropriate Hilbert space H:

$$\begin{cases} \dfrac{dU}{dt} + AU = F, \\ U(0) = U_0. \end{cases} \tag{2.1}$$

Here A is an unbounded operator with domain $D(A) \subset H$, the forcing $F \in H$ and the initial data $U_0 \in D(A)$ are given.

We define H by setting

$$H = H_u \times H_v \times H_\psi, \tag{2.2}$$

$$H_u = \Big\{ u \in L^2(\mathcal{M}) \ / \ \int_{-L_3}^{0} u(x,z)\,dz = 0 \text{ a.e. in } (0, L_1) \Big\},$$

$$H_v = H_\psi = L^2(\mathcal{M}),$$

where $\mathcal{M}$ is the 2D domain $(0, L_1) \times (-L_3, 0)$. We endow H with the scalar product[2]

$$(U, \tilde{U})_H = \int_{\mathcal{M}} (u\,\tilde{u} + v\,\tilde{v} + \frac{1}{N^2}\,\psi\,\tilde{\psi})\,d\mathcal{M}, \qquad \forall (U, \tilde{U}) \in H^2. \tag{2.3}$$

The space H_u is clearly closed in $L^2(\mathcal{M})$, and $H = H_u \times H_v \times H_\psi$ is a closed subspace of $(L^2(\mathcal{M}))^3$, which we endow with the scalar product and norm derived from (2.3) and equivalent to those of $(L^2(\mathcal{M}))^3$. We denote by P the orthogonal projector from $L^2(\mathcal{M})$ onto H_u. For every $g \in L^2(\mathcal{M})$,

$$P(g)(x,z) = g(x,z) - \frac{1}{L_3} \int_{-L_3}^{0} g(x,z')\,dz', \tag{2.4}$$

$$(I - P)(g)(x,z) = \frac{1}{L_3} \int_{-L_3}^{0} g(x,z')\,dz'. \tag{2.5}$$

It is easily checked that $Pg \in H_u$ and $(I-P)g \perp Pg$. Finally $H_u^{\perp}$ is identical to $L_x^2(0, L_1)$. Indeed for $g \in H_u^{\perp}$, $(I-P)\,g = g$, so that g does not depend on z and belongs to $L_x^2(0, L_1)$. Conversely if $h \in L_x^2(0, L_1)$, then for every $u \in H_u$, $(u,h)_{L^2(\mathcal{M})} = \int_0^{L_1} h(x) \int_{-L_3}^{0} u(x,z) dz\, dx = 0$ and $h \in H_u^{\perp}$.

[2] It is not surprising to have $1/N^2$ as a multiplicative coefficient in front of the last term of $(U, \tilde{U})_H$, since $\int_{\mathcal{M}} u^2 + v^2 d\mathcal{M}$ represents the kinetic energy whereas $1/N^2 \int_{\mathcal{M}} \psi^2\, d\mathcal{M}$ is the available potential energy.

We are now in position to define the operator A; its domain $D(A)$ is defined by

$$D(A) = \Big\{ U = (u, v, \psi) \in H \ / \ \left| \begin{array}{l} (u_x, v_x, \psi_x) \in L^2(\mathcal{M}) \\ (u, v, \psi) \text{ verify (2.7) and (2.8)} \end{array} \right. \Big\}. \tag{2.6}$$

Here and in the sequel u_x, u_z denote the partial derivatives $\partial u/\partial x$, $\partial u/\partial z$ of a function u.

The boundary conditions (2.7) and (2.8), identical to (1.44) and (1.46), are written in the following form[3]:

$$\left\{ \begin{array}{l} \displaystyle\int_{-L_3}^{0} u(0,z)\,\mathcal{U}_n(z)\,dz - \frac{1}{N}\int_{-L_3}^{0} \psi(0,z)\,\mathcal{W}_n(z)\,dz = 0, \\ \displaystyle\int_{-L_3}^{0} v(0,z)\,\mathcal{U}_n(z)\,dz = 0, \\ \displaystyle\int_{-L_3}^{0} u(L_1,z)\,\mathcal{U}_n(z)\,dz + \frac{1}{N}\int_{-L_3}^{0} \psi(L_1,z)\,\mathcal{W}_n(z)\,dz = 0, \end{array} \right. \qquad \forall\, 1 \le n \le n_c, \tag{2.7}$$

and

$$\left\{ \begin{array}{rcl} \displaystyle\int_{-L_3}^{0} u(0,z)\,\mathcal{U}_n(z)\,dz &=& 0, \\ \displaystyle\int_{-L_3}^{0} v(0,z)\,\mathcal{U}_n(z)\,dz &=& 0, \\ \displaystyle\int_{-L_3}^{0} \psi(0,z)\,\mathcal{W}_n(z)\,dz &=& 0, \end{array} \right. \qquad \forall n > n_c. \tag{2.8}$$

For every $U = (u, v, \psi) \in D(A)$, AU is given by:

$$AU = \begin{pmatrix} P\big[\,\overline{U_0}\,u_x - f\,v - \displaystyle\int_z^0 \psi_x(x,z')\,dz'\big] \\ \overline{U_0}\,v_x + f\,u \\ \overline{U_0}\,\psi_x + N^2\,w \end{pmatrix} \tag{2.9}$$

where $w = w(u)$ is given by (1.21).

We now intend to prove, in the context of the linear semi-group theory ([26], [2], [1], [5], [7], [13]), the well-posedness for equation (2.1), corresponding to the linearized PEs supplemented with the boundary conditions (2.7) and (2.8).

2.2. **The main result.**

To prove the well-posedness of the initial value problem (2.1), we will use the following version of the Hille-Yosida theorem borrowed from [1] (see also [2], [5],[7],[13],[26]):

Theorem 2.1. (Hille-Yosida Theorem) *Let H be a Hilbert space and let $A : D(A) \longrightarrow H$ be a linear unbounded operator, with domain $D(A) \subset H$. Assume the following :*
(i) $D(A)$ is dense in H and A is closed,
(ii) A is ≥ 0, i.e. $(AU, U)_H \geq 0, \quad \forall\, U \in D(A)$,
(iii) $\exists \mu_0 > 0$, such that $A + \mu_0 I$ is onto.

[3]We note that the boundary conditions on v do not depend on the modes (see also the boundary condition on the constant mode v_0 in Section 2.5 below), hence they could be written in the form $v(0,z) = 0$, $\forall z \in (-L_3, 0)$. However we keep the modal notation by analogy with the other functions u and ψ, and because this is the way this boundary condition is actually implemented in numerical simulations [19].

Then $-A$ is infinitesimal generator of a semigroup of contractions $\{S(t)\}_{t\geq 0}$ in H, and for every $U_0 \in H$ and $F \in L^1(0,T;H)$, there exists a unique solution $U \in \mathcal{C}([0,T];H)$ of (2.1),

$$U(t) = S(t)\,U_0 + \int_0^t S(t-s)\,F(s)\,ds. \tag{2.10}$$

If furthermore $\overline{U_0} \in D(A)$ and $F' = dF/dt \in L^1(0,T;H)$ then U satisfies (2.1) and

$$U \in \mathcal{C}([0,T];H) \cap L^\infty(0,T;D(A)), \;\; \frac{dU}{dt} \in L^\infty(0,T;H). \tag{2.11}$$

The hypotheses of Theorem 2.1 being proved in Section 2.3, Theorem 2.1 readily implies our main result for the homogeneous boundary conditions:

Theorem 2.2. *Let H be the Hilbert space defined in (2.2) and A be the linear operator defined in (2.9) corresponding to the linearized Primitive Equations with vanishing viscosity and homogeneous modal boundary conditions.*

Then the initial value problem (2.1), corresponding to equations (1.22)-(1.26) supplemented with the boundary conditions (2.7) and (2.8) is well-posed, that is for every initial data $U_0 \in D(A)$ and forcing $F \in L^1(0,T;H)$, there exists a unique solution $U \in \mathcal{C}([0,T];H)$ of (2.1).

2.3. **Proof of Theorem 2.2.**

We now want to apply Theorem 2.1 to equation (2.1). To this aim we verify the hypotheses (i), (ii) and (iii) of the Hille-Yosida theorem (Theorem 2.1); we start with (ii) and (iii), and postpone the proof of (i) to Lemma 2.3 below. We start with the proof of (ii):

Lemma 2.1. *For every $U \in D(A), (A\,U,U)_H \geq 0$.*

Proof. For any $U \in H$, let us compute the scalar product $(A\,U,U)_H$:

$$\begin{aligned}(A\,U,U)_H \;\; &= \;\; \int_{\mathcal{M}} P(\overline{U_0}\,u_x - f\,v - \int_z^0 \psi_x(x,z')\,dz')\,u\,d\mathcal{M} \\ &\quad + \int_{\mathcal{M}} (\overline{U_0}\,v_x + f\,u)\,v\,d\mathcal{M} + \int_{\mathcal{M}} (\overline{U_0}\,\psi_x + N^2\,w)\,\frac{\psi}{N^2}\,d\mathcal{M}.\end{aligned}$$

Since $u \in H_u$, we have, using (1.21):

$$\begin{aligned}(A\,U,U)_H \;\; &= \;\; \int_{\mathcal{M}} (\overline{U_0}\,u_x - f\,v - \int_z^0 \psi_x(x,z')\,dz')\,u\,d\mathcal{M} \\ &\quad + \int_{\mathcal{M}} (\overline{U_0}\,v_x + f\,u)\,v\,d\mathcal{M} + \int_{\mathcal{M}} (\overline{U_0}\,\psi_x + N^2\,w)\,\frac{\psi}{N^2}\,d\mathcal{M} \\ &= \;\; \int_{-L_3}^0 \frac{\overline{U_0}}{2}\Big(u^2(L_1) - u^2(0) + v^2(L_1) - v^2(0) + \frac{1}{N^2}\,\psi^2(L_1) - \frac{1}{N^2}\,\psi^2(0)\Big)\,dz \\ &\quad - \int_{\mathcal{M}} \Big\{u(x,z)\int_z^0 \psi_x(x,z')\,dz' - \psi(x,z)\int_z^0 u_x(x,z')\,dz'\Big\}\,dx\,dz.\end{aligned}$$

Here $u(L_1)$, $u(0)$ stands for $u(L_1,z)$, $u(0,z)$, etc. Using the expansion (1.28), (1.29) with (1.39), it is easy to check that:

$$
\left\{
\begin{aligned}
-\int_z^0 \psi_x(x,z')\,dz' &= \sum_{n\geq 1} \frac{\psi_{n\,x}(x)}{N\,\lambda_n}\,(1-\mathcal{U}_n(z)) \\
&= \theta(x) - \sum_{n\geq 1} \frac{\psi_{n\,x}(x)}{N\,\lambda_n}\,\mathcal{U}_n(z), \\
\int_z^0 u_x(x,z')\,dz' &= -\sum_{n\geq 1} \frac{u_{n\,x}(x)}{N\,\lambda_n}\,\mathcal{W}_n(z).
\end{aligned}
\right.
\tag{2.12}
$$

where $\theta=\theta(x)$ is an L^2-function depending only on x.

Using again the expansion (1.28), (1.29), and remembering that $u\in H_u$, the integral $\int_{\mathcal{M}} u\,\theta\,d\mathcal{M}$ vanishes and we find:

$$
\begin{aligned}
(A\,U,U)_H &= \sum_{n\geq 1} \frac{\overline{U_0}}{2}\Big(u_n^2(L_1)-u_n^2(0)+v_n^2(L_1)-v_n^2(0)+\frac{1}{N^2}\,\psi_n^2(L_1)-\frac{1}{N^2}\,\psi_n^2(0)\Big) \\
&\quad +\frac{\overline{U_0}}{2}\Big(v_0^2(L_1)-v_0^2(0)\Big)-\sum_{n\geq 1}\frac{1}{N\,\lambda_n}\int_0^{L_1}(\psi_{n\,x}\,u_n+\psi_n\,u_{n\,x})\,dx.
\end{aligned}
$$

Using the boundary conditions (2.7) for the subcritical modes and (2.8) for the supercritical ones, we find:

$$
\begin{aligned}
(A\,U,U)_H &= \sum_{1\leq n\leq n_c} \frac{\overline{U_0}}{2}\Big(u_n^2(L_1)-u_n^2(0)+v_n^2(L_1)+u_n^2(L_1)-u_n^2(0)\Big) \\
&\quad +\frac{\overline{U_0}}{2}\,v_0^2(L_1)+\sum_{1\leq n\leq n_c}\frac{1}{\lambda_n}\Big(u_n^2(L_1)+u_n^2(0)\Big) \\
&\quad +\sum_{n>n_c}\frac{\overline{U_0}}{2}\Big(u_n^2(L_1)+v_n^2(L_1)+\frac{1}{N^2}\,\psi_n^2(L_1)\Big) \\
&\quad -\sum_{n>n_c}\frac{1}{N\,\lambda_n}\,u_n(L_1)\,\psi_n(L_1).
\end{aligned}
$$

For every subcritical mode (when $n\leq n_c$):

$$
\begin{aligned}
&\overline{U_0}\Big(u_n^2(L_1)-u_n^2(0)+\frac{1}{2}\,v_n^2(L_1)\Big)+\frac{1}{\lambda_n}\Big(u_n^2(L_1)+u_n^2(0)\Big) \\
= \; &(\overline{U_0}+\frac{1}{\lambda_n})\,u_n^2(L_1)+\frac{\overline{U_0}}{2}\,v_n^2(L_1)+(\frac{1}{\lambda_n}-\overline{U_0})\,u_n^2(0)\geq 0;
\end{aligned}
$$

the latter quantity is nonnegative, thanks to the definition of n_c. For every supercritical mode (when $n>n_c$):

$$
\begin{aligned}
&\frac{\overline{U_0}}{2}\Big(u_n^2(L_1)+v_n^2(L_1)+\frac{1}{N^2}\,\psi_n^2(L_1)\Big)-\frac{1}{N\,\lambda_n}\,u_n(L_1)\,\psi_n(L_1) \\
= \; &\frac{\overline{U_0}}{2}\,v_n^2(L_1)+\frac{\overline{U_0}}{2}\Big(u_n(L_1)-\frac{1}{\overline{U_0}\,N\,\lambda_n}\,\psi_n(L_1)\Big)^2+\frac{\overline{U_0}}{2\,N^2}\Big(1-\frac{1}{\overline{U_0}^2\,\lambda_n^2}\Big)\,\psi_n^2(L_1)\geq 0.
\end{aligned}
$$

This quantity is also nonnegative, which achieves the proof of Lemma 2.1. $\square$

In order to simplify the following study, we now assume that $\overline{U_0}$ is not a critical value, that is

$$\overline{U_0} \notin \Big\{ \frac{1}{\lambda_n},\ n \geq 1 \Big\}, \text{ or equivalently } \frac{N\,L_3\,\overline{U_0}}{\pi} \notin \mathbb{N}. \tag{2.13}$$

The case where (2.13) is not satisfied ($\overline{U_0} = \lambda_n^{-1}$) is actually simpler and will be discussed in Remark 2.1 below. Assuming (2.13), we choose μ_0 such that:

$$\mu_0 \notin \Big\{ f^2\,(1 - \overline{U_0}^2\,\lambda_n^2),\ n \geq 1 \Big\}, \tag{2.14}$$

$$\mu_0 \notin \Big\{ f^2\,\overline{U_0}^2\,\lambda_n^2,\ n \geq 1 \Big\}. \tag{2.15}$$

With this choice of μ_0, we can prove the following lemma:

Lemma 2.2. *The operator $A + \mu_0 I$ is onto from $D(A)$ onto H, where μ_o satisfies* (2.14) *and* (2.15).

Proof. For μ_0 as indicated, we are given $F = (F_u, F_v, F_\psi)$ in H, and we look for $U = (u, v, \psi)$ in $D(A)$ such that $(A + \mu_0\,I)\,U = F$. Writing this equation componentwise, we find:

$$\left\{ \begin{array}{rcl} \overline{U_0}\,u_x(x,z) - f\,v(x,z) + \mu_0\,u(x,z) & & \\ \displaystyle -\int_z^0 \psi_x(x,z')\,dz' + \phi_s'(x) & = & F_u(x,z), \\ \overline{U_0}\,v_x(x,z) + f\,u(x,z) + \mu_0\,v(x,z) & = & F_v(x,z), \\ \overline{U_0}\,\psi_x(x,z) + N^2\,w(x,z) + \mu_0\,\psi(x,z) & = & F_\psi(x,z). \end{array} \right. \tag{2.16}$$

To obtain the modal equations corresponding to (2.16), we multiply the three equations by $\mathcal{U}_n$, $\mathcal{U}_n$ and $\mathcal{W}_n$ respectively, and integrate on $(-L_3, 0)$.

Of course, since $F = (F_u, F_v, F_\psi) \in H$, we also have the following modal decompositions:

$$\left\{ \begin{array}{rcl} F_u(x,z) & = & \displaystyle\sum_{n\geq 1} \mathcal{U}_n(z)\,\hat{F}_{u,n}(x), \\ F_v(x,z) & = & \displaystyle\sum_{n\geq 0} \mathcal{U}_n(z)\,\hat{F}_{v,n}(x), \\ F_\psi(x,z) & = & \displaystyle\sum_{n\geq 1} \mathcal{W}_n(z)\,\hat{F}_{\psi,n}(x). \end{array} \right. \tag{2.17}$$

Note that for F as for U, since $F_u \in H_u \subset L^2(\mathcal{M})$, $\hat{F}_{u,0} = 0$ and the decomposition of F_u starts from $n = 1$.

For the mode $n = 0$ (constant in the variable z), we only consider the first two equations, since multiplying the third one by $\mathcal{W}_0 = 0$ would be useless. Integrating the equation for v and reporting in the equation for u (in which $\hat{u}_0 = 0$, see above), we find v_0 (formerly denoted $\hat{v}_0$) and the surface pressure ϕ_s, up to an additive constant $\phi_s(0)$:

$$\left\{ \begin{array}{rcl} v_0(x) & = & \displaystyle\frac{1}{\overline{U_0}} \int_0^x F_{v\,0}(x')\,e^{(x'-x)\,\mu_0/\overline{U_0}}\,dx', \\ \phi_s(x) & = & \displaystyle\phi_s(0) + \int_0^x \Big(f\,v_0(x') - \frac{L_3^2}{\pi} \sum_{n\geq 1} \psi_{n\,x}(x') \Big)\,dx'. \end{array} \right. \tag{2.18}$$

We recall that the nth mode is now denoted by $(u_n, v_n, w_n, \psi_n, \phi_n)$ instead of $(\hat{u}_n,\ \hat{v}_n,\ \hat{w}_n,\ \hat{\psi}_n,\ \hat{\phi}_n)$. Naturally, the above expression of ϕ_s depends on the other modes $(n \geq 1)$. We now write the corresponding equations, derived from (2.16) mode by mode:

$$\left\{\begin{array}{rcl} \overline{U_0}\, u_{n\,x} - f\, v_n + \mu_0\, u_n - \dfrac{1}{N\,\lambda_n}\, \psi_{n\,x} & = & F_{u,n}, \\ \overline{U_0}\, v_{n\,x} + f\, u_n + \mu_0\, v_n & = & F_{v,n}, \\ \overline{U_0}\, \psi_{n\,x} - \dfrac{N}{\lambda_n}\, u_{n\,x} + \mu_0\, \psi_n & = & F_{\psi,n}. \end{array}\right. \tag{2.19}$$

We recall that the functions (u_n, v_n, ψ_n) only depend on the x variable. Hence (2.19) is just a linear system of ordinary differential equations for u_n, v_n, ψ_n.

As usual, to solve (2.19), we first consider the corresponding homogeneous system. Dropping the subscripts n for the moment, we write:

$$\left\{\begin{array}{ccccccccc} \overline{U_0}\, \dfrac{du}{dx} & - & f\, v & - & \dfrac{1}{N\,\lambda_n}\, \dfrac{d\psi}{dx} & + & \mu_0\, u & = & 0, \\ \overline{U_0}\, \dfrac{dv}{dx} & + & f\, u & & & + & \mu_0\, v & = & 0, \\ \overline{U_0}\, \dfrac{d\psi}{dx} & & & + & \dfrac{N}{\lambda_n}\, \dfrac{du}{dx} & + & \mu_0\, \psi & = & 0. \end{array}\right. \tag{2.20}$$

The general solution of this linear system is of the form

$$(u, v, \psi) = \sum_{i=1}^{3} (A_i, B_i, C_i)\, e^{R_i\, x} \tag{2.21}$$

where the coefficients R_i are as follows:

$$\left\{\begin{array}{rcl} R_1 & = & -\dfrac{\mu_0}{\overline{U_0}}, \\ R_2 & = & \dfrac{-\mu_0\, \overline{U_0} + \dfrac{1}{\lambda}\Big(\mu_0^2 - f^2\, (\overline{U_0}^2\, \lambda^2 - 1)\Big)^{1/2}}{\overline{U_0}^2 - \dfrac{1}{\lambda^2}}, \\ R_3 & = & \dfrac{-\mu_0\, \overline{U_0} - \dfrac{1}{\lambda}\Big(\mu_0^2 - f^2\, (\overline{U_0}^2\, \lambda^2 - 1)\Big)^{1/2}}{\overline{U_0}^2 - \dfrac{1}{\lambda^2}}. \end{array}\right. \tag{2.22}$$

The $(A_i, B_i, C_i)_{1 \le i \le 3}$ satisfy the equations:

$$\left\{\begin{array}{rcl} A_i & = & a_i\, B_i, \\ C_i & = & c_i\, B_i, \end{array}\right. \tag{2.23}$$

with

$$\left\{\begin{array}{rcl} a_1 & = & 0, \\ c_1 & = & -\dfrac{f\, N\, \lambda}{R_1}, \end{array}\right. \tag{2.24}$$

and, for $i = 2, 3$:

$$\left\{\begin{array}{rcl} a_i & = & -\dfrac{\overline{U_0}\, R_i + \mu_0}{f}, \\ c_i & = & \dfrac{N\, R_i}{\lambda\, (\overline{U_0}\, R_i + \mu_0)}. \end{array}\right. \tag{2.25}$$

Now, returning to the nonhomogeneous system (2.19), we look for a solution $(u_n, v_n, \psi_n) = (u, v, \psi)$ of the form:

$$Y = (u, v, \psi)^t = \sum_{i=1}^{3} (a_i, 1, c_i,)^t \, B_i(x) \, e^{R_i \, x}, \tag{2.26}$$

where the (a_i, c_i) and R_i have been defined above. Equation (2.19) reads then:

$$M \, Y' + N \, Y = F, \tag{2.27}$$

where $\lambda = \lambda_n$ and

$$M = \begin{pmatrix} \overline{U_0} & 0 & -\dfrac{1}{N\,\lambda} \\ 0 & \overline{U_0} & 0 \\ -\dfrac{N}{\lambda} & 0 & \overline{U_0} \end{pmatrix} \quad , \quad N = \begin{pmatrix} \mu_0 & -f & 0 \\ f & \mu_0 & 0 \\ 0 & 0 & \mu_0 \end{pmatrix}, \tag{2.28}$$

$$F = (F_u, F_v, F_\psi)^t. \tag{2.29}$$

Thanks to assumption (2.13), $\overline{U_0} \neq 1/\lambda_n$, the matrix M is regular and it can be inverted. Equation (2.27) then implies:

$$\sum_{i=1}^{3} (a_i, 1, c_i)^t \, B_i'(x) \, e^{R_i \, x} = M^{-1} F =: \tilde{F}. \tag{2.30}$$

We now write the latter equation component by component. We find:

$$\Lambda(x).(B_1'(x), B_2'(x), B_3'(x))^t = (\tilde{F}_1(x), \tilde{F}_2(x), \tilde{F}_3(x))^t, \tag{2.31}$$

with

$$\Lambda(x) = \begin{pmatrix} 0 & a_2 \, e^{R_2 \, x} & a_3 \, e^{R_3 \, x} \\ e^{R_1 \, x} & e^{R_2 \, x} & e^{R_3 \, x} \\ c_1 \, e^{R_1 \, x} & c_2 \, e^{R_2 \, x} & c_3 \, e^{R_3 \, x} \end{pmatrix}. \tag{2.32}$$

Let us check that the matrix $\Lambda(x)$ is regular for every $x \in \mathbb{R}$; it is clearly sufficient to do so for $x = 0$, for which

$$\Lambda(0) = \begin{pmatrix} 0 & a_2 & a_3 \\ 1 & 1 & 1 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

We call $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ the lines of $\Lambda(0)$. It is clear that $\mathcal{L}_1$ and $\mathcal{L}_2$ are linearly independent vectors. Then if $\Lambda(0)$ were not regular there would exist $(\alpha, \beta) \in \mathbb{R}^2$ such that $\mathcal{L}_3 = \alpha \, \mathcal{L}_1 + \beta \, \mathcal{L}_2$. After some easy computations we would find that necessarily:

$$a_3 \, (c_2 - c_1) = a_2 \, (c_3 - c_1), \tag{2.33}$$

which leads (see (2.24) and (2.25)) to:

$$\overline{U_0} \, (R_3 - R_2) \, f^2 \, \lambda^2 = -\mu_0 \, R_1 \, (R_3 - R_2). \tag{2.34}$$

From (2.14) we find that $R_2 \neq R_3$, and thanks to the definition of R_1 equation (2.34) becomes:

$$\overline{U_0}^2 \, f^2 \, \lambda^2 = \mu_0^2, \tag{2.35}$$

which contradicts (2.15). Thus the matrix $\Lambda(x)$ is regular for every $x \in \mathbb{R}$.

Back to equation (2.31), and thanks to the latter result, the functions $B_i'(x)$ are uniquely determined for $i = 1, 2, 3$. It remains to use the modal boundary conditions in order to determine the constants $B_i(0)$ and thus the functions $B_i(x)$.

At this point, it is desirable to reintroduce the indices n i.e. to return to the notation (u_n, v_n, ψ_n), since the boundary conditions depend on the mode considered. For the supercritical modes ($n > n_c$), the modal boundary condition is the one of (1.44). We thus look for the $B_i(0)$ satisfying:

$$\left\{ \begin{array}{rcrcrcl} & & a_2\,B_2(0) & + & a_3\,B_3(0) & = & 0, \\ B_1(0) & + & B_2(0) & + & B_3(0) & = & 0, \\ c_1\,B_1(0) & + & c_2\,B_2(0) & + & c_3\,B_3(0) & = & 0. \end{array} \right. \tag{2.36}$$

The matrix of this system is again $\Lambda(0)$ which was shown to be regular (see above). We conclude that the constants $B_i(0)$ are uniquely determined by (2.36) and equal to zero. The functions $B_i(x)$ for the supercritical modes ($n > n_c$) are now fully determined.

If $n \leq n_c$, the mode is subcritical and we consider the boundary condition (1.46). We thus want to solve the following system:

$$\left\{ \begin{array}{rcrcrcl} -N\,c_1\,B_1(0) & + & (a_2 - N\,c_2)\,B_2(0) & + & (a_3 - N\,c_3)\,B_3(0) & = & 0, \\ B_1(0) & + & B_2(0) & + & B_3(0) & = & 0, \\ N\,c_1\,B_1(0) & + & (a_2 + N\,c_2)\,B_2(0) & + & (a_3 + N\,c_3)\,B_3(0) & = & \Gamma, \end{array} \right. \tag{2.37}$$

where

$$\Gamma = -\sum_{i=1}^{3} \int_0^{L_1} (a_i + N\,c_i)\,B_i'(x)\,dx. \tag{2.38}$$

The quantity Γ depends only on the data and on the B_i', hence it is known at this stage. After some computations and using hypotheses (2.14) and (2.15), we check that the matrix of the linear system (2.37) is regular (same proof exactly as for $\Lambda(0)$). This achieves the determination of the B_i in the subcritical case, and the lemma is proved. $\square$

Remark 2.1. The case when there exists $n \geq 1$ such that $\overline{U_0} = 1/\lambda_n$ is slightly different and actually simpler since the third equation (1.45) becomes $\partial\eta_n(x,t)/\partial t = F_{\eta,n}(t)$, which can be integrated directly. We note that no boundary condition (neither in the subcritical case nor in the supercritical one) would then be required for η_n so that (2.7),(2.8) would have to be modified. But we do not want to go into the details since this nongeneric situation seldom occurs in numerical simulations.

To conclude there remains to prove the hypothesis (i) of the Hille-Yosida theorem, that is:

Lemma 2.3. *The domain $D(A)$ of A is dense in H, and the operator A is closed.*

Proof. We first verify that the orthogonal in H of $D(A)$, $D(A)^\perp$, is reduced to $\{0\}$.

Let v be an element of $D(A)^\perp$. Since $A + \mu_0 I$ is onto, there exists $u \in D(A)$ such that $(A + \mu_0 I)\,u = v$. Then:

$$0 = (v,u)_H = \Big((A + \mu_0 I)\,u, u\Big)_H \geq \mu_0 \|u\|_H^2;$$

hence $u = v = 0$, which implies that $D(A)^\perp = \{0\}$, and $D(A)$ is dense in H.

To show that A is closed, we consider a sequence $(u_j, v_j, \psi_j) = U_j$ of $D(A)$, such that :

$$U_j \longrightarrow U \text{ in } H, \tag{2.39}$$

$$A\,U_j = F_j \longrightarrow F \text{ in } H, \tag{2.40}$$

and we want to verify that $U=(u,v,\psi)\in D(A)$ and $F=AU$, so that the graph of A is closed.

Thanks to (2.39), we know that

$$u_j \longrightarrow u \text{ in } H_u\subset L^2(\mathcal{M}), \tag{2.41}$$

$$v_j \longrightarrow v \text{ in } L^2(\mathcal{M}). \tag{2.42}$$

We also find from (2.9) and (2.40) that

$$\overline{U_0}\,\frac{dv_j}{dx}+f\,u_j\longrightarrow F_2 \text{ in } L^2(\mathcal{M}). \tag{2.43}$$

Hence the sequence $(dv_j/dx)_{j\in\mathbb{N}}$ is bounded in $L^2(\mathcal{M})$, and thanks to (2.42) we obtain that $v_x\in L^2(\mathcal{M})$.

In view of proving that $(u_x,\psi_x)\in L^2(\mathcal{M})$, we consider the decomposition in normal modes, introduced in Section 1.2. Thanks to (2.39), we have for every $n\geq 1$:

$$\hat{u}_{j,n} \longrightarrow \hat{u}_n \text{ in } L^2(0,L_1), \tag{2.44}$$

$$\hat{v}_{j,n} \longrightarrow \hat{v}_n \text{ in } L^2(0,L_1), \tag{2.45}$$

$$\hat{\psi}_{j,n} \longrightarrow \hat{\psi}_n \text{ in } L^2(0,L_1), \tag{2.46}$$

and the quantities $\sum_{n\geq 1}|\hat{u}_{j\,n}|^2$, $\sum_{n\geq 1}|\hat{v}_{j\,n}|^2$ and $\sum_{n\geq 1}|\hat{\psi}_{j\,n}|^2$ are bounded uniformly in j.

Similarly, we infer from (2.40) that for every $n\geq 1$:

$$\overline{U_0}\,\frac{d\hat{u}_{j,n}}{dx}-f\,v_n-\frac{1}{N\,\lambda_n}\,\frac{d\hat{\psi}_{j,n}}{dx}=F^j_{u,n}\quad\longrightarrow\quad F_{u,n} \text{ in } L^2(0,L_1), \tag{2.47}$$

$$\overline{U_0}\,\frac{d\hat{v}_{j,n}}{dx}+f\,\hat{u}_n=F^j_{v,n}\quad\longrightarrow\quad F_{v,n} \text{ in } L^2(0,L_1), \tag{2.48}$$

$$\overline{U_0}\,\frac{d\hat{\psi}_{j,n}}{dx}-\frac{N}{\lambda_n}\,\frac{d\hat{u}_{j,n}}{dx}=F^j_{\psi,n}\quad\longrightarrow\quad F_{\psi,n} \text{ in } L^2(0,L_1), \tag{2.49}$$

and the quantities $\sum_{n\geq 1}|F^j_{u,n}|^2$, $\sum_{n\geq 1}|F^j_{v,n}|^2$ and $\sum_{n\geq 1}|F^j_{\psi,n}|^2$ are bounded uniformly in j.

Combining (2.47) and (2.49), we find that:

$$\frac{d\hat{u}_{j,n}}{dx}=\frac{1}{\overline{U_0}^2-1/\lambda_n^2}\,(\overline{U_0}\,F^j_{u,n}+f\,\overline{U_0}\,\hat{v}_{j,n}+\frac{F^j_{\psi,n}}{N\,\lambda_n}), \tag{2.50}$$

hence the $(d\hat{u}_{j,n}/dx)_{j\geq 1}$ are bounded in $L^2(0,L_1)$ and $(d\hat{u}_n/dx)\in L^2(0,L_1)$. Moreover, we find that[4]

$$\sum_{n\geq 1}\Big|\frac{d\hat{u}_{j,n}}{dx}\Big|^2\leq\frac{4}{\min\limits_{n\geq 1}|\overline{U_0}^2-1/\lambda_n^2|^2}\,\sum_{n\geq 1}(\overline{U_0}^2|F^j_{u,n}|^2+f^2\,\overline{U_0}^2\,|\hat{v}_{j,n}|^2+\Big|\frac{F^j_{\psi,n}}{N\,\lambda_n}\Big|^2), \tag{2.51}$$

so that the latter quantity is bounded uniformly in j. This guarantees that $u_x\in L^2(\mathcal{M})$. Following the same idea, and using either (2.47) or (2.49), we also prove that $\psi_x\in L^2(\mathcal{M})$.

[4]Thanks to (2.13), we know that $\min\limits_{n\geq 1}|\overline{U_0}^2-1/\lambda_n^2|>0$.

To insure that $U \in D(A)$, we need to verify that the modal boundary conditions (1.44) and (1.46) are satisfied by U. This is clear since the convergence of $(\hat{u}_{j,n}, \hat{v}_{j,n}, \hat{\psi}_{j,n})$ to $(\hat{u}_n, \hat{v}_n, \hat{\psi}_n)$ is in fact in $H^1(0, L_1)$, so that the boundary conditions pass to the limit.

Finally, let us show that $AU = F$. Thanks to (2.39), we find that $AU_j \to AU$ in $\mathcal{D}'(\mathcal{M})$, hence $AU = F$ in $\mathcal{D}'(\mathcal{M})$. We infer from $U \in D(A)$ that $AU \in L^2(\mathcal{M})$, and conclude that $AU = F$ in $L^2(\mathcal{M})$, which ends the proof of Lemma 2.3. □

2.4. **The case of nonhomogeneous boundary conditions.**

In practical simulations, we want to solve the PEs with nonhomogeneous boundary conditions on U at $x = 0$ and $x = L_1$, that is $U^{g,l}$ and $U^{g,r}$. The latter are boundary values derived from a solution $\tilde{U}$ computed on a domain $\tilde{\mathcal{M}}$ larger than $\mathcal{M}$[5] .

We discussed in Section 2.3 above the case when $U^{g,l} = U^{g,r} = 0$. The issue is now to determine which components of $U^{g,l}$ and $U^{g,r}$ are needed to obtain a well-posed problem. In this context all components of $U^{g,l}$ and $U^{g,r}$ are available but we know (or surmise at this point) that they will not be all used, those used depending on the mode that we consider.

Based on the data $U^{g,l}$,$U^{g,r}$, let us now construct the following $U^g = (u^g, v^g, \psi^g)$:

$$(u^g, v^g, \psi^g)(z,t) = \sum_{n \geq 1} \Big(u_n^g(t)\mathcal{U}_n(z), v_n^g(t)\mathcal{U}_n(z), \psi_n^g(t)\mathcal{W}_n(z) \Big), \tag{2.52}$$

where (u_n^g, v_n^g, ψ_n^g) are found using the boundary values $U^{g,l}$ and $U^{g,r}$ by:

$$\left\{ \begin{array}{rcl} u_n^g(t) - \dfrac{1}{N}\,\psi_n^g(t) & = & u_n^{g,l}(t) - \dfrac{1}{N}\,\psi_n^{g,l}(t), \\ v_n^g(t) & = & v_n^{g,l}(t), \qquad\qquad 1 \leq n \leq n_c, \\ u_n^g(t) + \dfrac{1}{N}\,\psi_n^g(t) & = & u_n^{g,r}(t) + \dfrac{1}{N}\,\psi_n^{g,r}(t), \end{array} \right. \tag{2.53}$$

$$\left\{ \begin{array}{rcl} u_n^g(t) & = & u_n^{g,l}(t), \\ v_n^g(t) & = & v_n^{g,l}(t), \qquad n > n_c. \\ \psi_n^g(t) & = & \psi_n^{g,l}(t), \end{array} \right. \tag{2.54}$$

We note that U^g is a function of z and t, and hence it does not depend on the horizontal coordinate x. Setting $F^\# = F - dU^g/dt$ and $U_0^\# = U_0 - U_0^g$ where $U_0^g = U^g(t=0)$, we will look for $U^\#$ solution of

$$\left\{ \begin{array}{l} \dfrac{dU^\#}{dt} + AU^\# = F^\#, \\ U^\#(t=0) = U_0^\#. \end{array} \right. \tag{2.55}$$

Like (2.1) this equation corresponds to the case with homogeneous boundary conditions, and Theorem 2.2 applies[6]. Writting $U = U^\# + U^g$, we find that U is solution of (1.22) -(1.27), and the boundary conditions of U at $x = 0$ and $x = L_1$ are those of U^g, that is for the subcritical modes $(1 \leq n \leq n_c)$:

[5]Assuming e.g. periodical boundary conditions for $\tilde{\mathcal{M}}$.

[6]We will state in Theorem 2.3 below some assumptions on $U^{g,l}$ and $U^{g,r}$ so that $U_0^\#$ and $f^\#$ are as in Theorem 2.2.

$$
\left\{
\begin{aligned}
&\int_{-L_3}^{0} u(0,z,t)\,\mathcal{U}_n(z)\,dz - \frac{1}{N}\int_{-L_3}^{0} \psi(0,z,t)\,\mathcal{W}_n(z)\,dz \\
&\qquad = \int_{-L_3}^{0} u^{g,l}(z,t)\,\mathcal{U}_n(z)\,dz - \frac{1}{N}\int_{-L_3}^{0} \psi^{g,l}(z,t)\,\mathcal{W}_n(z)\,dz, \\
&\int_{-L_3}^{0} v(0,z,t)\,\mathcal{U}_n(z)\,dz = \int_{-L_3}^{0} v^{g,l}(z,t)\,\mathcal{U}_n(z)\,dz, \\
&\int_{-L_3}^{0} u(L_1,z,t)\,\mathcal{U}_n(z)\,dz + \frac{1}{N}\int_{-L_3}^{0} \psi(L_1,z,t)\,\mathcal{W}_n(z)\,dz \\
&\qquad = \int_{-L_3}^{0} u^{g,r}(z,t)\,\mathcal{U}_n(z)\,dz + \frac{1}{N}\int_{-L_3}^{0} \psi^{g,r}(z,t)\,\mathcal{W}_n(z)\,dz,
\end{aligned}
\right. \tag{2.56}
$$

and for the supercritical ones ($n > n_c$):

$$
\left\{
\begin{aligned}
\int_{-L_3}^{0} u(0,z,t)\,\mathcal{U}_n(z)\,dz &= \int_{-L_3}^{0} u^{g,l}(z,t)\,\mathcal{U}_n(z)\,dz, \\
\int_{-L_3}^{0} v(0,z,t)\,\mathcal{U}_n(z)\,dz &= \int_{-L_3}^{0} v^{g,l}(z,t)\,\mathcal{U}_n(z)\,dz, \\
\int_{-L_3}^{0} \psi(0,z,t)\,\mathcal{W}_n(z)\,dz &= \int_{-L_3}^{0} \psi^{g,l}(z,t)\,\mathcal{W}_n(z)\,dz.
\end{aligned}
\right. \tag{2.57}
$$

Thus we have established the following result:

Theorem 2.3. *Let H be the Hilbert space defined in* (2.2) *and A be the linear operator defined in* (2.9) *corresponding to the linearized Primitive Equations with vanishing viscosity. We are given the boundary values $U^{g,l}$ and $U^{g,r}$ which are in $L^1\big(0,T;L^2(-L_3,0)^3\big)$, together with their first time derivative, F and $F' = dF/dt \in L^1(0,T;H)$.*

Then the initial value problem corresponding to equations (1.22)-(1.27), *supplemented with the boundary conditions* (2.56) *and* (2.57) *is well-posed, that is for every initial data $U_0 \in U_0^g + D(A)$*[7]*, there exists a unique solution $U \in \mathcal{C}([0,T];H)$ of* (1.22)-(1.27) *verifying* (2.56) *and* (2.57).

2.5. **The mode constant in z.**

We now return to the mode constant in z, when $n = 0$. This mode does not raise any mathematical difficulty, but it is fundamental in the numerical simulations, since it carries much energy.

Integrating (1.22),(1.23), and (1.26) on $(-L_3,0)$ we find:

$$\frac{\partial u_0}{\partial t} + \overline{U_0}\,\frac{\partial u_0}{\partial x} - f\,v_0 + \frac{\partial \phi_0}{\partial x} = F_{u,0}, \tag{2.58}$$

$$\frac{\partial v_0}{\partial t} + \overline{U_0}\,\frac{\partial v_0}{\partial x} + f\,u_0 = F_{v,0}, \tag{2.59}$$

$$\frac{\partial u_0}{\partial x} = 0. \tag{2.60}$$

We propose to supplement this system with the following boundary conditions:

$$u_0(0,t) = u_l(t), \tag{2.61}$$

$$v_0(0,t) = v_l(t), \tag{2.62}$$

[7] This means that U_0 has the same smoothness as a function of $D(A)$ and (2.56),(2.57) are satisfied at $t = 0$.

with u_l, v_l given (not necessarily zero, as in Section 2.4).

Then, since $\partial u_0/\partial x = 0$, u_0 does not depend on x, and it is thus equal to $u_l(t)$ everywhere, so that (2.61) means in fact that

$$u_0(x,t) = u_l(t), \quad \forall (x,t) \in (0,L_1) \times \mathbb{R}_+^*. \tag{2.63}$$

Introducing (2.63) in (2.59), we find that:

$$\frac{\partial v_0}{\partial t} + \overline{U_0}\,\frac{\partial v_0}{\partial x} = F_{v,0} - f\,(\overline{U_0} + u_l). \tag{2.64}$$

When we supplement (2.64) with the boundary condition (2.62), we have a simple well-posed problem and v_0 is given in terms of the data by integration along the characteristics.

Finally, since both u_0 and v_0 are known, equation (2.58) gives ϕ_0, up to an additive constant (as expected):

$$\begin{aligned} \phi_0(x,t) &= \phi_0(0,t) + \int_0^x \{f\,v_0(x',t) - \frac{\partial u_0}{\partial t}(x',t)\}dx' \\ &= \phi_0(0,t) - x\,u_l'(t) + f\int_0^x v_0(x',t)dx'. \end{aligned} \tag{2.65}$$

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