Some 2.5D Models for the Primitive Equations of the Ocean and the Atmosphere

Qingshan Chen, Jacques Laminie, Antoine Rousseau, Roger Temam, Joe Tribbia

SOME 2.5D MODELS FOR THE EQUATIONS OF THE OCEAN AND THE ATMOSPHERE

Q.S. CHEN, J. LAMINIE, A. ROUSSEAU, R. TEMAM, AND J. TRIBBIA

ABSTRACT. The primitive equations (PEs) of the atmosphere and the ocean without viscosity are considered. A 2.5D model is introduced, whose motivation is described in the Introduction. A set of nonlocal boundary conditions is proposed, and well-posedness is established for the flows linearized around a constant velocity stratified flow; homogeneous and nonhomogeneous boundary conditions are considered. A related model of dimension 2.5, of physical interest but with fewer degrees of freedom, is also considered at the end.

CONTENTS

Date: March 14, 2007.
2000 *Mathematics Subject Classification.* 35L50, 76N10, 47D06, 86A05.
Key words and phrases. Nonviscous Primitive Equations, limited domains, boundary conditions, transparent boundary conditions, compatibility conditions, well-posedness, semigroup theory.

1. Introduction

We pursue our work on the primitive equations (PEs) of the atmosphere and the ocean *in the absence of viscosity* [12, 13, 14, 15]. When the viscosity is present, the primitive equations have been the object of much attention since the works [6, 7]; review articles about the mathematical theory of the PEs with viscosity appear in [18] and in an updated form in [11].

In the absence of viscosity, little progress has been made on the analysis of the primitive equations since the negative result of Oliger and Sundstrom [8] showing that these equations are not well-posed for any set of local boundary conditions. In earlier works, three of the authors of the present article have investigated these equations in space dimension two, and an infinite set of boundary conditions has been proposed. Well-posedness of the corresponding linearized equations has been established in [14] and numerical simulations have been performed in [15] for the linearized equations and for the full nonlinear equations. Note that the nonoccurrence of blow-up in the latter case supports the (yet unproved) conjecture that the proposed nonlocal boundary conditions are also suitable for the nonlinear PEs. See also [1] for numerical issues concerning the boundary conditions for the primitive equations without viscosity.

The numerical simulations performed in [15] were mainly motivated by computational preoccupations and the need to support the idea that the proposed boundary conditions are computationally feasible and lead indeed to well-posedness. In view of performing (in dimension two) computations of physical significance, the last author expressed the wish that the flow was a perturbation of a geostrophic flow (which is not the case in [15]). Now, the geostrophic equation

$$p_y = -\rho f u, \tag{1.1}$$

implies that there does not exist any geostrophic solution depending only on x and z[1]. It is then necessary, even in dimension two, to introduce some y-dependence. A number of natural choices had to be abandoned, in particular the use of a few Fourier modes in y, which would produce the undesirable Gibbs phenomena when approximating the periodic extension of the function $\sigma(y) = y$ on $[0, L_2]$, this function being introduced in the model by (1.1). In this way we were led to choose, for the y-direction, a three-mode linear finite element model.

[1] Ox is the local west-east direction, Oy is the local south-north direction, and Oz is the ascendant vertical.

In this article, we present the full derivation of the model and study the well-posedness of the linearized equations.

This article is organized as follows. The model is derived in Section 2. We first derive the Galerkin finite element approximation based on the use of three piecewise linear elements in the direction y; we thus arrive at three coupled systems, each one similar to the 2D primitive equations in the variables x and z (and t). We then perform the normal mode decomposition of these equations in the direction z as in [17] (see also [13, 14, 15]), the normal modes in z being either sines or cosines (depending on the functions), and these sines and cosines are the eigenfunctions of a two-point boundary value Sturm-Liouville problem ([17]). At this stage, each mode consists in three coupled equations for the functions of the variables x and t (Section 2.2). We finally introduce, in Section 2.3, the boundary conditions for the latest systems in x and t, the boundary conditions depending on the nature of the mode (subcritical or supercritical), the subcritical modes being the mathematically most challenging and physically most relevant ones. In Section 3 the objectives are as follows: we first establish, in the absence of the zero mode, the well-posedness of the linearized PEs, all the non-zero modes taken into account. We then pay special attention to the mode zero (barotropic part), and finally consider the case of nonhomogeneous boundary conditions; we reduce this case to the homogeneous case by homogenization of the boundary conditions. We consider in Section 4 a related model, physically interesting but with fewer degrees of freedom. The well-posedness of this model is addressed in a similar way as in Sections 2 and 3 for the first model. For this model we only emphasize the parts of the proof and the discussions which are different from the first model. The actual numerical simulations will be performed and discussed in a separate work.

For this article, the thrust of the effort is due to the first author (Q.S.C.). A.R. has lent his expertise gained from the earlier works [13, 14, 15]. R.T. and J.T. have interactively developed the proposed model. Finally, taking advantage of his expertise with orthogonal wavelets [4], J.L. made the essential suggestion of using L^2-orthogonal finite elements. Without this simplification the equation would have been probably unmanageable, and this work could not exist.

2. The 2.5D primitive equations without Viscosity

The 3D primitive equations for the ocean and the atmosphere without viscosity read:

$$\begin{cases} \widetilde{\mathbf{v}}_t + (\widetilde{\mathbf{v}} \cdot \nabla)\widetilde{\mathbf{v}} + \widetilde{w}\mathbf{v}_z + fk \times \widetilde{\mathbf{v}} + \dfrac{1}{\rho_0}\nabla\widetilde{p} = F_{\widetilde{\mathbf{v}}}, \\ \widetilde{p}_z = -\widetilde{\rho}g, \\ \nabla \cdot \widetilde{\mathbf{v}} + \widetilde{w}_z = 0, \\ \widetilde{T}_t + (\widetilde{\mathbf{v}} \cdot \nabla)\widetilde{T} + \widetilde{w}\widetilde{T}_z = Q_{\widetilde{T}}, \\ \widetilde{\rho} = \rho_0(1 - \alpha(\widetilde{T} - T_0)). \end{cases} \tag{2.1}$$

Here $\widetilde{\mathbf{v}} = (\widetilde{u}, \widetilde{v})$ is the horizontal velocity, $\widetilde{w}$ the vertical velocity; $\widetilde{\rho}$ is the density, $\widetilde{p}$ the pressure, and $\widetilde{T}$ the temperature; ∇ denotes the horizontal gradient operator; $\widetilde{\mathbf{v}}_t = \partial\mathbf{v}/\partial t$, etc. The independent variables are $(x, y, z) \in \mathcal{M} = (0, L_1) \times (0, L_2) \times (-L_3, 0)$, and $t > 0$.

In the physical context the forcing terms $F_{\widetilde{\mathbf{v}}} = (F_{\widetilde{u}}, F_{\widetilde{v}})$ and $Q_{\widetilde{T}}$ vanish, but we introduce them here for mathematical generality. In the sequel, we will take the forcing terms to be zero except in the functional setting of the problem in Section 3.1. We consider, as a reference flow, a uniformly stratified flow with constant velocity $\bar{\mathbf{v}}_0 = (\bar{U}_0, 0)$, and the density, the pressure, and the temperature are of the form $\rho_0 + \bar{\rho}(z)$, $p_0 + \bar{p}(z)$ and $T_0 + \bar{T}(z)$, with

$$\begin{cases} \bar{\rho}(z) = -\rho_0 N^2 g^{-1} z, \\ \bar{T}(z) = N^2(\alpha g)^{-1} z, \\ \bar{p}_z(z) = -(\rho_0 + \bar{\rho})g. \end{cases} \tag{2.2}$$

Here N is the buoyancy frequency, assumed to be constant; ρ_0, p_0 and T_0 are respectively the reference density, pressure and temperature. We then decompose the unknowns of (2.1) in the following way:

$$\begin{cases} \widetilde{\mathbf{v}} = \bar{\mathbf{v}}_0 + \mathbf{v}(x, y, z, t), \\ \widetilde{w} = w(x, y, z, t), \\ \widetilde{\rho} = \rho_0 + \bar{\rho} + \rho(x, y, z, t), \\ \widetilde{T} = T_0 + \bar{T} + T(x, y, z, t), \\ \widetilde{p} = p_0 + \bar{p} + p(x, y, z, t). \end{cases} \tag{2.3}$$

We substitute (2.3) into the system (2.1) and obtain the new system for u, v, w, ρ, T and p:

$$\begin{cases} u_t + (\bar{U}_0 + u)u_x + vu_y + wu_z - fv + \dfrac{1}{\rho_0}p_x = 0, \\ v_t + (\bar{U}_0 + u)v_x + vv_y + wv_z + fu + \dfrac{1}{\rho_0}p_y = 0, \\ T_t + (\bar{U}_0 + u)T_x + vT_y + wT_z + \dfrac{N^2}{\alpha g}w = 0, \\ p_z = -\rho g, \\ u_x + v_y + w_z = 0, \\ \rho = -\alpha\rho_0 T. \end{cases} \tag{2.4}$$

In our model we assume that the perturbation variables u, v, w, ρ, T, p, as well as their first order derivatives, are small, and we then proceed to the linearization of the equations. We will also substitute $(2.4)_6$ into $(2.4)_4$, and set

$$\begin{cases} \phi = p/\rho_0, \\ \psi = \alpha T g (= \phi_z). \end{cases} \tag{2.5}$$

After these steps we reach the following system with five equations and five unknowns:

$$\begin{cases} u_t + \bar{U}_0 u_x - fv + \phi_x = 0, \\ v_t + \bar{U}_0 v_x + f(\bar{U}_0 + u) + \phi_y = 0, \\ \psi_t + \bar{U}_0 \psi_x + N^2 w = 0, \\ u_x + v_y + w_z = 0, \\ \phi_z = \psi. \end{cases} \tag{2.6}$$

2.1. **The finite element expansion in the y direction.** The aim is to find (and numerically study) a 2D version of (2.6) (and (2.4) for subsequent studies), which is physically interesting. For that purpose we want the flow to be close to geostrophic equilibrium, so that $u = u^g + u'$, or $\widetilde{u} = \bar{u} + u^g + u'$ [2], etc., where u^g, etc., and as well $\bar{u} + u^g$, etc., are geostrophic. The geostrophic equation

$$p_y^g = -\rho_0 f u^g \tag{2.7}$$

prevents us from taking functions $\widetilde{u} = u^g + u$, $\widetilde{p} = p^g + p$, ..., which are independent of y. Indeed if we consider a space periodic approximation

[2] The notations u', v', etc. are not used in the sequel.

with two or three modes of the Fourier series in y, (2.7) will introduce the Fourier series expansion of

$$h(y) = y, \qquad 0 < y < L_2, \tag{2.8}$$

and, as is well-known, the discontinuity of (the periodic extension of) h leads to numerical oscillations.

Hence our 2.5D model will allow linear variations in y, and, in view of (2.7) it is then natural to introduce piecewise linear finite elements in the y direction. We set $L_2 = 1$ for convenience, and introduce one middle point $\frac{1}{2}$: 0, 1 play the role of boundaries, and values at $\frac{1}{2}$ play the role of the flow "independent of y". We introduce three hat functions (finite elements) h_1, h_2, and h_3 (see Figure 2.1) corresponding to the points 0, $\frac{1}{2}$, and 1. Instead of the usual hat function $\widetilde{h}_2$ (see Figure 2.1), we use h_2 such that h_1, h_2, and h_3 are **orthogonal**.

We now look for approximate solutions of (2.6) of the form of

$$\begin{cases} u = u_1(x,z,t)h_1(y) + u_2(x,z,t)h_2(y) + u_3(x,z,t)h_3(y), \\ v = v_1(x,z,t)h_1(y) + v_2(x,z,t)h_2(y) + v_3(x,z,t)h_3(y), \\ w = w_1(x,z,t)h_1(y) + w_2(x,z,t)h_2(y) + w_3(x,z,t)h_3(y), \\ \phi = \phi_1(x,z,t)h_1(y) + \phi_2(x,z,t)h_2(y) + \phi_3(x,z,t)h_3(y), \\ \psi = \psi_1(x,z,t)h_1(y) + \psi_2(x,z,t)h_2(y) + \psi_3(x,z,t)h_3(y), \end{cases} \tag{2.9}$$

and consider the corresponding finite elements (Galerkin) approximation of (2.6). We introduce the expressions (2.9) for u, v, w, ϕ and ψ into the system (2.6), multiply each equation by h_1, h_2 and h_3 respectively, and integrate over $(0,1)$. Thanks to the orthogonality of h_1, h_2 and h_3, we obtain the following system:

$$\begin{cases} \boldsymbol{u}_t + \bar{U}_0 \boldsymbol{u}_x + \boldsymbol{\phi}_x - f\boldsymbol{v} = 0, \\ \boldsymbol{v}_t + \bar{U}_0 \boldsymbol{v}_x + f\boldsymbol{u} + \Lambda\boldsymbol{\phi} + \boldsymbol{f} = 0, \\ \boldsymbol{\psi}_t + \bar{U}_0 \boldsymbol{\psi}_x + N^2\boldsymbol{w} = 0, \\ \boldsymbol{u}_x + \Lambda\boldsymbol{v} + \boldsymbol{w}_z = 0, \\ \boldsymbol{\psi} = \boldsymbol{\phi}_z. \end{cases} \tag{2.10}$$

Here

$$\begin{aligned} \boldsymbol{u} &= (u_1, u_2, u_3)^T, \quad \boldsymbol{v} = (v_1, v_2, v_3)^T, \quad \boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)^T, \\ \boldsymbol{\psi} &= (\psi_1, \psi_2, \psi_3)^T, \quad \boldsymbol{w} = (w_1, w_2, w_3)^T, \end{aligned}$$

and

$$\Lambda = \begin{pmatrix} -3 & -9 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 9 & 3 \end{pmatrix} \quad \boldsymbol{f} = \begin{pmatrix} \frac{3}{2} f\bar{U}_0 \\ -\frac{1}{2} f\bar{U}_0 \\ \frac{3}{2} f\bar{U}_0 \end{pmatrix}. \tag{2.11}$$

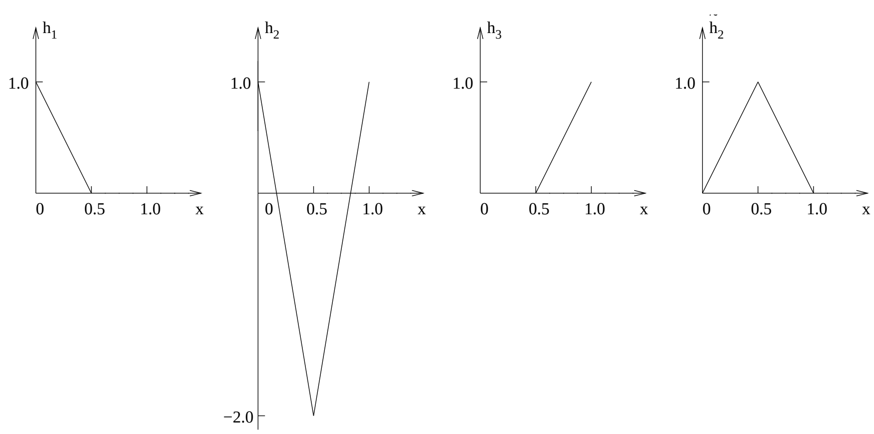

FIGURE 2.1. The hat functions h_1, h_2, h_3 and $\widetilde{h}_2$.

We denote by $\mathcal{M}' = (0, L_1) \times (-L_3, 0)$ the two-dimensional spatial domain for the system (2.10).

2.2. **The normal mode expansion.** As in the 2D case (see [14] and [17]), we consider a normal mode expansion of the solutions to the system (2.10). That is, we look for solutions of the system in the following form:

$$\begin{cases} (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\phi}) = \displaystyle\sum_{n\geq 0} \mathcal{U}_n(z)(\widehat{\boldsymbol{u}}_n, \widehat{\boldsymbol{v}}_n, \widehat{\boldsymbol{\phi}}_n)(x,t), \\ (\boldsymbol{w}, \boldsymbol{\psi}) = \displaystyle\sum_{n\geq 1} \mathcal{W}_n(z)(\widehat{\boldsymbol{w}}_n, \widehat{\boldsymbol{\psi}}_n)(x,t). \end{cases} \tag{2.12}$$

Here $\widehat{\boldsymbol{u}}_n$, $\widehat{\boldsymbol{v}}_n$, etc., are vector functions as $\boldsymbol{u}$, $\boldsymbol{v}$, etc. , but are independent of z. We refer the reader to [17] for the justification of the normal mode expansion. The specifications of the eigenfunctions $\mathcal{U}_n$ and $\mathcal{W}_n$ can be found in [14], and are repeated here for convenience:

$$\begin{cases} \mathcal{U}_0 = \sqrt{\dfrac{1}{L_3}}, \text{ and } \mathcal{U}_n = \sqrt{\dfrac{2}{L_3}}\cos(\lambda_n N z) \qquad \text{for } n \geq 1, \\ \mathcal{W}_n = \sqrt{\dfrac{2}{L_3}}\sin(\lambda_n N z) \qquad \text{for } n \geq 1, \end{cases} \tag{2.13}$$

where $\lambda_n = n\pi/NL_3$. We observe that, for $n, m \geq 1$,

$$\begin{cases} \displaystyle\int_{-L_3}^{0} \mathcal{U}_n(z)\mathcal{U}_m(z)\,\mathrm{d}z = \delta_{n,m}, \\ \displaystyle\int_{-L_3}^{0} \mathcal{W}_n(z)\mathcal{W}_m(z)\,\mathrm{d}z = \delta_{n,m}, \\ \displaystyle\int_{-L_3}^{0} \mathcal{U}_n(z)\mathcal{W}_m(z)\,\mathrm{d}z = 0, \\ \mathcal{U}_n'(z) = -N\lambda_n \mathcal{W}_n(z), \\ \mathcal{W}_n'(z) = N\lambda_n \mathcal{U}_n(z). \end{cases} \tag{2.14}$$

We then introduce (2.12) into the system (2.10). For each $n \geq 0$, we multiply each equation by $\mathcal{U}_n$ (or $\mathcal{W}_n$ for the 3rd and 5th equations), and integrate over $(-L_3, 0)$. From now on we will drop the hat for the modes, and write $\boldsymbol{u}_n$, $\boldsymbol{v}_n$, etc. When $n = 0$, we obtain a system for $\boldsymbol{u}_0$, $\boldsymbol{v}_0$ and $\boldsymbol{\phi}_0$ only:

$$\begin{cases} \boldsymbol{u}_{0t} + \bar{U}_0 \boldsymbol{u}_{0x} + \boldsymbol{\phi}_{0x} - f\boldsymbol{v}_0 = 0, \\ \boldsymbol{v}_{0t} + \bar{U}_0 \boldsymbol{v}_{0x} + f\boldsymbol{u}_0 + \Lambda\boldsymbol{\phi}_0 + \boldsymbol{f}_0 = 0, \\ \boldsymbol{u}_{0x} + \Lambda \boldsymbol{v}_0 = 0. \end{cases} \tag{2.15}$$

Here Λ is the same as before, and

$$\boldsymbol{f}_0 = \begin{pmatrix} \frac{3}{2} f \bar{U}_0 \sqrt{L_3} \\ -\frac{1}{2} f \bar{U}_0 \sqrt{L_3} \\ \frac{3}{2} f \bar{U}_0 \sqrt{L_3} \end{pmatrix}. \tag{2.16}$$

When $n \geq 1$, the corresponding system for each mode has the same form:

$$\begin{cases} \boldsymbol{u}_{nt} + \bar{U}_0 \boldsymbol{u}_{nx} + \boldsymbol{\phi}_{nx} - f \boldsymbol{v}_n = 0, \\ \boldsymbol{v}_{nt} + \bar{U}_0 \boldsymbol{v}_{nx} + f \boldsymbol{u}_n + \Lambda \boldsymbol{\phi}_n = 0, \\ \boldsymbol{\psi}_{nt} + \bar{U}_0 \boldsymbol{\psi}_{nx} + N^2 \boldsymbol{w}_n = 0, \\ \boldsymbol{u}_{nx} + \Lambda \boldsymbol{v}_n + N \lambda_n \boldsymbol{w}_n = 0, \\ - N \lambda_n \boldsymbol{\phi}_n = \boldsymbol{\psi}_n. \end{cases} \tag{2.17}$$

¿From the last two equations we notice that

$$\boldsymbol{\phi}_n = -\frac{1}{N\lambda_n} \boldsymbol{\psi}_n, \quad \boldsymbol{w}_n = -\frac{1}{N\lambda_n} (\boldsymbol{u}_{nx} + \Lambda \boldsymbol{v}_n), \tag{2.18}$$

which means that $\boldsymbol{\phi}_n$ and $\boldsymbol{w}_n$ are determined by the other three unknowns. Then we can eliminate $\boldsymbol{\phi}_n$ and $\boldsymbol{w}_n$ in (2.17), and obtain a system for $\boldsymbol{u}_n$, $\boldsymbol{v}_n$ and $\boldsymbol{\psi}_n$, for each $n \geq 1$:

$$\begin{cases} \boldsymbol{u}_{nt} + \bar{U}_0 \boldsymbol{u}_{nx} - \dfrac{1}{N\lambda_n} \boldsymbol{\psi}_{nx} - f \boldsymbol{v}_n = 0, \\ \boldsymbol{v}_{nt} + \bar{U}_0 \boldsymbol{v}_{nx} + f \boldsymbol{u}_n - \dfrac{1}{N\lambda_n} \Lambda \boldsymbol{\psi}_n = 0, \\ \boldsymbol{\psi}_{nt} - \dfrac{N}{\lambda_n} \boldsymbol{u}_{nx} + \bar{U}_0 \boldsymbol{\psi}_{nx} - \dfrac{N}{\lambda_n} \Lambda \boldsymbol{v}_n = 0. \end{cases} \tag{2.19}$$

In Section 2.3 we will present the boundary conditions at $x = 0$, L_1 for the modes $n \geq 1$. These boundary conditions will ensure the well-posedness of the system (2.10), which we are going to establish in Section 3. Due to its different, and somehow irregular form, the system (2.15) of the zero mode will be treated separately at the end of Section 3.

2.3. **Boundary conditions at $x = 0$, L_1.** The analysis by which we determine the boundary conditions for the systems (2.19), and ultimately for (2.10), is similar to that in the 2D case ([14]). We will review the spirit of the analysis here for the sake of completeness, and then list the boundary conditions that we propose for the systems (2.19).

The matrix associated with the coefficients of the first order terms with respect to x reads:

$$\begin{pmatrix} \bar{U}_0 & 0 & -\frac{1}{N\lambda_n} \\ 0 & \bar{U}_0 & 0 \\ -\frac{N}{\lambda_n} & 0 & \bar{U}_0 \end{pmatrix}.$$

There are three eigenvalues to this matrix, namely $\bar{U}_0 + 1/\lambda_n$, $\bar{U}_0$ and $\bar{U}_0 - 1/\lambda_n$. Because $\bar{U}_0$ and λ_n are positive, then each mode has at least two positive eigenvalues. The third eigenvalue $\bar{U}_0 - 1/\lambda_n$, however, can be either positive or negative, depending on n. We say that the corresponding mode is supercritical in the first case, and subcritical in the second case. The supercritical modes require three boundary conditions at $x = 0$, while the subcritical modes require two boundary conditions at $x = 0$ and one at $x = L_1$. This mandates that we impose different boundary conditions according to the type of the modes.

We first note that the sequence $\{ \lambda_n \}$ is monotone and $\lambda_n \longrightarrow \infty$ as $n \longrightarrow \infty$. Therefore there are only a finite number of subcritical modes, say n_c.

Definition 2.1. *We denote by n_c the number of subcritical modes, which is defined in the following way:*

$$
\begin{aligned}
\lambda_{n_c} &= \frac{n_c \pi}{N L_3} < \frac{1}{\bar{U}_0}, \\
\lambda_{n_c+1} &= \frac{(n_c+1)\pi}{N L_3} > \frac{1}{\bar{U}_0}.
\end{aligned}
$$

Remark 2.1. When $\bar{U}_0 - 1/\lambda_n = 0$ for some n, the mode will be neither subcritical, nor supercritical. But this can be easily avoided by modifying as necessary the velocity of the reference flow. For this reason we assume throughout this article that $\bar{U}_0 \neq 1/\lambda_n$ for all n's.

We also note here that though most of the modes are supercritical, the subcritical modes are particularly important since they carry most of the energy.

For the supercritical modes, i.e. when $n > n_c$, we take the natural boundary conditions:

$$
\begin{cases}
\boldsymbol{u}_n(0,t) = 0, \\
\boldsymbol{v}_n(0,t) = 0, \\
\boldsymbol{\psi}_n(0,t) = 0.
\end{cases} \tag{2.20}
$$

For the subcritical modes, i.e. when $1 \leq n \leq n_c$, we impose the boundary conditions in the following way:

$$
\begin{cases}
\boldsymbol{\xi}_n(0,t) = 0, \\
\boldsymbol{v}_n(0,t) = 0, \\
\boldsymbol{\eta}_n(L_1,t) = 0.
\end{cases} \tag{2.21}
$$

Here $\boldsymbol{\xi}_n = \boldsymbol{u}_n - \boldsymbol{\psi}_n/N$, $\boldsymbol{v}_n = \boldsymbol{v}_n$, and $\boldsymbol{\eta}_n = \boldsymbol{u}_n + \boldsymbol{\psi}_n/N$ are the three eigenvectors corresponding to $\bar{U}_0 + 1/\lambda_n$, $\bar{U}_0$ and $\bar{U}_0 - 1/\lambda_n$ respectively.

Remark 2.2. In this subsection the boundary conditions are given for each mode. The boundary conditions for the system (2.10) will come directly from (2.20) and (2.21), and will be presented later on (see (3.4) and (3.5)).

Remark 2.3. For most of the paper the boundary conditions will be homogeneous. But at the end we will explain how to handle the nonhomogeneous case. Some technicalities related to the so-called compatibility conditions will appear.

3. The Well-posedness of the Linear System

3.1. **The functional settings.** We want to write (2.10) (the zero mode excluded, see Remark 3.1 below) as an initial value problem of the form

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} + AU = F, \\ U(0) = U_0. \end{cases} \tag{3.1}$$

Here $U = U(t)$ stands for $(\boldsymbol{u}(t), \boldsymbol{v}(t), \boldsymbol{\psi}(t))$, and A is an unbounded operator in H with domain $D(A) \subset H$, and $U_0 \in D(A)$, $F \in H$. The space H is defined as follows:

$$H = H_{\boldsymbol{u}} \times H_{\boldsymbol{v}} \times H_{\boldsymbol{\psi}}, \tag{3.2}$$

where

$$\begin{aligned} H_{\boldsymbol{u}} &= H_{\boldsymbol{v}} = \Big\{ \boldsymbol{u} \in \mathbf{L}^2(\mathcal{M}') \ \Big| \ \int_{-L_3}^{0} \boldsymbol{u}(x,z)\,\mathrm{d}z = \mathbf{0}, \text{ for a.e. } x \in (0, L_1) \Big\}, \\ H_{\boldsymbol{\psi}} &= \mathbf{L}^2(\mathcal{M}'). \end{aligned}$$

In the definitions above the specification of the domain $\mathcal{M}'$ can be found at the end of Section 2.1. The convention that $\mathbf{L}^2(\mathcal{M}') = (L^2(\mathcal{M}'))^3$ has been used. Similarly, later in this article, we will use $\mathbf{H}^1(\mathcal{M}')$, $\boldsymbol{\mathcal{D}}(\mathcal{M}')$, etc., for the corresponding vector function spaces. H is endowed with the following scalar product:

$$(U, \widetilde{U})_H = \int_{\mathcal{M}'} (\boldsymbol{u} \cdot \widetilde{\boldsymbol{u}} + \boldsymbol{v} \cdot \widetilde{\boldsymbol{v}} + \frac{1}{N^2} \boldsymbol{\psi} \cdot \widetilde{\boldsymbol{\psi}})\,\mathrm{d}\mathcal{M}'.$$

Clearly H is a closed subspace of $(\mathbf{L}^2(\mathcal{M}'))^3$, and the norm of H derived from the scalar product $(\cdot,\cdot)_H$ is equivalent to that of $(\mathbf{L}^2(\mathcal{M}'))^3$.

We denote by P the orthogonal projector from $\mathbf{L}^2(\mathcal{M}')$ onto $H_{\boldsymbol{u}}$ (and also onto $H_{\boldsymbol{v}}$, since $H_{\boldsymbol{u}}$ and $H_{\boldsymbol{v}}$ are identical.) Hence, for each $\boldsymbol{g} \in \mathbf{L}^2(\mathcal{M}')$,

$$\begin{cases} P(\boldsymbol{g}) = \boldsymbol{g} - \dfrac{1}{L_3} \displaystyle\int_{-L_3}^{0} \boldsymbol{g}(x,z)\,\mathrm{d}z, \\ (I - P)(\boldsymbol{g}) = \dfrac{1}{L_3} \displaystyle\int_{-L_3}^{0} \boldsymbol{g}(x,z)\,\mathrm{d}z. \end{cases} \tag{3.3}$$

We can easily check that $P(\boldsymbol{g}) \in H_{\boldsymbol{u}}$, and $(I-P)(\boldsymbol{g}) \in H_{\boldsymbol{u}}^{\perp}$. We can also show that $H_{\boldsymbol{u}}^{\perp} = \mathbf{L}_x^2(0, L_1)$. Indeed, for each $\boldsymbol{f} \in \mathbf{L}_x^2(0, L_1)$, $P(\boldsymbol{f}) = \mathbf{0}$, and so $\boldsymbol{f} \in H_{\boldsymbol{u}}^{\perp}$. If, on the other hand, $\boldsymbol{f} \in H_{\boldsymbol{u}}^{\perp}$, then $(I-P)\boldsymbol{f} = \boldsymbol{f}$. Hence $\boldsymbol{f}$ is independent of z, and $\boldsymbol{f} \in \mathbf{L}_x^2(0, L_1)$.

The unknown U is subjected to modal boundary conditions, which are listed below. The parallelism between the modal boundary conditions for U and the boundary conditions for each mode (see (2.20) and (2.21)) is obvious.

For $n > n_c$ (i.e. for the the supercritical modes),

$$\begin{cases} \displaystyle\int_{-L_3}^{0} \boldsymbol{u}(0,z)\mathcal{U}_n(z)\,\mathrm{d}z = \mathbf{0}, \\ \displaystyle\int_{-L_3}^{0} \boldsymbol{v}(0,z)\mathcal{U}_n(z)\,\mathrm{d}z = \mathbf{0}, \\ \displaystyle\int_{-L_3}^{0} \boldsymbol{\psi}(0,z)\mathcal{W}_n(z)\,\mathrm{d}z = \mathbf{0}. \end{cases} \tag{3.4}$$

For $1 \le n \le n_c$ (i.e. for the subcritical modes),

$$\begin{cases} \displaystyle\int_{-L_3}^{0} \boldsymbol{u}(0,z)\mathcal{U}_n(z) - \frac{1}{N}\int_{-L_3}^{0} \boldsymbol{\psi}(0,z)\mathcal{W}_n(z)\,\mathrm{d}z = \mathbf{0}, \\ \displaystyle\int_{-L_3}^{0} \boldsymbol{v}(0,z)\mathcal{U}_n(z) = \mathbf{0}, \\ \displaystyle\int_{-L_3}^{0} \boldsymbol{u}(L_1,z)\mathcal{U}_n(z) + \frac{1}{N}\int_{-L_3}^{0} \boldsymbol{\psi}(L_1,z)\mathcal{W}_n(z)\,\mathrm{d}z = \mathbf{0}. \end{cases} \tag{3.5}$$

We now define $D(A)$ as follows:

$$\begin{aligned} D(A) = \{\, U \in H \mid U_x \in (\mathbf{L}^2(\mathcal{M}'))^3, \\ \textit{and } U \textit{ verifies the BC's (3.4) and (3.5)} \}. \end{aligned} \tag{3.6}$$

Then for every $U \in D(A)$, AU is defined by

$$AU = \begin{pmatrix} \bar{U}_0 \boldsymbol{u}_x - f\boldsymbol{v} - P[\int_z^0 \boldsymbol{\psi}_x(x,z')\,\mathrm{d}z'] \\ \bar{U}_0 \boldsymbol{v}_x + f\boldsymbol{u} - P[\int_z^0 \Lambda\boldsymbol{\psi}(x,z')\,\mathrm{d}z'] \\ \bar{U}_0 \boldsymbol{\psi}_x + N^2 \int_z^0 (\boldsymbol{u}_x + \Lambda \boldsymbol{v})\,\mathrm{d}z' \end{pmatrix}. \tag{3.7}$$

Remark 3.1. By the definition of the spaces H and $D(A)$, we include in the system (3.1) all the modes with $n \ge 1$. The zero mode ($n = 0$) is excluded from the system, and will be treated separately.

3.2. **The main result.** We will prove the well-posedness of the system (3.1) with the help of the following version of Hille-Yosida theorem:

Theorem 3.1. *Let H be a Hilbert space and let $A : D(A) \longrightarrow H$ be a linear unbounded operator, with domain $D(A) \subset H$. Assume the following:*
(i) $D(A)$ is dense in H and A is closed,
(ii) $A \geq 0$, i.e. $(AU, U)_H \geq 0$, $\forall U \in D(A)$,
(iii) $\exists\, \mu_0 > 0$ such that $A + \mu_0 I$ is onto.
Then $-A$ is the infinitesimal generator of a semigroup of contractions $\{S(t)\}_{t\geq 0}$ in H, and for every $U_0 \in H$ and $F \in L^1(0, T; H)$, there exists a unique solution $U \in C([0, T]; H)$ of (3.1), of the form

$$U(t) = S(t)U_0 + \int_0^t S(t-s)F(s)\,\mathrm{d}s. \tag{3.8}$$

If furthermore $U_0 \in D(A)$ and $F' = \mathrm{d}F/\,\mathrm{d}t \in L^1(0, T; H)$, then U satisfies (3.1) and

$$U \in C([0, T]; H) \cap L^\infty(0, T; D(A)), \quad \frac{\mathrm{d}U}{\mathrm{d}t} \in L^\infty(0, T; H). \tag{3.9}$$

Now we will state our main result concerning the system (3.1).

Theorem 3.2. *Let H, A and $D(A)$ be defined as in Section 3.1. Then the initial value problem (3.1) is well-posed. That is, for every $U_0 \in D(A)$, $F \in L^1(0, T; H)$, with $F' \in L^1(0, T; H)$, (3.1) has a unique solution U such that*

$$U \in C([0, T]; H) \cap L^\infty(0, T; D(A)), \quad \frac{\mathrm{d}U}{\mathrm{d}t} \in L^\infty(0, T; H). \tag{3.10}$$

3.3. **Proof of Theorem 3.2.** We first want to rewrite AU in another form, and we also want to introduce the adjoint A^* of A and its domain $D(A^*)$, which are needed in the course of the proof.

We want to express AU in terms of $\boldsymbol{u}_n$, $\boldsymbol{v}_n$ and $\boldsymbol{\psi}_n$. This form is more convenient for the calculations. To this end, we simply introduce the normal mode expansions of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{\psi}$, (2.12), into (3.7). Note that, since $\boldsymbol{u} \in H_{\boldsymbol{u}}$ and $\boldsymbol{v} \in H_{\boldsymbol{v}}$, $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$ vanish. After working out the integrations, and grouping the coefficients of the eigenfunctions we obtain

$$AU = \begin{pmatrix} \displaystyle\sum_{n\geq 1}(\bar{U}_0\boldsymbol{u}_{nx} - \frac{1}{N\lambda_n}\boldsymbol{\psi}_{nx} - f\boldsymbol{v}_n)\mathcal{U}_n \\ \displaystyle\sum_{n\geq 1}(\bar{U}_0\boldsymbol{v}_{nx} + f\boldsymbol{u}_n - \frac{1}{N\lambda_n}\Lambda\boldsymbol{\psi}_n)\mathcal{U}_n \\ \displaystyle\sum_{n\geq 1}(-\frac{N}{\lambda_n}\boldsymbol{u}_{nx} + \bar{U}_0\boldsymbol{\psi}_{nx} - \frac{N}{\lambda_n}\Lambda\boldsymbol{v}_n)\mathcal{W}_n \end{pmatrix}. \tag{3.11}$$

We recall (see e.g. [16]) that, given an unbounded operator A from $D(A)$ into H, and an element $\widetilde{U} \in H$, $\widetilde{U}$ is said to be in $D(A^*)$ if and only if $U \longrightarrow (AU, \widetilde{U})$ is a linear continuous functional K on $D(A)$, continuous for the norm of H, in which case $A^*\widetilde{U} = K$. The determination of A^* introduces the following boundary conditions for $\widetilde{U}$:
For the the supercritical modes, i.e. $n > n_c$,

$$\begin{cases} \widetilde{\boldsymbol{u}}_n(0,t) = 0, \\ \widetilde{\boldsymbol{v}}_n(0,t) = 0, \\ \widetilde{\boldsymbol{\psi}}_n(0,t) = 0. \end{cases} \tag{3.12}$$

For the the subcritical modes, i.e. $1 \leq n \leq n_c$,

$$\begin{cases} \widetilde{\boldsymbol{u}}_n(L_1) - \dfrac{1}{N}\widetilde{\boldsymbol{\psi}}_n(L_1) = 0, \\ \widetilde{\boldsymbol{v}}_n(L_1) = 0, \\ \widetilde{\boldsymbol{u}}_n(0) + \dfrac{1}{N}\widetilde{\boldsymbol{\psi}}_n(0) = 0. \end{cases} \tag{3.13}$$

A simple analysis, which we skip, shows that the domain $D(A^*)$ of A^* is as follows:

$$\begin{aligned} D(A^*) = \{\, \widetilde{U} \in H \mid \widetilde{U}_x \in (\mathbf{L}^2(\mathcal{M}')^3, \\ \textit{and } \widetilde{U} \textit{verifies the BC's } (3.12) \textit{ and } (3.13)\,\}. \end{aligned} \tag{3.14}$$

Furthermore for $\widetilde{U} \in D(A^*)$,

$$A^*\widetilde{U} = \begin{pmatrix} \sum_{n\geq 1}(-\bar{U}_0\widetilde{\boldsymbol{u}}_{nx} + \frac{1}{N\lambda_n}\widetilde{\boldsymbol{\psi}}_{nx} + f\widetilde{\boldsymbol{v}}_n)\mathcal{U}_n \\ \sum_{n\geq 1}(-\bar{U}_0\widetilde{\boldsymbol{v}}_{nx} - f\widetilde{\boldsymbol{u}}_n - \frac{1}{N\lambda_n}\Lambda^T\widetilde{\boldsymbol{\psi}}_n)\mathcal{U}_n \\ \sum_{n\geq 1}(\frac{N}{\lambda_n}\widetilde{\boldsymbol{u}}_{nx} - \bar{U}_0\widetilde{\boldsymbol{\psi}}_{nx} - \frac{N}{\lambda_n}\Lambda^T\widetilde{\boldsymbol{v}}_n)\mathcal{W}_n \end{pmatrix}. \tag{3.15}$$

The coefficient matrix Λ in (3.15) is the same as defined in (2.11), and Λ^T is its transpose.

The proof of Theorem 3.2 essentially consists of the verification of the hypotheses of Theorem 3.1. We will do it in the Lemmas 3.1-3.5. We will then summarize the proof of Theorem 3.2 at the end of this section.

Lemma 3.1. *There exists $\delta > 0$ such that $A + \delta I \geq 0$ and $A^* + \delta I \geq 0$, that is, $((A+\delta I)U, U)_H \geq 0$ for each $U \in D(A)$ and $((A^* + \delta I)\widetilde{U}, \widetilde{U})_H \geq 0$ for each $\widetilde{U} \in D(A^*)$.*

Proof. Let $U \in D(A)$. Then U has the normal mode expansion

$$U = \Big(\sum_{n\geq 1}\boldsymbol{u}_n\mathcal{U}_n, \sum_{n\geq 1}\boldsymbol{v}_n\mathcal{U}_n, \sum_{n\geq 1}\boldsymbol{\psi}_n\mathcal{W}_n\Big).$$

Using the expression (3.11) for AU, we compute,

$$
\begin{aligned}
(AU,U)_H &= \int_0^{L_1}\int_{-L_3}^{0} AU\cdot U \,\mathrm{d}z\,\mathrm{d}x \\
&= \int_0^{L_1} \Big[\sum_{n\geq 1}(\bar{U}_0\boldsymbol{u}_{nx}\cdot\boldsymbol{u}_n - \frac{1}{N\lambda_n}\boldsymbol{\psi}_{nx}\cdot\boldsymbol{u}_n - f\boldsymbol{v}_n\cdot\boldsymbol{u}_n) \\
&\quad + \sum_{n\geq 1}(\bar{U}_0\boldsymbol{v}_{nx}\cdot\boldsymbol{v}_n + f\boldsymbol{u}_n\cdot\boldsymbol{v}_n - \frac{1}{N\lambda_n}\Lambda\boldsymbol{\psi}_n\cdot\boldsymbol{v}_n) \\
&\quad + \sum_{n\geq 1}(-\frac{1}{N\lambda_n}\boldsymbol{u}_{nx}\cdot\boldsymbol{\psi}_n + \frac{\bar{U}_0}{N^2}\boldsymbol{\psi}_{nx}\cdot\boldsymbol{\psi}_n - \frac{1}{N\lambda_n}\Lambda\boldsymbol{v}_n\cdot\boldsymbol{\psi}_n)\Big]\,\mathrm{d}x \\
&= \sum_{n\geq 1}\Big[\frac{\bar{U}_0}{2}{\boldsymbol{u}_n}^2(L_1) - \frac{\bar{U}_0}{2}{\boldsymbol{u}_n}^2(0) + \frac{\bar{U}_0}{2}{\boldsymbol{v}_n}^2(L_1) - \frac{\bar{U}_0}{2}{\boldsymbol{v}_n}^2(0) \\
&\quad + \frac{\bar{U}_0}{2N^2}{\boldsymbol{\psi}_n}^2(L_1) - \frac{\bar{U}_0}{2N^2}{\boldsymbol{\psi}_n}^2(0) \\
&\quad - \frac{1}{N\lambda_n}\boldsymbol{u}_n(L_1)\cdot\boldsymbol{\psi}_n(L_1) + \frac{1}{N\lambda_n}\boldsymbol{u}_n(0)\cdot\boldsymbol{\psi}_n(0) \\
&\quad - \frac{1}{N\lambda_n}\int_0^{L_1}(\Lambda\boldsymbol{\psi}_n\cdot\boldsymbol{v}_n + \Lambda\boldsymbol{v}_n\cdot\boldsymbol{\psi}_n)\,\mathrm{d}x\Big]
\end{aligned}
$$

We now separate the supercritical and the subcritical modes, and drop those terms that vanish according to the boundary conditions (see (2.20), (2.21)). There remains:

$$
\begin{aligned}
(AU,U) = &\sum_{n>n_c}\Big[\frac{\bar{U}_0}{2}{\boldsymbol{u}_n}^2(L_1) + \frac{\bar{U}_0}{2}{\boldsymbol{v}_n}^2(L_1) + \frac{\bar{U}_0}{2N^2}{\boldsymbol{\psi}_n}^2(L_1) \\
&- \frac{1}{N\lambda_n}\boldsymbol{u}_n(L_1)\cdot\boldsymbol{\psi}_n(L_1)\Big] + \\
&\sum_{1\leq n\leq n_c}\Big[\frac{\bar{U}_0}{2}{\boldsymbol{u}_n}^2(L_1) - \frac{\bar{U}_0}{2}{\boldsymbol{u}_n}^2(0) + \frac{\bar{U}_0}{2}{\boldsymbol{v}_n}^2(L_1) \\
&+ \frac{\bar{U}_0}{2N^2}{\boldsymbol{\psi}_n}^2(L_1) - \frac{\bar{U}_0}{2N^2}{\boldsymbol{\psi}_n}^2(0) \\
&- \frac{1}{N\lambda_n}\boldsymbol{u}_n(L_1)\cdot\boldsymbol{\psi}_n(L_1) + \frac{1}{N\lambda_n}\boldsymbol{u}_n(0)\cdot\boldsymbol{\psi}_n(0)\Big] + \\
&\sum_{n\geq 1}\frac{1}{N\lambda_n}\int_0^{L_1}(-\Lambda\boldsymbol{\psi}_n\cdot\boldsymbol{v}_n - \Lambda\boldsymbol{v}_n\cdot\boldsymbol{\psi}_n)\,\mathrm{d}x.
\end{aligned}
$$

We then write

$$(AU,\, U) = \sum_{n>n_c} \mathrm{I}_n + \sum_{1\le n\le n_c} \mathrm{II}_n + \mathrm{III},$$

where

$$\begin{aligned}
\mathrm{I}_n =& \frac{\bar U_0}{2}\boldsymbol{u}_n{}^2(L_1) + \frac{\bar U_0}{2}\boldsymbol{v}_n{}^2(L_1) + \frac{\bar U_0}{2N^2}\boldsymbol{\psi}_n{}^2(L_1)\\
&- \frac{1}{N\lambda_n}\boldsymbol{u}_n(L_1)\cdot\boldsymbol{\psi}_n(L_1),\\
\mathrm{II}_n =& \frac{\bar U_0}{2}\boldsymbol{u}_n{}^2(L_1) - \frac{\bar U_0}{2}\boldsymbol{u}_n{}^2(0) + \frac{\bar U_0}{2}\boldsymbol{v}_n{}^2(L_1) + \frac{\bar U_0}{2N^2}\boldsymbol{\psi}_n{}^2(L_1)\\
&- \frac{\bar U_0}{2N^2}\boldsymbol{\psi}_n{}^2(0) - \frac{1}{N\lambda_n}\boldsymbol{u}_n(L_1)\cdot\boldsymbol{\psi}_n(L_1) + \frac{1}{N\lambda_n}\boldsymbol{u}_n(0)\cdot\boldsymbol{\psi}_n(0),\\
\mathrm{III} =& \sum_{n\ge 1}\frac{1}{N\lambda_n}\int_0^{L_1}(-\Lambda\boldsymbol{\psi}_n\cdot\boldsymbol{v}_n - \Lambda\boldsymbol{v}_n\cdot\boldsymbol{\psi}_n)\,\mathrm{d}x.
\end{aligned}$$

We see that I_n is the sum of a quadratic form and the positive term $\bar U_0\boldsymbol{v}_n^2(L_1)/2$. We find the determinant for the quadratic form to be $1/(N^2\lambda_n^2) - \bar U_0^2/N^2 = -(\bar U_0^2 - 1/\lambda_n^2)/N^2$, which is < 0, thanks to the fact that $\bar U_0 - 1/\lambda_n > 0$ for each supercritical mode. Hence we have

$$\mathrm{I}_n \ge 0, \qquad \text{for each } n > n_c.$$

Using the boundary conditions (see (2.21)) for the subcritical modes we find that

$$\begin{aligned}
\mathrm{II}_n =& \frac{\bar U_0}{2}\boldsymbol{v}_n^2(L_1) + (\bar U_0 + \frac{1}{\lambda_n})\boldsymbol{u}_n^2(L_1) + (\frac{1}{\lambda_n} - \bar U_0)\boldsymbol{u}_n^2(0)\\
\ge& \,0, \qquad \text{for each } 1\le n\le n_c.
\end{aligned}$$

The last inequality is due to the fact that $\bar U_0 - 1/\lambda_n < 0$ for each subcritical mode. We also find an upper bound on the absolute value of III:

$$\begin{aligned}
|\mathrm{III}| \le& \sum_{n\ge 1}\frac{1}{N\lambda_n}\int_0^{L_1}|(\Lambda\boldsymbol{\psi}_n\cdot\boldsymbol{v}_n + \Lambda\boldsymbol{v}_n\cdot\boldsymbol{\psi}_n)|\,\mathrm{d}x\\
\le& \sum_{n\ge 1}\frac{C}{\lambda_n}\left(\int_0^{L_1}|\frac{\boldsymbol{\psi}_n}{N}|^2\,\mathrm{d}x\right)^{\frac12}\left(\int_0^{L_1}|\boldsymbol{v}_n|^2\,\mathrm{d}x\right)^{\frac12}\\
\le& \frac{C}{2\lambda_1}\sum_{n\ge 1}\left(\int_0^{L_1}|\frac{\boldsymbol{\psi}_n}{N}|^2\,\mathrm{d}x + \int_0^{L_1}|\boldsymbol{v}_n|^2\,\mathrm{d}x\right),
\end{aligned}$$

where C is the norm of the matrix Λ. Hence, $\delta \geq C/2\lambda_n$ for every $n \geq 1$. Then

$$|\mathrm{III}| \leq \frac{C}{2\lambda_1}|U|_H^2,$$

and for any $\delta > \frac{C}{2\lambda_1}$ the operator $A + \delta I$ is positive.

That $A^* + \delta I \geq 0$ (possibly with a different δ) can be shown in a similar way. And we can always choose a constant δ such that the operators $A + \delta I$ and $A^* + \delta I$ are both positive. This completes the proof of Lemma 3.1.

□

Lemma 3.2. *$D(A)$ is dense in H, and A is a closed operator.*

Proof. To show that $D(A)$ is dense in H, consider $U = (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\psi}) \in H$. Since $(\boldsymbol{\mathcal{D}}(\mathcal{M}'))^3$ is dense in $\mathbf{L}^2(\mathcal{M}')$, U can be approximated in $\mathbf{L}^2(\mathcal{M}')$ by elements of $(\boldsymbol{\mathcal{D}}(\mathcal{M}'))^3$, say $\Phi^j = (\Phi^j_{\boldsymbol{u}}, \Phi^j_{\boldsymbol{v}}, \Phi^j_{\boldsymbol{\psi}})$, $1 \leq j \leq \infty$. Since P is continuous in $(\mathbf{L}^2(\mathcal{M}'))^3$, $(P\Phi^j_{\boldsymbol{u}}, P\Phi^j_{\boldsymbol{v}}, \Phi^j_{\boldsymbol{\psi}})$ also converge to $PU = U$ in H as $j \longrightarrow \infty$. The function Φ^j are compactly supported in $\mathcal{M}'$, and, by the definition of P, the functions $P\Phi^j$ are also compactly supported in $\mathcal{M}'$. The necessary boundary conditions are satisfied and it is then clear that $(P\Phi^j_{\boldsymbol{u}}, P\Phi^j_{\boldsymbol{v}}, \Phi^j_{\boldsymbol{\psi}})$ belong to $D(A)$.

To show that A is closed, we need to show that for a sequence $\{U^j\}_{j=1}^{\infty}$ in $D(A)$, such that

$$U^j \longrightarrow U \quad \text{in } H, \tag{3.16}$$

$$AU^j \longrightarrow F \quad \text{in } H, \tag{3.17}$$

with U, $F \in H$, then $U \in D(A)$ and $F = AU$. For each component of U^j, U and F, (3.16) and (3.17) mean

$$\begin{cases} \boldsymbol{u}^j \longrightarrow \boldsymbol{u} & \text{in } H_{\boldsymbol{u}}, \\ \boldsymbol{v}^j \longrightarrow \boldsymbol{v} & \text{in } H_{\boldsymbol{v}}, \\ \boldsymbol{\psi}^j \longrightarrow \boldsymbol{\psi} & \text{in } H_{\boldsymbol{\psi}}, \end{cases} \tag{3.18}$$

and

$$\begin{cases} \bar{U}_0 \boldsymbol{u}^j_x - f\boldsymbol{v}^j - P[\int_z^0 \boldsymbol{\psi}^j_x(x, z')\,\mathrm{d}z'] \longrightarrow F_{\boldsymbol{u}} & \text{in } H_{\boldsymbol{u}}, \\ \bar{U}_0 \boldsymbol{v}^j_x + f\boldsymbol{u}^j - P[\int_z^0 \Lambda\boldsymbol{\psi}^j(x, z')\,\mathrm{d}z'] \longrightarrow F_{\boldsymbol{v}} & \text{in } H_{\boldsymbol{v}}, \\ \bar{U}_0 \boldsymbol{\psi}^j_x + N^2 \int_z^0 (\boldsymbol{u}^j_x + \Lambda\boldsymbol{v}^j)\,\mathrm{d}z' \longrightarrow F_{\boldsymbol{\psi}} & \text{in } H_{\boldsymbol{\psi}}. \end{cases} \tag{3.19}$$

With regard to each mode, (3.18) implies that

$$\begin{cases} \boldsymbol{u}_n^j \longrightarrow \boldsymbol{u}_n & \text{in } \mathbf{L}^2(0, L_1), \\ \boldsymbol{v}_n^j \longrightarrow \boldsymbol{v}_n & \text{in } \mathbf{L}^2(0, L_1), \\ \boldsymbol{\psi}_n^j \longrightarrow \boldsymbol{\psi}_n & \text{in } \mathbf{L}^2(0, L_1). \end{cases} \tag{3.20}$$

(3.18) also implies that $\sum_{n\geq 1} |\boldsymbol{u}_n^j|^2_{L^2(0,L_1)}$, $\sum_{n\geq 1} |\boldsymbol{v}_n^j|^2_{L^2(0,L_1)}$, and $\sum_{n\geq 1} |\boldsymbol{\psi}_n^j|^2_{L^2(0,L_1)}$ are uniformly bounded in j, by a bound of the $|U^n|_H^2$. Similarly, (3.19) implies that

$$\begin{cases} \bar{U}_0 \boldsymbol{u}_{nx}^j - \dfrac{1}{N\lambda_n} \boldsymbol{\psi}_{nx}^j - f \boldsymbol{v}_n^j \equiv F_{\boldsymbol{u},n}^j \longrightarrow F_{\boldsymbol{u},n} & \text{in } \mathbf{L}^2(0, L_1), \\ \bar{U}_0 \boldsymbol{v}_{nx}^j + f \boldsymbol{u}_n^j - \dfrac{1}{N\lambda_n} \Lambda \boldsymbol{\psi}_n^j \equiv F_{\boldsymbol{v},n}^j \longrightarrow F_{\boldsymbol{v},n} & \text{in } \mathbf{L}^2(0, L_1), \\ \dfrac{N}{\lambda_n} \boldsymbol{u}_{nx}^j + \bar{U}_0 \boldsymbol{\psi}_{nx}^j - \dfrac{N}{\lambda_n} \Lambda \boldsymbol{v}_n^j \equiv F_{\boldsymbol{\psi},n}^j \longrightarrow F_{\boldsymbol{\psi},n} & \text{in } \mathbf{L}^2(0, L_1), \end{cases} \tag{3.21}$$

and that $\sum_{n\geq 1} |F_{\boldsymbol{u},n}^j|^2_{L^2(0,L_1)}$, $\sum_{n\geq 1} |F_{\boldsymbol{v},n}^j|^2_{L^2(0,L_1)}$, and $\sum_{n\geq 1} |F_{\boldsymbol{\psi},n}^j|^2_{L^2(0,L_1)}$ are uniformly bounded in j, by a bound of the $|AU^n|_H^2$. By the second convergence in (3.21) we have

$$\boldsymbol{v}_{nx}^j = \frac{1}{\bar{U}_0} \left(-f \boldsymbol{u}_n^j + \frac{1}{N\lambda_n} \Lambda \boldsymbol{\psi}_n^j + F_{\boldsymbol{v},n}^j \right). \tag{3.22}$$

Each term on the right hand side of (3.22) converges in $\mathbf{L}^2(0, L_1)$, and therefore $\boldsymbol{v}_{nx}^j$ also converges in $\mathbf{L}^2(0, L_1)$. In addition, since on the right hand side of (3.22), $\sum_{n\geq 1} |\boldsymbol{u}_n^j|^2_{L^2(0,L_1)}$, $\sum_{n\geq 1} |\boldsymbol{\psi}_n^j|^2_{L^2(0,L_1)}$, and $\sum_{n\geq 1} |F_{\boldsymbol{v},n}^j|^2_{L^2(0,L_1)}$ are all uniformly bounded in j, $\sum_{n\geq 1} |\boldsymbol{v}_{nx}^j|^2_{L^2(0,L_1)}$ is also uniformly bounded in j. These two facts imply that $\boldsymbol{v}_x^j$ converges in $\mathbf{L}^2(M')$. Combining this result with (3.18), we conclude that $\boldsymbol{v}_x$ belongs to $\mathbf{L}^2(\mathcal{M}')$, and

$$\boldsymbol{v}_x^j \longrightarrow \boldsymbol{v}_x \quad \text{in } \mathbf{L}^2(\mathcal{M}'). \tag{3.23}$$

By the first and third convergences in (3.21) we obtain

$$\boldsymbol{u}_{nx}^j = \frac{1}{\bar{U}_0^2 - 1/\lambda_n^2} \left[(\bar{U}_0 + \frac{1}{\lambda_n^2} \Lambda) \boldsymbol{v}_n^j + \bar{U}_0 F_{\boldsymbol{u},n}^j + \frac{1}{N\lambda_n} F_{\boldsymbol{\psi},n}^j \right], \tag{3.24}$$

$$\boldsymbol{\psi}_{nx}^j = \frac{1}{\bar{U}_0^2 - 1/\lambda_n^2} \left[\frac{N}{\lambda_n} (f + \bar{U}_0 \Lambda) \boldsymbol{v}_n^j + \frac{N}{\lambda_n} F_{\boldsymbol{u},n}^j + \bar{U}_0 F_{\boldsymbol{\psi},n}^j \right]. \tag{3.25}$$

Following the similar idea as for $\boldsymbol{v}_x$, we can show that $\boldsymbol{u}_x$ and $\boldsymbol{\psi}_x$ belong to $\mathbf{L}^2(\mathcal{M}')$, and

$$\boldsymbol{u}_x^j \longrightarrow \boldsymbol{u}_x \quad \text{in } \mathbf{L}^2(\mathcal{M}'), \tag{3.26}$$

$$\boldsymbol{\psi}_x^j \longrightarrow \boldsymbol{\psi}_x \quad \text{in } \mathbf{L}^2(\mathcal{M}'). \tag{3.27}$$

To finish the proof it remains to check that $U \in D(A)$ and $AU = F$. It is implied in the argument above that for each mode,

$$\begin{cases} \boldsymbol{u}_n^j \longrightarrow \boldsymbol{u}_n \quad \text{in } \mathbf{H}^1(0, L_1), \\ \boldsymbol{v}_n^j \longrightarrow \boldsymbol{v}_n \quad \text{in } \mathbf{H}^1(0, L_1), \\ \boldsymbol{\psi}_n^j \longrightarrow \boldsymbol{\psi}_n \quad \text{in } \mathbf{H}^1(0, L_1). \end{cases} \tag{3.28}$$

By the Sobolev embedding theorem, the convergences also hold in the space $\boldsymbol{C}([0, L_1])$. And so the boundary conditions pass to the limit. Hence $U \in D(A)$. We infer from (3.18), (3.23), (3.26), and (3.27) that $AU^j \longrightarrow AU$ in H. By (3.17), we have $AU = F$. This completes the proof of Lemma 3.2.

□

Lemma 3.3. *A^* is a closed operator.*

Proof. This is a consequence of Lemma 3.2 and of the following lemma.

□

Lemma 3.4. *If T is a densely defined operator in H, then T^* is a closed operator.*

The proof of Lemma 3.4 can be found in many functional analysis books, see, e.g., [16].

We now state and prove a simple lemma which we did not find available in the literature.

Lemma 3.5. *Let A and A^* be linear unbounded operators in H with domains $D(A)$ and $D(A^*)$ respectively, and let A^* be the adjoint operator of A (as an unbounded operator). It is also assumed that both $D(A)$ and $D(A^*)$ are dense in H. If furthermore A and A^* are both positive and closed, then $A + \mu_0 I$ and $A^* + \mu_0 I$ are onto for every $\mu_0 > 0$.*

Proof. Consider $\epsilon > 0$, which will eventually converge to zero. For each value of ϵ, we construct a bilinear form b_ϵ on $D(A)$:

$$b_\epsilon(U, \widetilde{U}) = \epsilon(AU, A\widetilde{U})_H + (AU, \widetilde{U})_H + \mu_0(U, \widetilde{U})_H. \tag{3.29}$$

It is easy to check that b_ϵ is bilinear, bounded and coercive on $D(A)$. Then, by the Lax-Milgram theorem, for any given $F \in H$, there exists a unique $U_\epsilon \in D(A)$, such that

$$\epsilon(AU_\epsilon, A\widetilde{U})_H + (AU_\epsilon, \widetilde{U})_H + \mu_0(U_\epsilon, \widetilde{U})_H = (F, \widetilde{U})_H \tag{3.30}$$

holds for any $\widetilde{U} \in D(A)$. For each ϵ, we observe that

$$\widetilde{U} \longrightarrow (AU_\epsilon, A\widetilde{U}) = \frac{1}{\epsilon}(F - \mu_0 U_\epsilon - AU_\epsilon, \widetilde{U}) \tag{3.31}$$

is a linear functional on $D(A)$ continuous for the norm of H. By the definition of the domain $D(A^*)$ of A^* (see e.g. [16]), this means that

$$AU_\epsilon \in D(A^*), \tag{3.32}$$

and

$$A^*AU_\epsilon = \frac{1}{\epsilon}(F - \mu_0 U_\epsilon - AU_\epsilon) \text{ in } H. \tag{3.33}$$

We then write (3.33) as

$$\epsilon A^*AU_\epsilon + AU_\epsilon + \mu_0 U_\epsilon = F. \tag{3.34}$$

Multiplying (3.34) by U_ϵ, we obtain

$$\epsilon(AU_\epsilon, AU_\epsilon)_H + (AU_\epsilon, U_\epsilon)_H + \mu_0(U_\epsilon, U_\epsilon)_H = (F, U_\epsilon)_H. \tag{3.35}$$

Since $(AU_\epsilon, AU_\epsilon)_H \geq 0$, and $(AU_\epsilon, U_\epsilon)_H \geq 0$ by the assumption that A is positive, we then have

$$|U_\epsilon| \leq C|F|_H, \tag{3.36}$$

where C is a constant independent of ϵ. Therefore there exists a subsequence $\epsilon' \longrightarrow 0$ such that

$$U_{\epsilon'} \rightharpoonup U \quad \text{weakly in } H, \tag{3.37}$$

for some $U \in H$. Multiplying (3.34) by AU_ϵ, we also obtain

$$\epsilon(A^*AU_\epsilon, AU_\epsilon)_H + (AU_\epsilon, AU_\epsilon)_H + \mu_0(AU_\epsilon, U_\epsilon)_H = (F, AU_\epsilon)_H. \tag{3.38}$$

Since $(A^*AU_\epsilon, AU_\epsilon)_H \geq 0$, and $(AU_\epsilon, U_\epsilon)_H \geq 0$, we have

$$|AU_\epsilon|_H \leq |F|_H. \tag{3.39}$$

This implies that there exists a subsequence, still denoted ϵ', such that

$$AU_{\epsilon'} \rightharpoonup \chi \quad \text{weakly in } H, \tag{3.40}$$

for some $\chi \in H$. By the assumption that the operator A is closed, and by (3.37) and (3.40), we see that

$$U \in D(A) \quad \text{and} \quad \chi = AU. \tag{3.41}$$

We find from (3.34) that

$$A^*(\epsilon AU_\epsilon) = F - AU_\epsilon - \mu_0 U_\epsilon. \tag{3.42}$$

Since both AU_ϵ and U_ϵ converge weakly in H,

$$A^*(\epsilon AU_\epsilon) \rightharpoonup \sigma = F - AU - \mu_0 U \quad \text{weakly in } H. \tag{3.43}$$

And since $|AU_\epsilon|_H$ is bounded independently of ϵ, we find that $\sigma = 0$, that is,

$$(A + \mu_0 I)U = F. \tag{3.44}$$

Thus the claim that $A + \mu_0 I$ is onto for any $\mu_0 > 0$ is proved. That $A^* + \mu_0 I$ is onto for any $\mu_0 > 0$ can be proved in a similar way.

□

Proof of Theorem 3.2. Let

$$U = e^{\delta t} U^b, \tag{3.45}$$

where δ is the positive constant chosen in the proof of Lemma 3.1. Inserting (3.45) into (3.1), we obtain an initial value problem for U^b:

$$\begin{cases} \dfrac{\mathrm{d}U^b}{\mathrm{d}t} + (A+\delta)U^b = \widetilde{F}, \\ U^b(0) = U_0. \end{cases} \tag{3.46}$$

If we can show that the system (3.46) is well-posed, and U^b satisfies (3.10), then, by the relation (3.45), U satisfies (3.10) too, and of course (3.1) is well-posed. We have in fact verified the hypotheses of Theorem 3.1 in Lemmas 3.1-3.5 for the operator $A + \delta I$ of Theorem 3.2. Now we readily apply Theorem 3.1 and complete the proof of Theorem 3.2. □

3.4. **The treatment of the zero mode ($n = 0$).** We now introduce (propose) the boundary conditions for the zero mode, which is important because it contains much energy. What follows is valid whether the boundary conditions are homogeneous or not for the modes $n \geq 1$.

The following technical point has no mathematical relevance, especially for the linearized equations for which the modes are decoupled; it has however a computational and physical importance, in particular in the nonlinear case when all the modes are coupled: the function $\boldsymbol{\phi}_0$ will be decomposed in the sum

$$\boldsymbol{\phi}_0 = \bar{\boldsymbol{\phi}}_0 + \boldsymbol{\phi}_0', \tag{3.47}$$

where $\bar{\boldsymbol{\phi}}_0$, which is not unique, is one of the constant solutions[3] of

$$\Lambda \bar{\boldsymbol{\phi}}_0 = -\boldsymbol{f}_0, \tag{3.48}$$

with Λ and $\boldsymbol{f}_0$ defined in (2.11) and (2.16) respectively, while $\boldsymbol{\phi}_0'$ needs to be determined.[4]

Then for $\boldsymbol{u}_0$, $\boldsymbol{v}_0$ and $\boldsymbol{\phi}_0'$ we propose the following boundary conditions:

$$\begin{cases} \boldsymbol{u}_0(0,t) = \boldsymbol{u}_l(t), \\ \boldsymbol{v}_0(0,t) = \boldsymbol{v}_l(t), \\ \boldsymbol{\phi}_0'(0,t) = \boldsymbol{\phi}_l'(t). \end{cases} \tag{3.49}$$

[3]$\mathrm{Det}\Lambda = 0$, and Λ is of rank 2.

[4]Essentially Λ is the mathematical representation of $\partial/\partial y$, and $\bar{\boldsymbol{\phi}}_0$ is the part of the basic geostrophic flow alluded to in (1.1).

Of course the third equation in (3.49) is the same as $\boldsymbol{\phi}_0(0,t) = \boldsymbol{\phi}_l(t) = \bar{\boldsymbol{\phi}}_0 + \boldsymbol{\phi}'_l(t)$. We can obtain $\boldsymbol{v}_{0x}(0,t)$ from $(2.15)_2$, that is

$$\boldsymbol{v}_{0x}(0,t) = -\frac{1}{\bar{U}_0}(\boldsymbol{v}_{lt} + f\boldsymbol{u}_l + \Lambda\boldsymbol{\phi}'_l). \tag{3.50}$$

We then multiply $(2.15)_1$ by Λ,

$$(\Lambda\boldsymbol{u}_0)_t + \bar{U}_0(\Lambda\boldsymbol{u}_0)_x + \Lambda\boldsymbol{\phi}'_{0x} - f\Lambda\boldsymbol{v}_0 = 0, \tag{3.51}$$

and differentiate $(2.15)_2$ with respect to x,

$$(\boldsymbol{v}_{0x})_t + \bar{U}_0\boldsymbol{v}_{0xx} + f\boldsymbol{u}_{0x} + \Lambda\boldsymbol{\phi}'_{0x} = 0. \tag{3.52}$$

By subtracting (3.52) from (3.51) we obtain, thanks to equation $(2.15)_3$,

$$(\Lambda\boldsymbol{u}_0 - \boldsymbol{v}_{0x})_t + \bar{U}_0(\Lambda\boldsymbol{u}_0 - \boldsymbol{v}_{0x})_x = 0. \tag{3.53}$$

The value of $\Lambda\boldsymbol{u}_0 - \boldsymbol{v}_{0x}$ at $x = 0$ is known, and therefore we can solve the equation above for $\Lambda\boldsymbol{u}_0 - \boldsymbol{v}_{0x}$. Once we have found $\Lambda\boldsymbol{u}_0 - \boldsymbol{v}_{0x}$, say $\Lambda\boldsymbol{u}_0 - \boldsymbol{v}_{0x} = k(x,t)$, then with $(2.15)_3$ we have

$$\begin{cases} \boldsymbol{u}_{0x} + \Lambda\boldsymbol{v}_0 = 0, \\ \boldsymbol{v}_{0x} - \Lambda\boldsymbol{u}_0 = -k(x,t). \end{cases} \tag{3.54}$$

We can solve this linear ODE system with the boundary conditions for $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$ at $x = 0$, which are given.

We now have $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$; we can solve $(2.15)_1$ for $\boldsymbol{\phi}_0 = \bar{\boldsymbol{\phi}}_0 + \boldsymbol{\phi}'_0$, since the boundary condition for $\boldsymbol{\phi}_0/\boldsymbol{\phi_0}'$ is also given at $x = 0$.

We leave as an exercise to the reader to find the suitable regularity assumptions for the data $\boldsymbol{u}_l$, $\boldsymbol{v}_l$ and $\boldsymbol{\phi}'_l$.

3.5. **The case of nonhomogeneous boundary conditions.** In practical simulations we want to be able to solve the Primitive Equations (2.10) with *nonhomogeneous* boundary conditions at $x = 0$ and L_1. We write (2.10) in a form similar to (3.1) corresponding to the elimination of $\boldsymbol{w}$ and $\boldsymbol{\phi}$ and the exclusion of the zero mode:

$$\begin{cases} \dfrac{\partial U}{\partial t} + \mathcal{A}U = F, \\ U(t=0) = U_0. \end{cases} \tag{3.55}$$

Here $U = (\boldsymbol{u},\, \boldsymbol{v},\, \boldsymbol{\psi})$ and $F = (F_{\boldsymbol{u}},\, F_{\boldsymbol{v}},\, F_{\boldsymbol{\psi}})$ like before, and $\mathcal{A}$ is the differential operator represented by the right-hand side of (3.7) or (3.11). The proposed boundary conditions for U at $x = 0$ and L_1 will be derived from given functions $U^{g,l}(z,t)$ and $U^{g.r}(z,t)$. In this subsection we will demonstrate how to derive from $U^{g,l}$ and $U^{g,r}$ the boundary conditions for U so that the initial boundary value problem corresponding to (3.55) is well-posed.

As pointed out in Section 2.3, the subcritical and the supercritical modes require different boundary conditions. ¿From the physical and computational points of view, we can assume that all the components of $U^{g,l}$ and $U^{g,r}$ are available. The mathematical issue is then to determine which components are needed for each mode. The normal mode expansions for $U^{g,l}$ and $U^{g,r}$ are written:

$$\begin{cases} U^{g,l}(z,t) = \Big(\sum_{n\geq 0} \boldsymbol{u}_n^{g,l}(t)\mathcal{U}_n(z), \sum_{n\geq 0} \boldsymbol{v}_n^{g,l}(t)\mathcal{U}_n(z), \sum_{n\geq 1} \boldsymbol{\psi}_n^{g,l}(t)\mathcal{W}_n(z) \Big), \\ U^{g,r}(z,t) = \Big(\sum_{n\geq 0} \boldsymbol{u}_n^{g,r}(t)\mathcal{U}_n(z), \sum_{n\geq 0} \boldsymbol{v}_n^{g,r}(t)\mathcal{U}_n(z), \sum_{n\geq 1} \boldsymbol{\psi}_n^{g,r}(t)\mathcal{W}_n(z) \Big). \end{cases} \tag{3.56}$$

¿From $U^{g,l}$ and $U^{g,r}$ we construct $U^g = U^g(z,t)$,

$$U^g(z,t) = \Big(\sum_{n\geq 1} \boldsymbol{u}_n^{g}(t)\mathcal{U}_n(z), \sum_{n\geq 1} \boldsymbol{v}_n^{g}(t)\mathcal{U}_n(z), \sum_{n\geq 1} \boldsymbol{\psi}_n^{g}(t)\mathcal{W}_n(z) \Big), \tag{3.57}$$

where, for each $n \geq 1$, $\boldsymbol{u}_n^g$, $\boldsymbol{v}_n^g$ and $\boldsymbol{\psi}_n^g$ are determined by the following equations:

$$\begin{cases} \boldsymbol{u}_n^g(t) - \dfrac{1}{N}\boldsymbol{\psi}_n^g(t) = \boldsymbol{u}_n^{g,l}(t) - \dfrac{1}{N}\boldsymbol{\psi}_n^{g,l}(t), \\ \boldsymbol{v}_n^g(t) = \boldsymbol{v}_n^{g,l}(t), \qquad\qquad\qquad\qquad \text{if } 1 \leq n \leq n_c, \\ \boldsymbol{u}_n^g(t) + \dfrac{1}{N}\boldsymbol{\psi}_n^g(t) = \boldsymbol{u}_n^{g,r}(t) + \dfrac{1}{N}\boldsymbol{\psi}_n^{g,r}(t), \end{cases} \tag{3.58}$$

that is

$$\begin{cases} \boldsymbol{u}_n^g(t) = \dfrac{1}{2}(\boldsymbol{u}_n^{g,l}(t) + \boldsymbol{u}_n^{g,r}(t)) + \dfrac{1}{2N}(\boldsymbol{\psi}_n^{g,r}(t) - \boldsymbol{\psi}_n^{g,l}(t)), \\ \boldsymbol{v}_n^g(t) = \boldsymbol{v}_n^{g,l}(t), \qquad\qquad\qquad\qquad \text{if } 1 \leq n \leq n_c, \\ \boldsymbol{\psi}_n^g(t) = \dfrac{N}{2}(\boldsymbol{u}_n^{g,r}(t) - \boldsymbol{u}_n^{g,l})(t) + \dfrac{1}{2}(\boldsymbol{\psi}_n^{g,l}(t) + \boldsymbol{\psi}_n^{g,r}(t)), \end{cases} \tag{3.59}$$

and

$$\begin{cases} \boldsymbol{u}_n^g(t) = \boldsymbol{u}_n^{g,l}(t), \\ \boldsymbol{v}_n^g(t) = \boldsymbol{v}_n^{g,l}(t), \qquad \text{if } n > n_c, \\ \boldsymbol{\psi}_n^g(t) = \boldsymbol{\psi}_n^{g,l}(t). \end{cases} \tag{3.60}$$

We observe here that U^g is independent of x, i.e., $\partial U^g/\partial x = 0$, and that, for a.e. $t \in (0,T)$, $U^g \in H$ provided that $U^{g,l}$ and $U^{g,r}$ are smooth enough: indeed $U^g \in \mathbf{L}^2(\mathcal{M}')^3$, and the integral conditions appearing in (3.2) are automatically satisfied since the mode 0 is not present here ($n \geq 1$).

Then we set

$$U = U^{\#} + U^{g}. \tag{3.61}$$

We observe that $U^{\#} \in D(A)$ (if $U^{\#}$ is smooth enough). Then setting $F^{\#} = F - \partial U^{g}/\partial t - \mathcal{A}U^{g}$ and $U_0^{\#} = U_0 - U^{g}|_{t=0}$, we see that $U^{\#}$ is the solution of the following problem:

$$\begin{cases} \dfrac{\mathrm{d}U^{\#}}{\mathrm{d}t} + AU^{\#} = F^{\#}, \\ U^{\#}(t=0) = U_0^{\#}. \end{cases} \tag{3.62}$$

Like (3.1), (3.62) corresponds to the case with homogeneous boundary conditions. In order to apply Theorem 3.2 to (3.62) we would need to have

$$U_0^{\#} = U_0 - U^{g}|_{t=0} \in D(A), \tag{3.63}$$

and

$$F^{\#}, \frac{\mathrm{d}F^{\#}}{\mathrm{d}t} \in L^1(0,T;H). \tag{3.64}$$

It is easily shown that (3.63) and (3.64) are satisfied if the following hypotheses are verified (up to (3.69), and see also (3.73)):

$$U_0 \in H, \quad \frac{\partial U_0}{\partial x} \in \mathbf{L}^2(\mathcal{M}')^3, \tag{3.65}$$

$$F \in L^1(0,T;H), \quad \frac{\partial F}{\partial t} \in L^1(0,T;H), \tag{3.66}$$

$$\frac{\partial^k U^{g,l}}{\partial t^k}, \frac{\partial^k U^{g,r}}{\partial t^k} \in L^1(0,T;\mathbf{L}^2(-L_3,0)^3) \quad \text{for } k = 0,1,2. \tag{3.67}$$

In addition we require that U_0, $U^{g,l}$ and $U^{g,r}$ satisfy certain compatibility conditions. Denoting the function U_0 of initial values by $(\widetilde{\boldsymbol{u}}_0, \widetilde{\boldsymbol{v}}_0, \widetilde{\boldsymbol{\psi}}_0)$, [5] the compatibility conditions for U_0, $U^{g,l}$ and $U^{g,r}$ are

[5] The tildes here on $\widetilde{\boldsymbol{u}}_0$, $\widetilde{\boldsymbol{v}}_0$ and $\widetilde{\boldsymbol{\psi}}_0$ are meant to distinguish these initial datas from the zero modes of $U(t)$, which do not appear in fact in this subsection.

written:

$$\begin{cases} \displaystyle\int_{-L_3}^{0} \widetilde{\boldsymbol{u}}_0(x=0,z)\mathcal{U}_n(z) - \frac{1}{N}\int_{-L_3}^{0} \widetilde{\boldsymbol{\psi}}_0(x=0,z)\mathcal{W}_n(z)\,\mathrm{d}z \\ \qquad \displaystyle = \int_{-L_3}^{0} \boldsymbol{u}^{g,l}(z,t=0)\mathcal{U}_n(z) - \frac{1}{N}\int_{-L_3}^{0} \boldsymbol{\psi}^{g,l}(z,t=0)\mathcal{W}_n(z)\,\mathrm{d}z, \\ \displaystyle\int_{-L_3}^{0} \widetilde{\boldsymbol{v}}_0(x=0,z)\mathcal{U}_n(z)\,\mathrm{d}z = \int_{-L_3}^{0} \boldsymbol{v}^{g,l}(z,t=0)\mathcal{U}_n(z)\,\mathrm{d}z, \qquad \text{for } 1 \le n \le n_c, \\ \displaystyle\int_{-L_3}^{0} \widetilde{\boldsymbol{u}}_0(x=L_1,z)\mathcal{U}_n(z) + \frac{1}{N}\int_{-L_3}^{0} \widetilde{\boldsymbol{\psi}}_0(x=L_1,z)\mathcal{W}_n(z)\,\mathrm{d}z \\ \qquad \displaystyle = \int_{-L_3}^{0} \boldsymbol{u}^{g,r}(z,t=0)\mathcal{U}_n(z) + \frac{1}{N}\int_{-L_3}^{0} \boldsymbol{\psi}^{g,r}(z,t=0)\mathcal{W}_n(z)\,\mathrm{d}z, \end{cases} \tag{3.68}$$

and

$$\begin{cases} \displaystyle\int_{-L_3}^{0} \widetilde{\boldsymbol{u}}_0(x=0,z)\mathcal{U}_n(z)\,\mathrm{d}z = \int_{-L_3}^{0} \boldsymbol{u}^{g,l}(z,t=0)\mathcal{U}_n(z)\,\mathrm{d}z, \\ \displaystyle\int_{-L_3}^{0} \widetilde{\boldsymbol{v}}_0(x=0,z)\mathcal{U}_n(z)\,\mathrm{d}z = \int_{-L_3}^{0} \boldsymbol{v}^{g,l}(z,t=0)\mathcal{U}_n(z)\,\mathrm{d}z, \qquad \text{for } n > n_c, \\ \displaystyle\int_{-L_3}^{0} \widetilde{\boldsymbol{\psi}}_0(x=0,z)\mathcal{W}_n(z)\,\mathrm{d}z = \int_{-L_3}^{0} \boldsymbol{\psi}^{g,l}(z,t=0)\mathcal{W}_n(z)\,\mathrm{d}z. \end{cases} \tag{3.69}$$

It should be noted here that (3.65)-(3.69) are sufficient conditions for (3.63) and (3.64). They have been chosen for their relative simplicity; (3.68) and (3.69) guarantee that the boundary conditions required in (3.63) ($U_0^{\#} \in D(A)$) are satisfied.

Now we can apply Theorem 3.2 to the system (3.62), and we obtain a unique solution $U^{\#}$ that satisfies the analogue of (3.10). We then recover U via (3.61), and we easily see that

$$U \in \mathcal{C}([0,T];H), \tag{3.70}$$

$$\frac{\partial U}{\partial x} \in L^{\infty}(0,T;\mathbf{L}^2(\mathcal{M}')^3). \tag{3.71}$$

We will also have

$$\frac{\partial U}{\partial t} \in L^{\infty}(0,T;H), \tag{3.72}$$

provided we further require that

$$\frac{\partial U^{g,l}}{\partial t}, \frac{\partial U^{g,r}}{\partial t} \in L^{\infty}(0,T;\mathbf{L}^2(-L_3,0)^3). \tag{3.73}$$

The boundary conditions that U satisfies, expressing the fact that $U^{\#}(t) = U(t) - U^g(t)$ belongs to $D(A)$ for a.e. t, are as follows:
For the subcritical modes $1 \le n \le n_c$:

$$\left\{\begin{aligned}
&\int_{-L_3}^{0} \boldsymbol{u}(0,z,t)\mathcal{U}_n(z) - \frac{1}{N}\int_{-L_3}^{0} \boldsymbol{\psi}(0,z,t)\mathcal{W}_n(z)\,\mathrm{d}z \\
&\qquad = \int_{-L_3}^{0} \boldsymbol{u}^{g,l}(z,t)\mathcal{U}_n(z) - \frac{1}{N}\int_{-L_3}^{0} \boldsymbol{\psi}^{g,l}(z,t)\mathcal{W}_n(z)\,\mathrm{d}z, \\
&\int_{-L_3}^{0} \boldsymbol{v}(0,z,t)\mathcal{U}_n(z)\,\mathrm{d}z = \int_{-L_3}^{0} \boldsymbol{v}^{g,l}(z,t)\mathcal{U}_n(z)\,\mathrm{d}z, \\
&\int_{-L_3}^{0} \boldsymbol{u}(L_1,z,t)\mathcal{U}_n(z) + \frac{1}{N}\int_{-L_3}^{0} \boldsymbol{\psi}(L_1,z,t)\mathcal{W}_n(z)\,\mathrm{d}z \\
&\qquad = \int_{-L_3}^{0} \boldsymbol{u}^{g,r}(z,t)\mathcal{U}_n(z) + \frac{1}{N}\int_{-L_3}^{0} \boldsymbol{\psi}^{g,r}(z,t)\mathcal{W}_n(z)\,\mathrm{d}z,
\end{aligned}\right. \tag{3.74}$$

and *for the supercritical mode* $n > n_c$:

$$\left\{\begin{aligned}
&\int_{-L_3}^{0} \boldsymbol{u}(0,z,t)\mathcal{U}_n(z)\,\mathrm{d}z = \int_{-L_3}^{0} \boldsymbol{u}^{g,l}(z,t)\mathcal{U}_n(z)\,\mathrm{d}z, \\
&\int_{-L_3}^{0} \boldsymbol{v}(0,z,t)\mathcal{U}_n(z)\,\mathrm{d}z = \int_{-L_3}^{0} \boldsymbol{v}^{g,l}(z,t)\mathcal{U}_n(z)\,\mathrm{d}z, \\
&\int_{-L_3}^{0} \boldsymbol{\psi}(0,z,t)\mathcal{W}_n(z)\,\mathrm{d}z = \int_{-L_3}^{0} \boldsymbol{\psi}^{g,l}(z,t)\mathcal{W}_n(z)\,\mathrm{d}z.
\end{aligned}\right. \tag{3.75}$$

We summarize the result concerning the case of nonhomogeneous boundary conditions in a theorem.

Theorem 3.3. *Let H be the Hilbert space defined in* (3.2)*, A be the linear operator defined in* (3.7)*,and $\mathcal{A}$ be the corresponding differential operator, and let $D(A)$ be the domain of the operator A in H. We assume that the data U_0, F, $U^{g,l}$ and $U^{g,r}$ satisfy the regularity conditions* (3.65)-(3.67)*, and in addition U_0, $U^{g,l}$ and $U^{g,r}$ satisfy the compatibility conditions* (3.68) *and* (3.69)*. Then the initial boundary value problem corresponding to* (3.55) *supplemented with the boundary conditions* (3.74) *and* (3.75) *has a unique solution U, and U satisfies* (3.70) *and* (3.71)*; U will also satisfy* (3.72) *if we furthermore assume* (3.73) *for $U^{g,l}$ and $U^{g,r}$.*

4. Another model

We now consider another interesting model with only one degree of freedom for v (one unknown component for v). In this case we require that v vanishes at $y = 0$ and 1, which in physics corresponds to impenetrable boundaries at the North and South. To impose this boundary condition, we use a single mode in y direction for v, namely $\widetilde{h}_2$ (see Fig. 2.1), and the other unknowns u, w, ϕ and ψ are decomposed as they were in Section 2.1; hence:

$$\begin{cases} u = u_1(x,z,t)h_1(y) + u_2(x,z,t)h_2(y) + u_3(x,z,t)h_3(y), \\ v = v_2(x,z,t)\widetilde{h}_2(y), \\ w = w_1(x,z,t)h_1(y) + w_2(x,z,t)h_2(y) + w_3(x,z,t)h_3(y), \\ \phi = \phi_1(x,z,t)h_1(y) + \phi_2(x,z,t)h_2(y) + \phi_3(x,z,t)h_3(y), \\ \psi = \psi_1(x,z,t)h_1(y) + \psi_2(x,z,t)h_2(y) + \psi_3(x,z,t)h_3(y). \end{cases} \tag{4.1}$$

Then we introduce (4.1) into (2.6). We perform the same operations as after (2.6), except that we multiply the equation for v, $(2.6)_2$, by $\widetilde{h}_2$, integrate over $(0,1)$, and divide it by $\int_0^1 \widetilde{h}_2^2 dy$. Thus we arrive at the following approximation of the system (2.6):

$$\begin{cases} \boldsymbol{u}_t + \bar{U}_0 \boldsymbol{u}_x + \boldsymbol{\phi}_x + f v_2 \boldsymbol{\sigma_1} = 0, \\ v_{2t} + \bar{U}_0 v_{2x} + f \boldsymbol{\sigma_3} \cdot \boldsymbol{u} + \boldsymbol{\sigma_4} \cdot \boldsymbol{\phi} + \dfrac{3}{2} f \bar{U}_0 = 0, \\ \boldsymbol{\psi}_t + \bar{U}_0 \boldsymbol{\psi}_x + N^2 \boldsymbol{w} = 0, \\ \boldsymbol{u}_x + v_2 \boldsymbol{\sigma_2} + \boldsymbol{w}_z = 0, \\ \boldsymbol{\psi} = \boldsymbol{\phi}_z. \end{cases} \tag{4.2}$$

The vector notation, i.e. $\boldsymbol{u} = (u_1,\, u_2,\, u_3)$, $\boldsymbol{w} = (w_1,\, w_2,\, w_3)$ etc., has been used, and

$$\boldsymbol{\sigma_1} = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \quad \boldsymbol{\sigma_2} = 3\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \boldsymbol{\sigma_3} = \frac{1}{4}\begin{pmatrix} 1 \\ -6 \\ 1 \end{pmatrix}, \quad \boldsymbol{\sigma_4} = \frac{3}{2}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}. \tag{4.3}$$

The dot in (4.2) represents the dot product in the Euclidean space. We note here that all the equations in (4.2) are vector equations except the second one, which is a scalar equation for the scalar unknown v_2.

¿From here on, we proceed essentially as in Sections 2.2, 2.3 and Section 3, and below we only highlight the differences with the previous case.

The normal mode expansion for $\boldsymbol{u}$, $\boldsymbol{w}$, $\boldsymbol{\phi}$, $\boldsymbol{\psi}$ are the same as before, and for v_2 it reads

$$v_2(x,z,t) = \sum_{n \geq 0} v_{2n}(x,t)\mathcal{U}_n(z). \tag{4.4}$$

The system for the zero mode is

$$\begin{cases} \boldsymbol{u}_{0t} + \bar{U}_0 \boldsymbol{u}_{0x} + \boldsymbol{\phi}_{0x} + f v_{20} \boldsymbol{\sigma_1} = 0, \\ v_{20t} + \bar{U}_0 v_{20x} + f \boldsymbol{\sigma_3} \cdot \boldsymbol{u}_0 + \boldsymbol{\sigma_4} \cdot \boldsymbol{\phi}_0 + \dfrac{3}{2} L_3^{\frac{1}{2}} f \bar{U}_0 = 0, \\ \boldsymbol{u}_{0x} + v_{20} \boldsymbol{\sigma_2} = 0. \end{cases} \tag{4.5}$$

For $n \geq 1$ the system is

$$\begin{cases} \boldsymbol{u}_{nt} + \bar{U}_0 \boldsymbol{u}_{nx} + \boldsymbol{\phi}_{nx} + f v_{2n} \boldsymbol{\sigma_1} = 0, \\ v_{2nt} + \bar{U}_0 v_{2nx} + f \boldsymbol{\sigma_3} \cdot \boldsymbol{u}_n + \boldsymbol{\sigma_4} \cdot \boldsymbol{\phi}_n = 0, \\ \boldsymbol{\psi}_{nt} + \bar{U}_0 \boldsymbol{\psi}_{nx} + N^2 \boldsymbol{w}_n = 0, \\ \boldsymbol{u}_{nx} + v_{2n} \boldsymbol{\sigma_2} + N \lambda_n \boldsymbol{w}_n = 0, \\ -N \lambda_n \boldsymbol{\phi}_n = \boldsymbol{\psi}_n. \end{cases} \tag{4.6}$$

Eliminating $\boldsymbol{\phi}_n$ and $\boldsymbol{w}_n$ from (4.6) we obtain a system for $\boldsymbol{u}_n$, $\boldsymbol{v}_n$ and $\boldsymbol{\psi}_n$ ($n \geq 1$), namely:

$$\begin{cases} \boldsymbol{u}_{nt} + \bar{U}_0 \boldsymbol{u}_{nx} - \dfrac{1}{N\lambda_n} \boldsymbol{\psi}_{nx} + f v_{2n} \boldsymbol{\sigma_1} = 0, \\ \boldsymbol{v}_{nt} + \bar{U}_0 \boldsymbol{v}_{nx} + f \boldsymbol{\sigma_3} \boldsymbol{u}_n - \dfrac{1}{N\lambda_n} \boldsymbol{\sigma_4} \cdot \boldsymbol{\psi}_n = 0, \\ \boldsymbol{\psi}_{nt} - \dfrac{N}{\lambda_n} \boldsymbol{u}_{nx} + \bar{U}_0 \boldsymbol{\psi}_{nx} - \dfrac{N}{\lambda_n} v_{2n} \boldsymbol{\sigma_2} = 0. \end{cases} \tag{4.7}$$

The coefficient matrix associated with the first order derivative (with respect to x) terms in the first and third equations in (4.7) is

$$\begin{pmatrix} \bar{U}_0 & -\dfrac{1}{N\lambda_n} \\ -\dfrac{N}{\lambda_n} & \bar{U}_0 \end{pmatrix}.$$

This matrix has two eigenvalues: $U_0 + 1/\lambda_n$ and $U_0 - 1/\lambda_n$. The first eigenvalue is always positive, while the second one could be positive, which corresponds to supercritical modes, or negative, which corresponds to subcritical modes. We let n_c denote the number of subcritical modes, and for each $n \geq 1$ we also introduce the variables $\boldsymbol{\xi}_n = \boldsymbol{u}_n - \boldsymbol{\psi}_n/N$, $\boldsymbol{\eta}_n = \boldsymbol{u}_n + \boldsymbol{\psi}_n/N$. By an analysis similar to that in Section 2.3 we are led to propose the following boundary conditions for the subcritical modes:

$$\begin{cases} \boldsymbol{\xi}_n(0,t) = 0, \\ v_{2n}(0,t) = 0, \qquad \text{for } 1 \leq n \leq n_c \\ \boldsymbol{\eta}_n(L_1,t) = 0, \end{cases} \tag{4.8}$$

and, for the supercritical modes:

$$\begin{cases} \boldsymbol{\xi}_n(0,t) = 0, \\ v_{2n}(0,t) = 0, \qquad\qquad \text{for } n > n_c. \\ \boldsymbol{\eta}_n(0,t) = 0. \end{cases} \tag{4.9}$$

Again we want to transform (4.2) (except the zero mode, which needs a separate treatment) into an abstract initial value problem of the form

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} + AU = F, \\ U(0) = U_0. \end{cases} \tag{4.10}$$

For this purpose we introduce the following function spaces:

$$H = H_{\boldsymbol{u}} \times H_{v_2} \times H_{\boldsymbol{\psi}}, \tag{4.11}$$

$$\begin{aligned} H_{\boldsymbol{u}} &= \Big\{ \boldsymbol{u} \in \mathbf{L}^2(\mathcal{M}') \mid \int_{-L_3}^{0} \boldsymbol{u}(x,z)\,\mathrm{d}z = \mathbf{0}, \text{ for a.e. } x \in (0,L_1) \Big\}, \\ H_{v_2} &= \Big\{ v_2 \in L^2(\mathcal{M}') \mid \int_{-L_3}^{0} v_2(x,z)\,\mathrm{d}z = 0, \text{ for a.e. } x \in (0,L_1) \Big\}, \\ H_{\boldsymbol{\psi}} &= \mathbf{L}^2(\mathcal{M}'). \end{aligned}$$

We endow H with the inner product

$$(U,\widetilde{U})_H = \int_{\mathcal{M}'} (\boldsymbol{u}\cdot\widetilde{\boldsymbol{u}} + v_2\widetilde{v}_2 + \frac{1}{N^2}\boldsymbol{\psi}\cdot\widetilde{\boldsymbol{\psi}})\,\mathrm{d}\mathcal{M}' \quad \text{for } U,\widetilde{U} \in H \tag{4.12}$$

With this inner product, H is a Hilbert space. We let P denote the orthogonal projector from $\mathbf{L}^2(\mathcal{M}')$ onto $H_{\boldsymbol{u}}$. For convenience we also use P for the orthogonal projector from $L^2(\mathcal{M}')$ onto H_{v_2}.

The boundary conditions for $\boldsymbol{u}$, v_2 and $\boldsymbol{\psi}$ follow those we chose above, mode by mode (see (4.8), (4.9)); hence:
For the subcritical modes $(1 \le n \le n_c)$,

$$\begin{cases} \displaystyle\int_{-L_3}^{0} \boldsymbol{u}(0,z)\mathcal{U}_n(z) - \frac{1}{N}\int_{-L_3}^{0} \boldsymbol{\psi}(0,z)\mathcal{W}_n(z)\,\mathrm{d}z = \mathbf{0}, \\ \displaystyle\int_{-L_3}^{0} v_2(0,z)\mathcal{U}_n(z) = \mathbf{0}, \\ \displaystyle\int_{-L_3}^{0} \boldsymbol{u}(L_1,z)\mathcal{U}_n(z) + \frac{1}{N}\int_{-L_3}^{0} \boldsymbol{\psi}(L_1,z)\mathcal{W}_n(z)\,\mathrm{d}z = \mathbf{0}, \end{cases} \tag{4.13}$$

and *for the supercritical modes* ($n > n_c$),

$$\begin{cases} \displaystyle\int_{-L_3}^{0} \boldsymbol{u}(0,z)\mathcal{U}_n(z)\,\mathrm{d}z = \mathbf{0}, \\ \displaystyle\int_{-L_3}^{0} v_2(0,z)\mathcal{U}_n(z)\,\mathrm{d}z = \mathbf{0}, \\ \displaystyle\int_{-L_3}^{0} \boldsymbol{\psi}(0,z)\mathcal{W}_n(z)\,\mathrm{d}z = \mathbf{0}. \end{cases} \tag{4.14}$$

The domain of the operator A is defined as

$$D(A) = \{\, U \in H \mid U_x \in \mathbf{L}^2(\mathcal{M}') \times L^2(\mathcal{M}') \times \mathbf{L}^2(\mathcal{M}'), \\ \textit{and } U \textit{ verifies the BC's (4.13) and (4.14)} \}. \tag{4.15}$$

For each $U \in D(A)$, AU is given by

$$AU = \begin{pmatrix} \bar{U}_0 \boldsymbol{u}_x + f v_2 \boldsymbol{\sigma_1} - P[\displaystyle\int_z^0 \boldsymbol{\psi}_x(x,z')\,\mathrm{d}z'] \\ \bar{U}_0 v_{2x} + f \boldsymbol{\sigma_3}\cdot\boldsymbol{u} - P[\displaystyle\int_z^0 \boldsymbol{\sigma_4}\cdot\boldsymbol{\psi}(x,z')\,\mathrm{d}z'] \\ \bar{U}_0 \boldsymbol{\psi}_x + N^2 \displaystyle\int_z^0 (\boldsymbol{u}_x + v_2\boldsymbol{\sigma_2})\,\mathrm{d}z' \end{pmatrix}. \tag{4.16}$$

In the process of establishing the well-posedness of the initial value problem associated with our new model we need to determine the adjoint operator A^* of A (as an unbounded operator in H), and its domain $D(A^*)$. We now list, without details of calculations, the definitions of the operator A^* and its domain $D(A^*)$. The functions $\widetilde{U} = (\widetilde{\boldsymbol{u}}, \widetilde{v}_2, \widetilde{\boldsymbol{\psi}})$ in $D(A^*)$ satisfy the following boundary conditions.
For the subcritical modes ($1 \le n \le n_c$):

$$\begin{cases} \displaystyle\int_{-L_3}^{0} \widetilde{\boldsymbol{u}}(L_1,z)\mathcal{U}_n(z) - \frac{1}{N}\int_{-L_3}^{0} \widetilde{\boldsymbol{\psi}}(L_1,z)\mathcal{W}_n(z)\,\mathrm{d}z = \mathbf{0}, \\ \displaystyle\int_{-L_3}^{0} \widetilde{v}_2(L_1,z)\mathcal{U}_n(z) = \mathbf{0}, \\ \displaystyle\int_{-L_3}^{0} \widetilde{\boldsymbol{u}}(0,z)\mathcal{U}_n(z) + \frac{1}{N}\int_{-L_3}^{0} \widetilde{\boldsymbol{\psi}}(0,z)\mathcal{W}_n(z)\,\mathrm{d}z = \mathbf{0}, \end{cases} \tag{4.17}$$

and *for the supercritical modes* ($n > n_c$):

$$\begin{cases} \displaystyle\int_{-L_3}^{0} \widetilde{\boldsymbol{u}}(L_1, z)\mathcal{U}_n(z)\,\mathrm{d}z = \mathbf{0}, \\ \displaystyle\int_{-L_3}^{0} \widetilde{v}_2(L_1, z)\mathcal{U}_n(z)\,\mathrm{d}z = \mathbf{0}, \\ \displaystyle\int_{-L_3}^{0} \widetilde{\boldsymbol{\psi}}(L_1, z)\mathcal{W}_n(z)\,\mathrm{d}z = \mathbf{0}. \end{cases} \tag{4.18}$$

The domain $D(A^*)$ is then defined as follows:

$$\begin{aligned} D(A^*) = \{\, \widetilde{U} \in H \mid \widetilde{U}_x \in (\mathbf{L}^2(\mathcal{M}') \times L^2(\mathcal{M}') \times \mathbf{L}^2(\mathcal{M}'), \\ \textit{and } \widetilde{U} \textit{ verifies the BC's } (4.17) \textit{ and } (4.18) \,\}. \end{aligned} \tag{4.19}$$

For each $\widetilde{U} \in D(A^*)$, $A^*\widetilde{U}$ is given by

$$A^*\widetilde{U} = \begin{pmatrix} \displaystyle\sum_{n\geq 1}(-\bar{U}_0\widetilde{\boldsymbol{u}}_{nx} + \frac{1}{N\lambda_n}\widetilde{\boldsymbol{\psi}}_{nx} + f\widetilde{v}_{2n}\boldsymbol{\sigma_3})\mathcal{U}_n \\ \displaystyle\sum_{n\geq 1}(-\bar{U}_0\widetilde{v}_{2nx} + f\widetilde{\boldsymbol{u}}_n\cdot\boldsymbol{\sigma_1} - \frac{1}{N\lambda_n}\widetilde{\boldsymbol{\psi}}_n\cdot\boldsymbol{\sigma_2})\mathcal{U}_n \\ \displaystyle\sum_{n\geq 1}(\frac{N}{\lambda_n}\widetilde{\boldsymbol{u}}_{nx} - \bar{U}_0\widetilde{\boldsymbol{\psi}}_{nx} - \frac{N}{\lambda_n}\widetilde{v}_{2n}\boldsymbol{\sigma_4})\mathcal{W}_n \end{pmatrix}. \tag{4.20}$$

where the $(\boldsymbol{u}_n, v_{2n}, \boldsymbol{\psi}_n)$, for $n \geq 1$, are the normal modes of $\widetilde{U}$.

The following theorem, which is a copy of Theorem 3.2 with minor modifications, gives the well-posedness result about the system (4.10).

Theorem 4.1. *Let H, A and $D(A)$ be defined as above. Then the initial value problem (4.10) is well-posed. That is, for every $U_0 \in D(A)$, and $F \in L^1(0,T;H)$, with $F' \in L^1(0,T;H)$, (4.10) has a unique solution U such that*

$$U \in C([0,T];H) \cap L^\infty(0,T;D(A)), \quad \frac{\mathrm{d}U}{\mathrm{d}t} \in L^\infty(0,T;H). \tag{4.21}$$

Theorem 4.1 is also a direct result of Theorem 3.1. The verification of the hypotheses of Theorem 3.1 can be done similarly as in Section 3.

For the system of the zero mode (4.5) we can also decompose $\boldsymbol{\phi}_0$ into two parts:

$$\boldsymbol{\phi}_0 = \bar{\boldsymbol{\phi}}_0 + \boldsymbol{\phi}_0', \tag{4.22}$$

where $\bar{\boldsymbol{\phi}}_0$ is one of the stationary solutions of the equation

$$\boldsymbol{\sigma_4} \cdot \bar{\boldsymbol{\phi}}_0 = -\frac{3}{2} L_3^{\frac{1}{2}} f \bar{U}_0. \tag{4.23}$$

Then we impose the boundary conditions on the left boundary:

$$\begin{cases} \boldsymbol{u}_0(0,t) = \boldsymbol{u}^l(t), \\ v_{20}(0,t) = v_2^l(t), \\ \boldsymbol{\phi}_0'(0,t) = \boldsymbol{\phi}'^l(t). \end{cases} \tag{4.24}$$

With the boundary conditions above, we can treat the zero mode in a way similar to that in Section 3.4. First, by combining the first and second equations (now with $\boldsymbol{\phi}_0'$) of (4.5), we find and then solve the resulting equation for $\boldsymbol{\sigma_4} \cdot \boldsymbol{u}_0 - v_{20x}$. Once $\boldsymbol{\sigma_4} \cdot \boldsymbol{u}_0 - v_{20x}$ is known, say $\boldsymbol{\sigma_4} \cdot \boldsymbol{u}_0 - v_{20x} = K(x,t)$, we can solve for $\boldsymbol{u}_0$ and v_{20} from this expression and the third equation of (4.5). Then when $\boldsymbol{u}_0$ and v_{20} are known, the first equation in (4.5) gives $\boldsymbol{\phi}_0'$. We leave it as an exercise for the reader to check the details, and to address the case of nonhomogeneous boundary conditions as in Section 3.5.

Acknowledgements

The authors thank Greg Duane for a careful review of this article and for pointing out various typos. This work was supported in part by NSF grants DMS0139874, DMS0305110 and DMS0604235, and by the Research Fund of Indiana University.

References

[1] Blayo, E., Debreu, L., *Revisiting open boundary conditions from the point of view of characteristic variables*, Ocean Modelling, **9** (2005), pp. 231-252.
[2] Burq, N., Gérard, P., *Contrôle optimal des équations aux dérivées partielles*, Ecole Polytechnique, Palaiseau, France, 2003.
[3] Cao, C., Titi E. S., *Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics*, arXiv:math.AP/0503028 **2** (2005).
[4] Debussche, A., Laminie, J., Zahrouni, E., *A dynamical multi-level scheme for the Burgers equation: wavelet and hierarchical finite element*, J. Sci. Comput. **25** (2005), no. 3 pp.445-497.
[5] Kobelkov, G. M., *Existence of a solution 'in the large' for the 3D large-scale ocean dynamics equations*, C.R. Acad. Sci. Paris, Ser. 343 (2006), pp. 283-286.
[6] Lions, J.L., Temam, R., Wang, S., *New Formulations of the primitive equations of the atmosphere and applications*, Nonlinearity **5** (1992), pp.273-288.
[7] Lions, J.L., Temam, R., Wang, S., *On the equations of the large-scale ocean*, Nonlinearity **5** (1992), pp.1007-1053.

[8] Oliger, J., Sundström, A., *Theoretical and practical aspects of some initial boundary value problems in fluid dynamics*, SIAM J. Appl. Math., **35(3)** (1978), pp.419-446.

[9] Pedlosky, J., *Geophysical fluid dynamics, 2nd edition*, Springer, 1987

[10] Petcu, M., Rousseau, A., *On the δ-primitive and Boussinesq type equations*, Advances in Differential Equations, **10(5)** (2005), pp.579-599.

[11] Petcu, M., Temam, R., Ziane, M., *Mathematical problems for the primitive equations with viscosity*, Handbook of Numerical Analysis (Ciarlet, P.G., ed.), special issue on Some Mathematical Problems in Geophysical Fluid Dynamics (Temam, R., Tribbia, J. guest eds.), Elsevier, New York, 2007.

[12] Rousseau, A., Temam, R., Tribbia, J., *Boundary layers in an ocean related system*, J. Sci. Comput., **21(3)** (2004), pp.405-432.

[13] Rousseau, A., Temam, R., Tribbia, J., *Boundary conditions for an ocean related system with a small parameter*, Nonlinear PDEs and Related Analysis (Chen, G., Gasper, G., Jerome, J. eds), Contemporary Mathematics, vol.371, AMS, Providence, 2005, pp.231-263.

[14] Rousseau, A., Temam, R., Tribbia, J. *Boundary Conditions for the 2D Linearized PEs of the Ocean in the absence of Viscosity*, Discrete And Continuous Dynamical Systems **13** (2005), pp.1257-1276.

[15] Rousseau, A., Temam, R., Tribbia, J. *Numerical simulations of the inviscid primitive equations in a limited domain*, Proceedings of Mathematical Methods for Hydrodynamics, University of Lille, (2005).

[16] Rudin, W., *Functional Analysis*, 2nd ed., McGraw-Hill, New York, 1991.

[17] Temam, R., Tribbia, J., *Open Boundary Conditions for the Primitive and Boussinesq Equations*, Journal of the Atmospheric Sciences **60** (2003), pp.2647-2660.

[18] Temam, R., Ziane, M., *Some mathematical problems in geophysical fluid dynamics*, Handbook of mathematical fluid dynamics, (Friedlander, S., Serre, D., eds), vol. 3, North Holland, 2004.

[19] Washington, W., Parkinson, C., *An introduction to three dimensional climate modelling*, Oxford Univ. Press, 1986.

[20] Warner, T.T., Peterson, R.A., Treadon, R.E., *A tutorial on lateral boundary conditions as a basic and potentially serious limitation to regional numerical weather prediction*, Bull. Amer. Meteor. Soc., **78(11)** (1997), pp.2599-2617.

(QSC) The institute for Scientific Computing and Applied Mathematics, Indiana University, Bloomington, IN 47405

(JL) CNRS, Mathematiques, Université Paris-Sud, Orsay, Paris, France

(JL) Université des Antilles et de la Guyane, Pointe à Pitre, GRIMAAG, Guadeloupe

(AR) Moïse Project, Inria Rhône-Alpes, 655 av. Europe, 38 330 Motbonnot, France

(RT) The institute for Scientific Computing and Applied Mathematics, Indiana University, Bloomington, IN 47405
E-mail address: temam@indiana.edu

(JT) National Center for Atmospheric Research, Boulder, Colorado, USA