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Abstract: This report is dedicated to the construction and analysis of so-called Generalized Impedance Boundary Conditions (GIBCs) used in electromagnetic scattering problems from imperfect conductors as higher order approximations of a perfect conductor condition. We consider here the 3-D case with Maxwell equations in a harmonic regime. The construction of GIBCs is based on a scaled asymptotic expansion with respect to the skin depth. The asymptotic expansion is theoretically justified at any order and we give explicit expressions till the third order. These expressions are used to derive the GIBCs. The associated boundary value problem is analyzed and error estimates are obtained in terms of the skin depth.

Key-words: GIBC, effective boundary conditions, highly conducting scatterers, electromagnetic scattering, Maxwell's equations, boundary layers, scaled asymptotic expansions, approximate models

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Résumé : Ce rapport est dédié à la construction et l'analyse des conditions d'impédances généralisées (GIBCs) utilisées en électromagnétisme comme approximations d'ordre supérieur à une simple condition de conducteur parfait. Nous nous intéressons ici au problème 3-D modélisé par les équations de Maxwell en régime harmonique. L'obtention des GIBCs se base sur un développement asymptotique par rapport à la profondeur de peau à l'intérieur de l'obstacle fortement conducteur. Ce développement est justifié à tout ordre par des estimations d'erreurs sur les séries tronquées, et nous donnons des formules explicites de ses termes jusqu'à l'ordre 3. Ces formules sont ensuite utilisées pour obtenir les expressions des GIBCs. Les problèmes aux limites associés aux GIBCs sont analysés et nous établirons des estimations d'erreurs relatives à ces modèles approchées en terme de la conductivité.

Mots-clés : GIBC, conditions aux limites effectives, forte conductivité, diffraction electromagnetique, equations de Maxwell, couches limites, développement asymptotique, modèles approchés

1 Introduction

Generalized Impedance Boundary Conditions (*GIBC*) have become a rather classical notion in the mathematical modeling of wave propagation phenomena (see for instance, [12] and, [15]). They are used in electromagnetism for time harmonic scattering problems from obstacles that are partially or totally penetrable. The general idea is to replace the use of an "exact model" inside (the penetrable part of) the obstacle by approximate boundary conditions (also called equivalent or effective conditions). This idea is pertinent if the boundary condition can be easily handled numerically, for instance when it is local. The diffraction problem of electromagnetic waves by perfectly conducting obstacles coated with a thin layer of dielectric material is well suited for the notion of impedance conditions: due to the small (typically with respect to the wavelength) thickness of the coating, the effect of the layer on the exterior medium is, as a first approximation, local (see for instance, [15], [12], [7], [3], [1]).

The application we consider here is the diffraction of waves by highly conducting materials in electromagnetism. In such a case, it is the well-known skin effect that creates a "thin layer" phenomenon. The high conductivity limitates the penetration of the wave to a boundary layer whose depth is inversely proportional to the square root of its magnitude. Then, here again, the effect of the obstacle is, as a first approximation, local.

The first effective boundary conditions for highly absorbing obstacles was proposed by Leontovich. This condition "sees" only locally the tangent plane to the frontier. Later, Rytov, [14], [15] proposed an extension of the Leontovitch condition, and his analysis was already based on the principle of asymptotic expansions with respect to the small parameter in the problem: the skin depth δ. More recently, Antoine-Barucq-Vernhet [2] proposed a new derivation of such conditions based on the technique of pseudo-differential operator expansions. However, in all these works, the rigorous mathematical justification of the resulting impedance conditions was not treated.

This paper is the continuation of the work in [11], in which we considered the case of the scalar wave equation. Our objective is to extend the results of to the case of 3D Maxwell's equations by constructing and analyzing GIBC's of order 1, 2 and 3 (with respect to the skin depth, the small parameter of the problem). These conditions are of impedance type (or $H-to-E$ nature): they relates the tangential traces of the electric and magnetic fields via a local impedance operator.

As in [11], the construction of the approximate conditions relies on an asymptotic expansion of the exact solution, based on a scaling technique and a boundary layer expansion in the neighborhood of the boundary of the scatterer. If the organization of this paper contents is similar to [11], its technicality is much higher. Moving from the scalar wave equation to the Maxwell system increases considerably the complexity of the problem at two levels.

- The first one is linked with the algebra involved in the formal construction of the asymptotic expansion of the exact solution (see Section 4). This is essentially due to the vectorial nature of the unknowns and the expression of the curl operator in a parametric coordinates system (see Section 4.1). The latter is based on the formulas proposed in [10] with some simplifications.
- The second one is related to the mathematical analysis on the GIBCs. This is not only due to the fact that we have to deal with usual functional analysis difficulties linked to Maxwell equations (in particular trace operators and compact embedding properties - see Sections 5 and 6 and Appendix A) but also because we have to face some new difficulties in the case of the third order condition. The tangential differential operators that would naturally appear in the construction of the third order condition have not the good "sign properties" to be able to guarantee the existence of the approximate solution and the convergence (at optimal order) to the true solution. This leads us to apply various regularization procedures to construct the modified third order conditions (see Section 3.2).

Our objectives in this work are essentially theoretical. The numerical pertinence of obtained conditions have already been demonstrated in [6] where, in particular, the interest of using a third order condition rather than a first or a second order condition is clearly shown.

The outline of the article is as follows. Section 2 contains a description of the physical and mathematical diffraction problem at study with some basic stability properties of the solutions and asymptotic estimates with respect to the conductivity. We state the main results of our paper in Section 3: the GIBCs are presented in Sections 3.1 and 3.2 while the corresponding error estimates are given in Section 3.3. The formal construction of the asymptotic expansion is given in Section 4. This construction is rigorously justified in Section 5 by proving optimal error estimates at each order. The last section is dedicated to the study of the boundary value problems associated with the GIBCs as well as the proof of optimal error estimates between these solutions and truncated asymptotic expansions. The main result of our paper is obtained as a combination of the results of Section 6 and Section 5. Some non standard technical results related to the $H(\text{curl})$ space (appropriate trace inequalities and special compact embedding properties) that may have their own interest have been gathered in Appendix A.

2 Description of the physical model

Let Ω_i be an open bounded domain in $\mathbb{R}^3$ with connected complement, occupied by a homogeneous conducting medium. We denote by Γ the boundary of Ω_i and assume that this boundary is a C^∞ manifold. We are interested in computing the electromagnetic diffracted wave when the conductivity of the medium, denoted by σ^δ, is sufficiently high (δ denotes a small parameter). More precisely we assume that $\sigma^\delta \to \infty$ as $\delta \to 0$ and would like to study the asymptotic behavior of the diffracted electromagnetic field as $\delta \to 0$ in order to derive efficient approximate models to compute the diffracted waves.

We assume that the exterior domain is homogeneous and the time and space scales are chosen such that the wave speed is 1 in this medium. The electromagnetic wave propagation is therefore governed by the following Maxwell's equations:

$$\begin{cases} \varepsilon(x)\dfrac{\partial \mathbf{E}^\delta}{\partial t} + \sigma^\delta(x)\mathbf{E}^\delta - \operatorname{curl}\mathbf{H}^\delta = F, & \text{in } \Omega, \\ \dfrac{\partial \mathbf{H}^\delta}{\partial t} + \operatorname{curl}\mathbf{E}^\delta = 0, & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ denotes the propagative medium that we shall assume to be bounded, regular and simply connected with connected boundary (for instance an open ball), the functions $\sigma^\delta(x)$ and $\varepsilon(x)$ are defined by:

$$(\varepsilon, \sigma^\delta)(x) = \begin{cases} (1, 0), & \text{in } \Omega_e, \\ (\varepsilon_r, \sigma^\delta), & \text{in } \Omega_i \end{cases}$$

where $\Omega_e = \Omega \setminus \overline{\Omega_i}$ and where $\varepsilon_r > 0$ denotes the relative electric permittivity of the conducting medium. The right-hand side F denotes some source term that we shall assume to be harmonic in time : $F(x,t) = \operatorname{Re}\{f(x)\exp(i\omega t)\}$, where $\omega > 0$ denotes a given frequency, and where $\operatorname{Re}(z)$ denotes the real part of z. Hence, the solutions are also time harmonic:

$$\mathbf{E}^\delta(x,t) = \operatorname{Re}\left\{E^\delta(x)\exp(i\omega t)\right\}, \quad \mathbf{H}^\delta(x,t) = \operatorname{Re}\left\{H^\delta(x)\exp(i\omega t)\right\},$$

where the electromagnetic field (E^δ, H^δ) is solution to the harmonic Maxwell system:

$$\begin{cases} (i) \quad (i\varepsilon\omega + \sigma^\delta)E^\delta - \operatorname{curl} H^\delta = f, & \text{in } \Omega, \\ (ii) \quad i\omega H^\delta + \operatorname{curl} E^\delta = 0, & \text{in } \Omega. \end{cases} \tag{1}$$

We assume that the support of the source term f does not touch Ω_i. The system of equations (1) has to be complemented with a boundary condition on the exterior boundary $\partial\Omega$, for instance we work with the following absorbing boundary condition

$$E_T^\delta - H^\delta \times n = g, \quad \text{on } \partial\Omega, \tag{2}$$

where $E_T := n \times (E \times n)$, n is a normal vector to $\partial\Omega$ directed to the exterior of Ω and g denoting some possible source term.

Remark 2.1 *According to (2), the boundary $\partial\Omega$ can be seen as a physical absorbing boundary where a standard impedance condition is applied. The problem (1, 2) can also be seen as an approximation in a bounded domain (namely Ω) of the scattering problem in $\mathbb{R}^3 \setminus \Omega_i$. In such a case, the boundary condition on $\partial\Omega$ has to be understood as an (low order) approximation of the outgoing radiation condition at infinity and g a source term linked to the incident field.*

Remark 2.2 *In this paper, we could have treated as well the scattering problem in $\mathbb{R}^3 \setminus \Omega_i$. The reader will easily be convinced that the obtained results can be extended to this case without any major difficulty. The only difference would lie in the reduction to a bounded domain. This additional difficulty is purely technical and not essential in the context of this paper whose main purpose is the treatment of the "interior boundary" Γ.*

As mentioned above we are interested in describing the asymptotic behavior of the solution for large σ^δ. As suggested by the expression of the analytic solution where Ω_i is the half space, the appropriate small length parameter can be defined as:

$$\delta := 1/\sqrt{\omega\sigma^\delta} \quad \Longleftrightarrow \quad \sigma^\delta = 1/(\omega\delta^2).$$

This small parameter defines the so-called skin depth: the "width" of the penetrable region inside the conducting medium is proportional to δ.

For the construction of approximate models in the exterior domain Ω_e it is useful to rewrite the problem (1-2) as a transmission problem between $(E_i^\delta, H_i^\delta) := (E^\delta, H^\delta)_{|\Omega_i}$ and $(E_e^\delta, H_e^\delta) = (E^\delta, H^\delta)_{|\Omega_e}$ as follows:

$$\begin{cases} i\omega E_e^\delta - \operatorname{curl} H_e^\delta = f, & \text{in } \Omega_e, \\ i\omega H_e^\delta + \operatorname{curl} E_e^\delta = 0, & \text{in } \Omega_e, \\ E_{e,T}^\delta - H_e^\delta \times n = g, & \text{on } \partial\Omega \\ E_e^\delta \times n = E_i^\delta \times n, & \text{on } \Gamma, \end{cases} \tag{3}$$

$$\begin{cases} (i\varepsilon_r\omega + \frac{1}{\omega\delta^2})E_i^\delta - \operatorname{curl} H_i^\delta = 0, & \text{in } \Omega_i, \\ i\omega H_i^\delta + \operatorname{curl} E_i^\delta = 0, & \text{in } \Omega_i, \\ H_i^\delta \times n = H_e^\delta \times n, & \text{on } \Gamma. \end{cases} \tag{4}$$

We have chosen to split the two transmission conditions (namely the continuity of the tangential electric and magnetic fields) in such a way that the first one appears as a boundary condition in (3) for the interior field while the second one appears as a boundary condition in (3) for the interior field. Roughly speaking, the approximate models are then obtained from replacing in system (4) the exact boundary condition on Γ by an approximate one, whose expression is derived from seeking appropriate asymptotic expansion of the solution in the boundary layer inside Ω_i.

2.1 Existence-Uniqueness-Stability

With $H(\operatorname{curl}, \mathcal{O})$ denoting the space of functions $V \in L^2(\mathcal{O})^3$ such that $\operatorname{curl} V \in L^2(U)^3$, where $\mathcal{O}$ is an open domain of $\mathbb{R}^3$, we define

$$\widetilde{H}(\operatorname{curl}, \mathcal{O}) = \{V \in H(\operatorname{curl}, \mathcal{O}) \;\; ; \;\; V_T \in L_t^2(\partial\mathcal{O})\} \tag{5}$$

where V_T is the tangential trace of V (cf. Section 4.1 for more details), $L^2_t(\partial\mathcal{O})$ denotes the space of functions $V \in L^2(\partial\mathcal{O})^3$ such that $V \cdot n = 0$ on $\partial\mathcal{O}$, where n denotes a normal to $\partial\mathcal{O}$. We recall that $\widetilde{H}(\mathrm{curl}, \mathcal{O})$ is a Hilbert space with scalar product

$$(U, V)_{\widetilde{H}(\mathrm{curl},\mathcal{O})} = (U, V)_{L^2(\mathcal{O})} + (\mathrm{curl}\, U, \mathrm{curl}\, V)_{L^2(\mathcal{O})} + (U_T, V_T)_{L^2_t(\partial\mathcal{O})}.$$

Theorem 2.1 *For given $f \in L^2(\Omega)^3$ and $g \in L^2_t(\partial\Omega)$ there exists an unique solution $(E^\delta, H^\delta) \in \widetilde{H}(\mathrm{curl}, \Omega) \times \widetilde{H}(\mathrm{curl}, \Omega)$ satisfying (1-2). Moreover, there exists a positive constant C independent of δ such that*

$$\|E^\delta\|_{\widetilde{H}(\mathrm{curl},\Omega)} + \frac{1}{\delta}\,\|E^\delta\|_{L^2(\Omega_i)} \le C\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2_t(\partial\Omega)}\right). \tag{6}$$

Proof. The proof of existence and uniqueness can be found in [13] (Theorem 4.17). The solution E^δ is constructed (using the Helmholtz decomposition) as

$$E^\delta = E^\delta_0 + \nabla p^\delta$$

where $p^\delta \in H^1_0(\Omega)$ and $E^\delta_0 \in \widetilde{H}^\delta_0(\mathrm{curl}, \Omega)$ where

$$\widetilde{H}^\delta_0(\mathrm{curl}, \Omega) = \{V \in \widetilde{H}(\mathrm{curl}, \Omega)\ ;\ \ \mathrm{div}((i\sigma^\delta - \varepsilon\omega^2)V) = 0\},$$

will be equipped with the norm of $\widetilde{H}(\mathrm{curl}, \Omega)$, of which it is a closed subspace.

Taking the divergence of (1-i) one easily see that the function p^δ is solution to

$$-\omega^2(\varepsilon\nabla p^\delta, \nabla\phi)_{L^2(\Omega)} + \frac{i}{\delta^2}(\nabla p^\delta, \nabla\phi)_{L^2(\Omega_i)} = i\omega(f, \nabla\phi)_{L^2(\Omega)} \tag{7}$$

for all $\phi \in H^1_0(\Omega)$. Choosing $\phi = -p^\delta$ then successively considering the real and imaginary parts of the resulting equality, one easily deduces that

$$\|\nabla p^\delta\|_{L^2(\Omega)} + \frac{1}{\delta}\,\|\nabla p^\delta\|_{L^2(\Omega_i)} \le C\,\|f\|_{L^2(\Omega)}\,, \tag{8}$$

for some positive constant C independent of δ, which proves estimate (6) for the gradient part of the solution. It remains to get an estimate for E^δ_0. The proof is based on the compactness embedding of Lemma A.5 (which is an adaptation of the proofs in [16] and [5]). The function E^δ_0 satisfies the variational formulation

$$\begin{aligned}&(\mathrm{curl}\, E^\delta_0, \mathrm{curl}\,\Phi)_{L^2(\Omega)} - \omega^2(\varepsilon\, E^\delta_0, \Phi)_{L^2(\Omega)} + \tfrac{i}{\delta^2}(E^\delta_0, \Phi)_{L^2(\Omega_i)} + i\omega(E^\delta_{0,T}, \Phi_T)_{L^2_t(\partial\Omega)}\\ &= i\omega\left\{(f, \Phi)_{L^2(\Omega_e)} - (g, \Phi_T)_{L^2_t(\partial\Omega)}\right\} + \omega^2(\varepsilon\,\nabla p^\delta, \Phi)_{L^2(\Omega)} - \tfrac{i}{\delta^2}(\nabla p^\delta, \Phi)_{L^2(\Omega_i)}\end{aligned} \tag{9}$$

for all $\Phi \in \widetilde{H}_0^\delta(\mathrm{curl},\Omega)$, where p^δ is the solution to (7). We first prove (by contradiction) that there exists a positive constant C independent of δ such that

$$\|E_0^\delta\|_{L^2(\Omega)} \le C\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2_t(\partial\Omega)}\right). \tag{10}$$

If not, then there would exist a sequence of data (f^δ) and (g^δ) such that

$$\|f^\delta\|_{L^2(\Omega)} + \|g^\delta\|_{L^2_t(\partial\Omega)} = 1$$

and such that the corresponding solution E_0^δ satisfies

$$\|E_0^\delta\|_{L^2(\Omega)} \to \infty \text{ as } \delta \to 0.$$

With an added tilda denoting a division by $\|E_0^\delta\|_{L^2(\Omega)}$, one observes that

$$\|\tilde{E}_0^\delta\|_{L^2(\Omega)} = 1, \tag{11}$$

and satisfies, for all $\Phi \in \widetilde{H}_0^\delta(\mathrm{curl},\Omega)$,

$$\begin{aligned}&(\mathrm{curl}\,\tilde{E}_0^\delta, \mathrm{curl}\,\Phi)_{L^2(\Omega)} - \omega^2(\varepsilon\,\tilde{E}_0^\delta,\Phi)_{L^2(\Omega)} + \tfrac{i}{\delta^2}(\tilde{E}_0^\delta,\Phi)_{L^2(\Omega_i)} + i\omega(\tilde{E}_{0,T}^\delta,\Phi_T)_{L^2_t(\partial\Omega)}\\ &= i\omega\left\{(\tilde{f}^\delta,\Phi)_{L^2(\Omega_e)} - (\tilde{g}^\delta,\Phi_T)_{L^2_t(\partial\Omega)}\right\} + \omega^2(\varepsilon\,\nabla\tilde{p}^\delta,\Phi)_{L^2(\Omega)} - \tfrac{i}{\delta^2}(\nabla\tilde{p}^\delta,\Phi)_{L^2(\Omega_i)}.\end{aligned} \tag{12}$$

where p^δ is the solution to (7) with f replaced by $\tilde{f}^\delta$. For $\Phi = \tilde{E}_0^\delta$, one gets

$$\begin{aligned}&\|\mathrm{curl}\,\tilde{E}_0^\delta\|^2_{L^2(\Omega)} - \omega^2(\varepsilon\,\tilde{E}_0^\delta,\tilde{E}_0^\delta)_{L^2(\Omega)} + \tfrac{i}{\delta^2}\|\tilde{E}_0^\delta\|^2_{L^2(\Omega_i)} + i\omega\|\tilde{E}_{0,T}^\delta\|^2_{L^2_t(\partial\Omega)}\\ &= i\omega\left\{(\tilde{f}^\delta,\tilde{E}_0^\delta)_{L^2(\Omega_e)} - (\tilde{g}^\delta,\tilde{E}_{0,T}^\delta)_{L^2_t(\partial\Omega)}\right\} + \omega^2(\varepsilon\,\nabla\tilde{p}^\delta,\tilde{E}_0^\delta)_{L^2(\Omega)} - \tfrac{i}{\delta^2}(\nabla\tilde{p}^\delta,\tilde{E}_0^\delta)_{L^2(\Omega_i)}.\end{aligned} \tag{13}$$

Taking first the imaginary part (13) and using estimate (8) and (11), one proves that

$$\frac{1}{\delta}\|\tilde{E}_0^\delta\|_{L^2(\Omega_i)} + \omega\|\tilde{E}_{0,T}^\delta\|_{L^2_t(\partial\Omega)} \le C\left(\|\tilde{f}^\delta\|_{L^2(\Omega)} + \|\tilde{g}^\delta\|_{L^2_t(\partial\Omega)}\right) \tag{14}$$

for some positive constant C independent of δ. One deduces after taking second the real part and using again estimate (8) and (11) that also

$$\|\mathrm{curl}\,\tilde{E}_0^\delta\|_{L^2(\Omega)} \le C_1 + C_2\left(\|\tilde{f}^\delta\|_{L^2(\Omega)} + \|\tilde{g}^\delta\|_{L^2_t(\partial\Omega)}\right) \tag{15}$$

for some different positive constants C_1 and C_2 independent of δ. It is then observed that the sequence $(\tilde{E}_0^\delta)$ is bounded in $\widetilde{H}_0^\delta(\mathrm{curl},\Omega)$.

Applying (the trace) Lemma A.1 (proved in the Appendix) to $\tilde{E}_0^\delta|_{\Omega_i}$ one deduces from (14) and (15) that

$$\|\tilde{E}_0^\delta \times n\|_{H^{-\frac{1}{2}}(\Gamma)} \to 0 \qquad \text{as } \delta \to 0.$$

Also from compactness theorems in Sobolev spaces and (15) one can assume (up to an extracted subsequence) that $E_0^\delta \times n$ is convergent in $H^{-\frac{1}{2}}(\partial\Omega)$. Hence, (the compactness) Lemma A.5 applied to the sequence $\tilde{E}_0^\delta|_{\Omega_e}$, shows that one can extract a subsequence (also denoted $\tilde{E}_0^\delta|_{\Omega_e}$) that converges to some $\tilde{E}$ weakly in $\widetilde{H}_0^\delta(\mathrm{curl}, \Omega_e)$ and strongly in $L^2(\Omega_e)^3$. Moreover

$$\tilde{E} \times n = 0 \ \text{ on } \Gamma. \tag{16}$$

Taking Φ with support inside Ω_e in (12) and taking the limit as $\delta \to 0$, show that $\tilde{E}$ satisfies (both $\tilde{f}^\delta$ and $\tilde{g}^\delta$ tend to 0)

$$\mathrm{curl}\,\mathrm{curl}\,\tilde{E} - \omega^2 \tilde{E} = 0 \qquad \text{in } \Omega_e \tag{17}$$

$$i\omega \tilde{E}_T + \mathrm{curl}\,\tilde{E} \times n = 0 \quad \text{on } \partial\Omega. \tag{18}$$

The uniqueness of the solution in $\widetilde{H}_0^\delta(\mathrm{curl}, \Omega)$ to (17-16) proves that $\tilde{E} = 0$ in Ω_e. Since estimate (14) also proves that $\|\tilde{E}_0^\delta\|_{L^2(\Omega_i)} \to 0$, we get that

$$1 = \|\tilde{E}_0^\delta\|_{L^2(\Omega)} \to \|\tilde{E}\|_{L^2(\Omega_e)}$$

which contradicts $\tilde{E} = 0$ in Ω_e and proves (10).

Now take $\Phi = E_0^\delta$ in (9) and use the imaginary part then the real part as previously done to deduce

$$\|E_0^\delta\|_{\tilde{H}(\mathrm{curl},\Omega)} + \frac{1}{\delta}\,\|E_0^\delta\|_{L^2(\Omega_i)} \leq C\left(\|f\|_{L^2(\Omega)} + \|g\|_{L_t^2(\partial\Omega)}\right). \tag{19}$$

Estimate (6) is a straightforward consequence of (19) and (8). □

2.2 Exponential interior decay of the solution

It is shown that the norm of the solution in a domain strictly interior to Ω_i goes to 0 faster than any power of δ. This is a first way to express how the interior solution concentrates near the boundary Γ. Let us indicate that this result will also be a consequence of the asymptotic analysis performed in next sections, however the methodology is more complex. The proof given here is a direct one and is independent of the subsequent analysis. The precise result is the following:

Theorem 2.2 *For any $\bar{\nu} > 0$ small enough so that $\Omega_{i,0}^{\bar{\nu}} := \{x \in \Omega_i; B(x, \bar{\nu}) \subset \Omega_i\}$ is a non-empty set, where $B(x, \bar{\nu})$ denotes the closed ball of center x and radius $\bar{\nu}$, there exist two positive constants $C_{\bar{\nu}}$ and $c_{\bar{\nu}}$ independent of δ such that*

$$\|E_i^\delta\|_{H(\mathrm{curl},\Omega_{i,0}^{\bar{\nu}})} + \|H_i^\delta\|_{H(\mathrm{curl},\Omega_{i,0}^{\bar{\nu}})} \leq C_{\bar{\nu}} \exp(-c_{\bar{\nu}}/\delta)(\|f\|_{L^2(\Omega)} + \|g\|_{L_t^2(\partial\Omega)}).$$

Proof. The proof of this result follows the same lines as the scalar case treated in [11]. For the reader convenience, we hereafter give the basic ideas.

We introduce a cut-off function $\phi_{\bar\nu} \in C^\infty(\Omega)$ such that

$$\phi_{\bar\nu}(x) = 0 \ \text{ in } \ \Omega_e \ , \qquad \phi_{\bar\nu}(x) = \beta^{\bar\nu} \ \text{ in } \ \Omega_{i,0}^{\bar\nu},$$

where the constant $\beta^{\bar\nu} > 0$ is chosen such that

$$\|\nabla\phi_{\bar\nu}\|_\infty < \frac{1}{4}. \tag{20}$$

We set $\tilde{\mathbf{E}} = \exp\big(\phi_{\bar\nu}(x)/\delta\big)E^\delta$. Straightforward calculations show that

$$\operatorname{curl} E^\delta = \exp\big(-\phi_{\bar\nu}(x)/\delta\big)\{\operatorname{curl}\tilde{\mathbf{E}} - \frac{1}{\delta}\nabla\phi_{\bar\nu} \times \tilde{\mathbf{E}}\}.$$

Hence,

$$\begin{aligned}\operatorname{curl}\operatorname{curl} E^\delta \quad = \quad \exp\big(-\phi_{\bar\nu}(x)/\delta\big)\ &\Big\{\operatorname{curl}\operatorname{curl}\tilde{\mathbf{E}} - \frac{1}{\delta}\nabla\phi_{\bar\nu} \times \operatorname{curl}\tilde{\mathbf{E}} \\ &-\frac{1}{\delta}\operatorname{curl}(\nabla\phi_{\bar\nu}\times\tilde{\mathbf{E}}) + \frac{1}{\delta^2}\nabla\phi_{\bar\nu}\times(\nabla\phi_{\bar\nu}\times\tilde{\mathbf{E}})\Big\}\end{aligned}$$

Therefore $\tilde{\mathbf{E}}$ satisfies, after multiplying the above equality by $\exp(\phi_{\bar\nu}/\delta)$

$$\begin{aligned}&\operatorname{curl}\operatorname{curl}\tilde{\mathbf{E}} - \frac{1}{\delta}\Big(\nabla\phi_{\bar\nu}\times\operatorname{curl}\tilde{\mathbf{E}} + \operatorname{curl}(\nabla\phi_{\bar\nu}\times\tilde{\mathbf{E}})\Big) \\ &+\Big((-\varepsilon\omega^2 + i\omega\sigma^\delta)\tilde{\mathbf{E}} + \frac{1}{\delta^2}\nabla\phi_{\bar\nu}\times(\nabla\phi_{\bar\nu}\times\tilde{\mathbf{E}})\Big) = i\omega f \quad \text{in} \ \ \Omega\end{aligned} \tag{21}$$

Taking the $L^2(\Omega)^3$ scalar product of this equation (21) by $\tilde{\mathbf{E}}$ and using Stokes formulas yields, (recall that $\tilde{\mathbf{E}} = E^\delta$ and $\phi_{\bar\nu} = 0$ in Ω_e)

$$\begin{aligned}&\|\operatorname{curl}\tilde{\mathbf{E}}\|^2_{L^2(\Omega_i)} - \tfrac{1}{\delta}\{(\nabla\phi_{\bar\nu}\times\operatorname{curl}\tilde{\mathbf{E}},\tilde{\mathbf{E}})_{L^2(\Omega_i)} + (\nabla\phi_{\bar\nu}\times\tilde{\mathbf{E}},\operatorname{curl}\tilde{\mathbf{E}})_{L^2(\Omega_i)}\} \\ &(-\varepsilon_r\omega^2 + \tfrac{i}{\delta^2})\|\tilde{\mathbf{E}}\|^2_{L^2(\Omega_i)} + \tfrac{1}{\delta^2}(\nabla\phi_{\bar\nu}\times(\nabla\phi_{\bar\nu}\times\tilde{\mathbf{E}}),\tilde{\mathbf{E}})^2_{L^2(\Omega_i)} \\ &= i\omega\Big\{(f,E^\delta)_{L^2(\Omega_e)} - (g,E^\delta)_{L^2_t(\partial\Omega)}\Big\} - \|\operatorname{curl} E^\delta\|^2_{L^2(\Omega_e)} \\ &\quad +\omega^2\,\|E^\delta\|^2_{L^2(\Omega_e)} - i\omega\|E^\delta_T\|^2_{L^2_t(\partial\Omega)}\end{aligned} \tag{22}$$

Let us denote by L_δ the right hand side of the previous equality. According to Theorem 2.1, there exists a constant C independent of δ such that

$$|L_\delta| \le C\,(\|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2_t(\partial\Omega)}).$$

On the other hand, thanks to inequality (20), and making use of the inequality $|b-a| \geq |b| - |a|$, we have the lower bound

$$\left|(-\varepsilon_r\omega^2 + \frac{i}{\delta^2})\|\tilde{\mathbf{E}}\|^2_{L^2(\Omega_i)} + \frac{1}{\delta^2}(\nabla\phi_{\bar\nu} \times (\nabla\phi_{\bar\nu} \times \tilde{\mathbf{E}}), \tilde{\mathbf{E}})^2_{L^2(\Omega_i)}\right|$$
$$\geq \left(\frac{1 - \|\nabla\phi_{\bar\nu}\|^2_\infty}{\delta^2} - \varepsilon_r\omega^2\right)\|\tilde{\mathbf{E}}\|^2_{L^2(\Omega_i)} \geq \frac{3}{4\delta^2}\ \|\tilde{\mathbf{E}}\|^2_{L^2(\Omega_i)},$$

for δ is sufficiently small. One therefore deduces from (22)

$$\|\mathrm{curl}\,\tilde{\mathbf{E}}\|^2_{L^2(\Omega_i)} + \frac{3}{4\delta^2}\ \|\tilde{\mathbf{E}}\|^2_{L^2(\Omega_i)} \leq \frac{2}{\delta}\ \|\nabla\phi_{\bar\nu}\|_\infty \|\tilde{\mathbf{E}}\|_{L^2(\Omega_i)}\|\mathrm{curl}\,\tilde{\mathbf{E}}\|_{L^2(\Omega_i)} + C\left(\|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2_t(\partial\Omega)}\right).$$

Then, using again (20) and inequality $2|a||b| \leq a^2 + b^2$ one gets

$$\|\mathrm{curl}\,\tilde{\mathbf{E}}\|^2_{L^2(\Omega_i)} + \frac{3}{4\delta^2}\|\tilde{\mathbf{E}}\|^2_{L^2(\Omega_i)} \leq 2C\left(\|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2_t(\partial\Omega)}\right).$$

The final estimate can now be easily deduced by noticing that $E^\delta = \exp\big(-\beta^{\bar\nu}/\delta\big)\tilde{\mathbf{E}}$ in $\Omega^{\bar\nu}_{i,0}$, and by using the Maxwell equations to get estimates on H^δ from those on E^δ. $\square$

3 Statement of the main results

We shall denote by $(E_e^{\delta,k}, H_e^{\delta,k})$, the approximate solutions in the exterior domain Ω_e, the presence of the integer k meaning that these fields will provide an approximation of order $O(\delta^{k+1})$ of the exact exterior electromagnetic field (E_e^δ, H_e^δ), in a sense that will be made precise by the error estimates (see Theorem 3.1). They are obtained by solving the standard Maxwell equations in the exterior domain Ω_e

$$\begin{cases} i\omega E_e^{\delta,k} - \mathrm{curl}\, H_e^{\delta,k} = f & \text{in } \Omega_e, \\ i\omega H_e^{\delta,k} + \mathrm{curl}\, E_e^{\delta,k} = 0 & \text{in } \Omega_e, \\ E_{e,T}^{\delta,k} - H_e^{\delta,k} \times n = g & \text{on } \partial\Omega, \end{cases} \tag{23}$$

where n denotes the normal to $\partial\Omega$ directed to the exterior of Ω, coupled with an appropriate GIBC on the interior boundary Γ of the form

$$E_e^{\delta,k} \times n + \omega\, \mathcal{D}^{\delta,k}(H_{e,T}^{\delta,k}) = 0, \tag{24}$$

where n denotes the normal to Γ directed to the exterior of Ω_e, $H_{e,T}^{\delta,k}$ is the tangential trace of $H_e^{\delta,k}$, and where $\mathcal{D}^{\delta,k}$ is an adequate local approximation of the *H-to-E* map for the Maxwell equations inside Ω_i, namely the operator:

$$\mathcal{D}^\delta : H^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \quad \longrightarrow \quad H^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$$

defined by

$$\mathcal{D}^\delta \varphi = -\frac{1}{\omega} E_i^\delta \times n|_\Gamma$$

where $\left(E_i^\delta(\varphi), H_i^\delta(\varphi)\right)$ is the solution of the *interior* boundary value problem

$$\begin{cases} (i\varepsilon_r\omega + \frac{1}{\omega\delta^2}) E_i^\delta(\varphi) - \operatorname{curl} H_i^\delta(\varphi) = 0, & \text{in } \Omega_i, \\ i\omega H_i^\delta(\varphi) + \operatorname{curl} E_i^\delta(\varphi) = 0, & \text{in } \Omega_i, \\ H_{i,T}^\delta(\varphi) = \varphi, & \text{on } \Gamma. \end{cases}$$

3.1 The "natural" GIBCs for $k = 0, 1, 2$.

The approach that we shall use in Section 4 for the formal derivation of the GIBCs leads to the following expressions of $\mathcal{D}^{\delta,k}$ (for $k = 0, 1, 2, 3$),

$$\begin{cases} \mathcal{D}^{\delta,0} = 0, \\ \mathcal{D}^{\delta,1} = \delta\sqrt{i}, \\ \mathcal{D}^{\delta,2} = \delta\sqrt{i} + \delta^2(\mathcal{H} - \mathcal{C}), \end{cases} \tag{25}$$

where, $\sqrt{i} := \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ denotes the complex square root of i with positive real part, and $\mathcal{C}$ and $\mathcal{H}$ are the curvature and mean curvature tensors of Γ (we refer to Section 4.1 for more details). Note that the condition of order 0 simply expresses the fact that the limit exterior problem when δ goes to 0 corresponds to the perfectly conducting boundary condition.

3.2 The modified third order GIBC

The same approach extended to $k = 3$ would suggest to take:

$$\mathcal{D}^{\delta,3} = \mathcal{D}_0^{\delta,3} \tag{26}$$

where by definition (we refer to Section 4.1 for the definition of the surface operators ∇_Γ, $\operatorname{div}_\Gamma$, $\operatorname{curl}_\Gamma$ and $\overrightarrow{\operatorname{curl}}_\Gamma$)

$$\mathcal{D}_0^{\delta,3} := \delta\sqrt{i} + \delta^2(\mathcal{H} - \mathcal{C}) + \frac{\delta^3}{2\sqrt{i}}\Big(\mathcal{C}^2 - \mathcal{H}^2 + \varepsilon_r\omega^2 + \nabla_\Gamma \operatorname{div}_\Gamma + \overrightarrow{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma\Big). \tag{27}$$

However, we did not succeed in proving that such a choice was mathematically sound due to the presence of the second order surface operator $\nabla_\Gamma \operatorname{div}_\Gamma + \overrightarrow{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma$. As a self-adjoint operator in $L_t^2(\Gamma)$, this operator (more precisely the associated quadratic form) has no fix sign. This induces difficulties in the study of the forward problem via variational techniques and, as a consequence, the well-posedness of the corresponding boundary value problem is not clear: this is a new difficulty with respect to the scalar wave equation.

This is why we propose hereafter another third order condition, that (formally) gives the same order of accuracy as the one in (25) but admits good mathematical properties with respect to stability and error estimates. The reader will easily notice that the proposed modifications are not the only possible ones (see for instance Remarks 3.1 and 3.2), we exhibit only one particular choice. We shall hereafter present the intuitive reasons that led us to introduce these modifications, postponing the rational justification to the error analysis of Section 6.3.

The first desirable (and probably necessary) property is the absorption property:

$$\mathcal{R}e \int_\Gamma \mathcal{D}^{\delta,3}\varphi \cdot \bar{\varphi}\, d\sigma \geq 0,$$

for any smooth tangential vector field φ on Γ. Such a property is satisfied by the exact DtN operator and expresses the absorbing nature of the conductive medium:

$$\mathcal{R}e \int_\Gamma \mathcal{D}^{\delta,3}\varphi \cdot \bar{\varphi}\, d\sigma = \frac{1}{\omega\delta^2} \int_{\Omega_i} |E_i^\delta(\varphi)|^2\, dx.$$

It will play an essential role in proving the *uniqueness* of solutions. One can observe that this condition is satisfied by $\mathcal{D}^{\delta,1}$ and $\mathcal{D}^{\delta,2}$. For $\mathcal{D}^{\delta,3}$, we see that

$$\mathcal{R}e\, \mathcal{D}_0^{\delta,3} = \delta\frac{\sqrt{2}}{2} + \delta^2(\mathcal{H}-\mathcal{C}) + \frac{\delta^3}{2\sqrt{2}}\left(\mathcal{C}^2 - \mathcal{H}^2 + \varepsilon_r\omega^2\right) + \frac{\delta^3}{2\sqrt{2}}\, \nabla_\Gamma \operatorname{div}_\Gamma\, + \frac{\delta^3}{2\sqrt{2}}\, \vec{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma .$$

The problem comes from the operator $\nabla_\Gamma \operatorname{div}_\Gamma$ which is negative in the L^2 sense. However, we can write formally

$$\left|\begin{aligned} \frac{\delta\sqrt{2}}{2} + \frac{\delta^3}{2\sqrt{2}}\, \nabla_\Gamma \operatorname{div}_\Gamma &= \frac{\delta}{2\sqrt{2}} + \frac{\delta}{2\sqrt{2}} + \frac{\delta^3}{2\sqrt{2}}\, \nabla_\Gamma \operatorname{div}_\Gamma \\ &= \frac{\delta}{2\sqrt{2}}\, (1 - \delta^2\, \nabla_\Gamma \operatorname{div}_\Gamma\,)^{-1} + O(\delta^5), \end{aligned}\right. \tag{28}$$

which suggests to define the real part of $\mathcal{D}^{\delta,3}$ as

$$\left|\begin{aligned} \mathcal{R}e\, \mathcal{D}^{\delta,3} \;&=\; \frac{\delta}{2\sqrt{2}} + \delta^2(\mathcal{H}-\mathcal{C}) + \frac{\delta^3}{2\sqrt{2}}\left(\mathcal{C}^2 - \mathcal{H}^2 + \varepsilon_r\omega^2\right) + \frac{\delta^3}{2\sqrt{2}}\, \vec{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma \\ &+\; \frac{\delta}{2\sqrt{2}}\, (1 - \delta^2\, \nabla_\Gamma \operatorname{div}_\Gamma\,)^{-1} \end{aligned}\right. \tag{29}$$

Remark 3.1 *The approximation process (28) is analogous to the process used in the construction of absorbing boundary conditions for the wave equations, see [8, 4] for instance, where the Padé approximations are preferred to Taylor approximations in order to enforce* *the stability of the resulting approximate problem.*

In (28), the splitting $\dfrac{\delta\sqrt{2}}{2} = \dfrac{\delta}{2\sqrt{2}} + \dfrac{\delta}{2\sqrt{2}}$ *is somewhat arbitrary and could be changed into*

$$\frac{\delta\sqrt{2}}{2} = (1-\alpha)\,\frac{\delta\sqrt{2}}{2} + \alpha\,\frac{\delta\sqrt{2}}{2}$$

for any $\alpha \in]0,1[$. *Our choice corresponds to* $\alpha = 1/2$.

The second modification was guided by the *existence* proof for the boundary value problem associated to the boundary condition (24). We realized that it was useful that the imaginary part of $\mathcal{D}^{\delta,3}$ satisfies a "Garding type" inequality, namely that the principal part of this operator be positive in the L^2 sense. This property is not satisfied by the imaginary part of $\mathcal{D}_0^{\delta,3}$:

$$\mathcal{I}m\;\mathcal{D}_0^{\delta,3} = \delta\frac{\sqrt{2}}{2} - \frac{\delta^3}{2\sqrt{2}}\left(\mathcal{C}^2 - \mathcal{H}^2 + \varepsilon_r\omega^2\right) - \frac{\delta^3}{2\sqrt{2}}\nabla_\Gamma\,\mathrm{div}_\Gamma\; - \frac{\delta^3}{2\sqrt{2}}\;\vec{\mathrm{curl}}_\Gamma\,\mathrm{curl}_\Gamma\;.$$

This time, the problem is due to the negative operator $-\vec{\mathrm{curl}}_\Gamma\,\mathrm{curl}_\Gamma$. The same manipulation as for the real part of $\mathcal{D}^{\delta,3}$ suggests to define the imaginary part of $\mathcal{D}_r^{\delta,3}$ as

$$\left|\begin{array}{rcl} \mathcal{I}m\;\mathcal{D}_r^{\delta,3} & = & \dfrac{\delta}{2\sqrt{2}} + \dfrac{\delta^3}{2\sqrt{2}}\left(\mathcal{C}^2 - \mathcal{H}^2 + \varepsilon_r\omega^2\right) - \dfrac{\delta^3}{2\sqrt{2}}\nabla_\Gamma\,\mathrm{div}_\Gamma \\ & + & \dfrac{\delta}{2\sqrt{2}}\,(1+\delta^2\;\vec{\mathrm{curl}}_\Gamma\,\mathrm{curl}_\Gamma\,)^{-1}. \end{array}\right. \tag{30}$$

Modifications (29) and (30) lead us to introduce the operator

$$\left|\begin{array}{rcl} \tilde{\mathcal{D}}^{\delta,3} & = & \delta\dfrac{\sqrt{i}}{2} + \delta^2(\mathcal{H}-\mathcal{C}) + \dfrac{\delta^3}{2\sqrt{i}}\left(\mathcal{C}^2 - \mathcal{H}^2 + \varepsilon_r\omega^2\right) \\ & & + \;\dfrac{\sqrt{2}}{4}\,\delta\,\Big(\,(1-\delta^2\;\nabla_\Gamma\,\mathrm{div}_\Gamma\,)^{-1} + \delta^2\;\vec{\mathrm{curl}}_\Gamma\,\mathrm{curl}_\Gamma\,\Big) \\ & & + \;i\;\dfrac{\sqrt{2}}{4}\,\delta\,\Big(\,(1+\delta^2\;\vec{\mathrm{curl}}_\Gamma\,\mathrm{curl}_\Gamma\,)^{-1} - \delta^2\;\nabla_\Gamma\,\mathrm{div}_\Gamma\,\Big). \end{array}\right. \tag{31}$$

which formally satisfies $\tilde{\mathcal{D}}^{\delta,3} = \mathcal{D}_0^{\delta,3} + O(\delta^5)$.

It turns out that even if this condition is suitable for variational study of existence and uniqueness of the resulting boundary value problem, it did not enable us to have a direct proof of optimal error estimates (although we think it can be achieved by constructing the full asymptotic expansion associated with the associated boundary value problem). We realized that the difficulties encountered in the analysis are related to the fact that the operator $\tilde{\mathcal{D}}^{\delta,3}$

is a pseudo-differential operator of order 2, whiles the exact impedance operator which maps continuously $H^{-1/2}(\mathrm{curl}_\Gamma\,,\Gamma)$ into $H^{-1/2}(\mathrm{div}_\Gamma\,,\Gamma)$ is more something between an operator of order -1 and an operator of order 1. This gave us the idea to force our approximate operator to be of order 0 by applying a *regularization* process (the Yosida regularization) to the operators $\vec{\mathrm{curl}}_\Gamma\,\mathrm{curl}_\Gamma$ and $\nabla_\Gamma\,\mathrm{div}_\Gamma$

$$\left|\begin{array}{ll} \vec{\mathrm{curl}}_\Gamma\,\mathrm{curl}_\Gamma \simeq \vec{\mathrm{curl}}_\Gamma\,\mathrm{curl}_\Gamma\ (1+\delta^2\,\vec{\mathrm{curl}}_\Gamma\,\mathrm{curl}_\Gamma\,)^{-1} & \text{in } O(\delta^2), \\ \nabla_\Gamma\,\mathrm{div}_\Gamma \simeq \nabla_\Gamma\,\mathrm{div}_\Gamma\ (1-\delta^2\,\nabla_\Gamma\,\mathrm{div}_\Gamma\,)^{-1}\big) & \text{in } O(\delta^2). \end{array}\right. \tag{32}$$

Such an approximation is consistent with the $O(\delta^5)$ accuracy provided by $\tilde{\mathcal{D}}^{\delta,3}$ since $\vec{\mathrm{curl}}_\Gamma\,\mathrm{curl}_\Gamma$ and $\nabla_\Gamma\,\mathrm{div}_\Gamma$ are multiplied by δ^3. Moreover, it does not affect the good sign properties of the real and imaginary parts of the operator since we "divide" by positive operators. Therefore, we propose for the third order condition the following expression

$$\left|\begin{array}{rcl} \mathcal{D}^{\delta,3} & := & \delta\dfrac{\sqrt{i}}{2}+\delta^2(\mathcal{H}-\mathcal{C})+\dfrac{\delta^3}{2\sqrt{i}}\left(\mathcal{C}^2-\mathcal{H}^2+\varepsilon_r\omega^2\right) \\ & & +\ \dfrac{\sqrt{2}}{4}\,\delta\,\Big(\,(1-\delta^2\,\nabla_\Gamma\,\mathrm{div}_\Gamma\,)^{-1}+\delta^2\,\vec{\mathrm{curl}}_\Gamma\,\mathrm{curl}_\Gamma\ (1+\delta^2\,\vec{\mathrm{curl}}_\Gamma\,\mathrm{curl}_\Gamma\,)^{-1}\Big) \\ & & +\ i\,\dfrac{\sqrt{2}}{4}\,\delta\,\Big(\,(1+\delta^2\,\vec{\mathrm{curl}}_\Gamma\,\mathrm{curl}_\Gamma\,)^{-1}-\delta^2\,\nabla_\Gamma\,\mathrm{div}_\Gamma\ (1-\delta^2\,\nabla_\Gamma\,\mathrm{div}_\Gamma\,)^{-1}\Big). \end{array}\right. \tag{33}$$

Remark 3.2 *The regularization process (32) can also be seen as an analogous to the stabilization process used in numerical methods such as stabilized finite elements or discontinuous Galerkin methods, in order to ensure optimal error estimates. Here again, in (32), we chosed arbitrarily equal to 1 the regularization constant in the term in factor of δ^2. We could have chosen for instance*

$$\vec{\mathrm{curl}}_\Gamma\,\mathrm{curl}_\Gamma \simeq \vec{\mathrm{curl}}_\Gamma\,\mathrm{curl}_\Gamma\ (1+\beta\,\delta^2\,\vec{\mathrm{curl}}_\Gamma\,\mathrm{curl}_\Gamma\,)^{-1}$$

where β is a positive constant. However, the interest of choosing $\beta = 1$, is that we make appear the same operators $(1+\delta^2\,\vec{\mathrm{curl}}_\Gamma\,\mathrm{curl}_\Gamma\,)^{-1}$ and $(1-\delta^2\,\nabla_\Gamma\,\mathrm{div}_\Gamma\,)^{-1})$ that are already present in the expression of $\tilde{\mathcal{D}}^{\delta,3}$), which is of interest from the numerical point of view.

The operator happens to have the good *consistency*, *coercivity* and continuity properties to lead to optimal error estimates. More precisely:

- One can check that

$$\mathcal{D}^{\delta,3}=\mathcal{D}_0^{\delta,3}+\delta^5\,\mathcal{R}^{\delta,3} \tag{34}$$

  where the operator $\mathcal{R}^{\delta,3}$, given by

$$\mathcal{R}^{\delta,3}=\frac{\sqrt{2}}{4}(1+i)\ \Big[\big(1-\delta^2\,\nabla_\Gamma\,\mathrm{div}_\Gamma\,\big)^{-1}\,(\nabla_\Gamma\,\mathrm{div}_\Gamma\,)^2+\big(1+\delta^2\,\vec{\mathrm{curl}}_\Gamma\,\mathrm{curl}_\Gamma\,\big)^{-1}\,(\vec{\mathrm{curl}}_\Gamma\,\mathrm{curl}_\Gamma\,)^2\Big],$$

maps continuously $H_t^{s+4}(\Gamma)$ in $H_t^s(\Gamma)$ and satisfies the uniform bound

$$\|\mathcal{R}^{\delta,3}\|_{\mathcal{L}\left(H_t^{s+4}(\Gamma);H_t^s(\Gamma)\right)} \;\leq\; 1. \tag{35}$$

- One can prove (see Lemma 6.1) that $\mathcal{D}^{\delta,3}$ has the following fundamental properties (obviously satisfied by $\mathcal{D}^{\delta,1}$ and $\mathcal{D}^{\delta,2}$)

$$\forall\varphi \in L_t^2(\Gamma), \quad \|\mathcal{D}^{\delta,k}\varphi\|_\Gamma \leq C_1\ \delta\ \|\varphi\|_\Gamma, \quad \mathcal{R}e\ (\mathcal{D}^{\delta,k}\varphi,\varphi)_\Gamma \geq C_2\ \delta\ \|\varphi\|_\Gamma^2.$$

with C_1 and C_2 strictly positive constants. These appear to be sufficient properties to transform the consistency properties $\mathcal{D}^{\delta,3}$ into optimal error estimates (see the proof of Lemma 6.2).

3.3 Existence, uniqueness and error estimates

The natural functional spaces for the solutions of the approximate problems vary according to the regularity of their traces on Γ. We shall distinguish the case $k = 0$ for which we set

$$\mathcal{V}_H^0 = \{H \in H(\mathrm{curl},\Omega_e)\ ;\ (H\times n)|_{\partial\Omega} \in L_t^2(\partial\Omega)\},\ \ \mathcal{V}_E^0 = \{E \in \mathcal{V}_H^0\ ;\ (E\times n)|_\Gamma = 0\},$$

from the case $k = 1$, 2 or 3 for which we set

$$\mathcal{V}_H^k = \mathcal{V}_E^k = \widetilde{H}(\mathrm{curl},\Omega_e)$$

(see (5) for the definition of $\widetilde{H}(\mathrm{curl},\Omega_e)$). Then we have the following central theorem, that uses and combines the partial results of Sections 4-6.

Theorem 3.1 *For $k = 0$, 1, 2 or 3, there exists δ_k such that for $\delta \leq \delta_k$, the boundary value problem ((23), (24)) has a unique solution $(E_e^{\delta,k}, H_e^{\delta,k}) \in \mathcal{V}_E^k \times \mathcal{V}_H^k$. Moreover, there exists a constant C_k, independent of δ, such that*

$$\|E_e^\delta - E_e^{\delta,k}\|_{H(\mathrm{curl},\Omega_e)} \leq C_k\ \delta^{k+1}.$$

Remark 3.3 *For $k = 0, 1$, the above theorem holds for all δ.*

4 Formal derivation of the GIBCs

4.1 Preliminary material

We recall in this section some well known facts about differential geometry and differential operators.

Local coordinates. Let n be the normal field defined on Γ and directed to the interior of Ω_i. For a sufficiently small positive constant $\bar{\nu}$ (see condition (39) below) we define

$$\Omega_i^{\bar{\nu}} = \{x \in \Omega_i \; ; \; \text{dist}(x, \partial\Omega_i) < \bar{\nu}\}.$$

To any $x \in \Omega_i^{\bar{\nu}}$ we uniquely associate the local parametric coordinates $(x_\Gamma, \nu) \in \Gamma \times (0, \bar{\nu})$ through

$$x = x_\Gamma + \nu\, n, \quad x \in \Omega_i^{\bar{\nu}}. \tag{36}$$

Tangential (or surface) differential operators. In what follows we deal with various fields defined on Γ: scalar fields φ (with values in $\mathbb{C}$), vector fields V (with values in $\mathbb{C}^3$) and matrix (or tensor) fields $\mathbf{A}$ (with values in $\mathcal{L}(\mathbb{C}^3)$). By definition:

- A vector field V is tangential if and only if $V \cdot n = 0$ (as a scalar field along Γ).
- A matrix field $\mathbf{A}$ is tangential if and only if $\mathbf{A}\, n = 0$ (as a vector field along Γ).

For simplicity, we assume that these fields have at least C^1 regularity, but this can be removed by interpreting the derivatives in the sense of distributions.

We recall that the surface gradient operator ∇_Γ is defined by:

$$\nabla_\Gamma\, \varphi(x_\Gamma) = \nabla\hat{\varphi}(x_\Gamma), \quad \forall \varphi : \Gamma \to \mathbb{R},$$

where $\hat{\varphi}$ is the 3-D vector field defined locally in $\Omega_i^{\bar{\nu}}$ by $\hat{\varphi}(x_\Gamma + \nu\, n) = \varphi(x_\Gamma)$. Note that $\nabla_\Gamma\, \varphi$ is a tangential vector field. We can define in the same way the surface gradient of a vector field as a tangential matrix field whose columns are the surface gradients of each component of the vector field.

We denote by $-\text{div}_\Gamma$ the $L^2(\Gamma)$- adjoint of ∇_Γ : $-\text{div}_\Gamma$ maps a tangential vector field into a scalar field. More generally, if $\mathbf{A}(x_\Gamma)$ is a tangential matrix field on Γ, we define the operator $\mathbf{A}\nabla_\Gamma$ for a scalar field $\varphi(x_\Gamma)$ by

$$(\mathbf{A}\nabla_\Gamma)u := \mathbf{A}(\nabla_\Gamma\, u).$$

In the same way, we define the operator $(\mathbf{A}\nabla_\Gamma)\cdot$ acting on a tangential vector field $V(x_\Gamma)$ as:

$$(\mathbf{A}\nabla_\Gamma) \cdot V := \sum_{i=1}^{3} (\mathbf{A}\nabla_\Gamma\, V_i)_i,$$

where the subscript i denotes the i^{th} component of a vector in the canonical basis of $\mathbb{R}^3$.

We then define the surface curl of a tangential vector filed $V(x_\Gamma)$ and the surface vector curl of a scalar function $\varphi(x_\Gamma)$ as

$$\text{curl}_\Gamma\, V := \text{div}_\Gamma\, (V \times n) \quad \text{and} \quad \vec{\text{curl}}_\Gamma\, \varphi := (\nabla_\Gamma\, \varphi) \times n.$$

Of course, the various operators ∇_Γ , div_Γ , $\mathcal{R}\nabla_\Gamma$, curl_Γ and $\vec{\mathrm{curl}}_\Gamma$ applies in principle to functions defined on Γ. However, they can obviously be understood as (partial) differential operators acting on fields defined in the (3-D) domain $\Gamma \times (0, \bar{\nu})$. For instance, if $\phi \in C^1(\Gamma \times (0,\bar{\nu}))$, we define $\nabla_\Gamma \phi \in C^0(\Gamma \times (0,\bar{\nu}))^3$ as follows (with obvious notation):

$$\forall\, \nu \in (0,\bar{\nu}), \quad \Big[\nabla_\Gamma\, \phi\Big](\cdot,\nu)] := \nabla_\Gamma \Big[\phi(\cdot,\nu)\Big].$$

We apply similar rules to div_Γ , $\mathcal{R}\nabla_\Gamma$, curl_Γ and $\vec{\mathrm{curl}}_\Gamma$. The extension of these definition in the sense of distributions is also elementary (as soon as Γ is C^∞).

Geometrical tools. In what follows, and for the sake of the notation conciseness, we shall most of time not explicitly indicate the dependence on x_Γ of the functions, except when we feel it necessary. We shall be more precise in mentioning the possible dependence with respect to the normal coordinate ν.

A particularly fundamental tensor field is the curvature tensor $\mathcal{C}$, defined by $\mathcal{C} := \nabla_\Gamma n$. We recall that $\mathcal{C}$ is symmetric and $\mathcal{C}\, n = 0$. We denote c_1, c_2 the other two eigenvalues of $\mathcal{C}$ (namely the *principal curvatures*) associated with tangential eigenvectors τ_1, τ_2 ($\tau_1 \cdot n = \tau_2 \cdot n = 0$). We also introduce

$$g := c_1 c_2 \quad \text{and} \quad h := \frac{1}{2}(c_1 + c_2) \tag{37}$$

which are respectively the *Gaussian* and *mean curvatures* of Γ, and also introduce the associated matrix fields:

$$\mathcal{H} = h\, I_\Gamma \quad \text{and} \quad \mathcal{G} = g\, I_\Gamma, \tag{38}$$

where $I_\Gamma(x_\Gamma)$ denotes the projection operator on the tangent plane to Γ at x_Γ.

Let us introduce (this is the Jacobian of the transformation $(x_\Gamma, \nu) \to x$ - see (36))

$$J(\nu)\ \big(= J(\nu, x_\Gamma)\big)\ := \det(I + \nu\, \mathcal{C}) = 1 + 2\nu h + \nu^2 g,$$

and we choose $\bar{\nu}$ sufficiently small in such a way that

$$\forall\, \nu < \bar{\nu}, \quad \forall\, x_\Gamma \in \Gamma, \quad J(\nu, x_\Gamma)) = 1 + 2\nu h(x_\Gamma) + \nu^2 g(x_\Gamma) > 0. \tag{39}$$

Thus, for each $\nu < \bar{\nu}$, there exists a tangential matrix field $x_\Gamma \to \mathcal{R}_\nu(x_\Gamma)$ such that

$$(I + \nu\, \mathcal{C}(x_\Gamma))\; \mathcal{R}_\nu(x_\Gamma) = I_\Gamma(x_\Gamma).$$

More precisely, there exists a tangential matrix field on Γ, $\mathcal{M}(x_\Gamma)$, such that:

$$I_\Gamma + \nu \mathcal{M} := J(\nu)\, \mathcal{R}_\nu, \quad \forall\, x_\Gamma \in \Gamma, \quad \forall\, \nu < \bar{\nu}.$$

One easily sees (using for instance the eigenbasis (τ_1, τ_2, n) of $\mathcal{C}$) that

$$\mathcal{M} = 2\mathcal{H} - \mathcal{C} \quad \text{and} \quad \mathcal{M}\, \mathcal{C} = \mathcal{G}.$$

The curl operator in local coordinates. The basic step of our forthcoming calculations will be to rewrite the Maxwell equations in the domain $\Omega_i^{\bar\nu}$,by using the local coordinates. For this, we need the expression of the curl operator in the variables (x_Γ, ν). It is shown in [10] that the curl of a 3-D vector field $V : \Omega_i^{\bar\nu} \to \mathbb{R}^3$ is given in parametric coordinates by :

$$\operatorname{curl} V = \Big[(\mathcal{R}_\nu \nabla_\Gamma\,) \cdot (\widehat{V} \times n)\Big]\, n + \Big[\mathcal{R}_\nu \nabla_\Gamma\, (\widehat{V} \cdot n)\Big] \times n - (\mathcal{R}_\nu \mathcal{C} \widehat{V}) \times n - \partial_\nu (\widehat{V} \times n),$$

where V and $\widehat{V}$ (defined on $\Gamma \times (0, \bar\nu)$) are related by

$$\widehat{V}(x_\Gamma, \nu) = V(x_\Gamma + \nu\, n).$$

This formula can be written in a more convenient form (for us), after multiplication by $J(\nu)$:

$$\left|\begin{array}{rcl} J(\nu)\, \operatorname{curl} V & = & \Big[\big((I + \nu\mathcal{M})\nabla_\Gamma\,\big) \cdot (\widehat{V} \times n)\Big]\, n + \Big[(I + \nu\mathcal{M})\nabla_\Gamma\, (\widehat{V} \cdot n)\Big] \times n \\[2ex] & - & \Big[(\mathcal{C} + \nu\mathcal{G})\widehat{V}\Big] \times n - J(\nu)\, \partial_\nu (\widehat{V} \times n), \end{array}\right.$$

or, in an equivalent form,

$$J(\nu)\, \operatorname{curl} V = \big(C_\Gamma + \nu\, C_\Gamma^M\big)\, \widehat{V} - J(\nu)\, \partial_\nu (\widehat{V} \times n) \tag{40}$$

where we have introduced the notation

$$\left\{\begin{array}{l} C_\Gamma \widehat{V} = \big(\operatorname{curl}_\Gamma \widehat{V}\big) n + \overrightarrow{\operatorname{curl}}_\Gamma\, (\widehat{V} \cdot n) - \mathcal{C}\widehat{V} \times n \\[2ex] C_\Gamma^M \widehat{V} = \big(\operatorname{curl}_\Gamma^M \widehat{V}\big) n + \overrightarrow{\operatorname{curl}}_\Gamma^M\, (\widehat{V} \cdot n) - \mathcal{G}\widehat{V} \times n \\[2ex] \overrightarrow{\operatorname{curl}}_\Gamma^M\, u := (\mathcal{M}\nabla_\Gamma\, u) \times n \quad \text{and} \quad \operatorname{curl}_\Gamma^M \widehat{V} = (\mathcal{M}\nabla_\Gamma\,) \cdot (\widehat{V} \times n). \end{array}\right. \tag{41}$$

This expression is convenient for the asymptotic matching procedure, described hereafter, because we made explicit the (polynomial) dependence of the operators with respect to ν.

Functional spaces on Γ and trace spaces.

We assume that the definition of $H^s(\Gamma)$ for any real s is well known. We shall denote by

$$(\cdot,\cdot)_\Gamma \quad \text{and} \quad \langle \cdot,\cdot \rangle_\Gamma\,,$$

respectively the inner product in $L^2(\Gamma)^3$ and the duality bracket $\mathcal{D}'(\Gamma)^3 - \mathcal{D}(\Gamma)^3$.

Next, we introduce some notation for spaces of tangent vector fields along Γ. For any $s \le 0$, we set:

$$\left\{\begin{array}{ll} H_t^s(\Gamma) = \{V \in H^s(\Gamma)^3 \;/\; V \cdot n = 0 \text{ on } \Gamma\} & \big(H_t^0(\Gamma) = L_t^2(\Gamma)\big) \\[2ex] H_t^{-s}(\Gamma) = \{V \in H^{-s}(\Gamma)^3 \;/\; \langle V, \varphi\, n\rangle_\Gamma = 0,\ \forall\, \varphi \in H^s(\Gamma)\} & \big(\equiv \big(H_t^s(\Gamma)\big)'\big) \end{array}\right.$$

as well as
$$\left\{\begin{array}{l} H^s(\mathrm{div}_\Gamma\,,\Gamma) = \{V \in H^s_t(\Gamma)^3 \;/\; \mathrm{div}_\Gamma\, V \in H^s(\Gamma)\} \\[2ex] H^s(\mathrm{curl}_\Gamma\,,\Gamma) = \{V \in H^s_t(\Gamma)^3 \;/\; \mathrm{curl}_\Gamma\, V \in H^s(\Gamma)\} \end{array}\right.$$
equipped with their natural graph norms (we notice that $H^0(\mathrm{div}_\Gamma\,,\Gamma)$ and $H^0(\mathrm{curl}_\Gamma\,,\Gamma)$ are often denoted by respectively $H(\mathrm{div}_\Gamma\,,\Gamma)$ and $H(\mathrm{curl}_\Gamma\,,\Gamma)$). Finally, we recall the well known trace theorems:

Theorem 4.1 *The two trace mappings*
$$\left\{\begin{array}{lll} u \in C^\infty(\Omega_e)^3 & \mapsto & u \times n|_\Gamma \\[2ex] u \in C^\infty(\Omega_e)^3 & \mapsto & u_T = u - (u\cdot n)n \quad \big(\equiv n \times (u \times n)\big) \end{array}\right.$$
can be extended as continuous and surjective linear applications from $H(\mathrm{curl},\Omega_e)$ *onto* $H^{-\frac12}(\mathrm{div}_\Gamma\,,\Gamma)$ *and* $H^{-\frac12}(\mathrm{curl}_\Gamma\,,\Gamma)$ *respectively. Moreover,* $H^{-\frac12}(\mathrm{div}_\Gamma\,,\Gamma)$ *is the dual of* $H^{-\frac12}(\mathrm{curl}_\Gamma\,,\Gamma)$ *and one has the Green's formula:*
$$\int_\Omega \big(\mathrm{curl}\, u \cdot v - u \cdot \mathrm{curl}\, v\big)\; dx = \langle u \times n, v_T\rangle_\Gamma = -\,\langle v \times n, u_T\rangle_\Gamma$$
$\forall\, (u,v) \in H(\mathrm{curl},\Omega_e)^2$.

4.2 The asymptotic ansatz.

To formulate our ansatz, it is useful to introduce a cutoff function $\chi \in C^\infty(\Omega_i)$ such that $\chi = 1$ in $\Omega_i^{\bar\nu}$ and $\chi = 0$ in $\Omega_i \setminus \Omega_i^{2\bar\nu}$ for a sufficiently small $\bar\nu > 0$. For this ansatz we are not interested in the part of the solution inside the support of $(1-\chi)$, since we already know that the norm of the solution in this part exponentially decay to 0 as δ goes to 0 (this is Theorem 2.2). For the remaining part of the solution, we postulate the following expansions:
$$\left|\begin{array}{ll} E_e^\delta(x) = E_e^0(x) + \delta\; E_e^1(x) + \delta^2\; E_e^2(x) + \cdots & \text{for } x \in \Omega_e, \\ H_e^\delta(x) = H_e^0(x) + \delta\; H_e^1(x) + \;\delta^2 H_e^2(x) + \cdots & \text{for } x \in \Omega_e, \end{array}\right. \tag{42}$$
where E_e^ℓ, H_e^ℓ, $\ell = 0,\, 1,\, \cdots$ are functions defined on Ω_e and
$$\left|\begin{array}{ll} \chi(x)E_i^\delta(x) = E_i^0(x_\Gamma,\nu/\delta) + \delta\; E_i^1(x_\Gamma,\nu/\delta) + \delta^2\; E_i^2(x_\Gamma,\nu/\delta) + \cdots & \text{for } x \in \Omega_i^{\bar\nu} \\ \chi(x)H_i^\delta(x) = H_i^0(x_\Gamma,\nu/\delta) + \delta\; H_i^1(x_\Gamma,\nu/\delta) + \delta^2\; H_i^2(x_\Gamma,\nu/\delta) + \cdots & \text{for } x \in \Omega_i^{\bar\nu} \end{array}\right. \tag{43}$$
where x, x_Γ and ν are as in (36) and where $E_i^\ell(x_\Gamma,\eta)$, $H_i^\ell(x_\Gamma,\eta) : \Gamma \times \mathbb{R}^+ \mapsto \mathbb{C}$ satisfy
$$\left|\begin{array}{ll} \text{For a.e. } x_\Gamma \in \Gamma, & \displaystyle\int_0^{+\infty} |E_i^\ell(x_\Gamma,\eta)|^2\; d\eta < +\infty, \\[3ex] \text{For a.e. } x_\Gamma \in \Gamma, & \displaystyle\int_0^{+\infty} |H_i^\ell(x_\Gamma,\eta)|^2\; d\eta < +\infty. \end{array}\right. \tag{44}$$

Remark 4.1 *The condition (44) will imply that E_i^ℓ and H_i^ℓ are exponentially decreasing with respect to η.*

$$\left| \begin{array}{lll} \text{For a.e. } x_\Gamma \in \Gamma, & \lim\limits_{\eta\to\infty} E_i^\ell(x_\Gamma,\eta) = 0, & \text{(exponentially fast)} \\ \text{For a.e. } x_\Gamma \in \Gamma, & \lim\limits_{\eta\to\infty} H_i^\ell(x_\Gamma,\eta) = 0, & \text{(exponentially fast)} \end{array} \right.$$

which is coherent with the existence of a boundary layer suggested by Theorem 2.2.

Remark 4.2 *Expansion (43) makes sense since the local coordinates (x_Γ, ν) can be used inside the support of χ.*

In the next step, we shall identify the set of equations satisfied by (E_e^ℓ, H_e^ℓ) and (E_i^ℓ, H_i^ℓ), $\ell \geq 0$ by writing, formally, that we want to solve the transmission problem (3)-(4).

In the sequel, it is useful to introduce the notation

$$\left| \begin{array}{l} \widetilde{E}_i^\delta(x_\Gamma,\eta) := E_i^0(x_\Gamma,\eta) + \delta\ E_i^1(x_\Gamma,\eta) + \delta^2\ E_i^2(x_\Gamma,\eta) + \cdots \quad (x_\Gamma,\eta) \in \Gamma\times\mathbb{R}^+, \\ \widetilde{H}_i^\delta(x_\Gamma,\eta) := H_i^0(x_\Gamma,\eta) + \delta\ H_i^1(x_\Gamma,\eta) + \delta^2\ H_i^2(x_\Gamma,\eta) + \cdots \quad (x_\Gamma,\eta) \in \Gamma\times\mathbb{R}^+. \end{array} \right. \tag{45}$$

so that ansatz (43) has to be understood as

$$\left| \begin{array}{l} \chi(x) E_i^\delta(x) = \widetilde{E}_i^\delta(x_\Gamma, \nu/\delta) + O(\delta^\infty) \quad \text{for } x \in \Omega_i^{\bar\nu}, \\ \chi(x) H_i^\delta(x) = \widetilde{H}_i^\delta(x_\Gamma, \nu/\delta) + O(\delta^\infty) \quad \text{for } x \in \Omega_i^{\bar\nu}. \end{array} \right. \tag{46}$$

4.3 The equations for the exterior fields.

This is the easy part of the job. The equations are directly derived from (3) and we obtain that (E_e^k, H_e^k) satisfy

$$\left\{ \begin{array}{lll} i\omega E_e^k - \operatorname{curl} H_e^k = f_k, & \text{in } \Omega_e, \\ i\omega H_e^k + \operatorname{curl} E_e^k = 0, & \text{in } \Omega_e, \\ E_{e\ |T}^k - H_e^k \times n = g_k, & \text{on } \partial\Omega \end{array} \right. \tag{47}$$

where we have set $f_0 = f$, $g_0 = g$ and $f_k = 0$, $g_k = 0$, for $k \geq 1$, (47) being complemented with the interface condition

$$E_e^k|_\Gamma(x_\Gamma) \times n = E_i^k(x_\Gamma, 0) \times n, \quad \text{for } x_\Gamma \in \Gamma, \tag{48}$$

which completely defines (E_e^k, H_e^k) if $E_i^k(x_\Gamma, 0) \times n$ is known.

4.4 The equations for the interior fields.

As indicated above, we need to compute the interior fields E_i^k. The principle consists into expressing this field in terms of the tangential boundary values of (H_e^ℓ), $\ell \leq k$ by solving the interior equations. More precisely, we now substitute the expansion (45, 46) into the system (4) and assume that the quantity:

$$H_e^\delta \times n = \sum_{k=0}^{+\infty} \delta^k \; H_e^k \times n$$

is known on Γ. We need of course to rewrite the equations of (4) in the local "scaled" coordinates

$$(x_\Gamma, \eta = \nu/\delta).$$

Using formula (40) with $\nu = \delta\eta$ we obtain:

$$\left\{ \begin{array}{ll} J(\delta\eta) \Big(i\varepsilon_r \omega + \dfrac{1}{\omega\delta^2} \Big) \widetilde{E}_i^\delta - \big(\mathbf{C}_\Gamma + \delta\eta \, \mathbf{C}_\Gamma^M \big) \widetilde{H}_i^\delta + \dfrac{J(\delta\eta)}{\delta} \, \partial_\eta \widetilde{H}_i^\delta \times n = O(\delta^\infty), & \text{in } \Gamma \times [0, \frac{\overline{\nu}}{\delta}), \\[2ex] iJ(\delta\eta) \, \omega \, \widetilde{H}_i^\delta + (\mathbf{C}_\Gamma + \delta\eta \, \mathbf{C}_\Gamma^M) \widetilde{E}_i^\delta - \dfrac{J(\delta\eta)}{\delta} \, \partial_\eta \widetilde{E}_i^\delta \times n = O(\delta^\infty), & \text{in } \Gamma \times [0, \frac{\overline{\nu}}{\delta}), \end{array} \right. \tag{49}$$

These equations are complemented by the boundary condition

$$\widetilde{H}_i^\delta(x_\Gamma, 0) \times n + O(\delta^\infty) = H_e^\delta \times n \, (x_\Gamma), \quad x_\Gamma \in \Gamma. \tag{50}$$

The substitution of (45, 46) into (49, 50) leads to a sequence of problems that enable us to inductively determine the fields (E_i^k, H_i^k). The computations are relatively delicate but straightforward. The most difficult task is to explain the recurrence properly, which is the aim of this section. In Section 4.5, we shall compute explicitly the first terms of the expansions.

It turns out to be very useful to make a change of unknown concerning the electric field. This is motivated by the observation that

$$E_i^0 = 0. \tag{51}$$

This fact can be explained along the following lines: indeed from (45) and (46) one deduces (at least formally) that:

$$\|E_i^\delta\|^2_{L^2(\Omega_i)} \sim \; \delta \int_\Gamma \int_0^{+\infty} |E_i^0(x_\Gamma, \eta)|^2 \, d\eta \, d\sigma.$$

Therefore, the a priori estimate (10), which says that $\|E_i^\delta\|^2_{L^2(\Omega_i)} = O(\delta^2)$, implies $E_i^0 = 0$.

The expansion for the electric field therefore starts with $\delta \; E_i^1$ while for the magnetic field

$H_i^0 \neq 0$. In some sense there is a natural shift of one power of δ between the expansions of the electric and magnetic fields. This is why we introduce the "normalized" electric field:

$$\widehat{\mathbb{E}}_i^\delta = \frac{1}{\delta}\, \widetilde{E}_i^\delta, \tag{52}$$

and we seek an expansion of the form

$$\widehat{\mathbb{E}}_i^\delta(x_\Gamma,\eta) := \widehat{\mathbb{E}}_i^0(x_\Gamma,\eta) + \delta\, \widehat{\mathbb{E}}_i^1(x_\Gamma,\eta) + \delta^2\, \widehat{\mathbb{E}}_i^2(x_\Gamma,\eta) + \cdots \quad (x_\Gamma,\eta) \in \Gamma \times \mathbb{R}^+, \tag{53}$$

with the correspondence

$$E_i^{k+1} = \widehat{\mathbb{E}}_i^k, \quad k \geq 1.$$

We then rewrite (49) as a system of equations for $(\widehat{\mathbb{E}}_i^\delta, \widetilde{H}_i^\delta)$ (we have multiplied the first equation by δ)

$$\begin{cases} J(\delta\eta)\left(i\varepsilon_r\delta\omega + \dfrac{1}{\omega}\right)\widehat{\mathbb{E}}_i^\delta - \left(\delta\,\mathbf{C}_\Gamma + \delta^2\eta\,\mathbf{C}_\Gamma^M\right)\widetilde{H}_i^\delta + J(\delta\eta)\,\partial_\eta\widetilde{H}_i^\delta \times n = 0, & \text{in } \Gamma\times[0,\frac{\bar{\nu}}{\delta}), \\ iJ(\delta\eta)\,\omega\,\widetilde{H}_i^\delta + (\delta\,\mathbf{C}_\Gamma + \delta^2\,\eta\,\mathbf{C}_\Gamma^M)\,\widehat{\mathbb{E}}_i^\delta - J(\delta\eta)\,\partial_\eta\widehat{\mathbb{E}}_i^\delta \times n = 0, & \text{in } \Gamma\times[0,\frac{\bar{\nu}}{\delta}), \end{cases}$$

that we can rewrite by separating the "δ-independent" part, that we keep in the left hand side, from the remaining terms, that we put in the right hand side, as follows:

$$\left\{\begin{aligned} \partial_\eta\widetilde{H}_i^\delta \times n + \frac{1}{\omega}\,\widehat{\mathbb{E}}_i^\delta &= \sum_{\ell=1}^{4}\delta^\ell A_H^{(\ell)}(\widehat{\mathbb{E}}_i^\delta,\widetilde{H}_i^\delta) \quad \text{in } \Gamma\times\mathbb{R}^+, \\ -\,\partial_\eta\widehat{\mathbb{E}}_i^\delta \times n + i\omega\,\widetilde{H}_i^\delta &= \sum_{\ell=1}^{2}\delta^\ell A_E^{(\ell)}(\widehat{\mathbb{E}}_i^\delta,\widetilde{H}_i^\delta), \quad \text{in } \Gamma\times\mathbb{R}^+. \end{aligned}\right. \tag{54}$$

The linear operators $\{A_H^{(\ell)},\ \ell = 1,2,3\}$ are given by:

$$\begin{cases} A_H^{(1)}\,(u,v) = \mathbf{C}_\Gamma\ v - 2h\eta\,\left(\partial_\eta v \times n + \dfrac{1}{\omega}\ u\right), \\ A_H^{(2)}\,(u,v) = -i\varepsilon_r\omega\ u + \eta\,\mathbf{C}_\Gamma^M\ v - g\eta^2\,\left(\partial_\eta v \times n + \dfrac{1}{\omega}\ u\right), \\ A_H^{(3)}\,(u,v) = -2\eta h\ i\varepsilon_r\omega\ u, \\ A_H^{(4)}\,(u,v) = -2\eta^2 g\ i\varepsilon_r\omega\ u, \end{cases}$$

and the linear operators $\{A_E^{(\ell)},\ \ell = 1,2\}$ are given by:

$$\begin{cases} A_E^{(1)}\,(u,v) = -\mathbf{C}_\Gamma\ u + 2h\eta\,\left(\partial_\eta u \times n - i\omega\ v\right), \\ A_E^{(2)}\,(u,v) = -\eta\,\mathbf{C}_\Gamma^M\ u + g\eta^2\,\left(\partial_\eta u \times n - i\omega\ v\right). \end{cases}$$

Substituting (45) and (53) into (54) then equating the same powers of δ leads to the following systems:

$$\left\{\begin{array}{rcll} \partial_\eta H_i^k \times n + \dfrac{1}{\omega}\,\widehat{\mathbb{E}}_i^k &=& \displaystyle\sum_{\ell=1}^{4} A_H^{(\ell)}(\widehat{\mathbb{E}}_i^{k-\ell}, H_i^{k-\ell}), & \text{in } \Gamma\times\mathbb{R}^+, \\ -\,\partial_\eta \widehat{\mathbb{E}}_i^k \times n + i\omega\, H_i^k &=& \displaystyle\sum_{\ell=1}^{2} A_E^{(\ell)}(\widehat{\mathbb{E}}_i^{k-\ell}, H_i^{k-\ell}), & \text{in } \Gamma\times\mathbb{R}^+, \end{array}\right. \tag{55}$$

for $k = 0, 1, 2, \cdots$, with the convention $\widehat{\mathbb{E}}_i^\ell = H_i^\ell = 0$ for $\ell < 0$.

Of course, these equations have to be complemented with the conditions (see (50) and (44))

$$\left\{\begin{array}{l} H_i^k(x_\Gamma, 0)\times n = H_e^k(x_\Gamma, 0)\times n, \\ \displaystyle\int_0^{+\infty} |H_i^k(x_\Gamma,\eta)|^2\, d\eta < +\infty, \quad \text{and} \quad \int_0^{+\infty} |\widehat{\mathbb{E}}_i^k(x_\Gamma,\eta)|^2\, d\eta < +\infty, \end{array}\right. \tag{56}$$

$\forall\, x_\Gamma \in \Gamma$.

The reader can already notice how the roles of the variables η and x_Γ have been separated. The variable x_Γ appears as parameter for determining $(\widehat{\mathbb{E}}_i^k, H_i^k)$ from the previous $(\widehat{\mathbb{E}}_i^\ell, H_i^\ell)$'s since, for each x_Γ, one simply has to solve an ordinary differential system in the variable η. The solutions to this inductive system of equations can be expressed in a general way using the result of the following technical lemma. For that purpose it is useful to introduce

$$\mathbf{P}_k(\Gamma, \mathbb{R}^+; \mathbb{C}^3) := \Big\{ u(x_\Gamma, \eta) = \sum_{j=1}^{k} a_j(x_\Gamma)\, \eta^j,\ a_j \in C^\infty(\Gamma; \mathbb{C}^3) \Big\}.$$

Lemma 4.1 *Let $(f, g) \in \mathbf{P}_k(\Gamma, \mathbb{R}^+; \mathbb{C}^3)^2$ and $\varphi \in C^\infty(\Gamma; \mathbb{R}^3)$, Then the problem,*

Find $(u, v) \in C^\infty(\Gamma; C^\infty(\mathbb{R}^+))^2$ such that,

$$\left\{\begin{array}{ll} \partial_\eta v \times n + \dfrac{1}{\omega}\, u = e^{-\sqrt{i}\,\eta}\, f(\eta, \cdot), & \textit{in } \Gamma\times\mathbb{R}^+, \\ -\,\partial_\eta u \times n + i\omega\, v = e^{-\sqrt{i}\,\eta}\, g(\eta, \cdot), & \textit{in } \Gamma\times\mathbb{R}^+. \end{array}\right. \tag{57}$$

with the conditions:

$$\left\{\begin{array}{ll} \forall\, x_\Gamma \in \Gamma, & u(x_\Gamma, 0)\times n = \varphi(x_\Gamma), \\ \forall\, x_\Gamma \in \Gamma, & \displaystyle\int_0^{+\infty} |u(x_\Gamma,\eta)|^2\, d\eta < +\infty, \quad \int_0^{+\infty} |v(x_\Gamma,\eta)|^2\, d\eta < +\infty, \end{array}\right. \tag{58}$$

has a unique solution, which is of the form

$$u(x_\Gamma, \eta) = e^{-\sqrt{i}\,\eta}\, p(x_\Gamma, \eta) \quad \textit{and} \quad v(x_\Gamma, \eta) = e^{-\sqrt{i}\,\eta}\, q(x_\Gamma, \eta) \tag{59}$$

with $(p, q) \in \mathbf{P}_{k+1}(\Gamma, \mathbb{R}^+; \mathbb{C}^3)^2$ *and with the square root definition* $\sqrt{i} := \frac{\sqrt{2}}{2}\,(1 + i)$.

Proof. This is a simple exercise on ordinary differential equations. For the uniqueness, assuming $\varphi = 0$ and $f = g = 0$, we can eliminate v between the two equations of (57) and obtain, with $u_T = n \times (u \times n) = u - (u \cdot n)\, n$:

$$\partial_{\eta\eta} u_T - i\, u_T = 0.$$

Since we also have $u_T(x_\Gamma, 0) = 0$, the only solution satisfying the second condition of (58) is $u_T = 0$. One deduces from the second equation of (57) that $v = 0$ which in turn implies $u = 0$ using the first equation of (57).

We proceed in the same way to prove the existence of solutions of the form (59). First, the projection on n of the two equations directly gives

$$\begin{cases} (u \cdot n) = \omega\, e^{-\sqrt{i}\,\eta}\, (f \cdot n), \\[2ex] (v \cdot n) = \omega\, e^{-\sqrt{i}\,\eta}\, (f \cdot n). \end{cases}$$

For the tangential components, one easily gets

$$\begin{cases} \partial_{\eta\eta} u_T - i\, u_T = \widetilde{f}_T\, e^{-\sqrt{i}\,\eta}, \\[2ex] \partial_{\eta\eta} v_T - i\, v_T = \widetilde{g}_T\, e^{-\sqrt{i}\,\eta}, \end{cases}$$

where

$$\begin{cases} \widetilde{f}_T := \big(\partial_\eta g - \sqrt{i}\,\eta g\big) \times n + i\omega f_T, & \in \mathbf{P}_k(\Gamma, \mathbb{R}^+; \mathbb{C}^3), \\[2ex] \widetilde{g}_T := n \times \big(\partial_\eta f - \sqrt{i}\,\eta f\big) - \dfrac{1}{\omega} g_T, & \in \mathbf{P}_k(\Gamma, \mathbb{R}^+; \mathbb{C}^3). \end{cases}$$

Thus, the key point is the resolution of the scalar differential equation:

$$\partial_{\eta\eta}\psi - i\psi = s(\eta)\, e^{-\sqrt{i}\,\eta}, \quad \text{in } \mathbb{R}^+, \tag{60}$$

with $\psi(0) = \psi_0$. The unknown ψ is sought in $L^2(\mathbb{R}^+)$ and $s(\eta)$ is given in P_m, the space of polynomials in η of degree less than m. We prove that the solution of this equation has the form:

$$\psi(\eta) = e^{-\sqrt{i}\,\eta}\, p(\eta) \quad \text{with } p \in P_{m+1}.$$

Indeed, after subtracting

$$\psi_0\, e^{-\sqrt{i}\,\eta},$$

we can assume that $\psi_0 = 0$. Introducing the space:

$$\widetilde{P}_{m+1} = \{p \in P_m \ / \ p(0) = 0\},$$

Observing that $[\partial_{\eta\eta} - i]\left\{e^{-\sqrt{i}\,\eta}\, x^\ell\right\} = e^{-\sqrt{i}\,\eta}\left\{\ell\, x^{\ell-2}\left[2\sqrt{i}\,x + \ell - 1\right]\right\}$, we deduce that

$$(\partial_{\eta\eta} - i) \in \mathcal{L}(e^{-\sqrt{i}\,\eta}\ \widetilde{P}_{m+1}, e^{-\sqrt{i}\,\eta}\ P_m).$$

From the uniqueness result for L^2 solutions of (60) with Dirichlet condition at $\eta = 0$, we deduce that the operator $(\partial_{\eta\eta} - i)$ is injective in the space $e^{-\sqrt{i}\,\eta}\ \widetilde{P}_{m+1}$. Since $e^{-\sqrt{i}\,\eta}\ P_m$ and $e^{-\sqrt{i}\,\eta}\ \widetilde{P}_{m+1}$ have the same dimension, this operator is an isomorphism from $e^{-\sqrt{i}\,\eta}\ P_m$ into $e^{-\sqrt{i}\,\eta}\ \widetilde{P}_{m+1}$, which concludes the proof. □

As an application of this lemma we obtain the following result.

Theorem 4.2 *The fields $H_e^k \times n \in C^\infty(\Gamma;\mathbb{C}^3)$ being given, there exists a unique sequence*

$$\left\{\ (\widehat{\mathbb{E}}_i^k, H_i^k) \in C^\infty(\Gamma;\mathbb{C}^3)^2,\ k = 0,1,2,\cdots\right\}$$

satisfying the sequence of problems (55)-(56). Moreover,

$$(e^{\sqrt{i}\,\eta}\ \widehat{\mathbb{E}}_i^k,\ e^{\sqrt{i}\,\eta}\ H_i^k)\ \in\ \mathbf{P}_k(\Gamma,\mathbb{R}^+;\mathbb{C}^3)^2. \tag{61}$$

Proof. We prove this theorem using an induction on k. Suppose that the existence and uniqueness of $(\widehat{\mathbb{E}}_i^\ell, H_i^\ell)$ and the property (61) have been guaranteed up to $k-1$ and

$$\left(e^{\sqrt{i}\,\eta}\ \widehat{\mathbb{E}}_i^m,\ e^{\sqrt{i}\,\eta} H_i^m\right) \in \mathbf{P}_m(\Gamma,\mathbb{R}^+;\mathbb{R}^3)^2, \quad m = 0,\cdots,k-1. \tag{62}$$

We also include in the recurrence the assumption (by convention $\mathbf{P}_{-1}(\Gamma,\mathbb{R}^+;\mathbb{R}^3) = 0$):

$$\left\{\begin{array}{l} e^{\sqrt{i}\,\eta}\left(-\ \partial_\eta \widehat{\mathbb{E}}_i^m \times n + i\omega H_i^m\right) \in \mathbf{P}_{m-1}(\Gamma,\mathbb{R}^+;\mathbb{C}^3)^2 \quad m = 0,\cdots,k-1, \\ \\ e^{\sqrt{i}\,\eta}\left(\partial_\eta H_i^m \times n + \dfrac{1}{\omega}\ \widehat{\mathbb{E}}_i^m\right) \in \mathbf{P}_{m-1}(\Gamma,\mathbb{R}^+;\mathbb{C}^3)^2, \quad m = 0,\cdots,k-1. \end{array}\right. \tag{63}$$

According to Lemma 4.1, to prove (62) and (63) for $m = k$, it suffices to show that the two right hand sides of (55) are of the form:

$$e^{-\sqrt{i}\,\eta}\ p \quad \text{with} \quad p \in \mathbf{P}_{k-1}(\Gamma,\mathbb{R}^+;\mathbb{C}^3)^2.$$

To verify this, let us consider (u,v) satisfying:

$$\left\{\begin{array}{l} \left(\ e^{\sqrt{i}\,\eta}\ u,\ e^{\sqrt{i}\,\eta}\ v\right) \in \mathbf{P}_m(\Gamma,\mathbb{R}^+;\mathbb{C}^3)^2, \\ \\ \left(\ e^{\sqrt{i}\,\eta}\ (-\ \partial_\eta u \times n + i\omega v),\ e^{\sqrt{i}\,\eta}\ (\partial_\eta v \times n + \frac{1}{\omega}\ u)\ \right) \in \mathbf{P}_{m-1}(\Gamma,\mathbb{R}^+;\mathbb{C}^3)^2. \end{array}\right. \tag{64}$$

The fact that the operators $A_H^{(\ell)}$, $\ell = 3, 4$, are polynomials of degree 2 in η, means that

$$e^{\sqrt{i}\,\eta}\ A_H^{(\ell)}\ (u,v) \in \mathbf{P}_{m+2}(\Gamma,\mathbb{R}^+;\mathbb{C}^3), \text{for } \ell = 3,\ 4. \tag{65}$$

Moreover, using assumption (64) and the explicit form of $A_H^{(\ell)}$ and $A_H^{(\ell)}$ for $\ell = 1, 2$, we observe that

$$\begin{cases} \left(\ e^{\sqrt{i}\,\eta}\ A_H^{(1)}\ (u,v),\ e^{\sqrt{i}\,\eta}\ A_E^{(1)}\ (u,v)\ \right) \in \mathbf{P}_m(\Gamma,\mathbb{R}^+;\mathbb{C}^3)^2, \\[2ex] \left(\ e^{\sqrt{i}\,\eta}\ A_H^{(2)}\ (u,v),\ e^{\sqrt{i}\,\eta}\ A_E^{(2)}\ (u,v)\ \right) \in \mathbf{P}_{m+1}(\Gamma,\mathbb{R}^+;\mathbb{C}^3)^2. \end{cases} \tag{66}$$

Therefore, since $e^{\sqrt{i}\,\eta}\ (\widehat{\mathbb{E}}_i^{k-\ell}, H_i^{k-\ell}) \in \mathbf{P}_{k-\ell}(\Gamma,\mathbb{R}^+;\mathbb{R}^3)^2$ (according to (62)), we deduce from (65) that

$$e^{\sqrt{i}\,\eta} \sum_{l=3}^{4} A_E^{(\ell)}\ (\widehat{\mathbb{E}}_i^{k-\ell}, H_i^{k-\ell}) \in \mathbf{P}_{k-1}(\Gamma,\mathbb{R}^+;\mathbb{C}^3)^2,$$

while, thanks to (63) and by noticing that $k - \ell + 2 \leq k - 1$ for $\ell \geq 3$, we deduce from (66) that

$$\begin{cases} e^{\sqrt{i}\,\eta} \displaystyle\sum_{l=1}^{2}\ A_E^{(\ell)}\ (\widehat{\mathbb{E}}_i^{k-\ell}, H_i^{k-\ell}) \in \mathbf{P}_{k-1}(\Gamma,\mathbb{R}^+;\mathbb{C}^3)^2, \\ e^{\sqrt{i}\,\eta} \displaystyle\sum_{l=1}^{2}\ A_H^{(\ell)}\ (\widehat{\mathbb{E}}_i^{k-\ell}, H_i^{k-\ell}) \in \mathbf{P}_{k-1}(\Gamma,\mathbb{R}^+;\mathbb{C}^3)^2. \end{cases}$$

This leads to the desired property.

To end the proof, it suffices to check the result for $k = 0$ and $k = 1$, which will be done in Section 4.5 (see Remarks 4.3-4.4). □

4.5 Explicit computation of the interior fields for $k = 1, 2, 3$

This section is devoted to the presentation of the technical details related to the computation of the asymptotic terms up to the order $k = 3$. In the sequel, we shall systematically use the following formulas, deduced from (41),

$$(\mathbf{C}_\Gamma\, V) \cdot n = \mathrm{curl}_\Gamma\, V_T, \quad \mathbf{C}_\Gamma\, V \times n = \vec{\mathrm{curl}}_\Gamma\, (V \cdot n) \times n - \big(\mathcal{C} V \times n\big) \times n, \tag{67}$$

$$(\mathbf{C}_\Gamma^M\, V) \cdot n = \mathrm{curl}_\Gamma^M\, V_T, \quad \mathbf{C}_\Gamma^M\, V \times n = \vec{\mathrm{curl}}_\Gamma^M\, (V \cdot n) \times n - \big(\mathcal{G} V \times n\big) \times n. \tag{68}$$

Computation of $(\widehat{\mathbb{E}}_i^0 \equiv E_i^1, H_i^0)$. For $k = 0$, (55) gives

$$\begin{cases} \partial_\eta H_i^0 \times n + \dfrac{1}{\omega}\ \widehat{\mathbb{E}}_i^0 & = \quad 0, \quad \text{in } \ \Gamma \times \mathbb{R}^+, \\[2ex] -\ \partial_\eta \widehat{\mathbb{E}}_i^0 \times n + i\omega\, H_i^0, & = \quad 0 \quad \text{in } \ \Gamma \times \mathbb{R}^+. \end{cases} \tag{69}$$

whose unique L^2 solution satisfying $H^0_{i,T}(x_\Gamma,\eta) = H^0_{e,T}(x_\Gamma)$ is given by:

$$\left\{\begin{array}{l} \widehat{\mathbb{E}}^0_i(x_\Gamma,\eta) \equiv E^1_i(x_\Gamma,\eta) = \sqrt{i}\,\omega\,\big(H^0_e\times n\big)(x_\Gamma)\,e^{-\sqrt{i}\,\eta}, \\[2ex] H^0_i(x_\Gamma,\eta) = H^0_{e,T}(x_\Gamma)\,e^{-\sqrt{i}\,\eta}, \end{array}\right. \tag{70}$$

from which we deduce the useful information for the construction of the GIBCs, namely:

$$E^1_i \times n\,(x_\Gamma,0) = -\sqrt{i}\,\omega\,H^0_{e,T}(x_\Gamma). \tag{71}$$

Remark 4.3 *Notice that (70) proves in particular Theorem 4.2 for $k=0$.*

Computation of $(\widehat{\mathbb{E}}^1_i \equiv E^2_i, H^1_i)$. For $k=1$, (55) gives, using (69)

$$\left\{\begin{array}{rcll} \partial_\eta H^1_i\times n + \dfrac{1}{\omega}\,\widehat{\mathbb{E}}^1_i & = & \mathbf{C}_\Gamma\,H^0_i, & \text{in } \Gamma\times\mathbb{R}^+, \\[2ex] -\,\partial_\eta\widehat{\mathbb{E}}^1_i\times n + i\omega\,H^1_i & = & -\mathbf{C}_\Gamma\,\widehat{\mathbb{E}}^0_i, & \text{in } \Gamma\times\mathbb{R}^+. \end{array}\right. \tag{72}$$

We project (72) on n, use (67) and the expressions (70) for $\widehat{\mathbb{E}}^0_i$ and H^0_i, to obtain:

$$\left\{\begin{array}{ll} \widehat{\mathbb{E}}^1_i\cdot n = \omega\,\mathbf{C}_\Gamma\,H^0_i\cdot n = \omega\big[\mathrm{curl}_\Gamma\,H^0_{e,T}\big](x_\Gamma)\,e^{-\sqrt{i}\,\eta}, & \text{in } \Gamma\times\mathbb{R}^+, \\[2ex] H^1_i\cdot n = \dfrac{i}{\omega}\,\mathbf{C}_\Gamma\,\widehat{\mathbb{E}}^0_i\cdot n = -\dfrac{1}{\sqrt{i}}\,\big[\mathrm{curl}_\Gamma\,\big(H^0_e\times n\big)\big](x_\Gamma)\,e^{-\sqrt{i}\,\eta} & \text{in } \Gamma\times\mathbb{R}^+. \end{array}\right. \tag{73}$$

Next, we eliminate $\widehat{\mathbb{E}}^1_i$ in (72) and get the following equation in $H^1_{i,T}$

$$\big(\partial^2_{\eta\eta} - i\big)\,H^1_{i,T} = n\times\partial_\eta\big[\mathbf{C}_\Gamma\,H^0_i\big] - \frac{1}{\omega}\,n\times\big(\mathbf{C}_\Gamma\,\widehat{\mathbb{E}}^0_i\times n\big)$$

We use again (67) and (70) to transform the right hand side. Using the following identity, that can easily be deduced from the definitions (37) and (38)

$$\big(\mathcal{C}(V\times n)\big)\times n - \mathcal{C}V = -2\mathcal{H}\,V \quad \text{for all } V\in\mathbb{R}^3,$$

we finally get after some easy manipulations

$$\big(\partial^2_{\eta\eta} - i\big)\,H^1_{i,T} = 2\sqrt{i}\,\mathcal{H}\,H^0_{e,T}(x_\Gamma)\,e^{-\sqrt{i}\,\eta},$$

whose unique L^2 solution satisfying $H^0_i(x_\Gamma,\eta) = H^0_{e,T}(x_\Gamma)$ is given by:

$$H^1_{i,T}(x_\Gamma,\eta) = \Big(H^1_{e,T}(x_\Gamma) - \eta\,\mathcal{H}\,H^0_{e,T}(x_\Gamma)\Big)\,e^{-\sqrt{i}\,\eta}. \tag{74}$$

Coming back to the first equation of (72), we get

$$\mathbb{E}_i^1 \times n\ (x_\Gamma, \eta) = \omega\ \Big(-\sqrt{i}\, H^1_{e,T}(x_\Gamma) + (\mathcal{C} - \mathcal{H})\, H^0_{e,T}(x_\Gamma) + \eta\ \sqrt{i}\ \mathcal{H}\ H^0_{e,T}(x_\Gamma) \Big)\ e^{-\sqrt{i}\,\eta}. \quad (75)$$

In particular

$$E_i^2 \times n\ (x_\Gamma, 0) = \omega\ \Big(-\sqrt{i}\ H^1_{e,T}(x_\Gamma) + (\mathcal{C} - \mathcal{H})\, H^0_{e,T}(x_\Gamma) \Big). \quad (76)$$

Remark 4.4 *Notice that (73), (74) and (75) prove in particular Theorem 4.2 for $k = 1$.*

Computation of $(\widehat{\mathbb{E}}_i^2 \equiv E_i^3, H_i^2)$. The calculations are much harder and tedious than for the two previous cases. That is why we shall restrict ourselves to the main steps. Also, for the sake of simplicity, we shall often omit to mention the dependence of the various quantities we manipulate with respect to x_Γ.

For $k = 2$, (55) gives, using (72)

$$\left\{ \begin{array}{rcll} \partial_\eta H_i^2 \times n + \dfrac{1}{\omega}\ \widehat{\mathbb{E}}_i^2 & = & r_H^2, & \text{in } \Gamma \times \mathbb{R}^+ \\[2ex] -\ \partial_\eta \widehat{\mathbb{E}}_i^2 \times n + i\omega\, H_i^2 & = & r_E^2, & \text{in } \Gamma \times \mathbb{R}^+, \end{array} \right. \quad (77)$$

where we have set

$$\left\{ \begin{array}{rcl} r_H^2 & = & \mathbf{C}_\Gamma\, H_i^1 - i\ \varepsilon_r \omega\, \mathbb{E}_i^0 + \eta\ \big(\mathbf{C}_\Gamma^M\ - 2h\ \mathbf{C}_\Gamma\big) H_i^0, \\[2ex] r_E^2 & = & -\mathbf{C}_\Gamma\, \widehat{\mathbb{E}}_i^1 - \eta\ \big(\mathbf{C}_\Gamma^M\ - 2h\ \mathbf{C}_\Gamma\big) \mathbb{E}_i^0. \end{array} \right.$$

We can go directly to the evaluation of $H^2_{i,T}$ which satisfies (apply $n \times \partial_\eta$ to the first equation of (77), divide the second equation by ω and add the two results)

$$\big(\partial^2_{\eta\eta} - i\big)\ H^2_{i,T} = n \times \partial_\eta\, r_H^2 - \frac{1}{\omega}\ r^2_{E,T}. \quad (78)$$

The next step consists in expressing the right hand side of (78) in terms of the previous $(\mathbb{E}_i^\ell, H_i^\ell)$'s. Using (67), (68) and the fact that $H_i^0 \cdot n = 0$, we first compute that

$$n \times r_H^2 = n \times \vec{\mathrm{curl}}_\Gamma\ \big(H_i^1 \cdot n\big) - \mathcal{C} H^1_{i,T} - \eta\ \big(\mathcal{G} - 2h\mathcal{C}\big) H^0_{i,T} - i\ \varepsilon_r \omega\ \big(n \times \mathbb{E}_i^0\big),$$

Next, we use the expressions (70), (73) and (74) and the identity

$$n \times \vec{\mathrm{curl}}_\Gamma\ \big(\mathrm{curl}_\Gamma\, (V \times n)\big) = -\nabla_\Gamma\ \big(\mathrm{div}_\Gamma\, V\big)$$

to obtain

$$n \times r_H^2 = \left[\frac{1}{\sqrt{i}}\ \big(\nabla_\Gamma\ \mathrm{div}_\Gamma\ + \varepsilon_r \omega^2\big) H^0_{e,T} - \mathcal{C} H^1_{e,T} \right]\ e^{-\sqrt{i}\eta} + \eta\ \big(3h\mathcal{C} - \mathcal{G}\big) H^0_{e,T}\ e^{-\sqrt{i}\eta}\ .$$

After differentiation, we get

$$\left|\begin{array}{rcl} n \times \partial_\eta r_H^2 & = & \big[-\big(\nabla_\Gamma\ \mathrm{div}_\Gamma\ + \varepsilon_r\omega^2\big) H^0_{e,T} + \big(3h\mathcal{C} - \mathcal{G}\big) H^0_{e,T} + \sqrt{i}\,\mathcal{C} H^1_{e,T}\big]\ e^{-\sqrt{i}\eta} \\ & & -\eta\,\sqrt{i}\,\big(3h\mathcal{C} - \mathcal{G}\big) H^0_{e,T}\ e^{-\sqrt{i}\eta}\ . \end{array}\right. \tag{79}$$

In the same way, using again (67), (68) and the fact that $\mathbb{E}_i^0 \cdot n = 0$, we calculate

$$r^2_{E,T} = n \times (r_E^2 \times n) = -\mathrm{curl}_\Gamma\,\big(\mathbb{E}_i^1 \cdot n\big) + \big(\mathcal{C}\mathbb{E}_i^1\big) \times n + \eta\,\big[\big(\mathcal{G} - 2h\mathcal{C}\big)\mathbb{E}_i^0\big] \times n.$$

Next, we notice that

$$\mathcal{C}V = -\mathcal{C}\big[(V \times n) \times n\big] \quad \text{(and the same with } \mathcal{G}\text{)}$$

and use the expressions (70), (73) and (75) respectively for $\mathbb{E}_i^0 \times n$, $\mathbb{E}_i^1 \cdot n$ and $\mathbb{E}_i^1 \times n$ to obtain

$$\left|\begin{array}{rcl} -\dfrac{1}{\omega}\, r^2_{E,T} & = & \Big[\ \vec{\mathrm{curl}}_\Gamma\,\big(\mathrm{curl}_\Gamma\, H^0_{e,T}\big) + \eta\,\sqrt{i}\,\Big(\big(\mathcal{G} - 3h\,\mathcal{C}\big) H^0_{e,T} \times n\Big) \times n\ \Big]\ e^{-\sqrt{i}\eta} \\ & + & \Big[\ \mathcal{C}\Big(\big(\mathcal{H} - \mathcal{C}\big) H^0_{e,T} \times n\Big) \times n - \sqrt{i}\,\Big(\mathcal{C}\big(H^1_{e,T} \times n\big)\Big) \times n\ \Big]\ e^{-\sqrt{i}\eta} \end{array}\right.$$

This can be written in a simplified form, using the following identities that hold for all $V \in \mathbb{R}^3$ and that are easily deduced from (37) and (38)

$$\left\{\begin{array}{l} \{\mathcal{C}((\mathcal{H} - \mathcal{C})V) \times n)\} \times n = (3h\mathcal{C} - \mathcal{C}^2 - 2\mathcal{H}^2)V, \\ (\mathcal{C}(V \times n)) \times n = (\mathcal{C} - 2\mathcal{H})V, \\ \{(3\mathcal{H}\mathcal{C} - \mathcal{G})(V \times n)\} \times n = (3\mathcal{H}\mathcal{C} + \mathcal{G} - 6\mathcal{H}^2)V. \end{array}\right.$$

We obtain

$$\left|\begin{array}{rcl} -\dfrac{1}{\omega}\, r^2_{E,T} & = & \Big[\vec{\mathrm{curl}}_\Gamma\,\big(\mathrm{curl}_\Gamma\, H^0_{e,T}\big) + \big(3\mathcal{H}\mathcal{C} - \mathcal{C}^2 - 2\mathcal{H}^2\big) H^0_{e,T}\Big]\ e^{-\sqrt{i}\eta} \\ & - & \sqrt{i}\,(\mathcal{C} - 2\mathcal{H}) H^1_{e,T}\ e^{-\sqrt{i}\eta} +\ \eta\,\sqrt{i}\,\big(3\mathcal{H}\mathcal{C} + \mathcal{G} - 6\mathcal{H}^2\big)\ H^0_{e,T}\ e^{-\sqrt{i}\eta}. \end{array}\right. \tag{80}$$

Substituting (79) and (80) in (78) leads to the following equation

$$\left|\begin{array}{rcl} \big(\partial^2_{\eta\eta} - i\big)\, H^2_{i,T} & = & e^{-\sqrt{i}\,\eta}\quad \big\{2\,\sqrt{i}\,\mathcal{H}\, H^1_{e,T} + (\mathcal{C}^2 + 2\mathcal{H}^2 - \mathcal{G})\, H^0_{e,T} \\ & & \qquad -(\vec{\Delta}_\Gamma\ + \varepsilon_r\omega^2)\ H^0_{e,T} - \eta\,\sqrt{i}\,\big(6\mathcal{H}^2 - 2\mathcal{G}\big)\ H^0_{e,T}\Big\}, \end{array}\right.$$

where $\vec{\Delta}_\Gamma\ := \nabla_\Gamma\,\mathrm{div}_\Gamma\ - \vec{\mathrm{curl}}_\Gamma\,\mathrm{curl}_\Gamma$ is the vectorial Laplace Beltrami operator.

Since the L^2 solution to

$$(\partial^2_{\eta\eta} - i)u = (a + b\,\eta)\, e^{-\sqrt{i}\,\eta} \quad \text{in } \mathbb{R}^+,$$

is given by

$$u(\eta) = \left(u(0) + \left(\frac{a}{2\sqrt{i}} - \frac{b}{4i}\right)\eta + \frac{b}{4\sqrt{i}}\,\eta^2\right)\, e^{-\sqrt{i}\,\eta},$$

we deduce that H^2_T is given by the expression

$$\left|\begin{aligned} H^2_{i,T}(x_\Gamma,\eta) = e^{-\sqrt{i}\,\eta}\quad &\Big\{ H^2_{e,T} - \eta\,\mathcal{H} H^1_{e,T} - \frac{\eta}{2\sqrt{i}}\,\left(\mathcal{C}^2 - \mathcal{H}^2\right)\, H^2_{e,T} \\ &+ \frac{\eta}{2\sqrt{i}}\,\left(\vec{\Delta}_\Gamma + \varepsilon_r\omega^2\right)\, H^2_{e,T} + \frac{\eta^2}{2}\,\left(3\mathcal{H}^2 - \mathcal{G}\right) H^1_{e,T}\Big\}. \end{aligned}\right.$$

Finally we go back to the first equation of to obtain, after lengthy calculations that we do not detail here,

$$\left|\begin{aligned} \mathbb{E}^2_{i,T} \times n = \omega\, e^{-\sqrt{i}\,\eta} \Big\{ &-\sqrt{i}\, H^2_{e,T} + (\mathcal{C} - \mathcal{H})\, H^1_{e,T} - \frac{1}{2\sqrt{i}}\,\left(\mathcal{C}^2 - \mathcal{H}^2\right)\, H^0_{e,T} \\ &- \frac{1}{2\sqrt{i}}\,\left(\varepsilon_r\omega^2 + \nabla_\Gamma\,\mathrm{div}_\Gamma + \vec{\mathrm{curl}}_\Gamma\,\mathrm{curl}_\Gamma\right) H^0_{e,T} \\ &+ \eta\,\left(\sqrt{i}\mathcal{H} H^1_{e,T} + \frac{1}{2}\left(5\mathcal{H}^2 - 6\mathcal{H}\mathcal{C} + \mathcal{C}^2 - \vec{\Delta}_\Gamma - \varepsilon_r\omega^2\right) H^0_{e,T}\right) \\ &- \eta^2\,\frac{\sqrt{i}}{2}\,\left(3\mathcal{H}^2 - \mathcal{G}\right) H^0_{e,T}\Big\}. \end{aligned}\right.$$

In particular, for $\eta = 0$,

$$\left|\begin{aligned} E^3_{i,T} \times n = \omega\, e^{-\sqrt{i}\,\eta} \Big\{ &-\sqrt{i}\, H^2_{e,T} + (\mathcal{C} - \mathcal{H})\, H^1_{e,T} - \frac{1}{2\sqrt{i}}\,\left(\mathcal{C}^2 - \mathcal{H}^2\right)\, H^0_{e,T} \\ &- \frac{1}{2\sqrt{i}}\,\left(\varepsilon_r\omega^2 + \nabla_\Gamma\,\mathrm{div}_\Gamma + \vec{\mathrm{curl}}_\Gamma\,\mathrm{curl}_\Gamma\right) H^0_{e,T}\Big\}. \end{aligned}\right. \tag{81}$$

4.6 Construction of the GIBCs

The GIBCS of order k is obtained by considering the truncated expansions in Ω_e

$$E^\delta_{e,k} := \sum_{\ell=0}^{k} \delta^\ell\, E^\ell_e \quad \text{and} \quad H^\delta_{e,k} := \sum_{\ell=0}^{k} \delta^\ell\, H^\ell_e$$

as (formal) approximations of order $k+1$ of E^δ_e and H^δ_e respectively (notice that k appears here as a subscript while it appears as an exponent in the notation of the solution of the approximate problem (23, 24)). Using the "second" interface condition, namely (48), one has

$$E^\delta_{e,k}|_\Gamma(x_\Gamma) \times n = \sum_{\ell=0}^{k} \delta^\ell \, E^\ell_i(x_\Gamma, 0) \times n \;\text{ for } x_\Gamma \in \Gamma. \tag{82}$$

Substituting into (82) the computed expressions of $E^\ell_i(x_\Gamma, 0)$ for $\ell = 1, 2$ and 3, respectively given by (71), (76) and (81)) leads to an identity of the form

$$E^\delta_{e,k} \times n + \omega \, \mathcal{D}^{\delta,k}\big[(H^\delta_{e,k})_T\big] = \delta^{k+1} \, \varphi^\delta_k \;\text{ on } \Gamma, \quad \text{ for } k = 0, 1, 2, \tag{83}$$

and

$$E^\delta_{e,3} \times n + \omega \, \mathcal{D}^{\delta,3}_0\big[(H^\delta_{e,3})_T\big] = \delta^4 \, \varphi^\delta_{3,0} \;\text{ on } \Gamma, \tag{84}$$

where $\mathcal{D}^{\delta,k}$, $k = 0, 1, 2$ are given by (25) and $\mathcal{D}^{\delta,3}_0$ is given by (27) and where $\varphi^\delta_k \in C^\infty(\Gamma)^3$, $k = 0, 1, 2$ are tangential vector fields given by

$$\begin{cases} \varphi^\delta_0 &= 0, \\[2ex] \varphi^\delta_1 &= \sqrt{i}\, \omega H^1_{e,T}, \\[2ex] \varphi^\delta_2 &= \sqrt{i}\, \omega \, H^2_{e,T} + \omega(\mathcal{C} - \mathcal{H})\big(H^1_{e,T} + \delta \, H^2_{e,T}\big), \end{cases} \tag{85}$$

and obviously satisfy the estimates (for δ small enough)

$$\|\varphi^\delta_k\|_{H^s_t(\Gamma)} \leq C_k(s), \quad k = 0, 1, 2 \tag{86}$$

where $C_k(s)$ is independent of δ, while $\varphi^\delta_{3,0} \in C^\infty(\Gamma)^3$ is given by

$$\begin{cases} \varphi^\delta_{3,0} &= \sqrt{i}\, \omega \, H^3_{e,T} + \omega(\mathcal{C} - \mathcal{H})\big(H^3_{e,T} + \delta \, H^2_{e,T}\big) \\[2ex] &+ \dfrac{1}{2\sqrt{i}} \left(\mathcal{C}^2 - \mathcal{H}^2\right) \left(H^1_{e,T} + \delta \, H^2_{e,T} + \delta^3 \, H^3_{e,T}\right) \\[2ex] &+ \dfrac{1}{2\sqrt{i}} \left(\varepsilon_r \omega^2 + \nabla_\Gamma \operatorname{div}_\Gamma + \vec{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma \right) \left(H^1_{e,T} + \delta \, H^2_{e,T} + \delta^3 \, H^3_{e,T}\right). \end{cases} \tag{87}$$

The GIBC (24) is obtained for $k = 0, 1, 2$ by neglecting the right-hand side of (83). For $k = 3$, the same process leads to the condition (26) that is modified according to the process explained in Section 3.2. Notice that according to that construction, we have

$$E^\delta_{e,3} \times n + \omega \, \mathcal{D}^{\delta,k}\big[(H^\delta_{e,3})_T\big] = \delta^4 \, \varphi^\delta_3 \;\text{ on } \Gamma, \quad \text{where } \varphi^\delta_3 = \varphi^\delta_{3,0} + \delta \, \mathcal{R}^{\delta,3}\big[(H^\delta_{e,3})_T\big], \tag{88}$$

and using the property (35) of $\mathcal{R}^{\delta,3}$,

$$\|\varphi^\delta_3\|_{H^s_t(\Gamma)} \leq C_3(s) \tag{89}$$

where $C_3(s)$ is independent of δ.

4.7 Towards the theoretical justification of the GIBCs

Our goal in the next two sections is to justify the GIBCs (24) by estimating the errors

$$E_e^\delta - E_e^{\delta,k} \quad \text{and} \quad H_e^\delta - H_e^{\delta,k},$$

where $(E_e^{\delta,k}, H_e^{\delta,k})$ is the solution of the approximate problem ((23), (24)), whose well-posedness will be shown in Section 6.1 (see Theorem 6.1). It appears non trivial to work directly with the differences $E_e^\delta - E_e^{\delta,k}$ and $H_e^\delta - H_e^{\delta,k}$, we shall use the truncated series $(E_{e,k}^\delta, H_{e,k}^\delta)$ introduced in Section 4.6 as intermediate quantities. Therefore, the error analysis is split into two steps:

1. Estimate the differences $E_e^\delta - E_{e,k}^\delta$ and $H_e^\delta - H_{e,k}^\delta$; this is done in Section 5, and more precisely in Lemma 5.1 and Corollary 5.1.
2. Estimate the difference $E_{e,k}^\delta - E_e^{\delta,k}$ and $H_{e,k}^\delta - H_e^{\delta,k}$; this is done in Section 6.2 and more precisely in Theorem 134.

Remark 4.5 *Notice that step 1 of the proof is completely independent on the GIBC and will be valid for any integer k. Also, for $k = 0$, the second step is useless since $\widetilde{E}^{\delta,0} = E^{\delta,0}$.*

5 Error estimates for the truncated expansions

5.1 Main results

Let us introduce the fields $E_\chi^{\delta,k}(x),\ H_\chi^{\delta,k}(x) : \Omega \mapsto \mathbb{C}^3$ such that

$$E_\chi^{\delta,k}(x) = \begin{cases} \displaystyle\sum_{\ell=0}^{k} \delta^\ell E_e^\ell(x) = E_{e,k}^\delta\ , & \text{for } x \in \Omega_e, \\ \displaystyle\chi(x) \sum_{\ell=0}^{k} \delta^\ell E_i^\ell(x_\Gamma, \nu/\delta) & \text{for } x \in \Omega_i, \end{cases}$$

$$H_\chi^{\delta,k}(x) = \begin{cases} \displaystyle\sum_{\ell=0}^{k} \delta^\ell H_e^\ell(x) = H_{e,k}^\delta\ , & \text{for } x \in \Omega_e, \\ \displaystyle\chi(x) \sum_{\ell=0}^{k} \delta^\ell H_i^\ell(x_\Gamma, \nu/\delta) & \text{for } x \in \Omega_i, \end{cases}$$

where the local coordinates x_Γ and ν are defined as in Section 4.1 and the cut-off function χ is defined as in section 4.2. These fields are good candidates to be good approximations of the exact fields (E^δ, H^δ). The main result of this section is:

Lemma 5.1 *For any integer k, there exists a constant C_k independent of δ such that*

$$\left\{\begin{array}{llcl} (i) & \|E^\delta - E_\chi^{\delta,k}\|_{H(\mathrm{curl},\Omega)} & \le & C_k\ \delta^{k+\frac{1}{2}}, \\ (ii) & \|E^\delta - E_\chi^{\delta,k}\|_{L^2(\Omega_i)} & \le & C_k\ \delta^{k+\frac{3}{2}}, \\ (iii) & \|E^\delta \times n - E_\chi^{\delta,k} \times n\|_{H^{-\frac{1}{2}}(\Gamma)} & \le & C_k\ \delta^{k+1}. \end{array}\right. \tag{90}$$

The proof of Lemma 5.1, postponed to Section 5.3, rely on a fundamental a priori estimates that we shall state and prove in Section 5.2. We first give a straightforward corollary of Lemma 5.1.

Corollary 5.1 *For any integer k, there exists a constant $\widetilde{C}_k$ independent of δ such that:*

$$\left\{\begin{array}{l} \|E_e^\delta - E_{e,k}^\delta\|_{H(\mathrm{curl},\Omega_e)} \le\ \widetilde{C}_k\ \delta^{k+1}, \\ \|H_e^\delta - H_{e,k}^\delta\|_{H(\mathrm{curl},\Omega_e)} \le\ \widetilde{C}_k\ \delta^{k+1}. \end{array}\right.$$

Proof. Simply write

$$E_e^\delta - E_{e,k}^\delta = E_e^\delta - E_{e,k+1}^\delta + \delta^{k+1} E_e^{k+1}$$

which yields, since $E_{e,k}^\delta = E_\chi^{\delta,k+1}$ in Ω_e,

$$\|E_e^\delta - E_{e,k}^\delta\|_{H(\mathrm{curl},\Omega_e)} \le \|E_e^\delta - E_\chi^{\delta,k+1}\|_{H(\mathrm{curl},\Omega_e)} + \delta^{k+1}\ \|E_e^{k+1}\|_{H(\mathrm{curl},\Omega_e)}.$$

Using the estimate (90-i) of Lemma 5.1, we get

$$\|E_e^\delta - E_{e,k}^\delta\|_{H(\mathrm{curl},\Omega_e)} \le C_k\ \delta^{k+\frac{3}{2}} +\ \delta^{k+1}\ \|E_e^{k+1}\|_{H(\mathrm{curl},\Omega_e)} \le \widetilde{C}_k\ \delta^{k+1}.$$

The estimates for $H_e^\delta - H_{e,k}^\delta$ is an immediate consequence of

$$\left\{\begin{array}{lll} -i\omega(H_e^\delta - H_{e,k}^\delta) + \mathrm{curl}(E_e^\delta - E_{e,k}^\delta) = 0 & \text{in} & \Omega_e, \\ i\omega(E_e^\delta - E_{e,k}^\delta) + \mathrm{curl}(H_e^\delta - H_{e,k}^\delta) = 0 & \text{in} & \Omega_e. \end{array}\right.$$

□

5.2 A fundamental a priori estimate

The proof of lemma 5.1 relies of the following fundamental technical lemma.

Lemma 5.2 *Assume that $\mathbf{E}^\delta \in H(\mathrm{curl},\Omega)$ satisfies*

$$\left\{\begin{array}{ll} \mathrm{curl}\,\mathrm{curl}\,\mathbf{E}^\delta - \omega^2\mathbf{E}^\delta = 0, & \text{in } \Omega_e, \\ i\omega\mathbf{E}_T^\delta - \mathrm{curl}\,\mathbf{E}^\delta \times n = 0, & \text{on } \partial\Omega, \end{array}\right. \tag{91}$$

together with the following inequality

$$
\begin{aligned}
&\left|\int_\Omega \left(|\operatorname{curl}\mathbf{E}^\delta|^2 - \omega^2|\mathbf{E}^\delta|^2\right)\,dx + i\omega\left(\int_{\partial\Omega}|\mathbf{E}^\delta\times n|^2\,ds + \frac{1}{\delta^2}\int_{\Omega_i}|\mathbf{E}^\delta|^2\,dx\right)\right| \\
&\qquad\qquad \leq A\left(\delta^{s+\frac{1}{2}}\,\|\mathbf{E}^\delta\times n\|_{H^{-\frac{1}{2}}(\Gamma)} + \delta^s\,\|\mathbf{E}^\delta\|_{L^2(\Omega_i)}\right),
\end{aligned}
\tag{92}
$$

for some non-negative constants A and s independent of δ. Then there exists a constant C independent of δ such that

$$
\|\mathbf{E}^\delta\|_{H(\mathrm{curl},\Omega)} \leq C\,\delta^{s+1}, \quad \|\mathbf{E}^\delta\|_{L^2(\Omega_i)} \leq C\,\delta^{s+2}, \quad \|\mathbf{E}^\delta\times n\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C\,\delta^{s+\frac{3}{2}}, \tag{93}
$$

for sufficiently small δ.

Proof. For convenience, we shall denote by C a positive constant whose value may change from one line to another but remains independent of δ. We divide the proof in two steps.

Step 1. We first prove by contradiction that $\|\mathbf{E}^\delta\|_{L^2(\Omega)} \leq C\,\delta^{s+1}$. This is the main step of the proof which will use two important technical lemmas A.1 and A.5, that are proven in the Appendix.

Assume that the positive quantity

$$
\lambda^\delta := \delta^{-(s+1)}\,\|\mathbf{E}^\delta\|_{L^2(\Omega)}
$$

is unbounded as $\delta \to 0$. After extraction of a subsequence, still denoted $\mathbf{E}^\delta$ with $\delta \to 0$, we can assume that $\lambda^\delta \to +\infty$. Let $\tilde{\mathbf{E}}^\delta = \mathbf{E}^\delta/\|\mathbf{E}^\delta\|_{L^2(\Omega)}$ (so that $\|\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega)} = 1$).

Our goal is to show that, up to the extraction of another subsequence, $\tilde{\mathbf{E}}^\delta$ converges strongly in $L^2(\Omega_e)$ and to obtain a contradiction by looking at the limit field $\tilde{\mathbf{E}}$.

To show this, we wish to apply to $\tilde{\mathbf{E}}^\delta$ the compactness result of Lemma A.5 with $\mathcal{O} = \Omega_e$. Since $\operatorname{div}\mathbf{E}^\delta = 0$ and since $\tilde{\mathbf{E}}^\delta$ is bounded in $L^2(\Omega_e)$, we only need to show that:

$$
\left|
\begin{array}{ll}
\textbf{(i)} & \operatorname{curl}\tilde{\mathbf{E}}^\delta \text{ is bounded in } L^2(\Omega_e), \\[2ex]
\textbf{(ii)} & \tilde{\mathbf{E}}^\delta\times n|_{\partial\Omega} \text{ converges in } H^{-\frac{1}{2}}(\partial\Omega), \\[2ex]
\textbf{(iii)} & \tilde{\mathbf{E}}^\delta\times n|_{\Gamma} \text{ converges in } H^{-\frac{1}{2}}(\Gamma).
\end{array}
\right.
\tag{94}
$$

We first notice that after division by $\|\mathbf{E}^\delta\|_{L^2(\Omega)}$, the inequality (92) yields

$$
\left|
\begin{aligned}
&\left|\int_\Omega \left(|\operatorname{curl}\tilde{\mathbf{E}}^\delta|^2 - \omega^2|\tilde{\mathbf{E}}^\delta|^2\right)dx + i\omega\left(\int_{\partial\Omega}|\tilde{\mathbf{E}}^\delta\times n|^2\,ds + \frac{1}{\delta^2}\int_{\Omega_i}|\tilde{\mathbf{E}}^\delta|^2\,dx\right)\right| \\
&\qquad\qquad\leq \frac{A}{\lambda^\delta}\left(\delta^{-1}\,\|\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega_i)} + \delta^{-\frac{1}{2}}\,\|\tilde{\mathbf{E}}^\delta\times n\|_{H^{-\frac{1}{2}}(\Gamma)}\right).
\end{aligned}
\right.
\tag{95}
$$

We shall now establish estimates on the two terms in the right hand side of (95) in terms of $\|\operatorname{curl} \tilde{\mathbf{E}}^\delta\|_{L^2(\Omega)}$ (namely inequalities (99) and (101)).

Considering the imaginary part of the left hand side of (95), we observe that since $1/\lambda^\delta$ is bounded,

$$\|\tilde{\mathbf{E}}^\delta\|^2_{L^2(\Omega_i)} \leq \ C\ \delta^{\frac{3}{2}}\ \|\tilde{\mathbf{E}}^\delta \times n\|_{H^{-\frac{1}{2}}(\Gamma)} + C\ \delta\ \|\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega_i)}.$$

Next, we use the trace inequality (143) of Lemma A.1 with $\mathcal{O} = \Omega_i$ to get

$$\|\tilde{\mathbf{E}}^\delta\|^2_{L^2(\Omega_i)} \leq \ C\ \delta^{\frac{3}{2}}\ \|\tilde{\mathbf{E}}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)}\ \left(\|\tilde{\mathbf{E}}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)} + \|\operatorname{curl} \tilde{\mathbf{E}}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)}\right) + C\ \delta\ \|\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega_i)},$$

which yields, after division by $\|\tilde{\mathbf{E}}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)}$,

$$\|\tilde{\mathbf{E}}^\delta\|^{\frac{3}{2}}_{L^2(\Omega_i)} \leq C_1\ \delta\ \|\tilde{\mathbf{E}}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)} + C_2\ \delta^{\frac{3}{2}}\ \|\operatorname{curl} \tilde{\mathbf{E}}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)}. \tag{96}$$

Let K be a positive constant to be fixed later. Using Young's inequality $ab \leq 2/3\ a^{3/2} + 1/3\ b^3$ with $a = K^{-1}\ \delta$ and $b = K\ \|\tilde{\mathbf{E}}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)}$,
we get

$$\delta\ \|\tilde{\mathbf{E}}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)} \leq \frac{2}{3}\ K^{-\frac{3}{2}}\ \delta^{\frac{3}{2}} + \frac{K^3}{3}\ \|\tilde{\mathbf{E}}^\delta\|^{\frac{3}{2}}_{L^2(\Omega_i)}. \tag{97}$$

Choosing $C_1 K^3 = 3/2$ and substituting (96) into (97), one deduces

$$\|\tilde{\mathbf{E}}^\delta\|^{\frac{3}{2}}_{L^2(\Omega_i)} \leq C\ \delta^{\frac{3}{2}}\ \left(1 + \|\operatorname{curl} \tilde{\mathbf{E}}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)}\right), \tag{98}$$

which yields

$$\delta^{-1}\ \|\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega_i)} \leq C\ \left(1 + \|\operatorname{curl} \tilde{\mathbf{E}}^\delta\|^{\frac{1}{3}}_{L^2(\Omega_i)}\right). \tag{99}$$

Now considering the real part of the left hand side of (95) and using the fact that $\|\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega)} = 1$, we observe that

$$\|\operatorname{curl} \tilde{\mathbf{E}}^\delta\|^2_{L^2(\Omega)} \leq C\ \left(1 + \delta^{-\frac{1}{2}}\ \|\tilde{\mathbf{E}}^\delta \times n\|_{H^{-\frac{1}{2}}(\Gamma)} + \delta^{-1}\ \|\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega_i)}\right). \tag{100}$$

On the other hand, after multiplication by $\delta^{-\frac{1}{2}}$, the trace inequality (143) applied to $\tilde{\mathbf{E}}^\delta$ is equivalent to

$$\delta^{-\frac{1}{2}}\ \|\tilde{\mathbf{E}}^\delta \times n\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C\ \delta^{\frac{1}{2}}\ \left\{\delta^{-1}\ \|\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega_i)}\right\} + C\ \left\{\delta^{-1}\|\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega_i)}\right\}^{\frac{1}{2}}\ \|\operatorname{curl} \tilde{\mathbf{E}}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)}.$$

After applying the Cauchy-Schwarz inequality to the second term of the right hand side of the above inequality, we easily get, since δ is bounded

$$\left|\begin{array}{lcl} \delta^{-\frac{1}{2}}\ \|\tilde{\mathbf{E}}^\delta \times n\|_{H^{-\frac{1}{2}}(\Gamma)} & \leq & C\ \left\{\delta^{-1}\|\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega_i)} +\ \|\operatorname{curl} \tilde{\mathbf{E}}^\delta\|_{L^2(\Omega_i)}\right\}, \\ & \leq & C\ \left\{1 + \|\operatorname{curl} \tilde{\mathbf{E}}^\delta\|^{\frac{1}{3}}_{L^2(\Omega_i)} + \|\operatorname{curl} \tilde{\mathbf{E}}^\delta\|_{L^2(\Omega_i)}\right\}, \end{array}\right. \tag{101}$$

where we used (99) for the second inequality. Substituting (101) into (100) shows that

$$\|\mathrm{curl}\,\tilde{\mathbf{E}}^\delta\|^2_{L^2(\Omega)} \le C\,\left(1 + \|\mathrm{curl}\,\tilde{\mathbf{E}}^\delta\|^{\frac{1}{3}}_{L^2(\Omega_i)} + \|\mathrm{curl}\,\tilde{\mathbf{E}}^\delta\|_{L^2(\Omega_i)}\right).$$

This proves (94-**(i)**). We also deduce thanks to (99) and (101) that

$$\delta^{-1}||\tilde{\mathbf{E}}^\delta||_{L^2(\Omega_i)} \;\text{ and }\; \delta^{-\frac{1}{2}}\,||\tilde{\mathbf{E}}^\delta \times n||_{H^{-\frac{1}{2}}(\Gamma)} \quad \text{are bounded,} \tag{102}$$

which proves in particular (94-**(ii)**) ($\tilde{\mathbf{E}}^\delta \times n$ converges to 0 in $H^{-\frac{1}{2}}(\Gamma)$). This also means that the right hand side of (95) remains bounded. Thus, going back to (95) shows that $||\tilde{\mathbf{E}}^\delta \times n||_{L^2(\partial\Omega)}$ is bounded, which proves (94-**(iii)**) by the compactness of the $L^2(\partial\Omega)$ embedding into $H^{-\frac{1}{2}}(\partial\Omega)$.

Now we shall conclude the proof of Step 1. From (94-**(i)**) one deduces that $\tilde{\mathbf{E}}^\delta$ is a bounded sequence in $H(\mathrm{curl},\Omega)$, therefore, up to extracted subsequence, we can assume that $\tilde{\mathbf{E}}^\delta$ weakly converges in $H(\mathrm{curl},\Omega)$ to some $\tilde{\mathbf{E}}$. Considering the restriction to Ω_e, thanks to (94) we can apply the compactness result of Lemma A.5 and deduce that an extracted subsequence of $\tilde{\mathbf{E}}^\delta$, denoted again by $\tilde{\mathbf{E}}^\delta$ for simplicity, strongly converges to $\tilde{\mathbf{E}}$ in $L^2(\Omega_e)$. On the other hand, we observe from (102) that $\tilde{\mathbf{E}}^\delta$ strongly converges to 0 in $L^2(\Omega_i)$, hence $\tilde{\mathbf{E}} = 0$ in Ω_i, which implies in particular

$$\tilde{\mathbf{E}} \times n = 0 \text{ on } \Gamma. \tag{103}$$

Passing to the weak limit in equations (91) one easily verify that

$$\begin{cases} \mathrm{curl}\,\mathrm{curl}\,\tilde{\mathbf{E}} - \omega^2\tilde{\mathbf{E}} = 0, & \text{in } \Omega_e, \\ i\omega\tilde{\mathbf{E}}_T - \mathrm{curl}\,\tilde{\mathbf{E}} \times n = 0, & \text{on } \partial\Omega, \end{cases} \tag{104}$$

The uniqueness of solutions to (104)-(103) in $H(\mathrm{curl},\Omega_e)$ implies that also $\tilde{\mathbf{E}} = 0$ in Ω_e. We therefore obtain that $\tilde{\mathbf{E}}^\delta$ converges to 0 in $L^2(\Omega)$ which is contradiction with $||\tilde{\mathbf{E}}^\delta||_{L^2(\Omega)} = 1$. Consequently λ^δ is bounded, that is to say

$$\|\mathbf{E}^\delta\|_{L^2(\Omega)} \le C\,\delta^{s+1}. \tag{105}$$

Step 2. We shall now proceed with the proof of estimates (90). Considering the imaginary part of the left hand side of estimate (92) and applying Lemma A.1 (with $\mathcal{O} = \Omega_i$) yields

$$\|\mathbf{E}^\delta\|^2_{L^2(\Omega_i)} \le C\left(\delta^{s+\frac{5}{2}}\,\|\mathrm{curl}\,\mathbf{E}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)}\,\|\mathbf{E}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)} + \delta^{s+2}\,\|\mathbf{E}^\delta\|_{L^2(\Omega_i)}\right).$$

Using two times the Young inequality $ab \le 1/2(a^2+b^2)$, the first time with

$$a = \delta^{\frac{1}{2}}\,\|\mathrm{curl}\,\mathbf{E}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)} \quad\text{ and }\quad b = \|\mathbf{E}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)}\,,$$

and the second time with

$$a = \|\mathbf{E}^\delta\|_{L^2(\Omega_i)} \quad \text{and} \quad b = \delta^{s+2},$$

leads to (we also use $\|\mathrm{curl}\,\mathbf{E}^\delta\|_{L^2(\Omega_i)} \leq \|\mathrm{curl}\,\mathbf{E}^\delta\|_{L^2(\Omega)}$)

$$\|\mathbf{E}^\delta\|^2_{L^2(\Omega_i)} \leq C\ \left(\delta^{2s+4} + \delta^{s+2}\left(\|\mathbf{E}^\delta\|_{L^2(\Omega_i)} + \delta\|\mathrm{curl}\,\mathbf{E}^\delta\|_{L^2(\Omega)}\right)\right). \tag{106}$$

On the other hand, considering this the real part of the left hand side of estimate (92) and using (105), we get

$$\|\mathrm{curl}\,\mathbf{E}^\delta\|^2_{L^2(\Omega)} \leq C\ \left(\delta^{2s+2} + \delta^{s+\frac{1}{2}}\ \|\mathrm{curl}\,\mathbf{E}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)}\ \|\mathbf{E}^\delta\|^{\frac{1}{2}}_{L^2(\Omega_i)} + \delta^s\ \|\mathbf{E}^\delta\|_{L^2(\Omega_i)}\right),$$

which gives, using Young's inequality once again,

$$\|\mathrm{curl}\,\mathbf{E}^\delta\|^2_{L^2(\Omega)} \leq C\ \left(\delta^{2s+2} + \delta^s\ \left(\|\mathbf{E}^\delta\|_{L^2(\Omega_i)} + \delta\ \|\mathrm{curl}\,\mathbf{E}^\delta\|_{L^2(\Omega)}\right)\right). \tag{107}$$

Combining (106) and (107) leads to

$$\|\mathbf{E}^\delta\|^2_{L^2(\Omega_i)} + \delta^2\ \|\mathrm{curl}\,\mathbf{E}^\delta\|^2_{L^2(\Omega)} \leq C\left(\delta^{2s+4} + \delta^{s+2}\ \left(\|\mathbf{E}^\delta\|_{L^2(\Omega_i)} + \delta\ \|\mathrm{curl}\,\mathbf{E}^\delta\|_{L^2(\Omega)}\right)\right),$$

which yields

$$\|\mathbf{E}^\delta\|_{L^2(\Omega_i)} + \delta\ \|\mathrm{curl}\,\mathbf{E}^\delta\|_{L^2(\Omega)} \leq C\ \delta^{s+2}$$

and in particular the second inequality of (93). The third inequality of (93) is a direct consequence of the first two ones and the application of Lemma A.1 in Ω_i. □

Remark 5.1 *Notice that since we simply used in the first step of the proof the fact that $1/\lambda^\delta$ is bounded, we have proved in fact that*

$$\lim_{\delta\to 0}\ \delta^{-(s+1)}\ \|E^\delta\|_{L^2(\Omega)} = 0.$$

5.3 The proof of Lemma 5.1.

Let us introduce, for each integer k, the *error fields*

$$\mathcal{E}^{\delta,k} = E^\delta_e - E^{\delta,k}_\chi, \quad \mathcal{H}^{\delta,k} = H^\delta_e - H^{\delta,k}_\chi. \tag{108}$$

The idea of the proof is to show that $\mathcal{E}^{\delta,k}$ satisfies an a priori estimate of the type (92) and then to use the stability lemma 5.2. To prove such an estimate, we shall use the equations satisfied by $(\mathcal{E}^{\delta,k}, \mathcal{H}^{\delta,k})$, respectively in Ω_i and Ω_e.

The equations in Ω_e. It is straightforward to check that in the exterior domain Ω_e, the errors

$$(\mathcal{E}^{\delta,k}_e, \mathcal{H}^{\delta,k}_e) := (\mathcal{E}^{\delta,k}|_{\Omega_e}, \mathcal{H}^{\delta,k}|_{\Omega_e})$$

satisfies the homogeneous equation:

$$\begin{cases} (i) \quad \operatorname{curl} \mathcal{H}_e^{\delta,k} + i\omega \mathcal{E}_e^{\delta,k} = 0, & \text{in } \Omega_e, \\ (ii) \quad \operatorname{curl} \mathcal{E}_e^{\delta,k} - i\omega \mathcal{H}_e^{\delta,k} = 0, & \text{in } \Omega_e, \end{cases} \tag{109}$$

and

$$(\mathcal{E}_e^{\delta,k})_T - \mathcal{H}_e^{\delta,k} \times n = 0, \quad \text{on } \partial\Omega. \tag{110}$$

Eliminating $\mathcal{H}_e^{\delta,k}$ in (109), we get

$$\begin{cases} \operatorname{curl}\left(\operatorname{curl} \mathcal{E}_e^{\delta,k}\right) - \omega^2 \mathcal{E}_e^{\delta,k} = 0, & \text{in } \Omega_e. \\ \operatorname{curl} \mathcal{E}_e^{\delta,k} \times n + i\omega \left(\mathcal{E}_e^{\delta,k}\right)_T = 0, & \text{on } \partial\Omega. \end{cases} \tag{111}$$

The equations in Ω_i. Now consider the restrictions to Ω_i and set

$$(\mathcal{E}_i^{\delta,k}, \mathcal{H}_i^{\delta,k}) := (\mathcal{E}^{\delta,k}|_{\Omega_i}, \mathcal{H}^{\delta,k}|_{\Omega_i}).$$

It is also useful to introduce the fields

$$E_{i,k}^{\delta}(x_\Gamma, \nu) := \sum_{\ell=0}^{k} \delta^\ell \; E_i^\ell(x_\Gamma, \frac{\nu}{\delta}), \quad H_{i,k}^{\delta}(x_\Gamma, \eta) := \sum_{\ell=0}^{k} \delta^\ell \; H_i^\ell(x_\Gamma, \frac{\nu}{\delta}),$$

so that using the local coordinates, we can write

$$E_\chi^{\delta,k}(x) = \chi \; E_{i,k}^{\delta}(x_\Gamma, \eta), \quad H_\chi^{\delta,k}(x) = \chi \; H_{i,k}^{\delta}(x_\Gamma, \eta) \quad \text{in } \Omega_i.$$

Our goal is to show that $(E_\chi^{\delta,k}, H_\chi^{\delta,k})$ satisfy the "interior equations" except that two *small* source terms appear at the right hand side, respectively due to the cut-off function χ and the truncation of the series at order k. We first compute that

$$\begin{cases} \operatorname{curl} H_\chi^{\delta,k} + i\omega E_\chi^{\delta,k} - \dfrac{1}{\omega\,\delta^2} E_\chi^{\delta,k} & = \chi \left(i\omega E_{i,k}^{\delta} + \operatorname{curl} H_{i,k}^{\delta} - \dfrac{1}{\omega\,\delta^2} E_{i,k}^{\delta} \right) + \nabla\chi \times H_{i,k}^{\delta}, \\ \operatorname{curl} E_\chi^{\delta,k} - i\omega H_\chi^{\delta,k} & = \chi \left(\operatorname{curl} E_{i,k}^{\delta} - i\omega H_{i,k}^{\delta} \right) + \nabla\chi \times E_{i,k}^{\delta}. \end{cases} \tag{112}$$

Thanks to the exponentially decaying nature of $E_{i,k}^{\delta}(x_\Gamma, \eta)$ and $H_{i,k}^{\delta}(x_\Gamma, \eta)$ with respect to η (cf. Theorem 4.2), the terms in factor of $\nabla\chi$ are exponentially small in δ.

It remains to compute the terms in factor of χ. These calculations are tedious, but the idea is simple and consists - in some sense - to do the same calculations as in Section 4.4 but in the reverse sense. According to (52) we define

$$\mathbb{E}_{i,k}^{\delta} := \frac{E_{i,k}^{\delta}}{\delta} = \sum_{p=0}^{k-1} \delta^p \; \mathbb{E}_i^p.$$

With the notation of Section 4.4, we have

$$\mathrm{curl} E^\delta_{i,k} - i\omega H^\delta_{i,k} = r_i^{\delta,k}(x_\Gamma, \nu/\delta) \tag{113}$$

where the function $r_i^{\delta,k}(x,\eta)$ is given by

$$r_i^{\delta,k} = \partial_\eta \mathbb{E}^\delta_{i,k} \times n - i\omega\, H^\delta_{i,k} + \sum_{\ell=1}^{2} \delta^\ell A_E^{(\ell)}(\mathbb{E}^\delta_{i,k}, H^\delta_{i,k}).$$

Replacing $\mathbb{E}^\delta_{i,k}$ and $H^\delta_{i,k}$ by their polynomial expansion in δ, we get

$$r_i^{\delta,k} = \sum_{p=0}^{k-1} \delta^p \left(\partial_\eta \mathbb{E}_i^p \times n - i\omega\, H_i^p\right) - i\omega\, \delta^k\, H_i^k + \sum_{\ell=1}^{2} \delta^\ell \sum_{p=0}^{k-1} \delta^p\, A_E^{(\ell)}(\mathbb{E}_i^p, H_i^p).$$

Using the equations (55) satisfied by the $\mathbb{E}_i^p$'s and H_i^p's, we get

$$r_i^{\delta,k} = -i\omega\, \delta^k\, H_i^k + \sum_{\ell=1}^{2} \sum_{p=0}^{k-1} \delta^{p+\ell}\, A_E^{(\ell)}(\mathbb{E}_i^p, H_i^p) - \sum_{\ell=1}^{2} \sum_{p=0}^{k-1} \delta^p\, A_E^{(\ell)}(\mathbb{E}_i^{p-\ell}, H_i^{p-\ell}).$$

Applying the change of index $p + l \to p$ in the first sum, we get

$$r_i^{\delta,k} = -i\omega\, \delta^k\, H_i^k + \sum_{\ell=1}^{2} \sum_{p=0}^{k-1+\ell} \delta^p\, A_E^{(\ell)}(\mathbb{E}_i^{p-\ell}, H_i^{p-\ell}) - \sum_{\ell=1}^{2} \sum_{p=0}^{k-1} \delta^p\, A_E^{(\ell)}(\mathbb{E}_i^{p-\ell}, H_i^{p-\ell}),$$

that is to say

$$r_i^{\delta,k} = -i\omega\, \delta^k\, H_i^k + \sum_{\ell=1}^{2} \sum_{p=k}^{k-1+\ell} \delta^p\, A_E^{(\ell)}(\mathbb{E}_i^{p-\ell}, H_i^{p-\ell}).$$

Paying attention to the above expression and using the form of the functions $\mathbb{E}_i^p$ and H_i^p (cf. Theorem 4.2), we see that

$$\mathrm{curl}\, \mathbb{E}^\delta_{i,k} - i\omega H^\delta_{i,k} = \delta^k \left(g^\delta_{k,0} + \delta\, g^\delta_{k,1}\right) \quad \text{in supp } \chi,$$

where the functions g_0^δ and g_1^δ are of the form

$$g^\delta_{k,q}(x) = p_{k,q}(x_\Gamma, \frac{\nu}{\delta})\, e^{-\sqrt{i}\,\frac{\nu}{\delta}}, \quad p_{k,q} \in \mathbf{P}_k(\Gamma, \mathbb{R}^+; \mathbb{C}^3), \quad q = 0, 1. \tag{114}$$

From (114), we easily deduce that

$$\|\chi\, g^\delta_{k,q}\|_{L^2(\Omega_i)} \leq C_{k,q}\, \delta^{\frac{1}{2}}, \quad \|\chi\, \mathrm{curl}\, g^\delta_{k,q}\|_{L^2(\Omega_i)} \leq C'_{k,q}\, \delta^{\frac{1}{2}}, \quad q = 0, 1. \tag{115}$$

In the same way, using again local coordinates, we have

$$i\omega E^\delta_{i,k} + \mathrm{curl} H^\delta_{i,k} - \frac{1}{\delta^2} E^\delta_{i,k} = \frac{1}{\delta}\, s_i^{\delta,k}(x, \nu/\delta),$$

with

$$s_i^{\delta,k} = \partial_\eta H_{i,k}^\delta \times n + \frac{1}{\omega}\,\mathbb{E}_{i,k}^\delta - \sum_{\ell=1}^{4} \delta^\ell\, A_H^{(\ell)}(\mathbb{E}_{i,k}^\delta, H_{i,k}^\delta)$$

Replacing $\mathbb{E}_{i,k}^\delta$ and $H_{i,k}^\delta$ by their polynomial expansion in δ we get

$$\left|\begin{array}{lcl} s_i^{\delta,k} & = & \displaystyle\sum_{p=0}^{k-1} \delta^p \left(\partial_\eta H_i^p \times n + \frac{1}{\omega}\,\mathbb{E}_i^p\right) - \sum_{\ell=1}^{4} \delta^\ell \sum_{p=0}^{k-1} \delta^p\, A_H^{(\ell)}(\mathbb{E}_i^p, H_i^p) \\ & + & \displaystyle\delta^k\, \partial_\eta H_i^p \times n - \sum_{\ell=1}^{4} \delta^{\ell+k}\, A_H^{(\ell)}(0, H_i^k). \end{array}\right.$$

Using equations (55) satisfied by the $\mathbb{E}_i^p$'s and H_i^p's, we get

$$\left|\begin{array}{lcl} s_i^{\delta,k} & = & \displaystyle\sum_{\ell=1}^{4} \sum_{p=0}^{k-1} \delta^p\, A_H^{(\ell)}(\mathbb{E}_i^{p-\ell}, H_i^{p-\ell}) - \sum_{\ell=1}^{4} \sum_{p=0}^{k-1} \delta^{p+\ell}\, A_H^{(\ell)}(\mathbb{E}_i^p, H_i^p) \\ & + & \displaystyle\delta^k\, \partial_\eta H_i^p \times n - \sum_{\ell=1}^{4} \delta^{\ell+k}\, A_H^{(\ell)}(0, H_i^k), \end{array}\right.$$

or equivalently

$$\left|\begin{array}{lcl} s_i^{\delta,k} & = & \displaystyle\sum_{p=k}^{k+\ell-1} \delta^p \sum_{\ell=1}^{4} A_H^{(\ell)}(\mathbb{E}_i^{p-\ell}, H_i^{p-\ell}) \\ & + & \displaystyle\delta^k\, \partial_\eta H_i^p \times n - \sum_{\ell=1}^{4} \delta^{\ell+k}\, A_H^{(\ell)}(0, H_i^k) \end{array}\right.$$

This time, we see that we can write

$$i\omega E_{i,k}^\delta + \mathrm{curl} H_{i,k}^\delta - \frac{1}{\delta^2} E_{i,k}^\delta = \frac{1}{\delta} \sum_{q=0}^{3} \delta^q\, h_{k,q}^\delta \quad \text{in supp}\ \chi,$$

where the expression of $h_{k,q}^\delta$ is similar to the $g_{k,q}$'s (see formula (114)) and implies in particular that

$$\|\chi\, h_{k,q}^\delta\|_{L^2(\Omega_i)} \leq C_{k,q}\, \delta^{\frac{1}{2}}, \quad q = 0,1,2,3. \tag{116}$$

In summary, taking the difference between (112) and (4) we have shown that

$$\left\{\begin{array}{ll} \mathrm{curl}\, \mathcal{H}_i^{\delta,k} + i\omega \mathcal{E}_i^{\delta,k} - \dfrac{1}{\omega\delta^2}\, \mathcal{E}_i^{\delta,k} & = \delta^{k-1}\, \chi \left(\displaystyle\sum_{q=0}^{3} \delta^q\, h_{k,q}^\delta\right) + \nabla\chi \times H_{i,k}^\delta, \\ \mathrm{curl}\, \mathcal{E}_i^{\delta,k} - i\omega \mathcal{H}_i^{\delta,k} & = \delta^k\, \chi \left(\displaystyle\sum_{q=0}^{1} \delta^q\, g_{k,q}^\delta\right) + \nabla\chi \times E_{i,k}^\delta, \end{array}\right.$$

where eliminating $\mathcal{H}_i^{\delta,k}$ we get

$$\operatorname{curl}\operatorname{curl}\mathcal{E}_i^{\delta,k} - \omega^2\,\mathcal{E}_i^{\delta,k} + \frac{i}{\delta^2}\,\mathcal{E}_i^{\delta,k} = f_k^\delta \tag{117}$$

with

$$\left|\begin{array}{rcl} f_k^\delta & := & \delta^k\,\chi\,\Big(\displaystyle\sum_{q=0}^{1}\delta^q\operatorname{curl} g_{k,q}^\delta\Big) + \nabla\chi\times\Big(\displaystyle\sum_{q=0}^{1}\delta^q\,g_{k,q}^\delta\Big) + \nabla\chi\times\operatorname{curl}\big(\nabla\chi\times E_{i,k}^\delta\big) \\ & - & i\omega\,\delta^{k-1}\,\chi\,\Big(\displaystyle\sum_{q=0}^{3}\delta^q\,h_{k,q}^\delta\Big)\;-\;i\omega\,\nabla\chi\times H_{i,k}^\delta. \end{array}\right.$$

Taking into account the form of the functions $g_{k,q}^\delta$ and the exponential decay of the the fields E_i^p and H_i^p (Theorem 4.2), and since the support of $\nabla\chi$ is separated from Γ, there exists a constant $\tau > 0$ such that:

$$\left|\begin{array}{l} \|\,\nabla\chi\times\operatorname{curl}\big(\nabla\chi\times E_{i,k}^\delta\big)\,\|_{L^2(\Omega_e)} \;\leq\; C_1(k)\,e^{-\tau\,\delta}, \\[2ex] \|\,\nabla\chi\times\Big(\displaystyle\sum_{q=0}^{1}\delta^q\,g_{k,q}^\delta\Big)\,\|_{L^2(\Omega_e)} \;\leq\; C_2(k)\,e^{-\tau\,\delta}, \\[2ex] \|\,\nabla\chi\times H_{i,k}^\delta\,\|_{L^2(\Omega_e)} \leq C_3(k)\,e^{-\tau\,\delta}. \end{array}\right.$$

Combining these inequalities with estimates (115) and (116), we see that:

$$\|f_k^\delta\|_{L^2(\Omega_e)} \;\leq\; C_k\,\delta^{k-\frac{1}{2}}\;. \tag{118}$$

Error estimates. We can now proceed with the final step of the proof. First, we multiply the equation (111) by $\overline{\mathcal{E}_e^{\delta,k}}$ and integrate over Ω_e. Using the Stokes formula and the boundary condition in (111), we get

$$\int_{\Omega_e}|\operatorname{curl}\mathcal{E}_e^{\delta,k}|^2\,dx - \omega^2\int_{\Omega_e}|\mathcal{E}_e^{\delta,k}|^2\,dx - i\omega\int_{\partial\Omega}|\mathcal{E}_e^{\delta,k}\times n|^2 + \Big\langle\operatorname{curl}\mathcal{E}_e^{\delta,k}\times n,\;\big(\overline{\mathcal{E}_e^{\delta,k}}\big)_T\Big\rangle_\Gamma = 0.$$

Next, we multiply the equation (117) by $\mathcal{E}_i^{\delta,k}$ and integrate over Ω_i. We get

$$\left|\begin{array}{l} \displaystyle\int_{\Omega_i}|\operatorname{curl}\mathcal{E}_i^{\delta,k}|^2\,dx - \omega^2\int_{\Omega_i}|\mathcal{E}_i^{\delta,k}|^2\,dx - \frac{1}{\delta^2}\int_{\Omega_i}|\mathcal{E}_i^{\delta,k}|^2 - \Big\langle\big(\operatorname{curl}\mathcal{E}_e^{\delta,k}\times n\big)\cdot\big(\overline{\mathcal{E}_i^{\delta,k}}\big)_T\Big\rangle_\Gamma \\ \hfill = \displaystyle\int_{\Omega_i} f_k^\delta\cdot\overline{\mathcal{E}_i^{\delta,k}}\,dx. \end{array}\right.$$

Adding the last two equalities and using the fact that $\mathcal{E}^{\delta,k}$ belongs to $H(\operatorname{curl};\Omega)$, we get

$$\left|\begin{array}{l} \displaystyle\int_\Omega|\operatorname{curl}\mathcal{E}^{\delta,k}|^2 - \omega^2\int_\Omega|\mathcal{E}^{\delta,k}|^2 - i\omega\Big(\int_{\partial\Omega}|\mathcal{E}^{\delta,k}\times n|^2 + \frac{1}{\delta^2}\int_{\Omega_i}|\mathcal{E}^{\delta,k}|^2\Big) \\ \qquad = \Big\langle\operatorname{curl}\mathcal{E}_e^{\delta,k}\times n - \operatorname{curl}\mathcal{E}_i^{\delta,k}\times n,\,\big(\overline{\mathcal{E}^{\delta,k}}\big)_T\Big\rangle_\Gamma + \displaystyle\int_{\Omega_i} f_k^\delta\cdot\overline{\mathcal{E}_i^{\delta,k}}\,dx \end{array}\right. \tag{119}$$

It remains to compute the jump

$$\operatorname{curl}\mathcal{E}_e^{\delta,k}\times n-\operatorname{curl}\mathcal{E}_i^{\delta,k}\times n\equiv\operatorname{curl}E_{e,k}^{\delta}\times n-\operatorname{curl}E_{i,k}^{\delta}\times n$$

across Γ. Taking the trace on Γ of equation (113), we get, with $\rho_i^{\delta,k}(x_\Gamma)=r_i^{\delta,k}(x_\Gamma,0)$,

$$\operatorname{curl}E_{i,k}^{\delta}\times n=i\omega H_{i,k}^{\delta}\times n+\rho_i^{\delta,k}\quad\text{on }\Gamma.$$

The function $\rho_i^{\delta,k}$ is not zero but small. In particular, according to , we have

$$\|\rho_i^{\delta,k}\|_{H^{\frac12}(\Gamma)}\le C_k\;\delta^k. \tag{120}$$

On the other hand, taking the trace on Γ of the first equation of (109) we get

$$\operatorname{curl}E_{e,k}^{\delta}\times n=i\omega H_{e,k}^{\delta}\times n,\quad\text{on }\Gamma$$

The continuity conditions (56) imply $H_{i,k}^{\delta}\times n=H_{e,k}^{\delta}\times n$ on Γ so that

$$\operatorname{curl}\mathcal{E}_e^{\delta,k}\times n-\operatorname{curl}\mathcal{E}_i^{\delta,k}\times n\equiv\operatorname{curl}E_{e,k}^{\delta}\times n-\operatorname{curl}E_{i,k}^{\delta}\times n=\rho_i^{\delta,k}. \tag{121}$$

Substituting (121) into (119) and using the estimates (118) and (120), we get

$$\left|\begin{array}{l}\displaystyle\left|\int_\Omega|\operatorname{curl}\mathcal{E}^{\delta,k}|^2-\omega^2\int_\Omega|\mathcal{E}^{\delta,k}|^2-i\omega\Big(\int_{\partial\Omega}|\mathcal{E}^{\delta,k}\times n|^2+\frac{1}{\delta^2}\int_{\Omega_i}|\mathcal{E}^{\delta,k}|^2\Big)\right|\\ \qquad\qquad\displaystyle\le\;C_k\;\Big(\;\delta^{k-\frac12}\;\|\mathcal{E}^{\delta,k}\times n\|_{L^2(\Omega_i)}+\;\delta^k\;\|\mathcal{E}^{\delta,k}\times n\|_{H^{-\frac12}(\Gamma)}\Big)\;.\end{array}\right.$$

We can finally apply Lemma 5.2 with $\mathbf{E}^\delta=\mathcal{E}^{\delta,k}$ and $s=k-\frac12$, which provides the desired estimates. □

6 Analysis of the GIBCs

6.1 Well-posedness of the approximate problems

We shall prove in this section that the approximate fields $(E^{\delta,k},H^{\delta,k})$, solution of (23, 24) for $k=0,\,1,\,2,\,3$, are well defined. In fact, for $k\le2$ this result is an application (or an adaptation) of classical results about Maxwell equations with an impedance boundary condition of the form

$$E\times n+\omega\,Z\,H_T=0\text{ on }\Gamma,$$

where Z is a function with positive real part (see for instance [13]). To include the case $k=3$ it is sufficient to extend these results to the cases where Z is a continuous operator form $L^2_t(\Gamma)$ into $L^2_t(\Gamma)$ with positive definite real part. More precisely we shall assume that there exists two positive constants z_* and z^* such that

$$\begin{cases} (i) & \|Z\varphi\|_\Gamma \leq z^* \ \|\varphi\|_\Gamma, \\ (ii) & \mathcal{R}e \ (Z\varphi, \varphi)_\Gamma \geq z_* \ \|\varphi\|^2_\Gamma, \end{cases} \tag{122}$$

for all $\varphi \in L^2_t(\Gamma)$. These properties are satisfied by the operators $\mathcal{D}^{\delta,k}$, $k = 1, 2, 3$ for δ sufficiently small, and can be seen as a special consequence of Lemma 6.1 (stated and proved in next section) where the dependence of the constants z_* and z^* in terms of δ is also given (which is important for error analysis). The functional space adapted to this type of boundary conditions is the same as for constant impedances, namely $\widetilde{H}(\mathrm{curl}, \Omega_e)$ (see (5) for the definition of this space).

Theorem 6.1 *Let $f \in L^2(\Omega_e)$ be compactly supported in Ω_e and $g \in L^2_t(\partial\Omega_e)$. Then the boundary value problem*

$$\begin{cases} \mathrm{curl}\, H + i\omega E = 0, & \textit{in } \Omega_e, \\ \mathrm{curl}\, E - i\omega H = 0, & \textit{in } \Omega_e, \\ E \times n + H_T = g, & \textit{on } \partial\Omega, \\ E \times n + \omega Z H_T = g, & \textit{on } \Gamma. \end{cases}$$

has a unique solution (E, H) in $\widetilde{H}(\mathrm{curl}, \Omega_e) \times \widetilde{H}(\mathrm{curl}, \Omega_e)$.

Proof. The proof uses basically the same arguments as for classical impedance conditions (see for instance [13]) and is detailed here for the reader convenience. The approach is divided in three steps:

1. One eliminates the electric field E and formulate a boundary value problem in H only, then writes the associated weak formulation in an appropriate functional framework.
2. One shows that the Fredholm alternative can be applied for solving this problem.
3. One shows the uniqueness of the solution which also implies the existence.

Step 1. The problem to be solved for $H \in \widetilde{H}(\mathrm{curl}, \Omega_e)$ is

$$\begin{cases} \mathrm{curl}\,\mathrm{curl}\, H - \omega^2 \ H = -\mathrm{curl}\, f & \text{in } \ \Omega_e, \\ \mathrm{curl}\, H \times n - i \ \omega^2 Z H_T = -i\omega g, & \text{on } \ \Gamma, \\ \mathrm{curl}\, H \times n - i \ \omega \ H_T = -i\omega g, & \text{on } \ \partial\Omega. \end{cases} \tag{123}$$

The equivalent weak formulation of the boundary value problem (123) can be written in the form:

$$\left\{\begin{array}{l} \text{Find } H \in \widetilde{H}(\mathrm{curl}, \Omega_e) \text{ such that,} \\ \displaystyle\int_{\Omega_e} \left(\mathrm{curl}\, H \cdot \overline{\mathrm{curl}\, H'} - \omega^2 H \cdot \overline{H'} \right) dx \;-\; i\,\omega (H_T, H'_T)_{\partial\Omega} -\; i\,\omega^2\,(Z H_T, H'_T)_\Gamma \\ \displaystyle\qquad\qquad = -\,i\,\omega (g, H'_T)_{\partial\Omega_e} + \int_{\Omega_e} f \cdot \overline{\mathrm{curl}\, H'}\, dx, \quad \forall\, H' \in \widetilde{H}(\mathrm{curl}, \Omega_e). \end{array}\right. \tag{124}$$

Next, we formulate a variational problem, equivalent to (124), posed in a subspace $\widetilde{H}_0^\delta(\mathrm{curl}, \Omega_e)$ of $\widetilde{H}(\mathrm{curl}, \Omega_e)$ having additional compactness properties, by using the Helmholtz decomposition. Let us consider the closed subspace of $H^1(\Omega)$,

$$S := \{v \in H^1(\Omega) \;/\; (u|_\Gamma, u|_{\partial\Omega}) \in P_0(\Gamma) \times P_0(\partial\Omega)\}$$

where $P_0(\Gamma)$ is the space of constant functions on Γ (the same for $P_0(\partial\Omega)$). Then we define $\widetilde{H}_0^\delta(\mathrm{curl}, \Omega_e)$ as

$$\widetilde{H}_0^\delta(\mathrm{curl}, \Omega_e) := \{H \in \widetilde{H}(\mathrm{curl}, \Omega_e) \;/\; \int_{\Omega_e} H \cdot \nabla v\; dx = 0, \forall\, v \in S\},$$

which forms a closed subset of $\widetilde{H}(\mathrm{curl}, \Omega_e)$. One also has the orthogonal decomposition (in $L^2(\Omega_e)$)

$$\widetilde{H}(\mathrm{curl}, \Omega_e) = \widetilde{H}_0^\delta(\mathrm{curl}, \Omega_e) \oplus \nabla S, \quad \text{where } \nabla S := \{\nabla v \;/\; v \in S\}. \tag{125}$$

Remark 6.1 *By Green's theorem, the reader will notice that* $\widetilde{H}_0^\delta(\mathrm{curl}, \Omega_e)$ *is nothing but the subspace of* $\widetilde{H}(\mathrm{curl}, \Omega_e)$ *made of vector fields whose divergence vanishes and whose normal trace on each connected component of* $\partial\Omega_e$*, namely* Γ *and* $\partial\Omega$*, has zero mean value.*

We claim that the problem (124) is equivalent to

$$\left\{\begin{array}{l} \text{Find } H \in \widetilde{H}_0^\delta(\mathrm{curl}, \Omega_e) \text{ such that,} \\ \displaystyle\int_{\Omega_e} \left(\mathrm{curl}\, H \cdot \overline{\mathrm{curl}\, H'} - \omega^2 H \cdot \overline{H'} \right) dx \;-\; i\,\omega (H_T, H'_T)_{\partial\Omega} -\; i\,\omega^2\,(Z H_T, H'_T)_\Gamma \\ \displaystyle\qquad\qquad = -\,i\,\omega (g, H'_T)_{\partial\Omega_e} + \int_{\Omega_e} f \cdot \overline{\mathrm{curl}\, H'}\, dx, \quad \forall\, H' \in \widetilde{H}_0^\delta(\mathrm{curl}, \Omega_e). \end{array}\right. \tag{126}$$

Indeed, let $H \in \widetilde{H}(\mathrm{curl}, \Omega_e)$ be a solution of (124). Since, $\widetilde{H}_0^\delta(\mathrm{curl}, \Omega_e) \subset \widetilde{H}(\mathrm{curl}, \Omega_e)$, to prove that H is solution of (126), we only have to prove that $H \in \widetilde{H}_0^\delta(\mathrm{curl}, \Omega_e)$. According to (125), we have:

$$H = H_0 + \nabla u, \quad H_0 \in \widetilde{H}_0^\delta(\mathrm{curl}, \Omega_e), \quad u \in S, \tag{127}$$

and we simply have to prove that $\nabla u = 0$. If we choose $H' = \nabla u$ with $v \in S$ in (124), remarking that $\operatorname{curl} H' = 0$ and $H'_T = 0$ on Γ and $\partial\Omega$, we get, using (127):

$$\omega^2 \int_{\Omega_e} |\nabla u|^2 \, dx = 0 \quad \Longrightarrow \quad \nabla u = 0.$$

Reciprocally, one proves that any solution of (126) is solution of (124) using the same type of argument (decompose the test function instead of the solution).

Step 2. We now rewrite (126) in the form

$$\left\{ \begin{array}{l} \text{Find } H \in \widetilde{H}_0^\delta(\operatorname{curl}, \Omega_e) \text{ such that,} \\ a(H, H') = b(H, H') + L(H'), \quad \forall\, H' \in \widetilde{H}_0^\delta(\operatorname{curl}, \Omega_e), \end{array} \right. \tag{128}$$

where we have defined:

$$\begin{array}{lcl} a(H, H') & = & \displaystyle\int_{\Omega_e} \left(\operatorname{curl} H \cdot \overline{\operatorname{curl} H'} + H \cdot \overline{H'} \right) dx - \; i\, \omega (H_T, H'_T)_{\partial\Omega} - \; i\, \omega^2 \, (Z H_T, H'_T), \\ b(H, H') & = & (1 + \omega^2) \displaystyle\int_{\Omega_e} H \cdot \overline{H'} \, dx, \\ L(H') & = & - \, i\, \omega (g, H'_T)_{\partial\Omega_e} + \displaystyle\int_{\Omega_e} f \cdot \overline{\operatorname{curl} H'} \, dx. \end{array}$$

According to (122), one has, with $\alpha < 1$ denoting an arbitrary positive constant

$$\begin{array}{rcl} |a(H, H)| & \geq & \frac{\alpha}{2} |\mathcal{R}e \; a(H, H)| + \frac{1}{2} |\mathcal{I}m \; a(H, H)| \\ & \geq & \frac{\alpha}{2} (\|H\|^2_{H(\operatorname{curl}, \Omega_e)} - z^* \|H_T\|^2_\Gamma) + \frac{\omega^2 z_*}{2} \|H_T\|^2_\Gamma + \frac{\omega}{2} \|H_T\|^2_{\partial\Omega}. \end{array}$$

Choosing α such that $\alpha z^* < \omega^2 z_*$ proves that the sesquilinear form a is coercive on $\widetilde{H}_0^\delta(\operatorname{curl}, \Omega_e)$. On the other hand the sesquilinear form b is continuous with respect to the $L^2(\Omega_e)$ norm, which compactly contains $\widetilde{H}_0^\delta(\operatorname{curl}, \Omega_e)$. Therefore the Fredholm alternative can be applied to (128): existence uniqueness and stability is equivalent to the uniqueness of solutions.

Step 3. We shall now prove the uniqueness of the solution of (124). Let H be a solution of (124) with $f = 0$ and $g = 0$. Let us take $H' = \overline{H}$ and condider the imaginary part of the resulting equality. We get

$$\omega \, |H|^2_{\partial\Omega} + \omega^2 \, \mathcal{R}e \, (Z H_T, H_T) = 0.$$

From (122) one deduces that $H_T = 0$ on $\partial\Omega$ and, using the boundary condition on $\partial\Omega$ that $\operatorname{curl} H \times n = 0$ on $\partial\Omega$ (this makes sense since, $\operatorname{curl} \operatorname{curl} H \; = \omega^2 \; H$, and therefore $\operatorname{curl} H$ belongs to $H(\operatorname{curl}, \Omega_e)$). One concludes using standard unique continuation theorems for Maxwell's equations (see [13]). $\square$

6.2 Error estimates for the GIBCs

The error estimates rely of some key properties of the boundary operator $\mathcal{D}^{\delta,k}$ that we shall summarize in the following lemma. We recall that $\mathcal{D}^{\delta,k} = 0$, $\delta\sqrt{i}$ and $\delta\sqrt{i} + \delta^2(\mathcal{H} - \mathcal{C})$ for $k = 0, 1$ and 2, respectively. For $k = 3$, we denote by A^δ and B^δ the two operators

$$A^\delta := \left(1 - \delta^2\, \nabla_\Gamma \operatorname{div}_\Gamma\right)^{-1}, \quad B^\delta = \left(1 + \delta^2\, \vec{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma\right)^{-1}.$$

By Lax-Milgram's Lemma these operators are well defined as continuous operators from $L^2_t(\Gamma)$ to respectively $H(\operatorname{div}_\Gamma, \Gamma)$ and $H(\operatorname{curl}_\Gamma, \Gamma)$. Setting

$$\alpha_\delta := \frac{1}{2\sqrt{2}} + \delta(\mathcal{H} - \mathcal{C}) + \frac{\delta^2}{2\sqrt{2}}\left(\mathcal{C}^2 - \mathcal{H}^2 + \varepsilon_r\omega^2\right),$$
$$\beta_\delta := \frac{1}{2\sqrt{2}} - \frac{\delta^2}{2\sqrt{2}}\left(\mathcal{C}^2 - \mathcal{H}^2 + \varepsilon_r\omega^2\right),$$

the expression (33) of $\mathcal{D}^{\delta,3}$ can be written in the form

$$\left\{\begin{array}{rcl} \mathcal{D}^{\delta,3}\varphi & = & \delta\,\alpha_\delta\varphi + \dfrac{\sqrt{2}}{4}\,\delta\left(A^\delta\varphi + \delta^2\vec{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma B^\delta\varphi\right) \\[2ex] & + & i\delta\,\beta_\delta\varphi + i\dfrac{\sqrt{2}}{4}\,\delta\left(B^\delta\varphi - \delta^2\nabla_\Gamma \operatorname{div}_\Gamma A^\delta\varphi\right). \end{array}\right.$$

The fundamental properties of the operators $\mathcal{D}^{\delta,k}$ are summarized in the following lemma.

Lemma 6.1 *Let $k = 1, 2$ or 3. There exist a constant $\delta_k > 0$ and two constants $C_1 > 0$ and $C_2 > 0$, independent of δ, such that*

$$\begin{array}{ll} (i) & \|\mathcal{D}^{\delta,k}\varphi\|_\Gamma \leq C_1\,\delta\,\|\varphi\|_\Gamma, \\ (ii) & \mathcal{R}e\,(\mathcal{D}^{\delta,k}\varphi,\varphi)_\Gamma \geq C_2\,\delta\,\|\varphi\|^2_\Gamma, \end{array} \tag{129}$$

for all $\varphi \in L^2_t(\Gamma)$ and $\delta \leq \delta_k$.

Proof. These properties are straightforward for $k = 1$ and 2. We shall concentrate on the case $k = 3$. We first observe that α_δ and β_δ are bounded functions on Γ, and if $\varphi \in L^2_t(\Gamma)$ then $\delta^2\nabla_\Gamma \operatorname{div}_\Gamma A^\delta\varphi = (A^\delta\varphi - \varphi) \in L^2_t(\Gamma)$ and $\delta^2\vec{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma B^\delta\varphi = (-B^\delta\varphi + \varphi) \in L^2_t(\Gamma)$. Therefore $\mathcal{D}^{\delta,3}\varphi \in L^2_t(\Gamma)$ and one has

$$\left\{\begin{array}{rcl} (\mathcal{D}^{\delta,3}\varphi,\psi)_\Gamma & = & \delta\,(\alpha_\delta\varphi,\psi)_\Gamma + \dfrac{\sqrt{2}}{4}\,\delta\left((A^\delta\varphi,\psi)_\Gamma + \delta^2\,(\vec{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma B^\delta\varphi,\psi)_\Gamma\right) \\[2ex] & + & i\delta\,(\beta_\delta\varphi,\psi)_\Gamma + i\dfrac{\sqrt{2}}{4}\,\delta\left((B^\delta\varphi,\psi)_\Gamma - \delta^2\,(\nabla_\Gamma \operatorname{div}_\Gamma A^\delta\varphi,\psi)_\Gamma\right) \end{array}\right. \tag{130}$$

for all $\varphi,\ \psi \in L^2_t(\Gamma)$. For δ sufficiently small, the functions α_δ and β_δ satisfy

$$0 < \alpha_* < |\alpha_\delta| < \alpha^* \quad \text{and} \quad 0 < \beta_* < |\beta_\delta| < \beta^* \tag{131}$$

for some positive constants α_*, α^*, β_* and β^* independent of δ. On the other hand, from the identities

$$(1 - \delta^2 \nabla_\Gamma \operatorname{div}_\Gamma) A^\delta \varphi = \varphi \quad \text{and} \quad (1 + \delta^2 \vec{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma) B^\delta \varphi = \varphi$$

one respectively deduces

$$\left\{ \begin{array}{rcl} (A^\delta \varphi, \varphi)_\Gamma & = & \|A^\delta \varphi\|^2_\Gamma + \delta^2 \|\operatorname{div}_\Gamma A^\delta \varphi\|^2_\Gamma \\ -(\nabla_\Gamma \operatorname{div}_\Gamma A^\delta \varphi, \varphi)_\Gamma & = & \|\operatorname{div}_\Gamma A^\delta \varphi\|^2_\Gamma + \delta^2 \|\nabla_\Gamma \operatorname{div}_\Gamma A^\delta \varphi\|^2_\Gamma \end{array} \right. \tag{132}$$

and

$$\left\{ \begin{array}{rcl} (B^\delta \varphi, \varphi)_\Gamma & = & \|B^\delta \varphi\|^2_\Gamma + \delta^2 \|\operatorname{curl}_\Gamma B^\delta \varphi\|^2_\Gamma \\ (\vec{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma B^\delta \varphi, \varphi)_\Gamma & = & \|\operatorname{curl}_\Gamma B^\delta \varphi\|^2_\Gamma + \delta^2 \|\vec{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma B^\delta \varphi\|^2_\Gamma \end{array} \right. \tag{133}$$

Property (ii) is obtained as an immediate consequence of (132), (133) and (131) when applied to (130) with $\psi = \varphi$. Identities (132) and (133) also respectively imply,

$$\|A^\delta \varphi\|_\Gamma \le \|\varphi\|_\Gamma, \quad \delta^2 \|\nabla_\Gamma \operatorname{div}_\Gamma A^\delta \varphi\|_\Gamma \le \|\varphi\|_\Gamma,$$

$$\|B^\delta \varphi\|_\Gamma \le \|\varphi\|_\Gamma \text{ and } \delta^2 \|\vec{\operatorname{curl}}_\Gamma \operatorname{curl}_\Gamma B^\delta \varphi\|_\Gamma \le \|\varphi\|_\Gamma.$$

Property (i) is then easily obtained from (130) with $\psi = \mathcal{D}^{\delta,3}\varphi$ and using these estimates, as well as (131). $\square$

6.3 Error estimates for the GIBCs

We shall set for $k = 0, 1, 2, 3$,

$$\left\{ \begin{array}{l} \widetilde{\mathcal{E}}^{\delta,k}_e = E^{\delta,k}_e - \sum_{\ell=0}^k E^\ell_e, \\ \widetilde{\mathcal{H}}^{\delta,k}_e = H^{\delta,k}_e - \sum_{\ell=0}^k H^\ell_e. \end{array} \right. \tag{134}$$

Using (83), together with (34) and (35) when $k = 3$, we see that $(\widetilde{\mathcal{E}}^{\delta,k}_e, \widetilde{\mathcal{H}}^{\delta,k}_e) \in \mathcal{V}^k_E \times \mathcal{V}^k_H$ is solution of the boundary value problem:

$$\left\{ \begin{array}{ll} \operatorname{curl} \widetilde{\mathcal{H}}^{\delta,k}_e + i\omega\, \widetilde{\mathcal{E}}^{\delta,k}_e = 0, & \text{in } \Omega_e, \\ \operatorname{curl} \widetilde{\mathcal{E}}^{\delta,k}_e - i\omega\, \widetilde{\mathcal{H}}^{\delta,k}_e = 0, & \text{in } \Omega_e, \\ \left(\widetilde{\mathcal{E}}^{\delta,k}_e\right)_T - \widetilde{\mathcal{H}}^{\delta,k}_e \times n = 0, & \text{on } \partial\Omega, \\ \widetilde{\mathcal{E}}^{\delta,k}_e \times n + \omega\, \mathcal{D}^{\delta,k} (\widetilde{\mathcal{H}}^{\delta,k}_e)_T = \delta^{k+1}\, \varphi^\delta_k & \text{on } \Gamma, \end{array} \right. \tag{135}$$

where the tangential vector fields φ_k^δ remain bounded with respect to δ in all spaces $H_t^s(\Gamma)^3$.

Eliminating $\widetilde{\mathcal{E}}_e^{\delta,k}$, we see that $\widetilde{\mathcal{H}}_e^{\delta,k} \in \mathcal{V}_H^k$ satisfies

$$\begin{cases} \operatorname{curl}\big(\operatorname{curl}\widetilde{\mathcal{H}}_e^{\delta,k}\big) - \omega^2\, \widetilde{\mathcal{H}}_e^{\delta,k} = 0 & \text{in } \Omega_e, \\ \operatorname{curl}\widetilde{\mathcal{H}}_e^{\delta,k} \times n - i\,\omega^2\, \mathcal{D}^{\delta,k}(\widetilde{\mathcal{H}}_e^{\delta,k})_T = \delta^{k+1}\,\varphi_k^\delta, & \text{on } \Gamma, \\ \operatorname{curl}\widetilde{\mathcal{H}}_e^{\delta,k} \times n - i\,\omega\,(\widetilde{\mathcal{H}}_e^{\delta,k})_T = 0, & \text{on } \partial\Omega. \end{cases}$$

The proof of error estimates is based on some key a priori estimates that we shall give hereafter. We multiply by $\overline{\widetilde{\mathcal{H}}_e^{\delta,k}}$ the equation satisfied by $\widetilde{\mathcal{H}}_e^{\delta,k}$ in Ω_e, integrate over Ω_e and use Green's formula to obtain, after having used the boundary conditions on $\partial\Omega$ and Γ:

$$\left| \begin{aligned} &\int_{\Omega_e} \big(|\operatorname{curl}\widetilde{\mathcal{H}}_e^{\delta,k}|^2 - \omega^2|\widetilde{\mathcal{H}}_e^{\delta,k}|^2\big)\, dx - i\int_{\partial\Omega} |(\widetilde{\mathcal{H}}_e^{\delta,k})_T|^2\, d\sigma \\ &\qquad - i\,\omega^2\,\big(\mathcal{D}^{\delta,k}(\widetilde{\mathcal{H}}_e^{\delta,k})_T, (\widetilde{\mathcal{H}}_e^{\delta,k})_T\big)_\Gamma \;= \delta^{k+1}\left\langle \varphi_k^\delta, (\widetilde{\mathcal{H}}_e^{\delta,k})_T \right\rangle_\Gamma, \end{aligned} \right. \tag{136}$$

where $\langle\ ,\ \rangle_\Gamma$ here denotes a duality pairing between $H^{-1/2}(\mathrm{div},\Gamma)$ and $H^{-1/2}(\mathrm{curl},\Gamma)$. Considering the imaginary part of (136) and using (129)-(ii) together with trace theorems in $H(\mathrm{curl},\Omega_e)$, one obtains the existence of two non negative constants C_1 and C_2 independent of δ such that

$$C_1\,\delta\,\|(\widetilde{\mathcal{H}}_e^{\delta,k})_T\|_\Gamma^2 + \|(\widetilde{\mathcal{H}}_e^{\delta,k})_T\|_{\partial\Omega}^2 \le C_2\,\delta^{k+1}\,\|\widetilde{\mathcal{H}}_e^{\delta,k}\|_{H(\mathrm{curl},\Omega_e)}. \tag{137}$$

More precisely we have $C_1 = 0$ for $k = 0$ and $C_1 > 0$ for $k \neq 0$. Using (129)-(i) and (137) one also deduces that

$$|\mathcal{I}m\,\big(\mathcal{D}^{\delta,k}(\widetilde{\mathcal{H}}_e^{\delta,k})_T, (\widetilde{\mathcal{H}}_e^{\delta,k})_T\big)_\Gamma| \le C_3\,\delta^{k+1}\,\|\widetilde{\mathcal{H}}_e^{\delta,k}\|_{H(\mathrm{curl},\Omega_e)}, \tag{138}$$

for some constant C_3 independent of δ. Now considering the real part of (136) and using (138) as well as the trace theorem in $H(\mathrm{curl},\Omega_e)$, one gets the existence of two positive constants C_4 and C_5 independent of δ such that

$$\|\widetilde{\mathcal{H}}_e^{\delta,k}\|_{H(\mathrm{curl},\Omega_e)}^2 \le C_4\,\delta^{k+1}\,\|\widetilde{\mathcal{H}}_e^{\delta,k}\|_{H(\mathrm{curl},\Omega_e)} + C_5\,\|\widetilde{\mathcal{H}}_e^{\delta,k}\|_{L^2(\Omega_e)}^2. \tag{139}$$

Based on these a priori estimates we are in position to prove the following result.

Lemma 6.2 *For $k = 0, 1, 2$ or 3, there exist a constant C independent of δ and $\delta_0 > 0$ such that*

$$\|\widetilde{\mathcal{E}}_e^{\delta,k}\|_{H(\mathrm{curl},\Omega_e)} + \|\widetilde{\mathcal{H}}_e^{\delta,k}\|_{H(\mathrm{curl},\Omega_e)} \le C_k\,\delta^{k+1}$$

for all $\delta \le \delta_0$.

Proof. According to estimate (139) and first two equations of (135) it is sufficient to prove the existence of a constant C independent of δ such that

$$\|\widetilde{\mathcal{H}}_e^{\delta,k}\|_{L^2(\Omega_e)} \leq C\,\delta^{k+1}. \tag{140}$$

Let us assume that (140) does not hold, i.e. $\lambda_\delta := \delta^{k+1}/\|\widetilde{\mathcal{H}}_e^{\delta,k}\|_{L^2(\Omega_e)}$ goes to 0 as $\delta \to 0$, and consider the scaled fields

$$h^\delta = \widetilde{\mathcal{H}}_e^{\delta,k}/\|\widetilde{\mathcal{H}}_e^{\delta,k}\|_{L^2(\Omega_e)} \quad \text{and} \quad e^\delta = \widetilde{\mathcal{E}}_e^{\delta,k}/\|\widetilde{\mathcal{H}}_e^{\delta,k}\|_{L^2(\Omega_e)}.$$

Dividing (139) by $\|\widetilde{\mathcal{H}}_e^{\delta,k}\|^2_{L^2(\Omega_e)}$ implies in particular that (h^δ) is a bounded sequence in $H(\mathrm{curl},\Omega_e)$. Dividing (137) by the same quantity and using the latter result shows that

$$C_1\ \delta\ \|h_T^\delta\|_\Gamma^2 + \|h_T^\delta\|_{\partial\Omega}^2 \to 0 \text{ as } \delta \to 0. \tag{141}$$

The last boundary condition in (135) combined with property (129)-(i) shows in particular that

$$\|e_T^\delta\|_\Gamma \leq C_6\ \delta\ \|h_T^\delta\|_\Gamma + \lambda_\delta\ \|\varphi_k^\delta\|_\Gamma$$

(with the alternative, C_6 and $C_1 > 0$ or $C_6 = C_1 = 0$). We therefore conclude that $\|e_T^\delta\|_\Gamma$ goes to 0 as $\delta \to 0$. The first three equations of (135) implies that e^δ is a bounded sequence in $\widetilde{H}_0^\delta(\mathrm{curl},\Omega_e)$ (see definition in the proof of Theorem 6.1). Therefore, up to extracted subsequence, one can assume that e^δ converges strongly in $L^2(\Omega_e)$ and weakly in $\widetilde{H}_0^\delta(\mathrm{curl},\Omega_e)$ to some $e \in \widetilde{H}_0^\delta(\mathrm{curl},\Omega_e)$. Passing to the limit as $\delta \to 0$ in (135), we observe that $e \in \widetilde{H}_0^\delta(\mathrm{curl},\Omega_e)$ is solution of

$$\begin{cases} \mathrm{curl}\,\mathrm{curl}\, e - \omega^2\, e = 0, & \text{in } \Omega_e, \\ \mathrm{curl}\, e \times n - i\omega e_T \times n = 0, & \text{on } \partial\Omega, \\ e \times n = 0 & \text{on } \Gamma, \end{cases}$$

and therefore $e = 0$. We then deduce that $\mathrm{curl}\, h^\delta$ strongly converges to 0 in $L^2(\Omega_e)$. Coming back to identity (136) and considering the real part, one deduces after division by $\|\widetilde{\mathcal{H}}_e^{\delta,k}\|^2_{L^2(\Omega_e)}$ that

$$\|h^\delta\|^2_{H(\mathrm{curl},\Omega_e)} \leq \lambda_\delta\, \tilde{C}_0\ \|h^\delta\|_{H(\mathrm{curl},\Omega_e)} + \tilde{C}_1\ \|\mathrm{curl} h^\delta\|^2_{L^2(\Omega_e)} + \tilde{C}_2; |\mathcal{I}m\ (\mathcal{D}^{\delta,k} h_T^\delta, h_T^\delta)|. \tag{142}$$

Property (129)-(i) shows that

$$|\mathcal{I}m\ (\mathcal{D}^{\delta,k} h_T^\delta, h_T^\delta)| \leq \tilde{C}_3 \delta |h_T^\delta|_\Gamma^2 \quad \to \quad 0$$

according to (141). Therefore, considering the limit as $\delta \to 0$ in (142) implies that h^δ strongly converges to 0 in $H(\mathrm{curl},\Omega_e)$, which contradicts $\|h^\delta\|_{L^2(\Omega_e)} = 1$. □

A Technical Lemmas

The first Lemma is a slight variation of the classical trace lemma in $H(\mathrm{curl}, O)$.

Lemma A.1 *Let O be a bounded open subset of $\mathbb{R}^3$ of class C^2 (and which is locally from one side of its normal). Then there exists a constant C depending only on O such that*

$$\|u \times n\|^2_{H^{-\frac{1}{2}}(\partial O)} \leq C\, \|u\|_{L^2(O)} \, \big(\|\mathrm{curl}\, u\|_{L^2(O)} + \|u\|_{L^2(O)}\big) \quad \forall u \in H(\mathrm{curl}, O). \tag{143}$$

Remark A.1 *When $O = \mathbb{R}^3_+ := \big\{y \in \mathbb{R}^3 \;/\; y_3 > 0\big\}$, one can easily check, using partial Fourier transform in the variables (y_1, y_2), that*

$$\forall\, u \in H(\mathrm{curl}, \mathbb{R}^3_+), \quad \|u \times n\|^2_{H^{-\frac{1}{2}}(\partial \mathbb{R}^3_+)} \leq\; 2\, \|u\|_{L^2(\mathbb{R}^3_+)} \, (\|\mathrm{curl}\, u\|_{L^2(\mathbb{R}^3_+)} + \|u\|_{L^2(\mathbb{R}^3_+)}).$$

Proof. The idea is to see how the proof for the half-space (cf. previous remark) is modified when the boundary of O is not flat.

Let us consider first the cases where there exists $a > 0,\ b > 0,\ \delta > 0$ and $h \in C^2(\mathbb{R}^2) \cap W^{2,\infty}(\mathbb{R}^2)$ such that

$$\left|\begin{array}{l} \partial O \cap \;\mathrm{supp}\; u \subset \Sigma := \Big\{\; \varphi(y_1, y_2, 0)\;;\; (y_1, y_2) \in]-a, a\,[\times]-b, b\,[\;\Big\}, \\[2mm] \varphi\Big(]-a, a[\times]-b, b[\times]0, \delta[\Big) \subset O, \\[2mm] u \text{ is compactly supported in } \varphi\Big(\;]-a, a\,[\times]-b, b\,[\times]0, \delta\,[\;\Big), \end{array}\right.$$

where

$$\varphi(y_1, y_2, y_3) = \big(y_1, y_2, y_3 + h(y_1, y_2)\big).$$

Setting $\tilde{u} := u \circ \varphi$ and $\tilde{n} := u \circ \varphi$, one has for $y_3 = 0$,

$$\tilde{u} \times \tilde{n} = f\,\Big(\tilde{u}_2 + \tilde{u}_3 \frac{\partial h}{\partial y_2}\,,\; \tilde{u}_3 \frac{\partial h}{\partial y_1} - \tilde{u}_1,\; - \tilde{u}_1 \frac{\partial h}{\partial y_2} + \tilde{u}_2 \frac{\partial}{\partial y_1} h\Big), \quad f := \big(1 + |Dh|^2\big)^{-\frac{1}{2}} \tag{144}$$

On the other hand, (e_1, e_2, e_3) denoting the canonical basis in the y-space,

$$\left|\begin{array}{llcl} (i) & (\mathrm{curl}\, u \circ \varphi) \cdot e_1 & = & \dfrac{\partial \tilde{u}_3}{\partial y_2} - \dfrac{\partial \tilde{u}_3}{\partial y_3}\dfrac{\partial h}{\partial y_2} - \dfrac{\partial \tilde{u}_2}{\partial y_3}, \\[3mm] (ii) & (\mathrm{curl}\, u \circ \varphi) \cdot e_2 & = & \dfrac{\partial \tilde{u}_1}{\partial y_3} - \dfrac{\partial \tilde{u}_3}{\partial y_1} + \dfrac{\partial \tilde{u}_3}{\partial y_3}\dfrac{\partial h}{\partial y_1}, \\[3mm] (iii) & (\mathrm{curl}\, u \circ \varphi) \cdot e_3 & = & \dfrac{\partial \tilde{u}_2}{\partial y_1} - \dfrac{\partial \tilde{u}_1}{\partial y_2} + \dfrac{\partial \tilde{u}_3}{\partial y_3}(\dfrac{\partial h}{\partial y_2} - \dfrac{\partial h}{\partial y_1}). \end{array}\right. \tag{145}$$

Let us set $u_2^* = \tilde{u}_2 + \tilde{u}_3 \dfrac{\partial h}{\partial y_2}$, since,

$$\frac{\partial u_2^*}{\partial y_3} = \frac{\partial \tilde{u}_2}{\partial y_3} + \frac{\partial \tilde{u}_3}{\partial y_3}\frac{\partial h}{\partial y_2} + \tilde{u}_3\ \frac{\partial^2 h}{\partial y_3 \partial y_2}$$

using the formula (145)-(i) one gets, setting $r := (\mathrm{curl}\, u \circ \varphi) = (r_1, r_2, r_3)$

$$\frac{\partial u_2^*}{\partial y_3} = -r_1 + \frac{\partial \tilde{u}_3}{\partial y_2} + \tilde{u}_3 \frac{\partial^2 h}{\partial y_3 \partial y_2} \tag{146}$$

In what follows, we use the Fourier transform $\mathcal{F}$ in the variables (y_1, y_2) and denote by (ξ_1, ξ_2) the dual variable, by $\widehat{u}_i$ the Fourier transform of $\widetilde{u}_i$ and $\widehat{u}_2^*$ the Fourier transform of u_2^*. By definition of the norm in $H^{-\frac{1}{2}}(\mathbb{R}^2)$,

$$\|u_2^*(\cdot,\cdot,0)\|^2_{H^{-\frac{1}{2}}} = \int_{\mathbb{R}^2} (1 + |\xi|^2)^{-\frac{1}{2}}\ |\widehat{u}_2^*(\xi_1, \xi_2, 0)|^2\ d\xi_1 d\xi_2$$

Since

$$|\widehat{u}_2^*(\xi_1, \xi_2, 0)|^2 = -2\mathcal{R}e \int_0^\delta \Big[\ \frac{\partial \widehat{u}_2^*}{\partial y_3}\ \overline{\widehat{u}_2^*}\ \Big](\xi_1, \xi_2, y_3)\ dy_3$$

using (146), we have, $\widehat{r}_1$ being the Fourier transform of r_1,

$$\left|\begin{array}{rcl} |\widehat{u}_2^*(\xi_1, \xi_2, 0)|^2 & = & 2\ \mathcal{R}e \displaystyle\int_0^\delta \Big[\ \widehat{r}_1\ \overline{\widehat{u}_2^*}\ \Big](\xi_1, \xi_2, y_3)\ dy_3 \\ & - & 2\ \mathcal{R}e \displaystyle\int_0^\delta i\xi_2 \big[\widetilde{u}_3\ \overline{\widehat{u}_2^*}\ \big](\xi_1, \xi_2, y_3)\ dy_3 \\ & - & 2\ \mathcal{R}e \displaystyle\int_0^\delta \Big[\ \mathcal{F}\big(\tilde{u}_3 \frac{\partial^2 h}{\partial y_3 \partial y_1}\big)\ \overline{\widehat{u}_2^*}\ \Big](\xi_1, \xi_2, y_3)\ dy_3. \end{array}\right.$$

We divide the above equality by $(1+|\xi|^2)^{-\frac{1}{2}}$ and integrate over ξ. Next we use $(1+|\xi|^2)^{-\frac{1}{2}} \leq 1$, $|\xi_2|\ (1 + |\xi|^2)^{-\frac{1}{2}} \leq 1$ and Plancherel's theorem to obtain, since $h \in W^{2,\infty}(\mathbb{R}^2)$,

$$\|u_2^*(\cdot,\cdot,0)\|^2_{H^{-\frac{1}{2}}} \leq 2\int_0^\delta \int_{\mathbb{R}^2} |r_1|\ |u_2^*|\ dy + C\int_0^\delta \int_{\mathbb{R}^2} |u_3|\ |u_2^*|\ dy.$$

Coming back to the variable x through the change of variable $x = \varphi(y)$, we easily get, since $|u_2^*| \leq |u_2| + C|u_3|$

$$\|u_2^*(\cdot,\cdot,0)\|^2_{H^{-\frac{1}{2}}} \leq C \left(\|u\|^2_{L^2(O)} + \|u\|_{L^2(O)} \|\mathrm{curl} u\|_{L^2(O)}\right)$$

where the constant C only depends on h.

Finally, using the lemma A.2 (notice that $f := \big(1+|Dh|^2\big)^{-\frac{1}{2}}$ belongs to $W^{1,\infty}$), we get

$$\|f\big(\tilde{u}_2+\tilde{u}_3\frac{\partial h}{\partial y_2}\big)\|^2_{H^{-\frac{1}{2}}} \leq C\left(\|u\|^2_{L^2(O)}+\|u\|_{L^2(O)}\|\mathrm{curl} u\|_{L^2(O)}\right) \tag{147}$$

In the same way, one obtains

$$\left\{\begin{array}{lcl} \|f\big(\tilde{u}_3\dfrac{\partial h}{\partial y_1}-\tilde{u}_1\big)\|_{H^{-\frac{1}{2}}(\Gamma)} & \leq & C\left(\|u\|^2_{L^2(O)}+\|u\|_{L^2(O)}\|\mathrm{curl} u\|_{L^2(O)}\right) \\ \|f\big(-\tilde{u}_1\dfrac{\partial h}{\partial y_2}+\tilde{u}_2\dfrac{\partial h}{\partial y_1}\big)\|_{H^{-\frac{1}{2}}(\Gamma)} & \leq & C\left(\|u\|^2_{L^2(O)}+\|u\|_{L^2(O)}\|\mathrm{curl} u\|_{L^2(O)}\right) \end{array}\right. \tag{148}$$

Thanks to (144) and by definition of the norm in $H^{-\frac{1}{2}}(\Gamma)$, estimates (147) and (148) lead to the desired inequality.

Obtaining the same inequality in the general case can be deduced by the using a partition of unity $(\varphi_i)_{i=1,\ldots,N}$ of O and noticing that

$$\|\mathrm{curl}\varphi_i u\|_{L^2(O)} = \|\varphi_i \ \mathrm{curl} u + \nabla\varphi_i \times u\|_{L^2(O)} \leq \|\varphi_i\|_\infty \ \|\mathrm{curl} u\|_{L^2(O)} + \|\nabla\varphi_i\|_\infty \ \|u\|_{L^2(O)}.$$

□

Lemma A.2 *Let $f \in W^{1,\infty}(R^n)$ and $g \in H^{-\frac{1}{2}}(R^n)$ then $f\,g \in H^{-\frac{1}{2}}(R^n)$ and one has,*

$$\|f\,g\|_{H^{-\frac{1}{2}}(R^n)} \leq 3^{\frac{1}{4}} \ \|f\|_{W^{1,\infty}(R^n)} \ \|g\|_{H^{-\frac{1}{2}}(R^n)}.$$

Proof. We first remark that if $\psi \in H^{\frac{1}{2}}(R^n)$, then

$$\|f\,\psi\|_{H^{\frac{1}{2}}(R^n)} \leq 3^{\frac{1}{4}} \ \|f\|_{W^{1,\infty}(R^n)} \ \|\psi\|_{H^{\frac{1}{2}}(R^n)}$$

which is deduced by interpolation from the (obvious) inequalities:

$$\left\{\begin{array}{ll} \forall\,\psi \in L^2(\mathbb{R}^n), & \|f\,\psi\|_{H^{\frac{1}{2}}(R^n)} \leq \|f\|_{L^\infty(R^n)} \ \|g\|_{L^2(R^n)}, \\[2ex] \forall\,\psi \in H^1(\mathbb{R}^n), & \|f\,\psi\|_{H^1(R^n)} \leq \left(2\ \|f\|^2_{L^\infty(R^n)} + \|f\|^2_{W^{1,\infty}(R^n)}\right)^{\frac{1}{2}} \ \|\psi\|_{H^1(R^n)}. \end{array}\right.$$

Next, if $g \in H^{-\frac{1}{2}}(R^n)$, we have, for any $\psi \in H^{\frac{1}{2}}(R^n)$,

$$|\,\langle fg,\psi\rangle\,| = |\,\langle g, f\psi\rangle\,| \leq \|g\|_{H^{-\frac{1}{2}}(R^n)} \ \|f\,\psi\|_{H^{\frac{1}{2}}(R^n)} \leq 3^{\frac{1}{4}} \ \|g\|_{H^{-\frac{1}{2}}(R^n)} \ \|f\|_{W^{1,\infty}(R^n)} \ \|\psi\|_{H^{-\frac{1}{2}}(R^n)},$$

from which one easily concludes. □

Lemma A.3 *Let $O \subset R^3$ be a bounded open set with a C^2 boundary Γ. There exists a constant C that depends only on Γ such that*

$$\|u\|_{H^{\frac{1}{2}}(\Gamma)} \leq C(\|\nabla_\Gamma u\|_{H^{-\frac{1}{2}}(\Gamma)} + \|u\|_{L^2(\Gamma)}) \qquad \forall\, u \in H^{\frac{1}{2}}(\Gamma).$$

Proof. In the case $O = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ / \ x_3 \geq 0\}$ one can check by using Fourier transform in the plane (x_1, x_2) that

$$\|u\|^2_{H^{\frac{1}{2}}(\Gamma)} = \|\nabla_\Gamma u\|^2_{H^{-\frac{1}{2}}(\Gamma)} + \|u\|^2_{L^2(\Gamma)}\,.$$

The inequality is therefore trivially verified in this case. The general case can be easily deduced by using local parameterizations of the boundary Γ. This is where the C^2-regularity of Γ is taken into account. □

Our next lemma is a sharper version of classical compact-embedding theorem for spaces of L^2 functions with bounded divergence and curl into L^2. We set

$$H(\mathrm{curl}, \mathrm{div}, O) := \{u \in L^2(O)^3 \ / \ \mathrm{curl}\, u \in L^2(O)^3 \text{ and } \mathrm{div}\, u \in L^2(O)\}$$

equipped with the norm

$$\|u\|^2_{H(\mathrm{curl},\mathrm{div},O)} = \|u\|^2_{L^2(O)} + \|\mathrm{curl}\, u\|^2_{L^2(O)} + \|\mathrm{div}\, u\|^2_{L^2(O)}.$$

Lemma A.4 *Let $O \subset R^3$ be a bounded simply connected open set with C^2 boundary Γ. Then every bounded sequence $(u_k)_{k\in\mathbb{N}}$ of $H(\mathrm{curl}, \mathrm{div}, O)$ such that*

$$(u_{k|\Gamma} \times n)_{k\in\mathbb{N}} \text{ is convergent in } H_t^{-\frac{1}{2}}(\Gamma) \tag{149}$$

has a convergent subsequence $(u_{k'})$ in $L^2(O)^3$.

Proof. Our proof is an adaptation of the proof given by Costabel in the case where, instead of (149), one has an L^2 control of the boundary term of the sequence (see Theorem 2 of [5]).

The idea is to make a Helmholtz decomposition of u_k of the form:

$$u_k = w_k + \nabla p_k, \quad (w_k, p_k) \in H^1(O) \times L^2(O), \quad \mathrm{div}\, w_k = 0, \tag{150}$$

constructed in such a way that:

(i) w_k is bounded in $H^1(O)$ (and thus admits a converging subsequence in $L^2(O)^3$) : this uses the fact that $\mathrm{curl}\, u_k$ is bounded in $L^2(O)^3$,

(i) ∇p_k admits a converging subsequence in $L^2(O)^3$: this uses the fact that $\mathrm{div}\, u_k$ is bounded in $L^2(O)^3$ and that $(u_k \times n)_{|\Gamma}$ converges according to (149).

In order to construct w_k from curl u_k we first construct an extension of curl u_k in $\mathbb{R}^3$ which has a compact support (independent of k) and is divergence free. For this, we choose a ball B containing $\overline{O}$ in its interior and will constitute the support of the extension of curl u_k.

First notice that since $\operatorname{curl} u_k \in H(\operatorname{div}; O)$, the trace $\operatorname{curl} u_k \cdot n_{|\Gamma}$ is well defined in $H^{-\frac{1}{2}}(\Gamma)$ and satisfies, since $\operatorname{div} \operatorname{curl} u_k = 0$:

$$\|\operatorname{curl} u_k \cdot n\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C \, \|\operatorname{curl} u_k\|_{L^2(O)}, \quad \langle \operatorname{curl} u_k \cdot n, 1\rangle_\Gamma = 0. \tag{151}$$

Therefore, the following Neumann problem in $B \setminus O$:

$$\begin{cases} \Delta\varphi_k = 0 & \text{in } B \setminus \overline{O}, \\ \partial_n \varphi_k = (\operatorname{curl} u_k) \cdot n & \text{on } \partial O, \\ \partial_n \varphi_k = 0 & \text{on } \partial B, \end{cases} \tag{152}$$

admits a solution $\varphi_k \in H^1(B \setminus O)$, which is unique if we impose in addition

$$\int_{B\setminus O} \varphi_k \, dx = 0.$$

Moreover, using the Poincare-Wirtinger inequality and (151)

$$\int_{B\setminus\overline{O}} |\nabla\varphi_k|^2 \, dx = \langle \, (\operatorname{curl} u_k) \cdot n, \overline{\varphi_k} \, \rangle_\Gamma \leq C \, \|\operatorname{curl} u_k\|_{L^2(O)} \, \|\nabla\varphi_k\|_{L^2(B\setminus\overline{O})}$$

from which we deduce

$$\|\nabla\varphi_k\|_{L^2(B\setminus\overline{O})} \leq C \, \|\operatorname{curl} u_k\|_{L^2(O)}. \tag{153}$$

We then introduce the extension χ_k of curl u_k as

$$\chi_k = \begin{cases} \text{curl } u_k & \text{in } O, \\ \nabla\varphi_k & \text{in } B \setminus \overline{O}, \\ 0 & \text{in } R^3 \setminus \overline{B}. \end{cases}$$

Of course, $\chi_k \in L^2(R^3)$, is compactly supported in $\overline{B}$ and satisfies, thanks to (153)

$$\|\chi_k\|_{L^2(\mathbb{R}^3)} \leq C \, \|\operatorname{curl} u_k\|_{L^2(O)} \, .$$

Moreover, the two boundary conditions in (152) have been chosen in order to enforce the continuity of $\chi_k \cdot n$ across ∂O and ∂B so that:

$$\operatorname{div} \chi_k = 0 \quad \text{in } \mathbb{R}^3.$$

Next, we introduce Ψ_k as the unique solution of he following Laplace problem in $\mathbb{R}^3$

$$-\Delta\Psi_k = \chi_k \quad \text{in } \mathbb{R}^3, \quad \Psi_k \in H^2_{loc}(\mathbb{R}^3)^3, \quad \nabla\Psi_k \in L^2(\mathbb{R}^3)^3. \tag{154}$$

It is well known that Ψ_k is given by:

$$\Psi_k = G * \chi_k, \quad G = \frac{1}{4\pi|x|}, \quad \text{(the fundamental solution of the Laplace operator)}$$

and satisfies in particular

$$\|\Psi_k\|_{H^1(O)} \leq C \, \|\chi_k\|_{L^2(\mathbb{R}^3)} \, . \tag{155}$$

Moreover,

$$\operatorname{div}\chi_k = 0 \quad \Longrightarrow \quad \operatorname{div}\Psi_k = 0, \tag{156}$$

Next we define

$$w_k = \operatorname{curl}\Psi_k, \quad \in L^2(\mathbb{R}^3) \tag{157}$$

whose restriction to O is the good candidate for (150). Indeed

$$\operatorname{curl} w_k = \operatorname{curl}(\operatorname{curl}\Psi_k) = \text{ grad }(\operatorname{div}\Psi_k) - \Delta\Psi_k = \chi_k \quad \text{in } \mathbb{R}^3, \quad \text{(see (154) and (156))}$$

which implies in particular

$$\operatorname{curl} w_k = \operatorname{curl} u_k \quad \text{in } O.$$

Before constructing p_k, we first check property (i). The fact that w_k is bounded in $L^2(O)^3$ results directly from (155). Next, we show that w_k is bounded in $H^1(\mathbb{R}^3)$, which implies in particular (i). Indeed using the Fourier transform in $\mathbb{R}^3$ we deduce from (157) and (154) that (ξ denotes the dual variable of x and $\widehat{u}$ the Fourier transform of u):

$$|\xi|^2 \, |\widehat{w}_k(\xi)|^2 = \frac{|\, \xi \times \widehat{\chi}_k(\xi) \,|^2}{|\xi|^2} \leq \, |\widehat{\chi}_k(\xi)|^2$$

which yields, by Plancherel's theorem

$$\|\nabla w_k\|^2_{L^2(\mathbb{R}^3)} \; \leq \; \|\chi_k\|^2_{L^2(\mathbb{R}^3)} \leq C \, \|\operatorname{curl} u_k\|^2_{L^2(O)}.$$

From now on, we can therefore assume that (up to extracted subsequence) w_k converges in $L^2(O)$.

Since $\operatorname{curl}(u_k - w_k) = 0$ and O is simply connected, one can construct p_k (unique up to an additive constant) such that $\nabla p_k = u_k - w_k$ (use for instance Theorem 2.9 of [9]). Fixing p_k by imposing that $\int_O p_k dx = 0$ gives raise to a bounded sequence p_k in $H^1(O)$ by the Poincaré-Wirtinger inequality. Since we further have that $(\operatorname{div} u_k)$ is bounded in $L^2(O)$ and (w_k) is bounded in $H^1(O)$, then, up to extracted subsequence, one can assume that $\operatorname{div} u_k$ is convergent in $H^{-1}(O)$, $w_k|_{\partial O}$ is convergent in $H^{-\frac{1}{2}}(\partial O)$ and $p_k|_{\partial O}$ is convergent in $L^2(\partial O)$. We shall deduce that p_k is strongly convergent in $H^1(O)$. We first observe that p_k satisfies

$$\begin{cases} -\Delta p_k = \operatorname{div} u_k, & \text{in } O, \\ \nabla p_k \times n = u_k \times n - w_k \times n, & \text{on } \partial O. \end{cases}$$

Let m and k be two indexes. From

$$\begin{cases} \Delta(p_k - p_m) = \operatorname{div}(u_k - u_m) & \text{in } O, \\ \nabla(p_k - p_m) \times n = (u_k - u_m) \times n - (w_k - w_m) \times n & \text{on } \partial O. \end{cases} \tag{158}$$

and using the classical theory for elliptic equations one gets the existence of a constant C_1 such that

$$\|\nabla p_k - \nabla p_m\|_{L^2(\Omega)} \leq C_1 \left(\|\operatorname{div}(u_k - u_m)\|_{H^{-1}(O)} + \|p_k - p_m\|_{H^{\frac{1}{2}}(\partial O)} \right). \tag{159}$$

On the other hand, using Lemma A.3 one has

$$\|p_k - p_m\|_{H^{\frac{1}{2}}(\partial O)} \leq C_2 \left(\|\nabla(p_k - p_m) \times n\|_{H^{-\frac{1}{2}}(\partial O)} + \|p_k - p_m\|_{L^2(\partial O)} \right). \tag{160}$$

From the second equation of (158), (159) and (160) it is easily seen that

$$\begin{aligned} \|\nabla p_k - \nabla p_m\|_{L^2(\Omega)} \leq \quad & C_3 \Big(\|\operatorname{div}(u_k - u_m)\|_{H^{-1}(O)} + \|(u_k - u_m) \times n\|_{H^{-\frac{1}{2}}(\partial O)} \\ + \quad & \|w_k - w_m\|_{H^{-\frac{1}{2}}(\partial O)} + \|p_k - p_m\|_{L^2(\partial O)} \Big). \end{aligned}$$

Using assumption (149) one concludes ∇p_k is a Cauchy sequence in $L^2(\partial O)$. The result of the lemma is then proved since $u_k = w_k + \nabla p_k$. □

Lemma A.4 also applies to domains Ω_i that are not simply connected. This is proved in the following lemma.

Lemma A.5 *The result of Lemma A.4 applies to bounded open domains $O \in R^3$ of class C^2.*

Proof. Let x be an arbitrary point in $\overline{O}$. If $x \in O$, one defines U_x as a ball centered at x such that $\overline{U}_x \subset O$. If not, one defines U_x as a neighborhood of x such that there exits a bijective map $\phi_x : Q \mapsto U_x$ such that

$$\phi_x \in C^1(\overline{Q}),\ \phi^{-1} \in C^1(\overline{U}_x),\ \phi(Q_+) = U_x \cap O,\ \text{et } \phi(Q_0) = U \cap \partial O,$$

where Q denotes the unit cube of R^3, $Q_+ := \{x \in Q \mid x_3 > 0\}$, and $Q_0 = \{x \in Q \mid x_3 = 0\}$.

With this definition one observes that $U_x \cap O$ is a simply connected domain for all $x \in \overline{O}$. By the compactness of $\overline{O}$ one can extract a finite covering of $\overline{O}$ from $\{U_x; x \in \overline{O}\}$. Let us denote by $\{U_i,\ i \in I\}$ this finite covering and consider a partition of unity $(\theta_i)_{i\in I} \subset C^\infty(R^3)$ subordinated to this covering, i.e.

$$\text{supp } \theta_i \subset U_i\ ,\ \sum_{i\in I} \theta_i = 1 \text{ on } \overline{O}.$$

Then define $u_n^i := \theta_i\, u_n$ for all $i \in I$. It is easy to see that for every i, the sequence (u_n^i) satisfies the hypotheses of Lemma A.4 with O replaced by U_i. Using a finite diagonal process, one can therefore assume that there exists a subsequence n_k such that

$$u_{n_k}^i \text{ converges in } L^2(U_i) \text{ for all } i \in I.$$

Consequently, the sequence $u_{n_k} = \sum_{i\in I} u_{n_k}^i$ is convergent in $L^2(O)$. □

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