Approximations for All-to-all Uniform Traffic Grooming on Unidirectional Ring

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Abstract

Traffic grooming in a WDM network consists of assigning to each request (lightpath) a wavelength with the constraint that a given wavelength can carry at most C requests or equivalently a request uses at most $1/C$ of the bandwidth. C is known as the grooming ratio. A request (lightpath) need two SONET add-drop multiplexers (ADMs) at each end node ; using grooming different requests can share the same ADM. The so called traffic grooming problem consists of minimizing the total number of ADMs to be used (in order to reduce the overall cost of the network). Here we consider the traffic grooming problem in WDM unidirectional rings with all-to-all uniform unitary traffic. This problem has been optimally solved for specific values of the grooming ratio, namely $C = 2, 3, 4, 5, 6$. In this paper we present various simple constructions for the grooming problem providing good approximation of the total number of ADMs. For that we use the fact that the problem corresponds to a partition of the edges of the complete graph into subgraphs, where each subgraph has at most C edges and where the total number of vertices has to be minimized.

Keywords: Traffic Grooming, WDM Networks, ADM, Unidirectional Rings, Approximation Factor.

1 Introduction

Traffic grooming is the generic term for packing low rate signals into higher speed streams (see the surveys [7, 15, 23, 25, 31]). By using traffic grooming, one can bypass the electronics in the nodes for which there is no traffic sourced or destinated to it and therefore reduce the cost of the network. Typically, in a WDM (Wavelength Division Multiplexing) network, instead of having one SONET Add Drop Multiplexer (ADM) on every wavelength at every node, it may be possible to have ADMs only for the wavelength used at that node (the other wavelengths being optically routed without electronic switching).

In SONET/WDM networks, we assign to each request $\{i, j\}$ a fraction of the bandwidth offered by a wavelength along a path from node i to node j. If a given wavelength can carry at most C requests we can assign to each request at most $\frac{1}{C}$ of the bandwidth. C is known as the grooming ratio. In the particular case of unidirectional rings, the routing is unique. If furthermore the traffic requirement is symmetric, then without loss of generality we will assign to each pair of symmetric requests, call a circle, a fraction of the bandwidth in the whole ring. In both cases, we need one ADM at node i and one at node j. Also, two requests with a common extremity assigned to the same wavelength shared an ADM. That is, if requests $\{1, 2\}$ and $\{2, 3\}$ are assigned to two different wavelengths, then we need 4 ADMs, while if they are assigned to the same wavelength we will need only 3 ADMs.

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The so called traffic grooming problem consists of minimizing the total number of ADMs to be used (in order to reduce the overall cost of the network). Here we study the problem for an unidirectional SONET ring with N nodes, a grooming ratio C, and an all-to-all uniform unitary traffic. This problem has been modeled as a graph partition problem in both [6] and [19]. The set of requests is modeled by a graph I where $I = K_N$ in the all-to-all case. To a wavelength w is associated a subgraph B_w in which each edge corresponds to a request and each node to an ADM. The grooming constraint, that a wavelength can carry at most C requests, corresponds to the fact that the number of edges $|E(B_w)|$ of each subgraph B_w is at most C. The objective is therefore to minimize the total number of vertices used in the subgraphs.

Problem 1 (Grooming on unidirectional cycle [6])
Given a number of nodes N and a grooming ratio C find a partition of the edges of $I = K_N$ into subgraphs B_w, $w = 1, \dots, W$ with $|E(B_w)| \leq C$ such that $\sum_{1\leq w\leq W} |V(B_w)|$ is minimum.(This minimum will be denoted $A(C, N)$).

The traffic grooming problem has recently been extensively studied on unidirectional WDM ring, primarily in the context of variable traffic requirements [11, 14, 19, 26, 29], but the case of fixed traffic requirements has served as an important special case [3, 4, 5, 6, 7, 8, 15, 17, 18, 20, 21, 23, 27, 30]. The problem has also been studied on the path [2].

With a general set of requests, $I \neq K_N$, the grooming problem has been proved NP-Complete in unidirectional ring with grooming factor $C \geq 1$ [11, 24] and in a directed path with $C \geq 2$ [22] and so of course in general. Then a first approximation algorithm for computing the total number of ADMs with approximation factor $\frac{C\left(1+\frac{1}{\lceil C/2 \rceil}\right)}{\left\lceil \frac{1+\sqrt{1+8C}}{2} \right\rceil}$, i.e. $\sim \sqrt{C}$, has been given in [19], and in [16] a $\log(C)$-approximation algorithm has been obtained. More recently, the grooming problem has been proved APX-Hard in [1] (i.e. there exists a constant c, such that Problem 1 can not be approximate within a factor c).

With the all-to-all set of request, $I = K_N$, optimal constructions for given grooming ratio C were obtained using tools of graph and design theory [12, 13], in particular for grooming ratio $C = 3$ [3], $C = 4$ [20, 8], $C = 5$ [5], $C = 6$ [4] and $C \geq N(N-1)/6$ [8]. However it will be a very long and difficult task to find optimal constructions for all grooming ratio. In this conditions, it is mandatory to produce good upper bounds.

Existing heuristic algorithms [18, 27, 30] as well as the approximation algorithm proposed in [19, 16] are not satisfactory for the all-to-all case.

In this paper, we will first present an asymptotical $1+\frac{4C}{N}+o\left(\frac{1}{N}\right)$-approximation algorithm; unfortunately the construction is valid only for large N. Then we present a very simple construction using bipartite graphs which provides a $\gamma(C, N)$-approximation for the total number of ADMs, where $\gamma(C, N)$ is at most $\sqrt{2}\frac{\sqrt{C}}{\lfloor \sqrt{C} \rfloor}$ and in many cases better (for example, for $C = 16$ $\gamma(16, N) = \frac{5}{4}$ and for $C = 64$ $\gamma(64, N) = \frac{4}{3}$). Then we show several improvements of this construction by using other bipartite graphs or tripartite graphs ; in that case $\gamma(C, N)$ is of order $\sqrt{\frac{3}{2}}$. Values of the approximation factor obtained with different constructions are given in Table 2 for realistic values of C.

2 Lower bound

A tight lower bound for Problem 1 has been given in [8] and is recalled in Theorem 2. The idea consists in using in the partition subgraphs which for a given number of edges (less than C) have the minimum number of vertices. So let us denote by $\rho(G)$ the ratio of a subgraph G, $\rho(G) = \frac{|E(G)|}{|V(G)|}$, and by $\rho(m)$ the maximum ratio of a subgraph with m edges. Let finally $\rho_{\max}(C)$ denote the maximum possible ratio among all the subgraphs with $m \leq C$ edges, that is:

$$\rho_{\max}(C) = \max\{\rho(G) \mid |E(G)| \leq C\} = \max_{m\leq C} \rho(m)$$

Recall that $A(C,N)$ is the minimum number of ADM's needed in an unidirectional ring with the all-to-all set of request ($I = K_N$) and with a grooming ratio C. As $A(C,N) = \sum_{1\leq w\leq W} |V(B_w)|$, $|V(B_w)| \leq \rho_{\max}(C)|E(B_w)|$ and $\sum_{1\leq w\leq W} |E(B_w)| = \frac{N(N-1)}{2}$ we get the following lower bound:

Theorem 2 (Lower Bound [6]) $A(C,N) \geq \frac{N(N-1)}{2\rho_{\max}(C)}$.

The value of $\rho_{\max}(C)$ has been evaluated in Proposition 3.

Proposition 3 ([6])

- *If* $\frac{x(x-1)}{2} \leq C \leq \frac{(x+1)(x-1)}{2}$, *then* $\rho_{\max}(C) = \frac{x-1}{2}$ *and the value is attained for* K_x.
- *If* $\frac{(x+1)(x-1)}{2} < C < \frac{(x+1)x}{2}$, *then* $\rho_{\max}(C) = \frac{C}{x+1}$ *and the value is attained for any graph with* C *edges and* $x+1$ *vertices.*

Values of $\rho_{\max}(C)$ are given in Table 2 for realistic values of C. The following corollary gives also a lower bound easier to manipulate.

Corollary 4 $\rho_{\max}(C) \leq \sqrt{\frac{C}{2}}$ *and so* $A(C,N) \geq \frac{N(N-1)}{\sqrt{2C}}$.

Proof : From Proposition 3 we know that $\rho_{\max}(C) = \frac{x-1}{2}$ (case 1) or $\frac{C}{x+1}$ (case 2), and we can observe that $x = \left\lfloor \frac{1+\sqrt{1+8C}}{2} \right\rfloor$. Thus we have

- case 1 : $2\rho_{\max}(C) = \left\lfloor \frac{1+\sqrt{1+8C}}{2} \right\rfloor - 1 \leq \left\lfloor \frac{\sqrt{1+8C}-1}{2} \right\rfloor \leq \frac{\sqrt{8C}}{2} \leq \sqrt{2C}$ and so $\rho_{\max}(C) \leq \sqrt{\frac{C}{2}}$
- case 2 : $\rho_{\max}(C) = \frac{C}{\left\lfloor \frac{1+\sqrt{1+8C}}{2} \right\rfloor} \leq \frac{C}{\frac{1+\sqrt{1+8C}}{2}} \leq \sqrt{\frac{C}{2}}$

So $A(C,N) \geq \frac{N(N-1)}{\sqrt{2C}}$. □

3 Asymptotic construction

It has been shown in [6] that design theory can help to solve the grooming problem. In particular, a G-design of order N (see [12] VI.24 or [9] or [10]) is nothing else than a partition of the edges of K_N into subgraphs isomorphic to a given graph G. The interest of the existence of a G-design is shown by the following immediate proposition.

Proposition 5 ([6]) *If there exists a* G*-design of order* N*, where* G *is a graph with at most* C *edges and with ratio* $\rho_{\max}(C)$*, then* $A(C,N) = \frac{N(N-1)}{2\rho_{\max}(C)}$.

Necessary conditions 6 (Existence of a G-design) *If there exists a* G*-design, then*

(i) $\frac{N(N-1)}{2}$ *should be a multiple of* $E(G)$

(ii) $N-1$ *should be a multiple of the greatest common divisor of the degrees of the vertices of* G.

Wilson [28] has shown that these necessary conditions are also sufficient when N is large enough. For example, we have

Theorem 7 ([6]) *When* N *is large enough, we have*

- $A(3,N) = \frac{N(N-1)}{2}$ *when* $N \equiv 1$ *or* $3 \bmod 6$
- $A(4,N) = \frac{N(N-1)}{2}$ *when* $N \equiv 0$ *or* $1 \bmod 8$
- $A(5,N) = \frac{2N(N-1)}{5}$ *when* $N \equiv 0$ *or* $1 \bmod 5$
- $A(6,N) = A(7,N) = \frac{N(N-1)}{3}$ *when* $N \equiv 1$ *or* $4 \bmod 12$
- $A(8,N) = \frac{5N(N-1)}{16}$ *when* $N \equiv 0$ *or* $1 \bmod 16$
- $A(9,N) = \frac{5N(N-1)}{18}$ *when* $N \equiv 0$ *or* $1 \bmod 9$
- $A(10,N) = \frac{N(N-1)}{4}$ *when* $N \equiv 1$ *or* $5 \bmod 20$
- $A(15,N) = A(16,N) = \frac{N(N-1)}{5}$ *when* $N \equiv 1$ *or* 6 *or* 16 *or* $21 \bmod 30$

Construction 8 *If N does not satisfy the necessary Conditions 6, we can find two integers N_1 and N_2 with $N_1 \leq N \leq N_2$ satisfying the necessary Conditions 6.*

We will obtain a valid construction for N by removing $N_2 - N$ nodes and the corresponding edges from an optimal construction for N_2.

Let $f(C,N)$ denotes the number of ADMs obtained by this construction and let $\gamma(C,N) = \frac{f(C,N)}{A(C,N)}$ be the approximation factor of this solution.

We have $A(C,N_1) \leq A(C,N) \leq f(C,N) \leq A(C,N_2)$ and $\gamma(C,N) = \frac{f(C,N)}{A(C,N)} \leq \frac{A(C,N_2)}{A(C,N_1)} = \frac{N_2(N_2-1)}{N_1(N_1-1)}$.

So we have interest to find values of N_1 and N_2 very near. A solution is given by the following lemma:

Lemma 9 *Let $\alpha(C)$ be defined as follows :*

- *If $\frac{x(x-1)}{2} \leq C \leq \frac{(x+1)(x-1)}{2}$, then $\alpha(C) = x(x-1)$.*
- *If $\frac{(x+1)(x-1)}{2} \leq C < \frac{(x+1)x}{2}$, then $\alpha(C) = 2C$*

Let $N_1 = \alpha(C)t + 1$ and $N_2 = \alpha(C)(t+1) + 1$ be such that $N_1 \leq N \leq N_2$.

There always exists a graph G with at most C edges and ratio $\rho_{\max}(C)$ which satisfies Conditions 6 for N_1 and N_2.

Proof : When $\frac{x(x-1)}{2} \leq C \leq \frac{(x+1)(x-1)}{2}$, then $\rho_{\max}(C)$ is attained for K_x, and so let $G = K_x$. Both $N_1 - 1$ and $N_2 - 1$ are multiple of $\alpha(C) = x(x-1)$; and so the number of edges of K_{N_1} (resp. K_{N_2}) $\frac{N_1(N_1-1)}{2}$ (resp. $\frac{N_2(N_2-1)}{2}$) is a multiple of $E(G) = \frac{x(x-1)}{2}$. Condition (ii) is also satisfied as the degree of a vertex of K_{N_1} (resp. K_{N_2}) $N_1 - 1$ (resp. $N_2 - 1$) is a multiple of $x - 1$ the degree of K_x.

When $\frac{(x+1)(x-1)}{2} \leq C < \frac{(x+1)x}{2}$, then $\rho_{\max}(C)$ is attained for any graph with C edges and $x+1$ vertices. Let $r = \frac{(x+1)x}{2} - C$. So $0 < r < x$. Let G be the graph obtained from K_{x+1} by removing the edges of a path of length r. G has C edges and so Condition (i) is satisfied as $\frac{N_1(N_1-1)}{2} = (2Ct+1)Ct$ and $\frac{N_2(N_2-1)}{2} = (2C(t+1)+1)C(t+1)$ are multiples of $E(G)$. As $0 < r \leq x - 1$, G has a vertex which is not in the path that have been removed; this vertex has degree x, and the extremities of the path have degree $x-1$, so the greatest common divisor of the degrees of the vertices of G is 1. Condition (ii) is trivially satisfied. □

Proposition 10 *When N is large enough to satisfy Wilson's Theorem, Construction 8 has an approximation factor $\gamma(C,N) \leq 1 + \frac{4C}{N} + o\left(\frac{1}{N}\right)$.*

Proof : From Construction 8, we have :

$\gamma(C,N) = \frac{f(C,N)}{A(C,N)} \leq \frac{A(C,N_2)}{A(C,N_1)} = \frac{N_2(N_2-1)}{N_1(N_1-1)} = 1 + \frac{2\alpha(C)}{N_1} + \frac{\alpha(C)(\alpha(C)+1)}{N_1(N_1-1)}$.

As in both cases $\alpha(C) \leq 2C$ and $N - N_1 \leq \alpha(C)$, we obtain $\gamma(C,N) \leq 1 + \frac{4C}{N} + +o\left(\frac{1}{N}\right)$. □

Unfortunately, the values of N for which Wilson's Theorem and so Lemma 10 applies are very large and furthermore the constructions are not explicit. So there is a need to find simpler and general constructions.

4 Construction using bipartite graphs

In this section, we first present a simple construction which gives an upper bound on the number of ADM's and we analyze it's approximation factor. Then, we present some improvements of this construction.

Basically our construction consists of partitioning the edges of K_N into a maximum number of bipartite graphs with at most C edges plus some small complete graphs. A complete-bipartite graph with 2 sets of p nodes each has p^2 edges and a ratio of $\frac{p}{2}$. Therefore choosing p^2 to be C or almost C we get a ratio of $\frac{\sqrt{C}}{2}$. As we will see in the proof of Lemma 12, the number of ADMs due to bipartite graphs dominates the total cost of the construction, and w.l.o.g we can estimate the number of ADMs by $\frac{N(N-1)}{\sqrt{C}}$. From Theorem 2 we know that the lower bound is larger than $\frac{N(N-1)}{2\rho_{\max}(C)}$. So our construction gives an approximation factor close to $\frac{2\rho_{\max}(C)}{\sqrt{C}} \leq \sqrt{2}$ by Corollary 4.

Several constructions are possible. We first present a simple construction (Construction 11) and then some improvements.

Construction 11 *Let $C = p^2 + p'$, $0 \leq p' \leq 2p$; let $N = qp + r$, $0 \leq r < p$, and let the vertices of K_N be $V = \cup_{i=1}^{q} V_i \cup V_{q+1}$ with $|V_i| = p$ and $|V_{q+1}| = r$.*

We partition the edges of K_N into $\frac{q(q-1)}{2}$ $K_{p,p}$ on $V_i \cup V_j, 1 \leq i < j \leq q$, plus q $K_{p,r}$ on $V_i \cup V_{q+1}, 1 \leq i \leq q$, plus q K_p on V_i and one K_r on V_{q+1} .

Lemma 12 *Construction 11 is valid and uses $(q+1)N$ ADMs.*

Proof : First all the subgraphs of the decomposition have at most $p^2 \leq C$ edges. Since a bipartite graph $K_{x,y}$ has $x + y$ vertices and a complete graph K_x has x vertices, we have $2p\frac{q(q-1)}{2} + (p + r)q + qp + r = (q + 1)(qp + r) = (q + 1)N$ ADMs. □

Corollary 13 *When $C = p^2 + p'$, $0 \leq p' \leq 2p$, and $N = qp + r$, $0 \leq r < p$, Construction 11 provides a $\frac{2\rho_{\max}(C)}{\lfloor\sqrt{C}\rfloor} + O\left(\frac{1}{N}\right) < \sqrt{2}\frac{\sqrt{C}}{\lfloor\sqrt{C}\rfloor} + O\left(\frac{1}{N}\right)$-approximation of the number of ADMs.*

Proof : Let $\gamma(C, N)$ be the approximation factor that is the ratio between the upper bound construction and the lower bound for a given grooming factor C. We know from Theorem 2 that $A(C, N) \geq \frac{N(N-1)}{2\rho_{\max}(C)}$. So

$$\gamma(C, N) = (q + 1)N\frac{2\rho_{\max}(C)}{N(N-1)} = 2\rho_{\max}(C)\frac{q+1}{N-1}$$

Since $C = p^2 + p'$, we have $p = \left\lfloor\sqrt{C}\right\rfloor$ and $q = \frac{N-r}{p} = \frac{N-r}{\lfloor\sqrt{C}\rfloor}$. Thus we obtain

$$\gamma(C, N) = \frac{2\rho_{\max}(C)}{\left\lfloor\sqrt{C}\right\rfloor}\left(1 + \frac{\left\lfloor\sqrt{C}\right\rfloor - r + 1}{N-1}\right)$$

□

The above construction is very simple and provides a better approximation factor than [16]. The values of the approximation factor for some values of C are indicated in Table 2. A first improvement can be obtained by noting that some bipartite subgraphs of the decomposition have strictly less than C edges and therefore we can add to them some edges of the K_p's and of the K_r. That is always the case for the $K_{p,r}$ as $pr < p^2 \leq C$ and also for the $K_{p,p}$ when $C > p^2$. Doing so we can get ride of the $O\left(\frac{1}{N}\right)$ in Corollary 13.

	Upper bound	Ratio Upper bound over Lower bound
$q+1 \equiv 1,3 \bmod 6$	$\frac{q+2}{2}N$	$\sqrt{\frac{3}{2}}\sqrt{1+\frac{p'}{3p^2}}\left(1+\frac{2p-r+1}{N-1}\right)$
$q+1 \equiv 5 \bmod 6$	$\frac{q+2}{2}N+2p$	$\sqrt{\frac{3}{2}}\sqrt{1+\frac{p'}{3p^2}}\left(1+\frac{2p-r+1}{N-1}+\frac{4p^2}{N(N-1)}\right)$
$q+1 \equiv 0,4 \bmod 12$	$\frac{q+2}{2}N+p\frac{q+1}{4}$	$\sqrt{\frac{3}{2}}\sqrt{1+\frac{p'}{3p^2}}\left(1+\frac{5p-2r+2}{2(N-1)}+\frac{p(p-r)}{2N(N-1)}\right)$
$q+1 \equiv 2 \bmod 6$	$\frac{q+2}{2}N+p\left\lceil\frac{q-1}{4}\right\rceil+2p$	$\sqrt{\frac{3}{2}}\sqrt{1+\frac{p'}{3p^2}}\left(1+\frac{2p-r+1}{N-1}+\frac{2p^2\left\lceil\frac{q+7}{4}\right\rceil}{N(N-1)}\right)$
$q+1 \equiv 6,10 \bmod 12$	$\frac{q+2}{2}N+p\frac{q+3}{4}$	$\sqrt{\frac{3}{2}}\sqrt{1+\frac{p'}{3p^2}}\left(1+\frac{5p-2r+2}{2(N-1)}+\frac{p(3p-r)}{2N(N-1)}\right)$

Table 1: Upper bound and ratio with lower bound for Construction 21

Construction 14 *Let $C=p^2$, $N=qp+r$, $0<r<p$ be such that $\frac{r(r-1)}{2}\le q\left(C-pr-\frac{p(p-1)}{2}\right)$.*

The construction consists of partitioning the edges of K_N into $\frac{q(q-1)}{2}$ $K_{p,p}$ on $V_i\cup V_j, 1\le i<j\le q$, plus q subgraphs on $V_i\cup V_{q+1}, 1\le i\le q$ containing the pr edges of the $K_{p,r}$ between V_i and V_{q+1} plus the $\frac{p(p-1)}{2}$ of the K_p on V_i and some edges of the K_r on V_{q+1}.

Lemma 15 *Let $C=p^2$, $N=qp+r$, $0<r<p$ be such that $\frac{r(r-1)}{2}\le q\left(C-pr-\frac{p(p-1)}{2}\right)$. Construction 14 is valid and provides a $\frac{2\rho_{\max}(C)}{\sqrt{C}}\le\sqrt{2}$-approximation of the total number of ADMs.*

Proof : The subgraphs $K_{p,p}$ have $p^2=C$ edges. Each other subgraph contains the pr edges of the $K_{p,r}$ between V_i and V_{q+1} plus the $\frac{p(p-1)}{2}$ of the K_p on V_i. So we can still use $C-pr-\frac{p(p-1)}{2}>0$ edges of the K_r on V_{q+1}; and altogether we can use all the edges of K_r as $\frac{r(r-1)}{2}\le q\left(C-pr-\frac{p(p-1)}{2}\right)$.

Construction 14 uses $q(q-1)p+q(p+r)=q(qp+r)=qN$ ADMs. So it has an approximation factor $\gamma(C,N)=qN\frac{2\rho_{\max}(C)}{N(N-1)}=\frac{2\rho_{\max}(C)}{p}\frac{N-r}{(N-1)}$. Since $C=p^2$, we have $\gamma(C,N)=\frac{2\rho_{\max}(C)}{\sqrt{C}}\left(1-\frac{r-1}{N-1}\right)$, and since $0<r$, we obtain $\gamma(C,N)\le\frac{2\rho_{\max}(C)}{\sqrt{C}}\le\sqrt{2}$. □

This strategy allows us to win a small amount of ADMs (at most N). For example, when $C=16$, $p=4$, $q=4$, $r=1$ and so $N=17$, Construction 11 use $5\times 17=85$ ADMs and Construction 14 use 68 ADMs, that is a saving of 17 ADMs.

Construction 16 *Let $C=p^2+p'$, $0<p'\le 2p$ and $N=qp+r$, $0\le r<p$, be such that $(q-1)p'\ge p(p-1)$.*

The construction consists of partitioning the edges of K_N into $\frac{q(q-1)}{2}$ subgraphs on $V_i\cup V_j, 1\le i<j\le q$ containing the p^2 edges of the $K_{p,p}$ between V_i and V_j plus some edges of one of the K_p, plus q subgraphs on $V_i\cup V_{q+1}, 1\le i\le q$ containing the pr edges of the $K_{p,r}$ between V_i and V_{q+1} plus some edges of the K_r on V_{q+1}.

Lemma 17 *Let $C=p^2+p'$, $0<p'\le 2p$ and $N=qp+r$, $0\le r<p$ be such that $(q-1)p'\ge p(p-1)$.*

Construction 16 is valid and provides a $\frac{2\rho_{\max}(C)}{\lfloor\sqrt{C}\rfloor}\le\sqrt{2}\frac{\sqrt{C}}{\lfloor\sqrt{C}\rfloor}$-approximation of the total number of ADMs.

Proof : The subgraphs on $V_i\cup V_j, 1\le i<j\le q$ use the p^2 edges of $K_{p,p}$ and p' edges of one of the K_p. Altogether we can use all the edges of the K_p as by the condition $\frac{q(q-1)}{2}p'\ge\frac{p(p-1)}{2}q$. In each $K_{p,r}$ we can use $p^2+p'-pr=p(p-r)+p'>p$ edges (since $r<p$) of the K_r and altogether all the edges of K_r.

Construction 16 uses $q(q-1)p+q(p+r)=q(qp+r)=qN$ ADMs. So it has the desired approximation factor. □

Note that in some cases the approximation factor can be strictly larger than $\sqrt{2}$ due to the integer part of $\sqrt{C}$. For example if $C = 8$, $\lfloor \sqrt{C} \rfloor = 2$ but $\rho_{\max}(C) = \frac{8}{5}$ and the approximation factor is $\frac{8}{5} = 1.6 > \sqrt{2}$. For $C = 15$, $\lfloor \sqrt{C} \rfloor = 3$, $\rho_{\max}(C) = \frac{5}{2}$ and the approximation factor is $\frac{5}{3}$. The next construction helps to deal with these cases where $C = p_1 p_2$.

il faut revoir l'ensemble de la construction et les calculs, car Benjamin a raison...

Construction 18 *Let $C = p_1p_2 + p'$, $p_1 \le p_2$ and $0 \le p' \le p_1p_2$; let also $N = qp_1p_2 + r$, $0 \le r < p_1p_2$. Let the vertices of K_N be $V = \cup_{i=1}^{q} V_i \cup V_{q+1}$ with $|V_i| = p_1p_2$ and $|V_{q+1}| = r$.*

We partition the edges of K_N into $\frac{q(q-1)}{2}$ $K_{p_1p_2,p_1p_2}$ on $V_i \cup V_j, 1 \le i < j \le q$, plus q $K_{p_1p_2,r}$ on $V_i \cup V_{q+1}, 1 \le i \le q$, plus q $K_{p_1p_2}$ on V_i and one K_r on V_{q+1}.

Then we partition each $K_{p_1p_2,p_1p_2}$ into p_1p_2 K_{p_1,p_2} and each $K_{p_1p_2,r}$, where $r = \alpha_1p_1 + \beta_1$, into $p_1\alpha_1$ K_{p_1,p_2} plus p_1 K_{β_1,p_2}.

Finally, we partition each $K_{p_1p_2}$ into $\frac{p_2(p_2-1)}{2}$ K_{p_1,p_1} plus p_2 K_{p_1}, and each K_r into $\frac{\alpha_1(\alpha_1-1)}{2}$ K_{p_1,p_1} plus α_1 K_{p_1,β_1} and α_1 K_{p_1} and 1 K_{β_1}. All this subgraphs have size $\le p_1^2 \le C$.

Lemma 19 *Let $C = p_1p_2 + p'$, and $N = qp_1p_2 + r$, $0 \le r < p_1p_2$, Construction 18 is valid and provides a $\frac{\rho_{\max}(C)(p_1+p_2)}{p_1p_2}$-approximation of the total number of ADMs.*

Proof : As $\beta_1 < p_1$ and so each K_{β_1,p_2} and K_{p_1,p_2} has at most $p_1p_2 \le C$ edges, the construction is valid.

The total number of ADMs is $\frac{q(q-1)}{2}p_1p_2(p_1+p_2) + q(\alpha_1p_1(p_1+p_2) + p_1(\beta_1+p_2)) + qp_1p_2^2 + \alpha_1(\alpha_1 - 1)p_1 + \alpha_1(2p_1+\beta_1) + \beta_1$. Using $N = qp_1p_2 + r$ we get $\frac{N(N-1)(p_1+p_2)}{p_1p_2} + O(N)$ ADMs and so the approximation factor. □

Remark : We can also modify the construction like we did before to include the edges of the $K_{p_1p_2}$ or K_r in the bipartite subgraphs, and therefore get ride in many cases of the $o\left(\frac{1}{N}\right)$.

Note that for Construction 18, we have many possible choices for p_1, p_2, and p' in the decomposition. Of course we have interest to choose p' as small as possible, but also to choose p_1 and p_2 in order to minimize $\frac{p_1+p_2}{p_1p_2}$; that can be achieved by choosing p_1 and p_2 near from each other but not necessarily equal.

For example, let $C = 32$. We can write $32 = 4 \times 8$, or $32 = 5 \times 5 + 7$, or $32 = 5 \times 6 + 2$. If we use Construction 11 with $C = 5 \times 5 + 7$ the approximation factor is $\frac{2}{5}\rho_{\max}(C)$; if we choose $32 = 4 \times 8$ in Construction 18 we get an approximation factor $\frac{12}{32}\rho_{\max}(C)$ which is better since $\frac{12}{32} < \frac{2}{5}$. But we can do better using $32 = 5 \times 6 + 2$ in Construction 18 getting an approximation factor $\frac{11}{30}\rho_{\max}(C)$.

It follows that for many values, Construction 18 is better than Construction 11 (or its variants) , which in fact corresponds to the particular case where $p_1 = p_2$.

For example if $C = 8 = 2 \times 4$, $\rho_{\max}(C) = \frac{8}{5}$ and the approximation factor is $\frac{8}{5}\frac{6}{8} = 1.2$ to be compared with $\frac{8}{5} = 1.6$. For $C = 15 = 3 \times 5$, $\rho_{\max}(C) = \frac{5}{2}$ and the approximation factor is $\frac{5}{2} \times \frac{8}{15} = \frac{4}{3}$ to be compared with $\frac{5}{3}$.

5 Construction with tripartite graphs

In the previous section we have shown that using a partition of K_N into small bipartite graphs, it is possible to obtain a $\frac{2\rho_{\max}(C)}{\lfloor \sqrt{C} \rfloor} + O\left(\frac{1}{N}\right)$-approximation of the total number of ADMs. We will now show that using small multipartite graphs it is possible to drastically improve the approximation factor.

We will use the optimal decomposition of K_N obtained in [3] for grooming factor $C = 3$, and reported here in Theorem 20, to obtain a $\frac{\rho_{\max}(C)}{\left\lfloor \sqrt{\frac{C}{3}} \right\rfloor} + O\left(\frac{1}{N}\right)$

Theorem 20 (Theorem 4 of [3]) *Let $n \ge 2$. There exist a partition of K_n using*

- *if* $n \equiv 1, 3 \bmod 6$, $\frac{n(n-1)}{6}$ K_3
- *if* $n \equiv 5 \bmod 6$, $\frac{n(n-1)-8}{6}$ K_3 *and 2* P_3
- *if* $n \equiv 0, 4 \bmod 12$, $\frac{n(n-1)}{6} - \frac{n}{4}$ K_3 *and* $\frac{n}{4}$ $K_{1,3}$
- *if* $n \equiv 2 \bmod 6$, $\frac{n(n-1)-2}{6} - \left\lceil \frac{n-2}{4} \right\rceil$ K_3, $\left\lceil \frac{n-2}{4} \right\rceil$ $K_{1,3}$ *and 1* e
- *if* $n \equiv 6, 10 \bmod 12$, $\frac{n(n-1)}{6} - \frac{n+2}{4}$ K_3, $\frac{n-2}{4}$ $K_{1,3}$ *and 1* P_4

where P_3 *is a path with 3 vertices,* P_4 *a path with 4 vertices,* e *a single edge and* $K_{1,3}$ *a complete bipartite graph between a set of size 1 and a set of size 3 (also call a* claw *or a 3-star).*

Construction 21 *Let* $C = 3p^2 + p'$, $0 \leq p' < 6p + 3$ *and* $N = qp + r$, $0 \leq r < p$, *and let the vertices of* K_N *be* $\cup_{i=1}^{q} V_i \cup V_{q+1}$, *with* $|V_i| = p$ *and* $|V_{q+1}| = r$.

Given a partition of K_{q+1} *obtained from Theorem 20, we associate to node* i *of* K_{q+1} *the set of nodes* V_i *when* $1 \leq i \leq q+1$ *and we replace each edge of the partition of* K_{q+1} *by the corresponding* $K_{p,p}$ *or* $K_{p,r}$. *By adding the* q K_p *plus the* K_r *corresponding to the groups* V_i *and* V_{q+1}, *we obtain a valid partition of* K_N.

Lemma 22 *Construction 21 provides a* $\frac{\rho_{\max}(C)}{\left\lfloor \sqrt{\frac{C}{3}} \right\rfloor} + O\left(\frac{1}{N}\right)$*-approximation of the total number of ADMs.*

Proof :

Let us analyze the number of ADMs of Construction 21.

- When $q+1 \equiv 1, 3 \bmod 6$, each node of K_{q+1} appears in $\frac{q}{2}$ K_3. So the partition uses $pq\frac{q}{2} + r\frac{q}{2} + qp + r = \frac{q+2}{2}N$ ADMs.
- When $q+1 \equiv 5 \bmod 6$ and according to the proof of Theorem 20 of [3], the 2 P_3 of the partition of K_{q+1} contains the edges $x - u, u - y$ and $x - v, v - y$. So nodes x and y appears one more time than the others. Assuming that these nodes are replaced by groups of size p, the partition of K_N uses $\frac{q+2}{2}N + 2p$ ADMs.
- When $q+1 \equiv 0, 4 \bmod 12$, $\frac{q+1}{4}$ nodes of K_{q+1} appears one more time than the others. Let us assume that these nodes are replaced by groups of size p. So we obtain $\frac{q+2}{2}N + p\frac{q+1}{4}$ ADMs.
- When $q+1 \equiv 2 \bmod 6$, $\left\lceil \frac{q-1}{4} \right\rceil$ nodes of K_{q+1} appears one more time than the others and two other nodes appears one more times. Let us assume that these nodes are replaced by groups of size p. So we obtain $\frac{q+2}{2}N + \left\lceil \frac{q-1}{4} \right\rceil p + 2p$ ADMs.
- When $q+1 \equiv 6, 10 \bmod 12$, $\frac{q+3}{4}$ nodes of K_{q+1} appears one more time than the others. Let us assume that these nodes are replaced by groups of size p. We obtain $\frac{q+2}{2}N + \frac{q+3}{4}p$ ADMs.

The results are summarized in Table 1 In all the cases the total number of ADMs is $\frac{N(N-1)}{2p} + O(N)$ giving the approximation factor $\frac{\rho_{\max}(C)}{\left\lfloor \sqrt{\frac{C}{3}} \right\rfloor}$ as $p = \left\lfloor \sqrt{\frac{C}{3}} \right\rfloor$ □

This approximation factor can be improved by using the optimal construction for grooming factor $C = 6$ presented in [4], that is a partition of K_N into quadripartite graphs and so on.

C	8	9	12	15	16	32	48	64	192
$\rho_{\max}(C)$	$\frac{8}{5}$	$\frac{9}{5}$	2	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{32}{9}$	$\frac{9}{2}$	$\frac{16}{3}$	$\frac{19}{2}$
γ for Cons. 11	1.6	1.2	1.33	1.67	1.25	1.42	1.5	1.33	1.46
γ for Cons. 18	1.2	1.2	1.17	1.33	1.25	1.3	1.31	1.33	1.39
γ for Cons. 21	1.6	1.8	1	1.25	1.25	1.19	1.13	1.33	1.19

Table 2: Approximation factor of the different constructions (up to $O(\frac{1}{N})$).

6 Construction with multipartite graphs

Here we give some constructions using decomposition of multipartite graphs which work for some values of N.

Construction 23 *Let $C = 3p^2$ and $N = 3^a p$, $a \geq 1$.*

From the existence of 3-GDD of type u^3 [12], that is a partition of the tripartite graph $K_{u,u,u}$ into K_3, $u \geq 1$, we know that $K_{up,up,up}$ can be partition into u^2 $K_{p,p,p}$. Thus, we partition the edges of K_N as follows

1. *If $N = 3p$ (i.e. $a = 1$), partition K_{3p} into one $K_{p,p,p}$ and 3 K_p.*
2. *Otherwise*
   (a) *Partition the edges of K_N into 3 $K_{3^{a-1}p}$ and one $K_{3^{a-1}p,3^{a-1}p,3^{a-1}p}$*
   (b) *Partition $K_{3^{a-1}p,3^{a-1}p,3^{a-1}p}$ into $(3^{a-1})^2$ $K_{p,p,p}$*
   (c) *Repeat the process on each $K_{3^{a-1}p}$*

One can check that we have partitioned K_N into $\sum_{i=0}^{a-1} 3^i(3^{a-i-1})^2 = \frac{3^a(3^a-1)}{6} = \frac{N(N-p)}{6p^2}$ $K_{p,p,p}$ and $3^a = \frac{N}{p}$ K_p.

Lemma 24 *Construction 23 uses $\frac{N(N+p)}{2p}$ ADMs and provide a $\frac{\rho_{\max}(C)}{\sqrt{\frac{C}{3}}} + O\left(\frac{1}{N}\right)$-approximation of the total number of ADMs.*

Proof : Construction 23 uses $3p\frac{N(N-p)}{6p^2} + p\frac{N}{p} = \frac{N(N+p)}{2p}$ ADMs. Thus it has approximation factor $\gamma(C,N) = \frac{N(N+p)}{p}\frac{\rho_{\max}(C)}{N(N-1)} = \frac{\rho_{\max}(C)}{\sqrt{\frac{C}{3}}}\left(1 + \frac{p+1}{N-1}\right)$. □

The same simple idea can be used when $C = 6p^2$ and $N = 4^a p$, using a partition of $K_{up,up,up,up}$ into u^2 $K_{p,p,p,p}$, and we will obtain a $\sqrt{\frac{4}{3}} + O\left(\frac{1}{N}\right)$-approximation. Similarly, when $C = 10p^2$ and $N = 5^a p$ we will obtain a $\sqrt{\frac{5}{4}} + O\left(\frac{1}{N}\right)$-approximation, and more generally, when $C = \frac{\alpha(\alpha-1)}{2}p^2$ and $N = \alpha^a p$ we will obtain a $\sqrt{\frac{\alpha}{\alpha-1}} + O\left(\frac{1}{N}\right)$-approximation.

Unfortunately, such construction applies only for a few values of N. So the constructions that we have presented in Section 4 and 5 are more interesting in practice.

Also, depending on the value of the grooming factor, one has to choose the most efficient construction. For example, for small values of C, Construction 18 performs better than the others, which is no longer the case for larger values of C as shown in Table 2.

7 Conclusion

In this paper, using tools of design theory, we have given different approximate constructions for all-to-all traffic grooming in unidirectional ring. These simple constructions might also be used to compute good solutions for very dense set of requests, i.e. instances that are almost all-to-all, for which only $O(\log C)$-approximation algorithms are known so far. The traffic grooming problem being APX-Hard [1], this work represents an important step toward the conception of tight approximation algorithms for practical instances.

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