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Abstract

Given a group action, known by its infinitesimal generators, we exhibit a complete set of syzygies on a generating set of differential invariants. For that we elaborate on the reinterpretation of Cartan's moving frame by Fels and Olver (1999). This provides constructive tools for exploring algebras of differential invariants.

Key words: Lie group actions, Differential invariants, Syzygies, Differential algebra, Symbolic Computation.
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Introduction

A great variety of group actions arise in mathematics, physics, science and engineering and their invariants, whether algebraic or differential, are commonly used for symmetry reduction or to solve equivalence problems and determining canonical forms. Classifying invariants is consequently an essential task. One needs to determine a generating set of invariants and their syzygies, i.e. the relations they satisfy.

With minimal amount of data on the group action, we shall characterize two generating sets of differential invariants. Though not computing them explicitly, we describe inductive processes to rewrite any differential invariants in terms of them and their invariant derivatives. For one of those generating set we determine a complete set of differential relationships, which we call syzygies. The other generating set is of bounded cardinality and a complete set of syzygies can be computed from the previous one by the generalized differential elimination scheme provided by Hubert (2005b).

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The results in this paper are constructive and our presentation describes very closely their symbolic implementation in AIDA (Hubert, 2007b). They are indeed part of a bigger project the aim of which is to develop the foundations for symmetry reduction of differential systems with a view towards differential elimination. This is outlined in the motivational example of Hubert (2005b). The computational requirements include four main components: the explicit computation of a generating set of invariants (1), and the relations among them (2); procedures for rewriting the problem in terms of the invariants (3); and finally procedures for computing in the algebra of invariants (4). In this paper we focus on (2) and (3) while (1) and (4) were consistently addressed by Hubert and Kogan (2007a,b) and Hubert (2005b) respectively. This paper thus completes an algorithmic suite. While component (4) has been implemented as a generalization of the MAPLE library *diffalg* (Boulier and Hubert, 1998; Hubert, 2005a), components (1-3) is implemented in our Maple package AIDA (Hubert, 2007b) that works on top of the MAPLE library *DifferentialGeometry* (Anderson and et al., 2007), as well as *diffalg* and *Groebner*. In this paper we also use component (4) to reduce the number of generators, while still providing the complete syzygies.

On one hand, the question of the finite generation of differential invariants was addressed by Tresse (1894); Kumpera (1974, 1975a,b); Muñoz et al. (2003), in the more general case of pseudo-groups - see also Ovsiannikov (1982); Olver (1995) for Lie groups. On the other hand, Griffiths's (1974) interpretation of Cartan's (1935; 1937; 1953) moving frame method solved equivalence problem in many geometries (Green, 1978; Jensen, 1977; Gardner, 1989; Ivey and Landsberg, 2003). Alternatively, the approach of Gardner (1989) and its recent symbolic implementation (Neut, 2003) lead to computational solutions for the classification of differential equations (Neut and Petitot, 2002; Dridi and Neut, 2006a,b). Besides Fels and Olver (1999) offered another interpretation of Cartan's moving frame method, the application of which goes beyond geometry (Olver, 2005). In particular it includes an explicit approach to the generation properties.

The main original contribution in this paper is to formalize the notion of differential syzygies for a generating set of differential invariants and prove the completeness of a finite set of those. To this end we redevelop the construction of normalized invariants and invariant derivations of Fels and Olver (1999) in a spirit we believe closer to the audience of this journal. We offer alternative proofs, and sometimes more general results. In particular we shall put the emphasis on derivations, rather than differential forms.

One is interested in the action (effective on subsets) of a group $\mathcal{G}$ on a manifold $\mathcal{X}\times\mathcal{U}$ and its prolongation to the higher order jets $\mathrm{J}^k(\mathcal{X},\mathcal{U})$. In other words, $\mathcal{X}$ is the space of independent variables while $\mathcal{U}$ is the set of dependent variables. The jet space is parameterized by the derivatives of the dependent variables with respect to the independent variables. At each order k, a local cross-section to the orbits defines a finite set of normalized invariants. Those latter form a generating set for differential invariants of order k, in a functional sense. Rewriting those latter in terms of the normalized invariants is furthermore a trivial substitution. We review this material in Section 2.3, following the presentation of Hubert and Kogan (2007b).

As the orbit dimension stabilizes at order s the action becomes locally free and, to any local cross-section, we can associate a *moving frame*, i.e. an equivariant map $\rho : \mathrm{J}^s(\mathcal{X},\mathcal{U}) \rightarrow \mathcal{G}$ (Fels and Olver, 1999). The moving frame defines in turn a basis of invariant derivations. The great value of this particular set of invariant derivations is the fact that we can write explicitly their action on invariantized functions. This is captured in the so called *recurrence formulae*. They are the key to proving generation, rewriting and syzygies. Fels and Olver (1999) gave the recurrence formulae for the normalized invariants in the case of a coordinate cross-section. We propose generalized recurrence formulae in the case of any cross-section and offer an alternate proof, close in spirit to the one of Mansfield (2008).

We can then show that normalized invariants of order $s + 1$ form a generating set with respect to those invariant derivations. Rewriting any differential invariant in terms of those and their derivative is a simple application of the recurrence formulae (Section 4). By exhibiting a canonical rewriting, we can prove the completeness of a set of differential syzygies for those differential invariants, after giving this concept a definition (Section 5).

We formalize the notion of syzygies through the introduction of the algebra of *monotone derivatives*. Along the lines of Hubert (2005b), this algebra is equipped with derivations that are defined inductively so as to encode the nontrivial commutation rules of the invariant derivations. The syzygies are the elements of the kernel of the differential morphism between the algebra of monotone derivatives and the algebra of differential invariants, equipped with the invariant derivations. The type of differential algebra introduced at this stage was shown to be a natural generalization of classical differential algebra (Ritt, 1950; Kolchin, 1973). In the polynomial case, it is indeed endowed with an effective differential elimination theory that has been implemented (Hubert, 2005a,b).

For cross-sections of minimal order we can also prove that the set of *edge invariants* is generating. This latter set has a cardinality bounded by $m\,r + d_0$, where m, r are the dimensions of $\mathcal{X}$ and $\mathcal{G}$ while d_0 is the codimension of the orbits on $\mathcal{X} \times \mathcal{U}$. This is a generalization of the result of Olver (2007b) that bears on coordinate cross-sections. The edge invariants then form a subset of the normalized invariants of order $s+1$. Fels and Olver (1999) first conjectured syzygies on this set of generating invariants. We feel that constructing directly a complete and finite set of syzygies for the set of edge invariants is challenging, the problem bearing a high combinatorial difficulty. To obtain those, we suggest to apply generalized differential elimination (Hubert, 2005a,b) on the set of syzygies for the normalized invariants. This is illustrated in the examples of Section 5, 6 and 7.

Similarly, to reduce further the number of generators for the differential invariants we can apply the same generalized differential elimination techniques to the syzygies. This substantially reduces the work of computing explicitly a generating set for a given action. This is an approach that was applied for surfaces in Euclidean, affine, conformal and projective geometry (Olver, 2007a; Hubert and Olver, 2007).

Let us stress here the minimal amount of data indeed needed for the determination of a generating set, the rewriting in terms of those and the differential syzygies. All is based on the recurrence formulae that can be written with only the knowledge of the infinitesimal generators of the action and the equations of the cross-section. Furthermore the operations needed consist of derivations, arithmetic operations and test to zero. Provided the coefficients of the infinitesimal generators are rational functions, which provide a general enough class, we are thus in the realm of symbolic computation since we can indeed always choose linear equations for the cross-section. On the other hand, the explicit expression of the invariant derivations, or the differential invariants, requires the knowledge of the moving frame. This latter is obtained by application of the implicit function theorem on the group action. This is therefore not constructive in general, but there are algorithms in the algebraic case (Hubert and Kogan, 2007a,b).

In Section 1 we extract from the books of Olver (1986, 1995) the essential material we need for describing actions and their prolongations. In Section 2 we define invariantization and normalized invariants for the action of a group on a manifold along the lines of Hubert and Kogan (2007b). We then extend those notions to differential invariants. In Section 3 we define invariant derivations as the derivations that commute with the infinitesimal generators of the action. We introduce the construction of invariant derivations of Fels and Olver (1999) based on the moving frame together with the *recurrence formulae*. We write those latter in a more general form (Theorem 3.6): the derivations of the invariantization of a function are given explicitly in terms of invariantizations. Section 4 discusses then the generation property of the normalized invariants and effective rewriting. We furthermore show the generalization of Olver (2007a), the generation property of the *edge invariants* in the case of minimal order cross-section. In Section 5 we emphasize the non uniqueness of the rewriting in terms of the normalized invariants. We then introduce the algebra of monotone derivatives, and the inductive derivations acting on it, in order to formalize the concept of syzygies. We can then write a finite set of syzygies and prove its completeness.

In the penultimate section we present geometric examples that many readers are familiar with in order to illustrate our general approach: the action of the Euclidean group on space curves and surfaces. In the last section we undertake the challenging analysis for the action of the indefinite orthogonal groups on three independent variables, and their affine extensions. To the best of our knowledge, the structure of their differential algebra had not been explored so far. Additional non trivial applications of the results in this paper, and the related software, were developed by Hubert and Olver (2007).

1 Group action and their prolongations

This is a preliminary section introducing the definition and notations for Lie group actions and their prolongation to derivatives. We essentially follow the books of Olver (1986, 1995).

1.1 Local action of a Lie group on a manifold

Pullbacks and push-forwards of maps

Consider a smooth manifold $\mathcal{M}$. $\mathcal{F}(\mathcal{M})$ denotes the ring of smooth functions on $\mathcal{M}$ while $\mathrm{Der}(\mathcal{M})$ denotes the $\mathcal{F}(\mathcal{M})$-module of derivations on $\mathcal{F}(\mathcal{M})$.

If $\mathcal{N}$ is another smooth manifold and $\phi : \mathcal{M} \to \mathcal{N}$ a smooth map, the *pull-back* of ϕ is the map $\phi^* : \mathcal{F}(\mathcal{N}) \to \mathcal{F}(\mathcal{M})$ defined by $\phi^* f = f \circ \phi$ i.e. $(\phi^* f)(z) = f(\phi(z))$ for all $z \in \mathcal{M}$. Through ϕ^*, $\mathcal{F}(\mathcal{N})$ can be viewed as a $\mathcal{F}(\mathcal{M})$ module.

A derivation $\mathrm{V} : \mathcal{F}(\mathcal{M}) \to \mathcal{F}(\mathcal{M})$ on $\mathcal{M}$ induces a derivation$\mathrm{V}|_z : \mathcal{F}(\mathcal{M}) \to \mathbb{R}$ at z defined by $\mathrm{V}|_z(f) = \mathrm{V}(f)(z)$. The set of derivations at a point $z \in \mathcal{M}$ is the tangent space of $\mathcal{M}$ at z. Vector fields on $\mathcal{M}$ can be understood as derivations.

The *push-forward* or *differential* of ϕ is defined by

$$(\phi_* \mathrm{V})(f)(\phi(x)) = \mathrm{V}(\phi^* f)(x)$$

The coordinate expression for $\phi_* \mathrm{V}$ is given by the chain rule. Yet this *star* formalism allows us to write formulae in a compact way and we shall use it extensively.

Local action on a manifold

We consider a connected Lie group $\mathcal{G}$ of dimension r. The multiplication of two elements $\lambda, \mu \in \mathcal{G}$ is denoted as $\lambda \cdot \mu$. An action of $\mathcal{G}$ on a manifold $\mathcal{M}$ is defined by a map $g : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ that satisfies $g(\lambda, g(\mu, z)) = g(\lambda \cdot \mu, z)$. We shall implicitly consider local actions, that is g is defined only on an open subset of $\mathcal{G} \times \mathcal{M}$ that contains $\{e\} \times \mathcal{M}$. We assume that $\mathcal{M}$ is made of a single coordinate chart. If $(z_1, \ldots, z_k)$ are the coordinate functions then $g^* z_i : \mathcal{G} \times \mathcal{M} \to \mathbb{R}$ represents the ith component of the map g.

There is a fine interplay of right and left invariant vector fields in the paper.

We thus detail what we mean there now. Given a group action $g : \mathcal{G}\times\mathcal{M} \rightarrow \mathcal{M}$ define, for $\lambda \in \mathcal{G}$, $g_\lambda : \mathcal{M} \rightarrow \mathcal{M}$ by $g_\lambda(z) = g(\lambda, z)$ for $z \in \mathcal{M}$. A vector field X on $\mathcal{M}$ is *$\mathcal{G}$-invariant* if $g_{\lambda *}X = X$ for all $\lambda \in \mathcal{G}$, that is

$$\forall f \in \mathcal{F}(\mathcal{M}),\ \forall z \in \mathcal{M},\ X(f \circ g_\lambda)(z) = X(f)(g_\lambda(z)).$$

A vector field on $\mathcal{G}$ is right invariant if it is invariant under the action of $\mathcal{G}$ on itself by right multiplication. In other words, if $r_\mu : \mathcal{G} \rightarrow \mathcal{G}$ is the right multiplication by μ^{-1}, $r_\mu(\lambda) = \lambda \cdot \mu^{-1}$, a vector field v on $\mathcal{G}$ is right invariant if

$$\mathrm{v}(f \circ r_\mu)(\lambda) = \mathrm{v}(f)(\lambda \cdot \mu^{-1}), \quad \forall f \in \mathcal{F}(\mathcal{G}).$$

For a right invariant vector field on $\mathcal{G}$, the *exponential* map $e^{\mathrm{v}} : \mathbb{R} \rightarrow \mathcal{G}$ is the flow of v such that $e^{\mathrm{v}}(0)$ is the identity. We write $e^{t\mathrm{v}}$ for $e^{\mathrm{v}}(t)$. The defining equation for e^{v} is

$$\mathrm{v}(f)(\lambda) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f\left(e^{t\mathrm{v}} \cdot \lambda\right).$$

Similarly the associated *infinitesimal generator* V of the action g of $\mathcal{G}$ on $\mathcal{M}$ is the vector field on $\mathcal{M}$ defined by

$$\mathrm{V}(f)(z) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(g(e^{t\mathrm{v}}, z)), \quad \forall f \in \mathcal{F}(\mathcal{M}). \tag{1.1}$$

Note that v is the infinitesimal generator for the action of $\mathcal{G}$ on $\mathcal{G}$ by left multiplication. The infinitesimal generator associated to v for the action of $\mathcal{G}$ on $\mathcal{G}$ by right multiplication, $r : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$, $r(\lambda, \mu) = \mu \cdot \lambda^{-1}$ is

$$\hat{\mathrm{v}}(f)(\lambda) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\lambda \cdot e^{-t\mathrm{v}}). \tag{1.2}$$

We can observe that $\hat{\mathrm{v}}$ is a left invariant vector field on $\mathcal{G}$.

A right invariant vector field on $\mathcal{G}$ is completely determined by its value at identity. We can thus find a basis $\mathrm{v} = (\mathrm{v}_1, \ldots, \mathrm{v}_r)$ for the derivations on $\mathcal{F}(\mathcal{G})$ made of right invariant vector fields. The associated left invariant vector fields $\hat{\mathrm{v}} = (\hat{\mathrm{v}}_1, \ldots, \hat{\mathrm{v}}_r)$ then also form a basis of derivations on $\mathcal{F}(\mathcal{G})$ (Olver, 1995, Chapter 2).

The following property is used for the proof of Theorem 3.4 and 3.6. What is used more precisely in Theorem 3.6 is the fact that $\mathrm{v}(g^*f)|_e = \mathrm{V}(f)$. This can also be deduced from Theorem 3.10 by Fels and Olver (1999). In our notations this latter reads as: $\mathrm{v}(g^*z_i) = g^*\mathrm{V}(z_i)$.

Proposition 1.1 *Let v be a right invariant vector field on $\mathcal{G}$, $\hat{\mathrm{v}}$ the associated infinitesimal generator for the action of $\mathcal{G}$ on $\mathcal{G}$ by right multiplication and V the associated infinitesimal generator of the action g of $\mathcal{G}$ on $\mathcal{M}$.*

When both $\hat{\mathrm{v}}$ *and* V *are considered as derivations on* $\mathcal{F}(\mathcal{G}\times\mathcal{M})$ *then*

$$\hat{\mathrm{v}}(g^*f)+\mathrm{V}(g^*f)=0 \quad \text{and} \quad \mathrm{V}(g^*f)(e,z)=\mathrm{V}(f)(z), \quad \forall f\in\mathcal{F}(\mathcal{M}).$$

As a particular case we have $\hat{\mathrm{v}}(f)(e)=-\mathrm{v}(f)(e)$.

PROOF: $\hat{\mathrm{v}}$ is a linear combination of derivations with respect to the group parameters, i.e. the coordinate functions on $\mathcal{G}$, while V is a combination of derivations with respect to the coordinate functions on $\mathcal{M}$. By (1.1) and (1.2) we have

$$\mathrm{V}(g^*f)(\lambda,z)=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(g^*f)(\lambda,g(e^{t\mathrm{v}},z))$$

and

$$\hat{\mathrm{v}}(g^*f)(\lambda,z)=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(g^*f)(\lambda\cdot e^{-t\mathrm{v}},z)=-\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(g^*f)(\lambda\cdot e^{t\mathrm{v}},z).$$

The conclusion follows from the group action property that imposes:

$$(g^*f)(\lambda,g(e^{t\mathrm{v}},z))=f(g(\lambda,g(e^{t\mathrm{v}},z))=f(g(\lambda\cdot e^{t\mathrm{v}},z))=(g^*f)(\lambda\cdot e^{t\mathrm{v}},z).$$

□

Example 1.2 *We consider the group* $\mathcal{G}=\mathbb{R}^*_{>0}\ltimes\mathbb{R}$ *with multiplication* $(\lambda_1,\lambda_2)\cdot(\mu_1,\mu_2)^{-1}=(\frac{\lambda_1}{\mu_1},-\lambda_1\frac{\mu_2}{\mu_1}+\lambda_2)$.
A basis of right invariant vector fields is given by (Olver, 1995, Example 2.46)

$$\mathrm{v}_1=\lambda_1\frac{\partial}{\partial\lambda_1}+\lambda_2\frac{\partial}{\partial\lambda_2}, \quad \mathrm{v}_2=\frac{\partial}{\partial\lambda_2}.$$

The associated left invariant vector fields, i.e. the infinitesimal generators for the action of $\mathcal{G}$ *on* $\mathcal{G}$ *by right multiplication, are:*

$$\hat{\mathrm{v}}_1=-\lambda_1\frac{\partial}{\partial\lambda_1}, \quad \hat{\mathrm{v}}_2=-\lambda_1\frac{\partial}{\partial\lambda_2}.$$

If we consider the action g *of* $\mathcal{G}$ *on* $\mathbb{R}$ *given by* $g^*x=\lambda_1x+\lambda_2$, *the associated infinitesimal generators for this action are*

$$\mathrm{V}_1=x\frac{\partial}{\partial x}, \quad \mathrm{V}_2=\frac{\partial}{\partial x}.$$

Note that $\hat{\mathrm{v}}_i(g^*x)=-\mathrm{V}_i(g^*x)$ *and* $\hat{\mathrm{v}}_i|_e=-\mathrm{v}_i|_e$.

1.2 Action prolongations

We shall consider now a manifold $\mathcal{X}\times\mathcal{U}$. We assume that $\mathcal{X}$ and $\mathcal{U}$ are covered by a single coordinate chart with respectively $x=(x_1,\ldots,x_m)$ and $u=$

$(u_1, \ldots, u_n)$ as coordinate functions. The x are considered as the independent variables and the u as dependent variables. We discuss briefly the prolongation of an action of $\mathcal{G}$ on $\mathcal{X} \times \mathcal{U}$ to its jet space following Olver (1986, 1995).

Notation 1.3 *The m-tuple with 1 at the i^{th} position and 0 otherwise is denoted by ϵ_i . For $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ we note $|\alpha| = \alpha_1 + \ldots + \alpha_m$. If $\mathrm{D}_1, \ldots, \mathrm{D}_m$ are derivations we write D^α for $\mathrm{D}_1^{\alpha_1} \ldots \mathrm{D}_m^{\alpha_m}$. Similarly u_α stands for $\frac{\partial^{|\alpha|} u}{\partial\, x^\alpha} = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}}$.*

Total derivations

The k-th order jet space is noted $\mathrm{J}^k(\mathcal{X},\mathcal{U})$, or J^k for short, while the infinite jet space is J. Besides x and u the coordinate functions of J^k are u_α for u in $\{u_1, \ldots, u_n\}$ and $\alpha \in \mathbb{N}^m$ with $|\alpha| \leq k$.

The *total derivations with respect to the independent variables* are the derivations on J defined by

$$\mathrm{D}_i = \frac{\partial}{\partial x_i} + \sum_{u \in \mathcal{U},\, \alpha \in \mathbb{N}^m} u_{\alpha+\epsilon_i} \frac{\partial}{\partial u_\alpha}, \text{ for } 1 \leq i \leq m. \tag{1.3}$$

In other words, D_i is such that for any $u \in \mathcal{U}$ and $\alpha \in \mathbb{N}^m$, $\mathrm{D}_i(u_\alpha) = u_{\alpha+\epsilon_i}$, while $\mathrm{D}_i x_j = 1$ or 0 according to whether $i = j$ or not.

Pragmatically the set of *total derivations* is the free $\mathcal{F}(\mathrm{J})$-module with basis $\mathrm{D} = \{\mathrm{D}_1, \ldots, \mathrm{D}_m\}$. Geometrically one defines total derivations as the derivations of $\mathcal{F}(\mathrm{J})$ that annihilate the contact forms (Olver, 1995). Alternatively they correspond to the formal derivations in (Kumpera, 1974, 1975a,b; Muñoz et al., 2003). A total derivation D is of order l if for all $f \in \mathcal{F}(\mathrm{J}^{l+k})$, $k \geq 0$, $\mathrm{D}(f) \in \mathcal{F}(\mathrm{J}^{l+k+1})$. The total derivations of order l form a $\mathcal{F}(\mathrm{J}^l)$-module.

Prolongation of vector fields

Vector fields on J^k form a free $\mathcal{F}(\mathrm{J}^k)$-module a basis of which is given by $\{\frac{\partial}{\partial x} \mid x \in \mathcal{X}\} \cup \{\frac{\partial}{\partial u_\alpha} \mid u \in \mathcal{U}, |\alpha| \leq k\}$.

Definition 1.4 *Let V^0 be a vector field on J^0. The k-th prolongation V^k, $k \geq 0$, is the unique vector field of $\mathcal{F}(\mathrm{J}^k)$ defined recursively by the conditions*

$$\mathrm{V}^{k+1}|_{\mathcal{F}(\mathrm{J}^k)} = \mathrm{V}^k, \;\; \textit{and} \;\; \mathrm{V}^{k+1} \circ \mathrm{D}_i - \mathrm{D}_i \circ \mathrm{V}^k \textit{ is a total derivation for all } 1 \leq i \leq m.$$

This definition is to be compared with (Olver, 1995, Proposition 4.33) given in terms of contact forms. The explicit form of the prolongations are given in Chapter 4 of Olver (1995).

Proposition 1.5 *The prolongations of a vector field* $\mathrm{V}^0 = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i} + \sum_{j=1}^n \eta_j \frac{\partial}{\partial u_j}$ *on* J^0 *are the appropriate restrictions of the vector field*

$$\mathrm{V} = \sum_{i=1}^n \xi_i \, \mathrm{D}_i + \sum_{1 \leq j \leq n, \, \alpha \in \mathbb{N}^m} \mathrm{D}^\alpha(\zeta_j) \frac{\partial}{\partial u_{j_\alpha}} \quad \textit{where} \quad \zeta_j = \eta_j - \sum_{i=1}^m \xi_i \, \mathrm{D}_i(u_j).$$

Furthermore $\mathrm{D}_j \circ \mathrm{V} - \mathrm{V} \circ \mathrm{D}_j = \sum_{i=1}^m \mathrm{D}_j(\xi_i)\mathrm{D}_i, \quad \forall j \in \{1, \ldots, m\}.$

Action prolongations

Consider a connected Lie group $\mathcal{G}$ of dimension r acting on $\mathrm{J}^0 = \mathcal{X} \times \mathcal{U}$.

An action of $\mathcal{G}$ on $\mathrm{J}^0 = \mathcal{X} \times \mathcal{U}$ can be prolonged in a unique way to an action $\mathcal{G} \times \mathrm{J}^\kappa \rightarrow \mathrm{J}^\kappa$ that defines a contact transformation for each $\lambda \in \mathcal{G}$. We shall write g as well for the action on any J^k. The explicit expressions for $g^* u_\alpha$ is obtained as follows (Olver, 1986, Chapter 4).

In order to obtain compact formulae we introduce vectorial notations. D denotes the vector of total derivations $\mathrm{D} = (\mathrm{D}_1, \ldots \mathrm{D}_m)^T$ on $\mathcal{F}(\mathrm{J})$. Define the vector $\tilde{\mathrm{D}} = (\tilde{\mathrm{D}}_1, \ldots, \tilde{\mathrm{D}}_m)^T$ of derivations on $\mathcal{F}(\mathcal{G} \times \mathrm{J})$ as

$$\tilde{\mathrm{D}} = A^{-1}\mathrm{D} \quad \text{where} \quad A = \left(\mathrm{D}_i(g^* x_j)\right)_{ij}. \tag{1.4}$$

The total derivations D are here implicitly extended to be derivations on functions of $\mathcal{G} \times \mathrm{J}$. The derivations $\tilde{\mathrm{D}}$ commute and are such that $\tilde{\mathrm{D}}_i(g^* x_j) = \delta_{ij}$ and $g^* u_\alpha = \tilde{\mathrm{D}}^\alpha(g^* u)$ (Olver, 1995, Chapter 4). The prolongations are then given by:

$$g^*(\mathrm{D} f) = \tilde{\mathrm{D}}(g^* f), \; \forall f \in \mathcal{F}(\mathrm{J}). \tag{1.5}$$

If $\mathrm{V}^0 = (\mathrm{V}_1^0, \ldots, \mathrm{V}_r^0)$ are the infinitesimal generators for the action of g on J^0 then their k-th prolongations $\mathrm{V}^k = (\mathrm{V}_1^k, \ldots, \mathrm{V}_r^k)$ are the infinitesimal generators for the action of g on J^k.

Example 1.6 *We consider the group of Example 1.2,* $\mathcal{G} = \mathbb{R}^*_{>0} \ltimes \mathbb{R}$ *and extend trivially its action on* $\mathcal{X}^1 \times \mathcal{U}^1$ *as follows:*

$$g^* x = \lambda_1 \, x + \lambda_2, \quad g^* u = u.$$

The derivation $\tilde{\mathrm{D}} = \frac{1}{\lambda_1}\mathrm{D}$ *allows to compute the prolongations of the action:* $g^* u_k = \frac{u_k}{\lambda_1^k}$. *The infinitesimal generators of the action were given in Example 1.2. Their prolongations are:*

$$\mathrm{V}_1 = x \, \mathrm{D} - \sum_{k \geq 0} \mathrm{D}^k(x \, u_1) \frac{\partial}{\partial u_k} = x \frac{\partial}{\partial x} - k \, u_k \frac{\partial}{\partial u_k}, \quad \mathrm{V}_2 = \frac{\partial}{\partial x}.$$

2 Local and differential invariants

We first define the normalized invariants in the context of a group action on a manifold $\mathcal{M}$. We then generalize those concepts to differential invariants. The material of this section is essentially borrowed from Fels and Olver (1999) and Hubert and Kogan (2007b), following closely this latter. We refer the readers to those papers for more details and a substantial set of examples.

2.1 Normalized invariants

We consider the action $g : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ of the r-dimensional Lie group $\mathcal{G}$ on the smooth manifold $\mathcal{M}$.

Definition 2.1 *A smooth function f, defined on an open subset of $\mathcal{M}$, is a* local invariant *if* $\mathrm{V}(f) = 0$ *for any infinitesimal generator* V *of the action g of $\mathcal{G}$ on $\mathcal{M}$. The set of local invariants is denoted $\mathcal{F}^{\mathcal{G}}(\mathcal{M})$.*

This is equivalent to say that, for z in the definition set of f, $g_\lambda^* f(z) = f(z)$ for all λ in a neighbourhood of the identity in $\mathcal{G}$.

The orbit of a point $z \in \mathcal{M}$ is the set of points $\mathcal{O}_z = \{g(\lambda, z) | \lambda \in \mathcal{G}\}$. The action is semi-regular if all the orbits have the same dimension, say d. For those a maximally independent set of local invariants is classically shown to exist by Frobenius theorem (Olver, 1995, Theorem 2.23 and 2.34). Alternatively, a geometric method was described for free action based on a *moving frame* by Fels and Olver (1999) and extended to semi-regular actions with the sole use of a cross-section by Hubert and Kogan (2007b).

Definition 2.2 *An embedded submanifold $\mathcal{P}$ of $\mathcal{M}$ is a* local cross-section *to the orbits if there is an open set $\mathcal{U}$ of $\mathcal{M}$ such that*

- *$\mathcal{P}$ intersects $\mathcal{O}_z^0 \cap \mathcal{U}$ at a unique point $\forall z \in \mathcal{U}$, where $\mathcal{O}_z^0$ is the connected component of $\mathcal{O}_z \cap \mathcal{U}$, containing z.*
- *for all $z \in \mathcal{P} \cap \mathcal{U}$, $\mathcal{O}_z^0$ and $\mathcal{P}$ are transversal and of complementary dimensions.*

Most of the results in this paper restrict to $\mathcal{U}$. We shall thus assume, with no loss, that $\mathcal{U} = \mathcal{M}$.

An embedded submanifold of codimension d can be locally defined as the zero set of a map $P : \mathcal{M} \to \mathbb{R}^d$ where the components $(p_1, \ldots, p_d)$ are independent functions along $\mathcal{P}$. The transversality and dimension condition in the definition induce the following necessary condition for P to define a local cross-section $\mathcal{P}$:

$$\text{the rank of the } r \times d \text{ matrix } \left(\mathrm{V}_i(p_j)\right)_{i=1..r}^{j=1..d} \text{ equals to } d \text{ on } \mathcal{P}. \qquad (2.1)$$

When $\mathcal{G}$ acts semi-regularly on $\mathcal{M}$ there is a lot of freedom in choosing a cross-section. In particular we can always choose a coordinate cross-section (Hubert and Kogan, 2007b, Theorem 5.6).

A cross-section on $\mathcal{M}$ defines an invariantization process that is a projection from $\mathcal{F}(\mathcal{M})$ to $\mathcal{F}^{\mathcal{G}}(\mathcal{M})$.

Definition 2.3 *Let $\mathcal{P}$ be a local cross-section to the orbits of the action $g : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$. Let f be a smooth function on $\mathcal{M}$. The* invariantization *$\bar{\iota} f$ of f is the function defined by $\bar{\iota} f(z) = f(z_0)$ for each $z \in \mathcal{M}$, where $z_0 = \mathcal{O}_z^0 \cap \mathcal{P}$.*

The invariantization of the coordinate functions on $\mathcal{M}$ are the *normalized invariants*. Fels and Olver (1999, Definition 4.9) explain how invariantization actually ties in with the normalization procedure in Cartan's work. The following theorem (Hubert and Kogan, 2007b, Theorem 1.8) entails that normalized invariants form a generating set that is equipped with a trivial rewriting process.

Theorem 2.4 *Let a Lie group $\mathcal{G}$ act semi-regularly on a manifold $\mathcal{M}$, and let $\mathcal{P}$ be a local cross-section to the orbits. Then the invariantization $\bar{\iota} f$ of $f : \mathcal{M} \to \mathbb{R}$ is the unique local invariant whose restriction to $\mathcal{P}$ is equal to the restriction of f to $\mathcal{P}$. In other words $\bar{\iota} f|_{\mathcal{P}} = f|_{\mathcal{P}}$.*

Contained in this theorem as well is the fact that two local invariants are equal if and only if they have the same restriction on $\mathcal{P}$. In particular if $f \in \mathcal{F}^{\mathcal{G}}(\mathcal{M})$ then $\bar{\iota} f = f$. Now, by comparing the values of the functions involved at the cross-section, it is furthermore easy to check that:

Corollary 2.5 *For $f \in \mathcal{F}(\mathcal{M})$, $\bar{\iota} f(z_1, \ldots, z_n) = f(\bar{\iota} z_1, \ldots, \bar{\iota} z_n)$.*

Thus for $f \in \mathcal{F}^{\mathcal{G}}(\mathcal{M})$ we have $f(z_1, \ldots, z_n) = f(\bar{\iota} z_1, \ldots, \bar{\iota} z_n)$. Therefore the normalized invariants $\{\bar{\iota} z_1, \ldots, \bar{\iota} z_n\}$ form a generating set of local invariants: any local invariant can be written as a function of those. The rewriting is furthermore a simple replacement: we substitute the coordinate functions by their invariantizations.

The normalized invariants are nonetheless not functionally independent. Characterizing the functions that vanish on $(\bar{\iota} z_1, \ldots, \bar{\iota} z_n)$ amounts to characterize the functions the invariantization of which is zero. The functions that cut out the cross-section are an example of those.

Proposition 2.6 *Assume the cross-section $\mathcal{P}$ is the zero set of the map $P = (p_1, \ldots, p_d) : \mathcal{M} \to \mathbb{R}^d$ which is of maximal rank d along $\mathcal{P}$. The invariantization of $f \in \mathcal{F}(\mathcal{M})$ is zero if and only if, in a neighbourhood of each point of $\mathcal{P}$, there exist $a_1, \ldots, a_d \in \mathcal{F}(\mathcal{M})$ such that $f = \sum_{i=1}^{d} a_i \, p_i$.*

PROOF: Taylor's formula with integral remainder shows the following (Bourbaki, 1967, Paragraph 2.5). For a smooth function f on an open set $I_1 \times \ldots \times I_d \times U \subset \mathbb{R}^d \times \mathbb{R}^l$, where the I_i are intervals of $\mathbb{R}$ that contain zero, there are smooth functions f_0 on U, and f_i on $I_1 \times \ldots \times I_i \times U$, $1 \leq i \leq d$ such that $f(t_1, \ldots, t_d, x) = f_0(x) + \sum_{j=1}^{l} t_j\, f_j(t_1, \ldots, t_j, x)$ where $f_0(x) = f(0, \ldots, 0, x)$.

Assume that $\bar{\iota} f = 0 \quad \Leftrightarrow \quad f|_{\mathcal{P}} = 0$. Since $(p_1, \ldots, p_d)$ is of rank d along $\mathcal{P}$ we can find, in the neighbourhood of each point of $\mathcal{P}$, $x_{d+1}, \ldots, x_n \in \mathcal{F}(\mathcal{M})$ such that $(p_1, \ldots, p_d, x_{d+1}, \ldots, x_n)$ is a coordinate system. In this coordinate system we have $f(0, \ldots, 0, x_{d+1}, \ldots, x_n) = 0$. The result therefore follows from the above Taylor formula. $\square$

When $\mathcal{G}$ is an algebraic group and g a rational action, the normalized invariants $(\bar{\iota} z_1, \ldots, \bar{\iota} z_n)$ are algebraic functions and their defining ideal can be computed effectively (Hubert and Kogan, 2007b, Theorem 3.6). The method of Fels and Olver (1999) proceed through the moving frame.

2.2 Moving frames

Invariantization was first defined by Fels and Olver (1999) in terms of an $\mathcal{G}$-equivariant map $\rho : \mathcal{M} \to \mathcal{G}$ called a *moving frame* in reference to the *repère mobile* of Cartan (1935, 1937) of which they offer a new interpretation. As noted already by Griffiths (1974); Green (1978); Jensen (1977); Ivey and Landsberg (2003), the geometric idea of classical moving frames, like the Frenet frame for space curves in Euclidean geometry, can indeed be understood as maps to the group.

An action of a Lie group $\mathcal{G}$ on a manifold $\mathcal{M}$ is *locally free* if for every point $z \in \mathcal{M}$ its isotropy group $\mathcal{G}_z = \{\lambda \in \mathcal{G} \,|\, \lambda \cdot z = z\}$ is discrete. Local freeness implies semi-regularity with the dimension of each orbit being equal to the dimension of the group. Fels and Olver (1999, Theorem 4.4) established the existence of moving frames for actions with this property. It can indeed then be defined by a cross-section to the orbits.

If the action is locally free and $\mathcal{P}$ is a local cross-section on $\mathcal{M}$, then the equation

$$g(\rho(z), z) \in \mathcal{P} \text{ for } z \in \mathcal{M} \text{ and } \rho(z) = e, \forall z \in \mathcal{P} \tag{2.2}$$

uniquely defines a smooth map $\rho : \mathcal{M} \to \mathcal{G}$ in a sufficiently small neighborhood of any point of the cross-section. This map is seen to be equivariant: $\rho(\lambda \cdot z) = \rho(z) \cdot \lambda^{-1}$ for λ sufficiently close to the identity.

If $\mathcal{P}$ is the zero set of the map $P = (p_1, \ldots, p_r) : \mathcal{M} \to \mathbb{R}^r$ then $p_1(g(\rho, z)) = 0, \ldots, p_r(g(\rho, z)) = 0$ are implicit equations for the moving frame . If we can solve those, ρ provides an explicit construction for the invariantization process.

To make that explicit let us introduce the following maps.

$$\begin{array}{rlcrl} \sigma : \mathcal{M} \rightarrow & \mathcal{G} \times \mathcal{M} & \text{and} & \pi = g \circ \sigma : \mathcal{M} \rightarrow & \mathcal{M} \\ z \mapsto & (\rho(z),\, z) & & z \mapsto & g(\rho(z),\, z) \end{array} \qquad (2.3)$$

Proposition 1.16 of Hubert and Kogan (2007b) can be restated as:

Proposition 2.7 *$\bar{\iota} f = \pi^* f$, that is $\bar{\iota} f(z) = f(g(\rho(z), z))$ for all $z \in \mathcal{M}$.*

2.3 Differential invariants

We consider an action g of $\mathcal{G}$ on $\mathrm{J}^0 = \mathcal{X} \times \mathcal{U}$ and its prolongations to the jet spaces J^k. The prolongations of the infinitesimal generators on J^k are denoted $\mathrm{V}^k = (\mathrm{V}_1^k, \ldots, \mathrm{V}_r^k)$ while their prolongations to J are denoted $\mathrm{V} = (\mathrm{V}_1, \ldots, \mathrm{V}_r)$.

Definition 2.8 *A differential invariant of order k is a function f of $\mathcal{F}(\mathrm{J}^k)$ such that $\mathrm{V}_1^k(f) = 0, \ldots, \mathrm{V}_r^k(f) = 0$.*

A differential invariant of order k is thus a local invariant of the action prolonged to J^k. The ring of differential invariants of order k is accordingly denoted by $\mathcal{F}^{\mathcal{G}}(\mathrm{J}^k)$. The ring of differential invariants of any order is $\mathcal{F}^{\mathcal{G}}(\mathrm{J})$.

The maximal dimension of the orbits can only increase as the action is prolonged to higher order jets. It can not go beyond the dimension of the group though. The stabilization order is the order at which the maximal dimension of the orbits becomes stationary. If the action on J^0 is locally effective on subsets (Fels and Olver, 1999, Definition 2.2), i.e. the global isotropy group of any open set is discrete, then, for s greater than the stabilization order, the action on J^s is locally free on an open subset of J^s (Olver, 1995, Theorem 5.11). We shall make this assumption of an action that acts locally effectively on subsets. The dimension of the orbits in J^s is then r, the dimension of the group.

For any k, a cross-section to the orbits of g in J^k defines an invariantization and a set of normalized invariants on an open set of J^k. As previously we tacitly restrict to this open set though we keep the global notation J^k. Let s be equal to or bigger than the stabilization order and $\mathcal{P}^s$ a cross-section to the orbits in J^s. Its pre-image $\mathcal{P}^{s+k}$ in J^{s+k} by the projection map $\pi_s^{s+k} : \mathrm{J}^{s+k} \rightarrow \mathrm{J}^s$ is a cross-section to the orbits in J^{s+k}. It defines an invariantization $\bar{\iota} : \mathcal{F}(\mathrm{J}^{s+k}) \rightarrow \mathcal{F}^{\mathcal{G}}(\mathrm{J}^{s+k})$. The *normalized invariants of order* $s+k$ are the invariantizations of the coordinate functions on J^{s+k}. We note the set of those:

$$\mathcal{I}^{s+k} = \{\bar{\iota} x_1, \ldots, \bar{\iota} x_m\} \cup \{\bar{\iota} u_\alpha \,|\, u \in \mathcal{U},\ |\alpha| \leq s+k\}.$$

We can immediately extend Theorem 2.4 and its Corollary 2.5 to show that $\mathcal{I}^{s+k}$ is a generating set of differential invariants of order $s+k$ endowed with a trivial rewriting.

Theorem 2.9 *Let s be equal to or greater than the stabilization order and let $\mathcal{P}^s$ be a cross-section in J^s. For $f \in \mathcal{F}(\mathrm{J}^{s+k})$, $k \in \mathbb{N}$, $\bar{\iota} f$ is the unique differential invariant (of order $s+k$) whose restriction to $\mathcal{P}^{s+k}$ is equal to the restriction of f to $\mathcal{P}^{s+k}$.*

Corollary 2.10 *For $f \in \mathcal{F}(\mathrm{J}^{s+k})$, $\bar{\iota} f(x, u_\alpha) = f(\bar{\iota} x, \bar{\iota} u_\alpha)$.*

In particular, if $f \in \mathcal{F}^{\mathcal{G}}(\mathrm{J}^{s+k})$ then $\bar{\iota} f = f$ and $f(x, u_\alpha) = f(\bar{\iota} x, \bar{\iota} u_\alpha)$.

We furthermore know the functional relationships among the elements in $\mathcal{I}^{s+k}$. They are given by the functions the invariantization of which is zero. Those are essentially characterized by Proposition 2.6.

Proposition 2.11 *Let s be equal to or greater than the stabilization order. Consider the cross-section $\mathcal{P}^s$ in J^s that we assume given as the zero set of $P = (p_1, \ldots, p_r) : \mathrm{J}^s \to \mathbb{R}^r$, a map of maximal rank r along $\mathcal{P}^s$. The invariantization of $f \in \mathcal{F}(\mathrm{J}^{s+k})$, for $k \in \mathbb{N}$, is zero iff, in the neighbourhood of each point of $\mathcal{P}^{s+k}$, there exists $a_1, \ldots, a_r \in \mathcal{F}(\mathrm{J}^{s+k})$ such that $f = \sum_{i=1}^{r} a_i\, p_i$.*

Example 2.12 *We carry on with Example 1.6.*

We can choose $P = (x, u_1 - 1)$ as cross-section in J^1. This already implies that $\bar{\iota} x = 0, \bar{\iota} u_0 = u_0, \bar{\iota} u_1 = 1$. The associated moving frame $\rho : \mathrm{J}^1 \to \mathcal{G}$ is then defined by $\rho^* \lambda_1 = u_1, \rho^* \lambda_2 = -x\, u_1$ so that $\bar{\iota} u_i = \frac{u_i}{u_1^i}$ since $g^* u_i = \frac{u_i}{\lambda_1^i}$.

Example 2.13 *We consider the action of $\mathcal{G} = \mathbb{R}^*_{>0} \ltimes \mathbb{R}^2$ on $\mathrm{J}^0 = \mathcal{X}^2 \times \mathcal{U}^1$, with coordinate (x, y, u), given by:*

$$g^* x_1 = \lambda_1\, x_1 + \lambda_2, \quad g^* x_2 = \lambda_1\, x_2 + \lambda_3, \quad g^* u = u.$$

The derivations $\tilde{\mathrm{D}}_1 = \frac{1}{\lambda_1} \mathrm{D}_1$ and $\tilde{\mathrm{D}}_2 = \frac{1}{\lambda_1} \mathrm{D}_2$ allow to compute its prolongations:

$$g^* u_{ij} = \frac{u_{ij}}{\lambda_1^{i+j}}.$$

The action is locally free on $\mathrm{J}^1 \setminus \mathcal{S}$ where $\mathcal{S}$ are the points where both u_{10} and u_{01} are zero. The moving frame associated with the cross-section defined by $P = (x_1, x_2, u_{10} - 1)$ is $\rho^* \lambda_1 = u_{10}$, $\rho^* \lambda_2 = -x_1\, u_{10}$, $\rho^* \lambda_3 = -x_2\, u_{10}$. It is defined only on a proper subset of $\mathrm{J}^1 \setminus \mathcal{S}$, as are the normalized invariants: $\bar{\iota} u_{ij} = \frac{u_{ij}}{u_{10}^{i+j}}$

On the other hand, if we choose the cross-section defined by

$$P = \left(\, x_1, \; x_2, \; \frac{1}{2} - \frac{1}{2}(u_{10}^2 + u_{01}^2) \right)$$

the associated moving frame is well defined on the whole of $\mathrm{J}^1 \setminus \mathcal{S}$:

$$\rho^*\lambda_1 = \sqrt{u_{10}^2 + u_{01}^2}, \quad \rho^*\lambda_2 = -x_1\sqrt{u_{10}^2 + u_{01}^2}, \quad \rho^*\lambda_3 = -x_2\sqrt{u_{10}^2 + u_{01}^2}.$$

as are the normalized invariants:

$$\bar{\iota}x_1 = 0, \quad \bar{\iota}x_2 = 0, \text{ and } \bar{\iota}u_{ij} = \frac{u_{ij}}{(u_{10}^2 + u_{01}^2)^{\frac{i+j}{2}}}.$$

This shows that a nonlinear cross-section might have some desirable properties.

3 Invariant derivations

An invariant derivation is a total derivation that commutes with the infinitesimal generators. It maps differential invariants of order k to differential invariant of order $k+1$, for k large enough. Classically a basis of commuting invariant derivations is constructed with the use of sufficiently many differential invariants (Olver, 1995; Ovsiannikov, 1982; Kumpera, 1974, 1975a,b; Muñoz et al., 2003). The novel construction proposed by Fels and Olver (1999) is based on a moving frame. The constructed invariant derivations do not commute in general. Their principal benefit is that they bring an explicit formula for the derivation of normalized invariants. This has been known as the *recurrence formulae* (Fels and Olver, 1999, Section 13). They are the key to most results about generation and syzygies in this paper. All the algebraic and algorithmic treatments of differential invariants and their applications (Mansfield, 2001; Olver, 2007a; Hubert and Olver, 2007; Hubert, 2008) come as an exploitation of those formulae.

In Theorem 3.6 we present the derivation formulae for any invariantized functions. For the proof we take the dual approach of the one of Fels and Olver (1999) which is therefore close in essence to the one presented by Mansfield (2008), based on the application of the chain rule.

We always consider the action g of a connected r-dimensional Lie group $\mathcal{G}$ on $\mathrm{J}^0 = \mathcal{X} \times \mathcal{U}$ and its prolongations. We make use of a basis of right invariant vector fields $\mathrm{v} = (\mathrm{v}_1, \ldots, \mathrm{v}_r)$ on $\mathcal{G}$, and the associated infinitesimal generators:

- $\mathbf{V} = (\mathbf{V}_1, \ldots, \mathbf{V}_r)^T$ is the vector of infinitesimal generators for the action g of $\mathcal{G}$ on J
- $\hat{\mathrm{v}} = (\hat{\mathrm{v}}_1, \ldots, \hat{\mathrm{v}}_r)^T$ is the vector of infinitesimal generators for the action of $\mathcal{G}$ on itself by right multiplication.

Recall from Section 1.2 that total derivations are the derivations on J that belong to the $\mathcal{F}(\mathrm{J})$-module with basis $(\mathrm{D}_1, \ldots, \mathrm{D}_m)$, the total derivations with respect to the independent variables $x_1, \ldots, x_m$.

Definition 3.1 *An invariant derivation $\mathcal{D}$ is a total derivation that commutes with any infinitesimal generator V of the group action: $\mathcal{D} \circ \mathrm{V} = \mathrm{V} \circ \mathcal{D}$.*

As an immediate consequence of this definition we see that if f is a differential invariant and $\mathcal{D}$ an invariant derivation then $\mathcal{D}(f)$ is an differential invariant.

Proposition 3.2 *Let $A = (a_{ij})$ be an invertible $m \times m$ matrix with entries in $\mathcal{F}(\mathrm{J})$. A vector of total derivations $\mathcal{D} = (\mathcal{D}_1, \ldots, \mathcal{D}_m)^T$ defined by $\mathcal{D} = A^{-1}\,\mathrm{D}$ is a vector of invariant derivations if and only if, for all infinitesimal generator V of the action,*

$$\mathrm{V}(a_{ij}) + \sum_{k=1}^{m} \mathrm{D}_i(\xi_k)\, a_{kj} = 0, \qquad \textit{where } \xi_k = \mathrm{V}(x_k), \quad 1 \leq i, j \leq m.$$

PROOF: For all i we have $\mathrm{D}_i = \sum_{j=1}^{m} a_{ij}\, \mathcal{D}_j$. By expanding the equality $[\mathrm{D}_i, \mathrm{V}] = \sum_{k=1}^{m} \mathrm{D}_i(\xi_k)\, \mathrm{D}_k$ (Proposition 1.5) we obtain, for all i,

$$\sum_{j=1}^{m} a_{ij}\, [\mathcal{D}_j, V] = \sum_{j=1}^{m} \left(V(a_{ij}) + \sum_{k=1}^{m} \mathrm{D}_i(\xi_k) a_{kj} \right) \mathcal{D}_j$$

Since A is of non-zero determinant $[\mathcal{D}_j, V] = 0$ for all j if and only if $V(a_{ij}) + \sum_{k=1}^{m} \mathrm{D}_i(\xi_k) a_{kj} = 0, \forall i, j$. □

As illustration, a classical construction of invariant derivations is given by the following proposition (Kumpera, 1974, 1975a,b; Olver, 1995; Ovsiannikov, 1982; Muñoz et al., 2003):

Proposition 3.3 *If $f_1, \ldots, f_m$ are differential invariants such that the matrix $A = (\mathrm{D}_i(f_j))_{i,j}$ is invertible then the derivations $\mathcal{D} = A^{-1}\mathrm{D}$ are invariant derivations.*

PROOF: If $a_{ij} = \mathrm{D}_i(f_j)$ then, by Proposition 1.5,

$$V(a_{ij}) = V(\mathrm{D}_j(f_i)) = \mathrm{D}_j(V(f_i)) - \sum_k \mathrm{D}_j(\xi_k)\, \mathrm{D}_k(f_i) = \mathrm{D}_j(V(f_i)) - \sum_k \mathrm{D}_j(\xi_k)\, a_{ik}.$$

By hypothesis $V(f_i) = 0$ so that the result follows from Proposition 3.2. □

The above derivations commute. They can be understood as derivations with respect to the new *independent variables* $f_1, \ldots, f_m$.

As a side remark, note that Definition 3.1 is dual to the infinitesimal condition for a 1-form to be contact invariant (Olver, 1995, Theorem 2.91). The invariant derivations of Proposition 3.3 are dual to the contact invariant 1-forms $\mathrm{d}_H f_1, \ldots, \mathrm{d}_H f_m$.

3.2 Moving frame construction of invariant derivations

Assume that there exists on J^s a moving frame $\rho : \mathrm{J}^s \to \mathcal{G}$. As in Section 2 we construct the additional maps

$$\begin{array}{rcl} \sigma : \mathrm{J}^{s+k} & \to & \mathcal{G} \times \mathrm{J}^{s+k} \\ z & \mapsto & (\rho(z),\, z) \end{array} \quad \text{and} \quad \begin{array}{rcl} \pi = g \circ \sigma : \mathrm{J}^{s+k} & \to & \mathrm{J}^{s+k} \\ z & \mapsto & g(\rho(z),\, z) \end{array} \tag{3.1}$$

Theorem 3.4 *The vector of derivations $\mathcal{D} = (\sigma^* A)^{-1}\,\mathrm{D}$, where A is the $m \times m$ matrix $\left(\mathrm{D}_i(g^* x_j)\right)_{ij}$, is a vector of invariant derivations.*

The matrix A has entries in $\mathcal{F}(\mathcal{G} \times \mathrm{J}^1)$. Its pull back $\sigma^* A$ has entries in $\mathcal{F}(\mathrm{J}^s)$. The above result is proved by checking that the formula of Proposition 1.5 holds.

PROOF: The equivariance of ρ implies $\rho(g(e^{t\mathrm{v}}, z)) = \rho(z)\cdot e^{-t\mathrm{v}}$ so that $\rho_* \mathrm{V} = \hat{\mathrm{v}}$. Thus $\sigma_* V = \hat{\mathrm{v}} + \mathrm{V}$ that is $\sigma_* V(a_{ij}) = \hat{\mathrm{v}}(\mathrm{D}_i(g^* x_j)) + V(\mathrm{D}_i(g^* x_j))$. As derivations on $\mathcal{F}(\mathcal{G} \times \mathrm{J}^s)$, D_i and $\hat{\mathrm{v}}$ commute while the commutator of D_i and V is given by Proposition 1.5. It follows that $\sigma_* V(a_{ij}) = \mathrm{D}_i(\hat{\mathrm{v}}(g^* x_j)) + \mathrm{D}_i(\mathrm{V}(g^* x_j)) - \sum_{k=1}^m \mathrm{D}_i(\xi_k)\,\mathrm{D}_k(g^* x_j)$. By Proposition 1.1 the two first terms cancel and since $V(\sigma^* a_{ij}) = \sigma^*(\sigma_* V)(a_{ij})$ we have $V(\sigma^* a_{ij}) = -\sum_{k=1}^m \mathrm{D}_i(\xi_k)\,\sigma^* a_{kj}$. We can conclude with Proposition 3.2. □

Example 3.5 *We carry on with Example 1.6 and 2.12.*

We found that the equivariant map associated to $P = (x, u_1 - 1)$ is given by $\rho^* \lambda_1 = u_1, \rho^* \lambda_2 = -x\, u_1$. In addition $\tilde{\mathrm{D}} = \frac{1}{\lambda_1}\mathrm{D}$ while $\mathrm{V}_1 = x\,\frac{\partial}{\partial u} - \sum_{k \geq 0} k\, u_k \frac{\partial}{\partial u_k}$ and $\mathrm{V}_2 = \frac{\partial}{\partial x}$.

Accordingly define $\mathcal{D} = \frac{1}{u_1}\mathrm{D}$. We can then verify that $[\mathrm{V}_1, \mathcal{D}] = 0$ and $[\mathrm{V}_2, \mathcal{D}] = 0$. The application of $\mathcal{D}$ to a differential invariant thus produces a differential invariant. For instance

$$\mathcal{D}\left(\frac{u_i}{u_1^i}\right) = \frac{u_{i+1}}{u_1^{i+1}} - \frac{u_i}{u_1^{i+2}} u_2 = \frac{u_{i+1}}{u_1^{i+1}} - \frac{u_i}{u_1^i}\frac{u_2}{u_1^2}.$$

Remembering that $\bar{\iota} u_i = \frac{u_i}{u_1^i}$ we can observe that $\mathcal{D}(\bar{\iota} u_i) = \bar{\iota} u_{i+1} - \bar{\iota} u_2\, \bar{\iota} u_i$. This shows that $\mathcal{D}(\bar{\iota} u_i) \neq \bar{\iota} u_{i+1}$ in general. The relationship between these two quantities is the subject of Theorem 3.6 below. We shall furthermore observe that nonetheless $\mathcal{D}(\frac{u_i}{u_1^i}) = \bar{\iota}(\mathrm{D}(\frac{u_i}{u_1^i}))$ (Corollary 3.7).

An essential property of the invariant derivations of Theorem 3.4 is that we can write explicitly their action on the invariantized functions. Theorem 3.6 below is a general form for the recurrence formulae of Fels and Olver (1999, Equation 13.7).

Assume that the action of g on J^s is locally free and that $P = (p_1, \ldots, p_r)$ defines the cross-section $\mathcal{P}$. Let $\rho : \mathrm{J}^s \to \mathcal{G}$ be the associated moving frame. We construct the vector of invariant derivations $\mathcal{D} = (\mathcal{D}_1, \ldots, \mathcal{D}_m)$ as in Theorem 3.4.

Denote by $\mathrm{D}(P)$ the $m \times r$ matrix $\left(\mathrm{D}_i(p_j)\right)_{i,j}$ with entries in $\mathcal{F}(\mathrm{J}^{s+1})$ while $\mathrm{V}(P)$ is the $r \times r$ matrix $\left(\mathrm{V}_i(p_j)\right)_{i,j}$ with entries in $\mathcal{F}(\mathrm{J}^s)$. As $\mathcal{P}$ is transverse to the orbits of the action of $\mathcal{G}$ on J^s, the matrix $\mathrm{V}(P)$ has non zero determinant along $\mathcal{P}$ and therefore in a neighborhood of each of its points.

Theorem 3.6 *Let $P = (p_1, \ldots, p_r)$ define a cross-section $\mathcal{P}$ to the orbits in J^s, where s is equal to or greater than the stabilization order. Consider $\rho : \mathrm{J}^s \to \mathcal{G}$ the associated moving frame and $\bar\iota : \mathcal{F}(\mathrm{J}) \to \mathcal{F}^{\mathcal{G}}(\mathrm{J})$ the associated invariantization. Consider $\mathcal{D} = (\mathcal{D}_1, \ldots, \mathcal{D}_m)^T$ the vector of invariant derivations constructed in Theorem 3.4. Let K be the $m \times r$ matrix obtained by invariantizing the entries of $\mathrm{D}(P)\,\mathrm{V}(P)^{-1}$. Then*

$$\mathcal{D}(\bar\iota f) = \bar\iota(\mathrm{D} f) - K\,\bar\iota(\mathrm{V}(f)).$$

PROOF: From the definition of $\sigma : z \mapsto (\,\rho(z), z\,)$ and the chain rule we have

$$\mathcal{D}(\bar\iota f)(z) = \mathcal{D}(\sigma^* g^* f)(z) = \mathcal{D}(g^* f)(\rho(z), z) + (\rho_* \mathcal{D})(g^* f)(\rho(z), z). \tag{3.2}$$

Recall the definition of $\tilde{\mathrm{D}}$ in Section 1.2 that satisfies $\tilde{\mathrm{D}}_j(g^* f) = g^*(\mathrm{D}_j f)$ for all $f \in \mathcal{F}(\mathrm{J})$. We have $\mathcal{D}(g^* f)(\rho(z), z) = (\sigma^* \tilde{\mathrm{D}}(g^* f))(z) = \sigma^* g^* (\mathrm{D} f)(z) = \bar\iota(\mathrm{D} f)(z)$ and (3.2) becomes

$$\mathcal{D}(\bar\iota f)(z) = \bar\iota(\mathrm{D} f)(z) + \sigma^*(\rho_* \mathcal{D})(g^* f)(z). \tag{3.3}$$

Since $\hat{\mathrm{v}} = (\hat{\mathrm{v}}_1, \ldots, \hat{\mathrm{v}}_r)$ form a basis for the derivations on $\mathcal{G}$ there is a matrix[1] $\tilde{K}$ with entries in $\mathcal{F}(\mathcal{G} \times \mathrm{J}^s)$ such that $\rho_* \mathcal{D} = \tilde{K}\,\hat{\mathrm{v}}$.

We can write (3.3) as $\mathcal{D}(\bar\iota f)(z) = \bar\iota(\mathrm{D} f)(z) + \sigma^* \left(\tilde{K} \hat{\mathrm{v}}(g^* f)\right)(z)$ so that, by

[1] With $\mathcal{D}$ known explicitly, we can write $\tilde{K}$ explicitly in terms of coordinates $\lambda = (\lambda_1, \ldots, \lambda_r)$. $\tilde{K}$ is the matrix obtained by multiplying the matrix $\mathcal{D}(\rho) = \left(\mathcal{D}_j(\rho^* \lambda_i)\right)$ with the inverse of $\hat{\mathrm{v}}(\lambda) = (\hat{\mathrm{v}}_i(\lambda_j))$. Yet $\sigma^* \tilde{K}$ needs not have differential invariants as entries and we shall seek $\bar\iota(\sigma^* \tilde{K})$ in a more direct way. See Example 3.9.

Proposition 1.1,

$$\mathcal{D}(\bar{\iota}f)(z) = \bar{\iota}(\mathrm{D}f)(z) - \sigma^*\left(\tilde{K}\mathrm{V}(g^*f)\right)(z). \tag{3.4}$$

This latter equation shows that $\sigma^*\left(\tilde{K}\mathrm{V}(g^*f)\right) = \bar{\iota}(\mathrm{D}f) - \mathcal{D}(\bar{\iota}f)$ is a differential invariant. As such it is equal to its invariantization and thus

$$\sigma^*\left(\tilde{K}\mathrm{V}(g^*f)\right) = \bar{\iota}(\sigma^*\tilde{K})\,\bar{\iota}(\sigma^*\mathrm{V}(g^*f)).$$

For all $z \in \mathcal{P}$, $\rho(z) = e$ and therefore $\sigma^*\mathrm{V}(g^*f)$ and $\mathrm{V}(f)$ agree on $\mathcal{P}$: for all $z \in \mathcal{P}$, $\sigma^*\mathrm{V}(g^*f)\,(z) = \mathrm{V}(g^*f)\,(e, z) = \mathrm{V}(f)(z)$ by Proposition 1.1. It follows that $\bar{\iota}(\sigma^*\mathrm{V}(g^*f)) = \bar{\iota}(\mathrm{V}(f))$ so that (3.4) becomes

$$\mathcal{D}(\bar{\iota}f)(z) = \bar{\iota}(\mathrm{D}f)(z) - \bar{\iota}(\sigma^*\tilde{K})\,\bar{\iota}(\mathrm{V}(f)). \tag{3.5}$$

To find the matrix $K = \bar{\iota}(\sigma^*\tilde{K})$ we use the fact that $\bar{\iota}p_i = 0$ for all $1 \leq i \leq r$. Applying $\mathcal{D}$ and (3.5) to this equality we obtain: $\bar{\iota}(\mathrm{D}p_i) = K\,\bar{\iota}(\mathrm{V}(p_i))$ so that $\bar{\iota}(\mathrm{D}(P)) = K\,\bar{\iota}(\mathrm{V}(P))$. The transversality of $\mathcal{P}$ imposes that $\mathrm{V}(P)$ is invertible along $\mathcal{P}$, and thus so is $\bar{\iota}(\mathrm{V}(P))$.

We thus have proved that $\mathcal{D}(\bar{\iota}f) = \bar{\iota}(\mathrm{D}f) - K\,\bar{\iota}(\mathrm{V}(f))$ where $K = \bar{\iota}(\sigma^*\tilde{K}) = \bar{\iota}(\mathrm{D}(P)V(P)^{-1})$. □

If f is a differential invariant, $\mathcal{D}(f)$ is also a differential invariant, while $\mathrm{D}(f)$ need not be. But if we invariantize this latter though we find nothing else than $\mathcal{D}(f)$. This follows immediately from the above way of writing the *recurrence formulae* yet we have not seen the following corollary in previous papers on the subject.

Corollary 3.7 *If f is a differential invariant then $\mathcal{D}(f) = \bar{\iota}(\mathrm{D}(f))$.*

PROOF: If f is a differential invariant then $\bar{\iota}f = f$ and $\mathrm{V}(f) = 0$. The result thus follows from the above theorem. □

By deriving a recurrence formula for forms, (Fels and Olver, 1999, Section 13) derived explicitly the commutators of the invariant derivations . It can actually be derived directly from Theorem 3.6 through the use of *formal invariant derivations* (Hubert, 2008).

Proposition 3.8 *For all $1 \leq i, j \leq m$, $\left[\mathcal{D}_i, \mathcal{D}_j\right] = \sum_{k=1}^{m} \Lambda_{ijk}\,\mathcal{D}_k$ where*

$$\Lambda_{ijk} = \sum_{c=1}^{r} K_{ic}\,\bar{\iota}(\mathrm{D}_j(\xi_{ck})) - K_{jc}\,\bar{\iota}(\mathrm{D}_i(\xi_{ck})) \;\in\; \mathcal{F}^{\mathcal{G}}(\mathrm{J}^{s+1}),$$

$K = \bar{\iota}\,(D(P)\,V(P)^{-1})$, and $\xi_{ck} = \mathrm{V}_c(x_k)$.

Example 3.9 *We carry on with Example 1.6, 2.12, and 3.5.*
We chose $P = (x, u_1 - 1)$ *and showed that* $\mathcal{D} = \frac{1}{u_1}\mathrm{D}$ *while* $\bar{\iota}u_i = \frac{u_i}{u_1^i}$. *We computed*

$$\mathcal{D}(\bar{\iota}u_i) = \frac{u_{i+1}}{u_1^{i+1}} - i\,\frac{u_i}{u_1^i}\frac{u_2}{u_1^2} = \bar{\iota}u_{i+1} - i\,\bar{\iota}u_2\,\bar{\iota}u_i.$$

We have $\mathrm{D}(P) = \begin{pmatrix} 1 & u_2 \end{pmatrix}$ *and* $\mathrm{V}(P) = \begin{pmatrix} x & -u_1 \\ 1 & 0 \end{pmatrix}$. *The matrix K of Theorem 3.6 is thus* $K = \bar{\iota}\left(D(P)\,V(P)^{-1}\right) = \begin{pmatrix} -\bar{\iota}u_2 & 1 \end{pmatrix}$ *and the formula is verified:*

$$\mathcal{D}(\bar{\iota}u_i) = \bar{\iota}u_{i+1} - \begin{pmatrix} -\bar{\iota}u_2 & 1 \end{pmatrix}\begin{pmatrix} \bar{\iota}\mathrm{V}_1(u_i) \\ \bar{\iota}\mathrm{V}_2(u_i) \end{pmatrix} \quad \text{since} \quad \bar{\iota}\mathrm{V}(u_i) = \begin{pmatrix} -i\,u_i & 0 \end{pmatrix}^T.$$

What we shall do next is illustrate the proof by exhibiting the matrix $\tilde{K}$ that arises there. It is defined by $\rho_*\mathcal{D} = \tilde{K}\,\hat{\mathrm{v}}$ *and the fact that* $\sigma^*\tilde{K}\mathrm{V}(g^*f)$ *is an invariant for any* $f \in \mathcal{F}(\mathrm{J})$.
We have $\hat{\mathrm{v}}_1 = -\lambda_1\frac{\partial}{\partial\lambda_1}$, $\hat{\mathrm{v}}_2 = -\lambda_1\frac{\partial}{\partial\lambda_2}$ *and saw that* $\rho^*\lambda_1 = u_1$ *and* $\rho^*\lambda_2 = -xu_1$. *Thus*

$$\rho_*\mathcal{D} = \begin{pmatrix} \mathcal{D}(\rho^*\lambda_1) & \mathcal{D}(\rho^*\lambda_2) \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial\lambda_1} \\ \frac{\partial}{\partial\lambda_2} \end{pmatrix} = \begin{pmatrix} -\frac{u_2}{u_1}\frac{1}{\lambda_1} & \frac{u_1+x\,u_2}{u_1}\frac{1}{\lambda_1} \end{pmatrix}\begin{pmatrix} \hat{\mathrm{v}}_1 \\ \hat{\mathrm{v}}_2 \end{pmatrix}.$$

So here $\sigma^*\tilde{K} = \left(-\frac{u_2}{u_1^2}, \frac{u_1+xu_2}{u_1^2}\right)$. *We indeed have that* $\bar{\iota}\sigma^*\tilde{K} = K$ *as used in the proof. We verify here that* $\sigma^*\left(\tilde{K}\mathrm{V}(g^*f)\right)$ *is a vector of differential invariants. We have*

$$\mathrm{V}(g^*x) = \begin{pmatrix} \lambda_1\,x \\ \lambda_1 \end{pmatrix}, \; \mathrm{V}(g^*u_i) = \begin{pmatrix} -i\,\frac{u_i}{\lambda_1^i} \\ 0 \end{pmatrix}$$

so that $\sigma^*\tilde{K}\mathrm{V}(g^*x) = 1$ *and* $\sigma^*\tilde{K}\mathrm{V}(g^*u_i) = i\,\frac{u_2}{u_1^2}\,\frac{u_1}{u_1^i} = i\,\bar{\iota}u_2\,\bar{\iota}u_i$.

Example 3.10 *We carry on with Example 2.13.*
We chose

$$P = \left(x_1, x_2, \frac{1}{2} - \frac{1}{2}(u_{10}^2 + u_{01}^2)\right).$$

On one hand the prolongations of the infinitesimal generators to J *are*

$$\mathrm{V}_1 = \frac{\partial}{\partial x_1}, \quad \mathrm{V}_2 = \frac{\partial}{\partial x_2}, \quad \mathrm{V}_3 = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} - \sum_{i,j\geq 0}(i+j)\,u_{ij}\frac{\partial}{\partial u_{ij}}$$

so that

$$\mathrm{V}(P) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_1 & x_2 & u_{10}^2 + u_{01}^2 \end{pmatrix} \text{ while } \mathrm{D}(P) = \begin{pmatrix} 1 & 0 & v \\ 0 & 1 & w \end{pmatrix}$$

where

$$v = -(u_{10}u_{20} + u_{01}u_{11}) \text{ and } w = -(u_{10}u_{11} + u_{01}u_{02}).$$

Since $\bar{\iota}x_1 = 0$, $\bar{\iota}x_2 = 0$ *and* $\bar{\iota}(u_{10}^2 + u_{01}^2) = 1$, $\bar{\iota}\mathrm{V}(P)$ *is the identity matrix so that*

$$K = \bar{\iota}(\mathrm{D}(P)\mathrm{V}(P)^{-1}) = \begin{pmatrix} 1 & 0 & \bar{\iota}v \\ 0 & 1 & \bar{\iota}w \end{pmatrix}.$$

On the other hand the normalized invariants and invariant derivations are

$$\bar{\iota}u_{ij} = \frac{u_{ij}}{(u_{10}^2 + u_{01}^2)^{\frac{i+j}{2}}}, \ \forall i, j; \qquad \mathcal{D}_i = \frac{1}{\sqrt{u_{10}^2 + u_{01}^2}}\mathrm{D}_i, \ i = 1, 2.$$

We can thus check that

$$\begin{pmatrix} \mathcal{D}_1(\bar{\iota}u_{ij}) \\ \mathcal{D}_2(\bar{\iota}u_{ij}) \end{pmatrix} = \begin{pmatrix} \bar{\iota}(u_{i+1,j}) \\ \bar{\iota}(u_{i,j+1}) \end{pmatrix} - K \begin{pmatrix} 0 \\ 0 \\ -(i+j)\,\bar{\iota}u_{ij} \end{pmatrix},$$

as predicted by Theorem 3.6, and that $[\mathcal{D}_2, \mathcal{D}_1] = \bar{\iota}w\,\mathcal{D}_1 - \bar{\iota}v\,\mathcal{D}_2$, *as predicted by Proposition 3.8.*

4 Finite generation and rewriting

The recurrence formulae, Theorem 3.6, together with the replacement theorem, Theorem 2.10, show that any differential invariant can be written in terms of the normalized invariants of order $s+1$, where s is the order of the moving frame, and their invariant derivatives. The rewriting is effective.

In the case of a cross-section of minimal order, we exhibit another generating set of differential invariants with bounded cardinality. This bound is mr in the case of an action transitive on J^0. When in addition we choose a coordinate cross-section, this set consists of normalized invariants and we retrieve the result of Olver (2007b). This was incorrectly stated for any cross-section by Fels and Olver (1999, Theorem 13.3).

4.1 Rewriting in terms of normalized invariants of order $s+1$

Let s be equal to or greater then the stabilization order and let $\mathcal{P}$ be a cross-section to the orbits in J^s defined by $P = (p_1, \ldots, p_r)$ with $p_i \in \mathcal{F}(\mathrm{J}^s)$. Recall from Section 2.3 that

$$\mathcal{I}^{s+k} = \{\bar{\iota}x_1, \ldots, \bar{\iota}x_m\} \cup \{\bar{\iota}u_\alpha \,|\, u \in \mathcal{U},\ |\alpha| \leq s+k\},$$

where $\bar{\iota} : \mathcal{F}(\mathrm{J}^{s+k}) \to \mathcal{F}^{\mathcal{G}}(\mathrm{J}^{s+k})$ is the invariantization associated to $\mathcal{P}$, forms a generating set of local invariants for the action of g on J^{s+k}. Those invariants have additional very desirable properties: we can trivially rewrite any differential invariants of order $s+k$ in terms of them. Yet it is even more desirable to describe the differential invariants of all order in finite terms.

Theorem 3.6 implies in particular that

$$\bar{\iota}(\mathrm{D}_i u_\alpha) = \mathcal{D}_i(\bar{\iota}u_\alpha) + \sum_{a=1}^{r} K_{ia}\,\bar{\iota}(\mathrm{V}_a(u_\alpha))$$

where $K = \bar{\iota}(\mathrm{D}(P)\mathrm{V}(P)^{-1})$ has entries that are function of $\mathcal{I}^{s+1}$. It is then an easy inductive argument to show that any $\bar{\iota}u_\alpha$ can be written as a function of $\mathcal{I}^{s+1}$ and their derivatives of order $\max(0, |\alpha| - s - 1)$. Combining with the replacement property, Theorem 2.10, we have a constructive way of rewriting any differential invariants in terms of the elements of $\mathcal{I}^{s+1}$ and their derivatives: A differential invariant of order k is first trivially rewritten in terms of $\mathcal{I}^k$ by Theorem 2.10. If $k \leq s+1$ we are done. Otherwise, any element $\bar{\iota}u_\alpha$ of $\mathcal{I}^k$ with $|\alpha| = k$ is a $\bar{\iota}(\mathrm{D}_i u_\beta)$, for some $1 \leq i \leq m$ and $|\beta| = k-1$. We can thus write it as:

$$\bar{\iota}u_\alpha = \bar{\iota}(\mathrm{D}_i u_\beta) = \mathcal{D}_i(\bar{\iota}u_\beta) + \sum_a K_{ia}\,\bar{\iota}\left(\mathrm{V}_a(u_\beta)\right).$$

This involves only elements of $\mathcal{I}^{k-1}$ and their derivatives. Carrying on recursively we can rewrite everything in terms of the elements of $\mathcal{I}^{s+1}$ and their derivatives.

This leads to the following result that will be refined in Section 5. Indeed the rewriting is not unique: at each step there might be several choices of pairs (i, β) such that $u_\alpha = \mathrm{D}_i u_\beta$.

Theorem 4.1 *Any differential invariant of order $s+k$ can be written in terms of the elements of $\mathcal{I}^{s+1}$ and their derivatives of order $k-1$ and less.*

4.2 Case of minimal order cross-section

A natural question is to determine a smaller set of differential invariants that is generating. Olver (2007b) proved that when choosing a coordinate cross-section of *minimal order* the normalized invariants corresponding to the derivatives of the coordinates functions which are set to constant form a generating set of differential invariants. Here we generalize the result to noncoordinate cross-sections. The proof is based on the same idea.

Let s be equal to or greater than the stabilization order. A local cross-section $\mathcal{P}$ in J^s is of *minimal order* if its projection on J^k, for all $k \leq s$, is a local cross-section to the orbits of the action of g on J^k (Olver, 2007b). Assume $P = (p_1, \ldots, p_r)$ defines a cross-section $\mathcal{P}$ of minimal order. Without loss of generality we can assume that $P_k = (p_1, \ldots, p_{r_k})$ where r_k is the dimension of the orbits of the action of g on J^k, defines the projection of $\mathcal{P}$ on J^k.

Theorem 4.2 *If $P = (p_1, \ldots, p_r)$ defines a cross-section for the action of g on J such that $P_k = (p_1, \ldots, p_{r_k})$ defines a cross-section for the action of g on J^k, for all k, then $\mathcal{E} = \{\bar{\iota}(\mathrm{D}_i(p_j)) \,|\, 1 \leq i \leq m,\ 1 \leq j \leq r\}$ together with $\mathcal{I}^0$ form a generating set of differential invariants.*

PROOF: The minimal order condition imposes that the $r \times r_k$ matrix $\mathrm{V}(P_k)$ has maximal rank r_k on $\mathcal{P}$, and therefore on an open neighborhood of each point of $\mathcal{P}$. As V^k has rank r_k, for any f in $\mathcal{F}(\mathrm{J}^k)$, $\mathrm{V}(f)$ is linearly dependent on $\mathrm{V}(p_1), \ldots, \mathrm{V}(p_{r_k})$. In a neighborhood of each point of $\mathcal{P}^k$ there is thus a relation

$$\mathrm{V}(f) = \sum_{i=1}^{r_k} a_i \, \mathrm{V}(p_i), \text{ where } a_i \in \mathcal{F}(\mathrm{J}^k).$$

On one hand, by Theorem 3.6, we have $\bar{\iota}(\mathrm{D}f) = \mathcal{D}(\bar{\iota}f) + K\,\bar{\iota}(V(f))$ so that $\bar{\iota}(\mathrm{D}f) = \mathcal{D}(\bar{\iota}f) + \sum_{i=1}^{r_k} \bar{\iota}(a_i)\, K\,\bar{\iota}(\mathrm{V}(p_i))$. On the other hand $\bar{\iota}(p_i) = 0$ so that $\bar{\iota}(\mathrm{D}p_i) = K\bar{\iota}(\mathrm{V}(p_i))$. It follows that

$$\bar{\iota}(\mathrm{D}f) = \mathcal{D}(\bar{\iota}f) + \sum_{i=1}^{r_k} \bar{\iota}(a_i)\, \bar{\iota}(\mathrm{D}p_i).$$

Note that $\bar{\iota}(a_i)$ can be written in terms of the $\bar{\iota}(u_\beta)$ with $|\beta| \leq k$. So the formula implies that any $\bar{\iota}u_\alpha$, with $|\alpha| = k + 1$, can be written in terms of $\{\, \bar{\iota}(\mathrm{D}p_i) \mid 1 \leq i \leq r_k \,\}$ and $\{\, \bar{\iota}(u_\beta) \mid |\beta| \leq k \,\}$ together with their derivatives with respect to the invariant derivations $\mathcal{D}$. By induction, it follows that any $\bar{\iota}u_\alpha$ can be written in terms of the zero-th order normalized invariants together with the elements of $\mathcal{E}$ and their derivatives. $\square$

In the case of a coordinate cross-section $\mathcal{E}$ is a subset of the normalized invariants $\mathcal{I}^{s+1}$ that Olver (2007b) named the *edge invariants* for the representation of the derivatives of a dependent function on a lattice. We shall extend this name in the case of non coordinate cross-section though the pictorial representation is no longer valid.

Minimality is necessary for the edge invariants to be generating in general. Olver (2007a) exhibits a choice of non minimal (coordinate) cross-section for which the edge invariants are not generating. We review this example in Section 6.2.

A consequence of Theorem 4.2 is that we can bound the number of differential invariants necessary to form a generating set. The bound is $m\,r + d_0$, where $d_0 = m + n - r_0$ is the codimension of the orbits of the action of g on J^0. Transitive actions on J^0 are of particular interest. There $d_0 = 0$ and the bound is simply $m\,r$. Hubert (2007a) exhibits a generating set of such cardinality even in the case of non minimal cross-section.

Example 4.3 *Consider Example 3.10 again. The chosen cross-section is of minimal order. Specializing Theorem 3.6 we obtained*

$$\begin{pmatrix} \bar\iota(u_{i+1,j}) \\ \bar\iota(u_{i,j+1}) \end{pmatrix} = \begin{pmatrix} \mathcal{D}_1(\bar\iota u_{ij}) \\ \mathcal{D}_2(\bar\iota u_{ij}) \end{pmatrix} - (i+j)\,\bar\iota u_{ij} \begin{pmatrix} \bar\iota v \\ \bar\iota w \end{pmatrix}$$

from which it is clear that all the normalized invariants can be inductively written in terms of $\bar\iota u_{00}, \bar\iota v$ and $\bar\iota w$, i.e the non constant elements of $\mathcal{I}^0 \cup \mathcal{E}$, and their derivatives.

5 Syzygies

Loosely speaking, a *differential syzygy* is a relationship among a (generating) set of differential invariants and their derivatives. A set of differential syzygies is complete if any other syzygies is inferred by those and their derivatives. In this section we formalize a definition of syzygies by introducing the appropriate differential algebra. We then show the completeness of a finite set of differential syzygies on the normalized invariants of order $s+1$.

Fels and Olver (1999, Theorem 13.2) claimed a complete set of syzygies for edge invariants, in the case of coordinate cross-section. It has so far remained unproven [2]. As we are finishing this paper Olver and Pohjanpelto (2007) announce a syzygy theorem for pseudo-groups. The *symbol module* of the infinitesimal determining system takes there a prominent place: on one hand it dictates the coordinate cross-section to be used and, on the other hand, its (algebraic) syzygies prescribe the syzygies on the differential invariants. Let us note here two immediate advantages of our result for Lie group actions: we do not need to have any side algebraic computations (over a ring of functions) nor are we restricted in our choice of cross-section. In particular we are neither restricted to minimal order nor coordinate cross-section. Even if those latter are often the best choice, there are needs for more options. Such is the case in the symmetry reduction considered by Mansfield (2001). Also in Example 2.13

[2] An necessary amendment of the statement is that K might be taken as the empty set in (iii).

the nonlinear cross-section is defined for the whole open set where the action is regular, while a linear cross-section is only defined for a subset.

The commutation rules, Theorem 3.8, imply infinitely many relationships on derivatives of normalized invariants. Fels and Olver (1999), as well as Olver and Pohjanpelto (2007), considered those as syzygies. Our approach is in the line of Hubert (2005b). We encapsulate those relationships in a recursive definition of the derivations to work exclusively with *monotone derivatives*. The differential algebra of monotone derivatives that arises there is a generalization of the differential polynomial rings considered by Ritt (1950) and Kolchin (1973) to model nonlinear differential equations. Of great importance is the fact that it is endowed with a proper differential elimination theory (Hubert, 2005b). This generalization is effective and has been implemented (Hubert, 2005a).

Refining the discussion of Section 4, we first observe that any differential invariant can be written in terms of the monotone derivatives of the normalized invariants of order $s+1$. The rewriting is nonetheless not unique in general. The syzygies can be understood as the relationships among the monotone derivatives that govern this indeterminacy.

For the normalized invariants of order $s+1$ we introduce the concept of *normal derivatives*. They provide a canonical rewriting of any differential invariant. The set of differential relationships that allows one to rewrite any monotone derivative in terms of normal derivatives is then shown to be a complete set of syzygies for the normalized invariants of order $s+1$ (Theorem 5.14).

To prove these results we formalize the notion of syzygies by introducing the algebra of monotone derivatives. We endow this algebra with derivations so as to have a differential morphism onto the algebra of differential invariants. The syzygies are the elements of the kernel of this morphism. It is a differential ideal and Theorem 5.14 actually exhibits a set of generators.

5.1 Monotone and normal derivatives

In Section 4 we showed that any differential invariant can be written in terms of $\mathcal{I}^{s+1}$ and its derivatives. However, this rewriting is not unique. We can actually restrict the derivatives to be used in this rewriting, first to *monotone derivatives*, then to *normal derivatives*. Normal derivatives provide a canonical rewriting.

Definition 5.1 *An invariant derivation operator $\mathcal{D}_{j_1} \dots \mathcal{D}_{j_k}$ is monotone if $j_1 \leq \dots \leq j_k$. Such a monotone derivation operator is noted $\mathcal{D}^\alpha$ where $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$ and α_i is the cardinality of $\{j_l \mid j_l = i\}$.*

There is an inductive process to rewrite any normalized invariants, and therefore any differential invariants, in terms of the monotone derivatives of $\mathcal{I}^{s+1}$. For the inductive rewriting of $\bar{\iota} u_\beta$, for $|\beta| > s+1$, in terms of the monotone derivatives of $\mathcal{I}^{s+1}$ we can proceed as follows: split β in $\beta = \hat{\beta} + \bar{\beta}$ where $|\bar{\beta}| = s+1$ and then rewrite $\bar{\iota}u_\beta - \mathcal{D}^{\hat{\beta}}(\bar{\iota}u_{\bar{\beta}})$ which is of lower order. There might be several ways to split β, each leading to a different rewriting. The following definition imposes a single choice of splitting [3].

Notation 5.2 *For $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{N}^m$, we denote*

$$\bar{\beta} = \begin{cases} \beta \text{ if } |\beta| \leq s+1 \\ (0, \ldots, 0, \beta_i', \beta_{i+1}, \ldots, \beta_m) \text{ otherwise} \\ \quad \text{with } i = \max\{j \mid \beta_j + \ldots + \beta_m \geq s+1\} \\ \quad \text{and } \beta_i' = (s+1) - \beta_{i+1} - \ldots - \beta_m \end{cases}$$

and $\hat{\beta} = \beta - \bar{\beta}$.

With those notations, $\hat{\beta} = 0$ when $|\beta| \leq s+1$ and $|\bar{\beta}|$ is always less or equal to $s+1$.

Definition 5.3 *The normal derivatives of $\mathcal{I}^{s+1}$ are the elements of the set*

$$\mathcal{N} = \mathcal{I}^{s+1} \cup \left\{ \mathcal{D}^{\hat{\beta}}(\bar{\iota}u_{\bar{\beta}}) \mid \beta \in \mathbb{N}^m,\ |\beta| > s+1 \right\}.$$

The set $\mathcal{N}^k$ of the normal derivatives of order k is the subset thereof with $|\hat{\beta}| \leq k$.

We introduce a further notation to deal with tuples that is used in the coming inductive proofs and in the description of a complete set of syzygies in Theorem 5.14.

Notation 5.4 *For $\beta \in \mathbb{N}^m$, $|\beta| > 0$, we define $f(\beta)$ and $l(\beta)$ respectively as the first and last non zero component of β, i.e.*

$$f(\beta) = \min\{j \mid \beta_j \neq 0\} \quad \text{and} \quad l(\beta) = \max\{j \mid \beta_j \neq 0\}.$$

Note that the splitting of Notation 5.2 is such that $l(\hat{\beta}) \leq f(\bar{\beta})$ for all $\beta \neq 0$.

Proposition 5.5 *Any differential invariant is a function of the normal derivatives $\mathcal{N}$ of $\mathcal{I}^{s+1}$.*

This result follows from an easy inductive argument on the following lemma.

[3] The idea is reminiscent of *involutive division*. Originally introduced by Riquier (1910) and Janet (1929) for the completion of partial differential systems, generalizations and algorithmic refinements have been worked out by several authors in the past decade for polynomial systems as well within the framework of computer algebra.

Lemma 5.6 *For all $\beta \in \mathbb{N}^m$, $\beta \neq 0$, $\bar\iota u_\beta - \mathcal{D}^{\hat\beta}(\bar\iota u_{\bar\beta}) \in \mathcal{F}^{\mathcal{G}}(\mathrm{J}^{|\beta|-1})$.*

PROOF: This is trivially true for $|\beta| \leq s+1$ since then $\hat\beta = (0,\dots,0)$. We proceed by induction for $|\beta| > s+1$.

Assume the statement is true for all β with $s+1 \leq |\beta| \leq k$. Take β with $|\beta| = k+1$. Let $i = f(\beta)$ and $\beta' = \beta - \epsilon_i$. We have $\bar\beta' = \bar\beta$, $\hat\beta' = \hat\beta - \epsilon_i$ and $\mathcal{D}^{\hat\beta} = \mathcal{D}_i \mathcal{D}^{\hat\beta'}$ so that $\bar\iota u_\beta - \mathcal{D}^{\hat\beta}(\bar\iota u_{\bar\beta}) = \bar\iota(\mathrm{D}_i(u_{\beta'})) - \mathcal{D}_i\mathcal{D}^{\hat\beta'}(\bar\iota u_{\bar\beta'})$. Thus, by Theorem 3.6,

$$\bar\iota u_\beta - \mathcal{D}^{\hat\beta}(\bar\iota u_{\bar\beta}) = \mathcal{D}_i\left(\bar\iota u_{\beta'} - \mathcal{D}^{\hat\beta'}(u_{\bar\beta'})\right) + \sum_{a=1}^{r} K_{ia}\,\bar\iota\left(\mathrm{V}_a(u_{\beta'})\right).$$

The entries of K are functions of $\mathcal{I}^{s+1}$, while the entries of $\bar\iota\left(\mathrm{V}(u_{\beta'})\right)$ are functions of $\mathcal{I}^k$. By induction hypothesis $\mathcal{D}^{\hat\beta'}(\bar\iota u_{\bar\beta'}) - \bar\iota u_{\beta'} \in \mathcal{F}^{\mathcal{G}}(\mathrm{J}^{k-1})$ and thus $\mathcal{D}_i\left(\mathcal{D}^{\hat\beta'}(u_{\bar\beta'}) - \bar\iota u_{\beta'}\right) \in \mathcal{F}^{\mathcal{G}}(\mathrm{J}^k)$. □

Following the induction on Lemma 5.6, rewriting any $\bar\iota u_\beta$ in terms of the normal derivatives of $\mathcal{I}^{s+1}$ is an effective process. Now, the normalized invariants $\bar\iota u_\beta$ are in one-to-one correspondence with the normal derivatives $\mathcal{D}^{\hat\beta}(\bar\iota u_{\bar\beta})$ of $\mathcal{I}^{s+1}$. Extending Proposition 2.11, which bears on normalized invariants, we show that the rewriting of any differential invariants in terms of normal derivatives $\mathcal{N}$ of $\mathcal{I}^{s+1}$ is unique, modulo P.

Proposition 5.7 *Assume $P = (p_1,\dots,p_r)$ are the r independent functions of $\mathcal{F}(\mathrm{J}^s)$ that cut out the cross-section $\mathcal{P}^s$ to the orbits on J^s. Let $F \in \mathcal{F}(\mathrm{J}^{s+k})$ be a function such that $F(\bar\iota x, \mathcal{D}^{\hat\beta}(\bar\iota u_{\bar\beta})) = 0$. Then, in the neighborhood of each point of $\mathcal{P}$, there exist $a_1,\dots,a_r \in \mathcal{F}(\mathrm{J}^{s+k})$ such that $F = \sum_{i=1}^{r} a_i\, p_i$.*

PROOF: By Lemma 5.6, for $|\beta| \leq s+k$, there exists ζ_β in $\mathcal{F}(\mathrm{J}^{|\beta|-1})$ such that $\mathcal{D}^{\hat\beta}(\bar\iota u_{\bar\beta}) - \bar\iota u_\beta = \bar\iota\zeta_\beta$. We choose such a family of ζ_β with $\zeta_\beta = 0$ for $|\beta| \leq s+1$. The map $\theta : \mathcal{F}(\mathrm{J}^{s+k}) \rightarrow \mathcal{F}(\mathrm{J}^{s+k})$ then defined by $\theta(u_\beta) = u_\beta + \zeta_\beta$ is an automorphism of $\mathcal{F}(\mathrm{J}^{s+k})$. It satisfies $F(\bar\iota x, \mathcal{D}^{\hat\beta}\bar\iota u_{\bar\beta}) = \theta(F)(\bar\iota x, \bar\iota u_\beta)$ and its restriction to $\mathcal{F}(\mathrm{J}^{s+1})$ is the identity. In particular $\theta(p_i) = p_i$.

If $F\left(\bar\iota x, \mathcal{D}^{\hat\beta}(\bar\iota u_{\bar\beta})\right) = 0$ then, by Proposition 2.11, there exist $b_1,\dots,b_r \in \mathcal{F}(\mathrm{J}^{s+k})$ such that $\theta(F) = \sum_{i=1}^{r} b_i\, p_i$ in the neighborhood of each point of $\mathcal{P}$. Let $a_i \in \mathcal{F}(\mathrm{J}^{s+k})$ be such that $b_i = \theta(a_i)$. We have $F = \sum_{i=1}^{r} a_i\, p_i$. □

5.2 The differential algebra of monotone derivatives

When we apply the invariant derivation $\mathcal{D}_i$ to a monotone derivative $\mathcal{D}^\beta(\bar\iota u_\alpha)$ we do not obtain a monotone derivative unless $i \leq f(\beta)$. Yet the obtained result can be written in terms of monotone derivatives. This comes as a result of the general Proposition 5.5, but we could also deduce it from the commutation rules on the derivations, Proposition 3.8. This is detailed by Hubert (2005b) and leads to an appropriate definition of differential algebra in the presence of non trivial commutation rules for the derivations. [4]

We shall accordingly define a differential algebra where the differential indeterminates are in one-to-one correspondence with the elements of $\mathcal{I}^{s+1} = \{\bar{\iota}x_1, \ldots, \bar{\iota}x_m\} \cup \{\bar{\iota}u_\alpha \,|\, u \in \mathcal{U},\ |\alpha| \leq s+1\}$. They are noted $\{\mathfrak{x}_1, \ldots, \mathfrak{x}_m\} \cup \{\mathfrak{u}_\alpha \,|\, |\alpha| \leq s+1\}$. The monotone derivatives $\mathcal{D}^\beta(\bar{\iota}x_i)$ and $\mathcal{D}^\beta(\bar{\iota}u_\alpha)$ are then represented by the double-scripted indeterminates $\mathfrak{x}_i^\beta$ and $\mathfrak{u}_\alpha^\beta$. The correspondence is encoded with a natural morphism from this differential algebra to $\mathcal{F}^{\mathcal{G}}(\mathrm{J})$ given by $\mathfrak{x}_i^\beta \mapsto \mathcal{D}^\beta(\bar{\iota}x_i)$ and $\mathfrak{u}_\beta^\alpha \mapsto \mathcal{D}^\beta(\bar{\iota}u_\alpha)$. We shall then define $\mathfrak{D}_1, \ldots, \mathfrak{D}_m$ acting on the $\mathfrak{x}_i^\beta$ and $\mathfrak{u}_\beta^\alpha$ so that this becomes a differential morphism, i.e. $\mathfrak{D}_j\mathfrak{x}_i^\beta \mapsto \mathcal{D}_j\mathcal{D}^\beta(\bar{\iota}x_i)$ and $\mathfrak{D}_j\mathfrak{u}_\beta^\alpha \mapsto \mathcal{D}_j\mathcal{D}^\beta(\bar{\iota}u_\alpha)$. The key idea comes from Hubert (2005b): the formal invariant derivations $\mathfrak{D}_1, \ldots, \mathfrak{D}_m$ are given a recursive definition.

We develop here the formalism to incorporate the functional aspect, as opposed to the polynomial case developed by Hubert (2005b). We thus define first a sequence $(\mathfrak{A}^k)_k$ of manifolds [5] that correspond to the spaces of the monotone derivatives of $\mathcal{I}^{s+1}$ of order k. $\mathfrak{A}^0$ is isomorphic to J^{s+1} and therefore of dimension $N = m + n\binom{m+s+1}{s+1}$. The coordinate function on $\mathfrak{A}^0$ are noted $\{\mathfrak{x}_1^0, \ldots, \mathfrak{x}_m^0\} \cup \{\mathfrak{u}_\alpha^0 \,|\, |\alpha| \leq s+1\}$. Then, for each k, $\mathfrak{A}^k$ is a submanifold of $\mathfrak{A}^{k+1}$ and $\mathfrak{A}^k$ is of dimension $N\binom{k+m}{m}$. A coordinate system is given by $\{\mathfrak{x}^\beta \,|\, |\beta| \leq k\} \cup \{\mathfrak{u}_\alpha^\beta \,|\, |\beta| \leq k,\ |\alpha| \leq s+1\}$. We actually focus on the algebras of smooth functions $\mathcal{F}(\mathfrak{A}^k)$ and $\mathcal{F}(\mathfrak{A})$, where $\mathfrak{A} = \bigcup_{k\geq 0} \mathfrak{A}^k$.

We can go back and forth from $\mathcal{F}(\mathfrak{A})$ to $\mathcal{F}(\mathrm{J})$ and this is expressed with the maps ϕ and ψ introduced in the next proposition. This latter is nothing else than the statement that any differential invariants can be written in terms of the monotone derivatives of $\mathcal{I}^{s+1}$ (Proposition 5.5).

Proposition 5.8 *On one hand the ring morphism $\phi : \mathcal{F}(\mathfrak{A}^k) \to \mathcal{F}^{\mathcal{G}}(\mathrm{J}^{s+k+1})$ defined by*

$$\phi(\mathfrak{x}^\alpha) = \mathcal{D}^\alpha(\bar{\iota}x) \quad \text{and} \quad \phi(\mathfrak{u}_\beta^\alpha) = \mathcal{D}^\alpha(\bar{\iota}u_\beta),\ \text{for all } \alpha \in \mathbb{N}^m \text{ and } |\beta| \leq s+1,$$

is surjective.

On the other hand there exists a ring morphism $\psi : \mathcal{F}(\mathrm{J}^{s+1+k}) \to \mathcal{F}(\mathfrak{A}^k)$ such that $\phi \circ \psi(u_\alpha) = \bar{\iota}u_\alpha$. We can furthermore choose ψ so that $\psi(x_i) = \mathfrak{x}_i^0$ and $\psi(u_\alpha) = \mathfrak{u}_\alpha^0$, for $|\alpha| \leq s+1$.

[4] The difficulty, and major difference, compared with the case considered for instance by Kolchin (1973) or Yaffe (2001) is that the coefficients of the commutation rules are themselves in the polynomial ring to be defined as opposed as to be in the base field.

[5] We shall simply think of them as open subsets of $\mathbb{R}^l$ for the right l.

In other words, $\psi(u_\alpha)$ is a function that allows one to rewrite $\bar{\iota} u_\alpha$ in terms of the monotone derivatives of $\mathcal{I}^{s+1}$.

We proceed now to define on $\mathcal{F}(\mathfrak{A})$ the derivations $\mathfrak{D}_1, \ldots, \mathfrak{D}_m$ that will turn ϕ into a differential morphism.

Definition 5.9 *Consider the maps ϕ and ψ as in Proposition 5.8. We define the* formal invariant derivations *$\mathfrak{D}_1, \ldots, \mathfrak{D}_m$ from $\mathcal{F}(\mathfrak{A}^k)$ to $\mathcal{F}(\mathfrak{A}^{k+1})$ by the following inductive process:*

$$\mathfrak{D}_i(\mathfrak{z}^\beta) = \begin{cases} \mathfrak{z}^{\beta+\epsilon_i}, & \text{if } i \leq f(\beta) \\ \mathfrak{D}_f \mathfrak{D}_i(\mathfrak{z}^{\beta-\epsilon_f}) + \sum_{l=1}^{m} c_{ifl}\, \mathfrak{D}_l(\mathfrak{z}^{\beta-\epsilon_f}), & \text{where } f = f(\beta), \text{ otherwise,} \end{cases}$$

where

- *$\mathfrak{z}$ ranges over the* differential indeterminates $\{\mathfrak{x}_1, \ldots, \mathfrak{x}_m\} \cup \{\mathfrak{u}_\alpha \,|\, |\alpha| \leq s+1\}$
- *$c_{ijk} = \psi(\Lambda_{ijl}) \in \mathcal{F}(\mathfrak{A}^1)$, for all $1 \leq i, j, l \leq m$, where $\{\Lambda_{ijl}\}_{1 \leq i,j,l \leq m}$ are the commutator invariants defined in Proposition 3.8.*

Endowed with the derivations $(\mathfrak{D}_1, \ldots, \mathfrak{D}_m)$, $\mathcal{F}(\mathfrak{A})$ is the differential algebra of monotone derivatives of $\mathcal{I}^{s+1}$.

Taking the notation $\mathfrak{D}^\alpha = \mathfrak{D}_1^{\alpha_1} \ldots \mathfrak{D}_m^{\alpha_m}$ of Definition 5.1 we have $\mathfrak{D}^\alpha(\mathfrak{z}^0) = \mathfrak{z}^\alpha$ but in general $\mathfrak{D}^\alpha(\mathfrak{z}^\beta) \neq \mathfrak{z}^{\alpha+\beta}$, unless $l(\alpha) \leq f(\beta)$. We nonetheless have the following property[6] that allows to show that ϕ is a differential morphism, thus justifying the definition of the formal invariant derivations. The proofs of the two next results are reasonably straightforward inductions exploiting the definition of the derivations.

Lemma 5.10 *$\mathfrak{D}^\alpha(\mathfrak{z}^\beta) - \mathfrak{z}^{\alpha+\beta} \in \mathcal{F}(\mathfrak{A}^{|\alpha+\beta|-1})$, for any $\mathfrak{z} \in \{\mathfrak{x}_1, \ldots, \mathfrak{x}_m\} \cup \{\mathfrak{u}_\alpha \,|\, |\alpha| \leq s+1\}$.*

PROOF: By definition of the derivations $\mathfrak{D}$, this is true whenever α or β is zero and when $l(\alpha) \leq f(\beta)$. It is in particular true when $l(\alpha) = 1$ or $f(\beta) = m$. The result is then proved by induction along the well-founded pre-order:

$$(\alpha', \beta') \prec (\alpha, \beta) \;\Leftrightarrow\; \begin{cases} \beta' \prec_f \beta \text{ or} \\ f(\beta') = f(\beta) = f \text{ and } \beta'_f = \beta_f \text{ and } \alpha' \prec_l \alpha \end{cases}$$

where

$$\beta' \prec_f \beta \;\Leftrightarrow\; \begin{cases} f(\beta') > f(\beta) \text{ or} \\ f(\beta') = f(\beta) = f \text{ and } \beta'_f < \beta_f \end{cases}$$

[6] which is expected for a differential elimination theory.

and

$$\alpha' \prec_l \alpha \;\Leftrightarrow\; \begin{cases} l(\alpha') < l(\alpha) \text{ or} \\ l(\alpha') = l(\alpha) = l \text{ and } \alpha'_l < \alpha_l. \end{cases}$$

Assume the result is true for all $(\alpha', \beta') \prec (\alpha, \beta)$. We only need to scrutinize the case $l = l(\alpha) > f(\beta) = f$. By definition of $\mathfrak{D}$ then:

$$\mathfrak{D}^\alpha(\mathfrak{z}^\beta) = \mathfrak{D}^{\alpha-\epsilon_l}\left(\mathfrak{D}_f\mathfrak{D}_l(\mathfrak{z}^{\beta-\epsilon_f})\right) + \sum_k c_{lfk}\mathfrak{D}_k(\mathfrak{z}^{\beta-\epsilon_f}).$$

We have $\beta-\epsilon_f \prec_f \beta$ and thus, by induction hypothesis, $\mathfrak{D}_k(\mathfrak{z}^{\beta-\epsilon_f}) = \mathfrak{z}^{\beta-\epsilon_f+\epsilon_k} + F$ where $F \in \mathcal{F}(\mathfrak{A}^{|\beta|})$, for all k, and in particular for $k = l$. We apply then the induction hypothesis on $\mathfrak{D}_f(\mathfrak{z}^{\beta-\epsilon_f+\epsilon_l})$ and on $\mathfrak{D}^{\alpha-\epsilon_l}(\mathfrak{z}^{\beta+\epsilon_l})$, observing that $\beta - \epsilon_f + \epsilon_l \prec_f \beta$ while $\alpha - \epsilon_l \prec_l \alpha$. □

Proposition 5.11 *The map $\phi : \mathcal{F}(\mathfrak{A}) \to \mathcal{F}^{\mathcal{G}}(\mathrm{J})$ defined in Proposition 5.8 is a morphism of differential algebras i.e. $\phi \circ \mathfrak{D}_i = \mathcal{D}_i \circ \phi$, for all $1 \le i \le m$.*

PROOF: We need to prove that

$$H(i, \alpha) : \quad \phi(\mathfrak{D}_i(\mathfrak{z}^\alpha)) = \mathcal{D}_i(\phi(\mathfrak{z}^\alpha))$$

for all $\alpha \in \mathbb{N}^m$. If this is true for all $|\alpha| \le k$ then $\phi(\mathfrak{D}_i(F)) = \mathcal{D}_i(\phi(F))$ for all $F \in \mathcal{F}(\mathfrak{A}^k)$. The proof is an induction along the well founded pre-order:

$$(j, \beta) \prec (i, \alpha) \;\Leftrightarrow\; \begin{cases} |\beta| < |\alpha| \text{ or} \\ |\beta| = |\alpha| \text{ and } j < i. \end{cases}$$

$H(i, \alpha)$ is trivially true when α is zero or when $i \le f(\alpha)$. It is therefore true whenever $i = 1$.

Assume $H(j, \beta)$ holds for any $(j, \beta) \prec (i, \alpha)$. Only the case $i > f(\alpha) = f$ needs scrutiny. We have $\mathfrak{D}_i(\mathfrak{z}^\alpha) = \mathfrak{D}_f\left(\mathfrak{D}_i(\mathfrak{z}^{\alpha-\epsilon_f})\right) + \sum_k c_{ifk}\mathfrak{D}_k(\mathfrak{z}^{\alpha-\epsilon_f})$. Since $\mathfrak{D}_i(\mathfrak{z}^{\alpha-\epsilon_f}) \in \mathcal{F}(\mathfrak{A}^{|\alpha|})$ while $f < i$, the induction hypothesis implies that $\phi\left(\mathfrak{D}_f\left(\mathfrak{D}_i(\mathfrak{z}^{\alpha-\epsilon_f})\right)\right) = \mathcal{D}_f\left(\phi\left(\mathfrak{D}_i(\mathfrak{z}^{\alpha-\epsilon_f})\right)\right)$. And since $|\alpha - \epsilon_f| < |\alpha|$, $\phi\left(\mathfrak{D}_k(\mathfrak{z}^{\alpha-\epsilon_f})\right) = \mathcal{D}_k(\phi\left(\mathfrak{z}^{\alpha-\epsilon_f}\right))$, for any k and in particular for $k = i$. Therefore

$$\phi\left(\mathfrak{D}_i(\mathfrak{z}^\alpha)\right) = \mathcal{D}_f\mathcal{D}_i\left(\phi(\mathfrak{z}^\alpha)\right) + \sum_k \Lambda_{ifk}\mathcal{D}_k\left(\phi(\mathfrak{z}^{\alpha-\epsilon_f})\right).$$

This is equal to $\mathcal{D}_i\left(\phi(\mathfrak{z}^\alpha)\right)$ by Proposition 3.8. □

Example 5.12 *We carry on with Example 2.13, 3.10 and 4.3.*

The stabilization order was $s = 1$ and we took a cross-section of that order.

According to Theorem 4.1, or Proposition 5.5, the set $\mathcal{I}^2$ below forms a generating set of differential invariants:

$$\mathcal{I}^2 = \{\bar{\iota}x_1, \bar{\iota}x_2, \bar{\iota}u_{00}, \bar{\iota}u_{10}, \bar{\iota}u_{01}, \bar{\iota}u_{20}, \bar{\iota}u_{11}, \bar{\iota}u_{02}\}.$$

We accordingly introduce $\mathfrak{A}^0$ with coordinates

$$\mathfrak{A}^0 : \quad (\mathfrak{x}_1^{00}, \mathfrak{x}_2^{00}, \mathfrak{u}_{00}^{00}, \mathfrak{u}_{10}^{00}, \mathfrak{u}_{01}^{00}, \mathfrak{u}_{20}^{00}, \mathfrak{u}_{11}^{00}, \mathfrak{u}_{02}^{00}).$$

The coordinates on $\mathfrak{A}^k$ are the $\mathfrak{z}^{ij}$ where $i+j \leq k$ and $\mathfrak{z}$ ranges over the differential indeterminates $\{\, \mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{u}_{00}, \mathfrak{u}_{10}, \mathfrak{u}_{01}, \mathfrak{u}_{20}, \mathfrak{u}_{11}, \mathfrak{u}_{02}\,\}$:

$$\mathfrak{A}^k : \quad (\mathfrak{x}_1^{ij}, \mathfrak{x}_2^{ij}, \mathfrak{u}_{00}^{ij}, \mathfrak{u}_{10}^{ij}, \mathfrak{u}_{01}^{ij}, \mathfrak{u}_{20}^{ij}, \mathfrak{u}_{11}^{ij}, \mathfrak{u}_{02}^{ij}),\ i+j \leq k.$$

Their images through $\phi : \mathcal{F}(\mathfrak{A}) \to \mathcal{F}^{\mathcal{G}}(\mathrm{J})$ are the monotone derivatives of $\mathcal{I}^2$:

$$\phi(\mathfrak{x}_1^{ij}) = \mathcal{D}_1^i\mathcal{D}_2^j(\bar{\iota}x_1),\ \phi(\mathfrak{x}_2^{ij}) = \mathcal{D}_1^i\mathcal{D}_2^j(\bar{\iota}x_2),\ \phi(\mathfrak{u}_{00}^{ij}) = \mathcal{D}_1^i\mathcal{D}_2^j(\bar{\iota}u_{00}),$$
$$\phi(\mathfrak{u}_{10}^{ij}) = \mathcal{D}_1^i\mathcal{D}_2^j(\bar{\iota}u_{10}),\ \ldots,\ \phi(\mathfrak{u}_{02}^{ij}) = \mathcal{D}_1^i\mathcal{D}_2^j(\bar{\iota}u_{02}).$$

Given that $[\mathcal{D}_2, \mathcal{D}_1] = (\bar{\iota}u_{10}\bar{\iota}u_{20} + \bar{\iota}u_{01}\bar{\iota}u_{11})\,\mathcal{D}_2 - (\bar{\iota}u_{10}\bar{\iota}u_{11} + \bar{\iota}u_{01}\bar{\iota}u_{02})\,\mathcal{D}_1$ we define on $\mathcal{F}(\mathfrak{A})$ the derivations $\mathfrak{D}_1$ and $\mathfrak{D}_2$ recursively as follows.

$$\begin{aligned}
\mathfrak{D}_1(\mathfrak{z}^{i,j}) &= \mathfrak{z}^{i+1,j},\\
\mathfrak{D}_2(\mathfrak{z}^{0,j}) &= \mathfrak{z}^{0,j+1},\\
\mathfrak{D}_2(\mathfrak{z}^{i+1,j}) &= \mathfrak{D}_1\mathfrak{D}_2(\mathfrak{z}^{i,j}) + (\mathfrak{u}_{10}^{00}\mathfrak{u}_{20}^{00} + \mathfrak{u}_{01}^{00}\mathfrak{u}_{11}^{00})\,\mathfrak{D}_2(\mathfrak{z}^{i,j}) - (\mathfrak{u}_{10}^{00}\mathfrak{u}_{11}^{00} + \mathfrak{u}_{01}^{00}\mathfrak{u}_{02}^{00})\,\mathfrak{D}_1(\mathfrak{z}^{i,j}).
\end{aligned}$$

According to Proposition 5.11, $\phi \circ \mathfrak{D}_i = \mathcal{D}_i \circ \phi$. We have for instance, with $a+b \leq 2$:

$$\phi(\mathfrak{D}_2(\mathfrak{u}_{ab}^{kl})) = \mathcal{D}_2\mathcal{D}_1^k\mathcal{D}_2^l(\bar{\iota}u_{ab})$$

while

$$\phi(\mathfrak{D}_1(\mathfrak{u}_{ab}^{kl})) = \mathcal{D}_1\mathcal{D}_1^k\mathcal{D}_2^l(\bar{\iota}u_{ab}) = \mathcal{D}_1^{k+1}\mathcal{D}_2^l(\bar{\iota}u_{ab}) = \phi(\mathfrak{u}_{ab}^{k+1,l}).$$

5.3 Complete set of syzygies

As a rather immediate consequence of Theorem 3.6, the following differential relationships hold among the first order derivatives of $\mathcal{I}^{s+1}$:

$$\begin{aligned}
\mathcal{D}_i(\bar{\iota}x_j) &= \delta_{ij} - \sum_{a=1}^{r} K_{ia}\,\bar{\iota}\,(\mathrm{V}_a(x_j))\,, && 1 \leq i,j, \leq m\\
\mathcal{D}_i(\bar{\iota}u_\alpha) &= \bar{\iota}u_{\alpha+\epsilon_i} - \sum_{a=1}^{r} K_{ia}\,\bar{\iota}\,(\mathrm{V}_a(u_\alpha))\,, && |\alpha| \leq s\\
\mathcal{D}_i(\bar{\iota}u_\alpha) - \mathcal{D}_j(\bar{\iota}u_\beta) &= \sum_{a=1}^{r} K_{ja}\bar{\iota}\,(\mathrm{V}_a(u_\beta)) - K_{ia}\bar{\iota}\,(\mathrm{V}_a(u_\alpha))\,, && \alpha+\epsilon_i = \beta+\epsilon_j,\\
& && |\alpha| = |\beta| = s+1,
\end{aligned}$$

where $\delta_{ij} = 1$ or 0 according to whether $i = j$ or not while ϵ_i was defined in Notation 1.3.

The first two sets of equations describe how the invariant derivations act on the elements of $\mathcal{I}^s$ in terms of $\mathcal{I}^{s+1}$. The last set of equations describes the cross-derivatives of the elements of $\mathcal{I}^{s+1}\backslash\mathcal{I}^s$. The indices α and β and the derivations $\mathcal{D}_i$ and $\mathcal{D}_j$ are chosen so that u_α and u_β have a common derivative $u_\gamma = u_{\alpha+\epsilon_i} = u_{\beta+\epsilon_j}$. The idea here is that there are more than one way to rewrite $\bar{\iota}u_\gamma$ in terms of the monotone derivatives of $\mathcal{I}^{s+1}$: on one hand $\bar{\iota}u_\gamma = \mathcal{D}_i(\bar{\iota}u_\alpha)+K_{ia}\bar{\iota}\left(\mathrm{V}_a(u_\alpha)\right)$ and on the other hand $\bar{\iota}u_\gamma = \mathcal{D}_j(\bar{\iota}u_\beta) + \sum_{a=1}^{r} K_{ja}\bar{\iota}\left(\mathrm{V}_a(u_\beta)\right)$; both should be equivalent.

Using the setting introduced in the previous subsection we formalize and prove that those relationships form a complete set of differential syzygies for $\mathcal{I}^{s+1}$. We actually prove the result for a subset obtained by restricting the range of (i,j) for the third type of relationships which bears on $\mathcal{I}^{s+1}\setminus\mathcal{I}^s$. Indeed, some of those relationships can be deduced from the others. More specifically, if we write $T^{\alpha,i}_{\beta,j}$ for this latter relationship and if $\gamma+\epsilon_k = \alpha+\epsilon_i = \beta+\epsilon_j$ then $T^{\alpha,i}_{\beta,j} = T^{\alpha,i}_{\gamma,k} - T^{\beta,j}_{\gamma,k}$.

Definition 5.13 *Let $\phi : \mathcal{F}(\mathfrak{A}^k) \to \mathcal{F}^{\mathcal{G}}(\mathrm{J}^{s+k+1})$ be as in Proposition 5.8. An element of $\mathcal{F}(\mathfrak{A}^k)$ is a* (differential) syzygy *on the monotone derivatives of $\mathcal{I}^{s+1}$ if its image by ϕ is zero on the cross-section in J^k.*

Since differential invariants are locally determined by their restriction to the cross-section, this is the same as requesting that the image is zero on an open set that contains the cross-section. Furthermore, by Proposition 5.11, the set of syzygies is a differential ideal: if f is a syzygy then so is $\mathfrak{D}_i(f)$, for all $1 \leq i \leq m$.

Theorem 5.14 *Let s be greater or equal to the stabilization order*[7] *and assume a cross-section is defined as the zero set of $P = (p_1,\ldots,p_r) : \mathrm{J}^s \to \mathbb{R}^r$. Let $\mathcal{F}(\mathfrak{A}) = \cup_{k\geq 0}\mathcal{F}(\mathfrak{A}^k)$ be the differential algebra of monotone derivatives of $\mathcal{I}^{s+1}$, the normalized invariants of order $s+1$.*

Consider the map $\phi : \mathcal{F}(\mathfrak{A}) \to \mathcal{F}^{\mathcal{G}}(\mathrm{J})$ defined by $\phi\left(\mathfrak{x}^\alpha\right) = \mathcal{D}^\alpha(\bar{\iota}x)$, and $\phi\left(\mathfrak{u}^\alpha_\beta\right) = \mathcal{D}^\alpha(\bar{\iota}u_\beta), \forall \alpha,\beta \in \mathbb{N}^m, |\beta| \leq s+1$. It is surjective and its kernel is a differential ideal for the formal invariant derivations, $\mathfrak{D}_1,\ldots,\mathfrak{D}_m$ (Definition 5.9). Let $\psi : \mathcal{F}(\mathrm{J}^{s+1}) \to \mathcal{F}(\mathfrak{A}^0)$ be the morphism define by $\psi(x) = \mathfrak{x}^0$, $\psi(u_\beta) = \mathfrak{u}^0_\beta$. A generating set for the kernel of ϕ is given by the union of the three following finite subsets of $\mathcal{F}(\mathfrak{A}^1)$

- $\mathfrak{R} = \left\{\, p_1(\mathfrak{x}^0,\mathfrak{u}^0_\alpha),\ldots,p_r(\mathfrak{x}^0,\mathfrak{u}^0_\alpha)\,\right\} \subset \mathcal{F}(\mathfrak{A}^0)$

[7] Under our assumption of a locally effective action on J^0, the generic orbits in J^s are of the same dimension r as the group.

- $\mathfrak{S} = \{\, S^i_{x_j} \,|\, 1 \leq i,j \leq m \,\} \cup \{\, S^i_{u_\alpha} \,|\, |\alpha| \leq s,\ 1 \leq i \leq m \,\} \ \subset \mathcal{F}(\mathfrak{A}^1)$ *where*

$$S^i_{x_j} = \mathfrak{x}^{\epsilon_i}_j - \delta_{ij} - \sum_{a=1}^{r} \psi\left(K_{ia} \mathrm{V}_a(x_j)\right)$$

and

$$S^i_{u_\alpha} = \mathfrak{u}^{\epsilon_i}_\alpha - \mathfrak{u}^0_{\alpha+\epsilon_i} - \sum_{a=1}^{r} \psi\left(K_{ia} \mathrm{V}_a(u_\alpha)\right)$$

- $\mathfrak{T} = \{\, T^i_{u_\beta} \,|\, |\beta| = s+1$ *and* $f(\beta) < i \leq m \,\} \ \subset \mathcal{F}(\mathfrak{A}^1)$ *where, with* $f = f(\beta)$,

$$T^i_{u_\beta} = \mathfrak{u}^{\epsilon_i}_\beta - \mathfrak{u}^{\epsilon_f}_{\beta+\epsilon_i-\epsilon_f} - \sum_{a=1}^{r} \psi\left(K_{ia} \mathrm{V}_a(u_{\beta+\epsilon_i-\epsilon_f}) - K_{fa} \mathrm{V}_a(u_\beta)\right).$$

The result is deduced from the following lemma. It shows that any monotone derivative of $\mathcal{I}^{s+1}$ can be rewritten in terms of the normal derivatives modulo $\mathfrak{S} \cup \mathfrak{T}$.

Lemma 5.15 *For any* $\alpha \in \mathbb{N}^m$ *and* $|\gamma| \leq s+1$ *there exists a linear operator* $L^\alpha_{u_\gamma}$ *of order* $|\alpha|-1$ *in* $\mathfrak{D}_1, \ldots, \mathfrak{D}_m$ *such that, for* $\beta = \alpha + \gamma$,

$$\mathfrak{u}^\alpha_\gamma - \mathfrak{u}^{\hat{\beta}}_{\bar{\beta}} - L^\alpha_{u_\gamma}(\mathfrak{S}, \mathfrak{T}) \in \mathcal{F}(\mathfrak{A}^{|\alpha|-1}).$$

PROOF: We consider first the case where $|\gamma| = s+1$ and prove that there exists a homogeneous linear operator $H^\alpha_{u_\gamma}$ of order $|\alpha|-1$ in $\mathfrak{D}_1, \ldots, \mathfrak{D}_m$ such that $\mathfrak{u}^\alpha_\gamma - \mathfrak{u}^{\hat{\beta}}_{\bar{\beta}} - H^\alpha_{u_\beta}(\mathfrak{T}) \in \mathcal{F}(\mathfrak{A}^{|\beta|-1})$. The proof is by induction along the following well founded pre-order on $\mathbb{N}^m$:

$$\gamma \prec \gamma' \ \Leftrightarrow \ \begin{cases} |\gamma| < |\gamma'| \\ \text{ or } |\gamma| = |\gamma'| \text{ and } l(\gamma) < l(\gamma') \\ \text{ or } |\gamma| = |\gamma'| \text{ and } l = l(\gamma) = l(\gamma') \text{ and } \gamma_l < \gamma'_l \end{cases}$$

Let $E_\beta = \{\gamma' \,|\ |\gamma'| = s+1,\ \exists \alpha' \text{ such that } \alpha' + \gamma' = \beta\}$. Note that $\gamma \in E_\beta$ and that $\hat{\beta}$ is the minimal element of E_β according to $\prec$.

If $l(\alpha) \leq f(\gamma)$ then $\hat{\beta} = \alpha$ and $\bar{\beta} = \gamma$ and the result needs no further argument.

Otherwise assume the result is true for all $\gamma' \in E_\beta$ with $\gamma' \prec \gamma$. Let $l = l(\alpha) > f(\gamma) = f$. We have:

$$\begin{aligned} \mathfrak{u}^\alpha_\gamma &= \mathfrak{D}^{\alpha-\epsilon_l}(\mathfrak{u}^{\epsilon_l}_\gamma) \\ &= \mathfrak{D}^{\alpha-\epsilon_l}\left(\mathfrak{u}^{\epsilon_f}_{\gamma-\epsilon_f+\epsilon_l} + T^l_{u_\gamma} + \textstyle\sum_{a=1}^{r} \psi\left(K_{la} \mathrm{V}_a(u_{\gamma-\epsilon_f+\epsilon_l}) - K_{fa} \mathrm{V}_a(u_\gamma)\right)\right). \end{aligned}$$

On one hand, the argument of ψ belongs to $\mathcal{F}(\mathrm{J}^{s+1})$ so that its image belongs to $\mathcal{F}(\mathfrak{A}^0)$. On the other hand $\mathfrak{D}^{\alpha-\epsilon_l}\left(\mathfrak{u}^{\epsilon_f}_{\gamma-\epsilon_f+\epsilon_l}\right)-\mathfrak{u}^{\alpha+\epsilon_f-\epsilon_l}_{\gamma-\epsilon_f+\epsilon_l} \in \mathcal{F}(\mathfrak{A}^{|\alpha|-1})$ according to Lemma 5.10. Thus

$$\mathfrak{u}^{\alpha}_{\gamma} - \mathfrak{u}^{\alpha+\epsilon_f-\epsilon_l}_{\gamma-\epsilon_f+\epsilon_l} - \mathfrak{D}^{\alpha-\epsilon_l}\left(T^l_{u_\gamma}\right) \in \mathcal{F}(\mathfrak{A}^{|\alpha|-1}).$$

Since $\gamma - \epsilon_f + \epsilon_l \prec \gamma$ we can conclude our induction argument.

We are left to prove that, for all $|\gamma| \leq s$ and $\alpha \in \mathbb{N}^m$, there is a $\mu \in \mathbb{N}^m$ with $|\mu| = s+1-|\gamma|$ and a differential operator $L^{\alpha}_{u_\gamma}$ such that

$$\mathfrak{u}^{\alpha}_{\gamma} - \mathfrak{u}^{\alpha-\mu}_{\gamma+\mu} - L^{\alpha}_{u_\gamma}(\mathfrak{S}) \in \mathcal{F}(\mathfrak{A}^{|\alpha|-1}).$$

For that it is sufficient to lead an inductive argument on the fact that

$$\mathfrak{u}^{\alpha}_{\gamma} = \mathfrak{D}^{\alpha-\epsilon_l}\left(\mathfrak{u}^{\epsilon_l}_{\gamma}\right) = \mathfrak{u}^{\alpha-\epsilon_l}_{\gamma+\epsilon_l} + \mathfrak{D}^{\alpha-\epsilon_l}\left(S^l_{u_\gamma} + \sum_{a=1}^{r} \psi\left(K_{la}\mathrm{V}_a(u_\gamma)\right)\right),$$

where $l = l(\alpha)$. $\square$

PROOF: (of the theorem). Taylor's formula with integral remainder shows the following (Bourbaki, 1967, Paragraph 2.5). For a smooth function f on an open set $U \times I_1 \times \ldots \times I_l \subset \mathbb{R}^k \times \mathbb{R}^l$, where the I_i are intervals of $\mathbb{R}$ that contain zero, there are smooth functions f_0 on U, and f_i on $U \times I_1 \times \ldots \times I_i$, $1 \leq i \leq l$ such that $f(x, t_1, \ldots, t_l) = f_0(x) + \sum_{j=1}^{l} t_j\, f_j(x, t_1, \ldots, t_j)$.

Let us restrict the $\mathfrak{A}^k$ to appropriate neighborhoods of the zero set of $\mathfrak{S}$, $\mathfrak{T}$ and their derivatives. Take $f \in \mathcal{F}(\mathfrak{A}^{k+1})$. By first applying Lemma 5.15 for $|\alpha+\gamma| = k+1$, we can first write it as:

$$f(\mathfrak{u}^{\alpha}_{\gamma}, \mathfrak{u}^{\alpha'}_{\gamma'}) = f_1(\mathfrak{u}^{\hat{\beta}}_{\bar{\beta}}, \mathfrak{u}^{\alpha'}_{\gamma'}) + \sum_{|\alpha+\gamma|=k+1} L^{\alpha}_{u_\gamma}(\mathfrak{S}, \mathfrak{T})\, F^{\alpha}_{u_\gamma}$$

where (γ, α) range over $|\alpha+\gamma| = k+1$ while (γ', α') range over $|\alpha'+\gamma'| \leq k$ so that β ranges over $|\beta| = k+1$ and $F^{\alpha}_{u_\gamma} \in \mathcal{F}(\mathfrak{A}^{k+1})$. We can iterate this process on the $\mathfrak{u}^{\alpha'}_{\gamma'}$, with $|\alpha'+\gamma'| = k$, in f_1. Induction then shows that

$$f(\mathfrak{u}^{\alpha}_{\gamma}) = F(\mathfrak{u}^{\hat{\beta}}_{\bar{\beta}}) + \sum_{|\alpha+\gamma| \leq k+1} L^{\alpha}_{u_\gamma} F^{\alpha}_{u_\gamma}$$

where now (α, γ) range over $|\alpha+\gamma| \leq k+1$ and β over $|\beta| \leq k+1$.

Thus $\phi(f) = \phi(F)$. By Lemma 5.7, if f belongs to the kernel of ϕ then F is a linear combination of elements of $\mathfrak{R}$. $\square$

Example 5.16 *We carry on with Example 2.13, 3.10, 4.3 and 5.12.*

Recall that

$$\mathrm{V}_1 = \frac{\partial}{\partial x_1}, \quad \mathrm{V}_2 = \frac{\partial}{\partial x_2}, \quad \mathrm{V}_3 = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} - \sum_{i,j\geq 0}(i+j)\,u_{ij}\frac{\partial}{\partial u_{ij}}$$

while $P = \left(x_1, x_2, \frac{1}{2} - \frac{1}{2}(u_{10}^2 + u_{01}^2)\right)$, *so that*

$$K = \begin{pmatrix} 1 & 0 & \bar{\iota}v \\ 0 & 1 & \bar{\iota}w \end{pmatrix} \quad \text{where } v = -(u_{10}u_{20} + u_{01}u_{11}) \text{ and } w = -(u_{10}u_{11} + u_{01}u_{02}).$$

According to Theorem 5.14 a complete set of syzygies on $\mathcal{I}^2$, i.e. a basis for the kernel of $\phi : \mathcal{F}(\mathfrak{A}) \to \mathcal{F}^{\mathcal{G}}(\mathrm{J})$, consists of the following elements.

$\mathfrak{R}$**,** *the functional relationships implied by the choice of the cross-section.*

$$\mathfrak{x}_1^{00},\ \mathfrak{x}_2^{00},\ \frac{1}{2} - \frac{1}{2}\left((\mathfrak{u}_{10}^{00})^2 + (\mathfrak{u}_{01}^{00})^2\right)$$

$\mathfrak{S}$**,** *the relationships describing the derivations of the elements of $\mathcal{I}^s$:*

$$\begin{array}{llll}
S^1_{x_1} & : \mathfrak{x}_1^{10} & S^2_{x_1} & : \mathfrak{x}_1^{01} \\
S^1_{x_2} & : \mathfrak{x}_2^{10} & S^2_{x_2} & : \mathfrak{x}_2^{01} \\
S^1_{u_{00}} & : \mathfrak{u}_{00}^{10} - \mathfrak{u}_{10}^{00}, & S^2_{u_{00}} & : \mathfrak{u}_{00}^{01} - \mathfrak{u}_{01}^{00}, \\
S^1_{u_{10}} & : \mathfrak{u}_{10}^{10} - \mathfrak{u}_{20}^{00} + \mathfrak{u}_{10}^{00}\,\mathfrak{v}, & S^2_{u_{10}} & : \mathfrak{u}_{10}^{01} - \mathfrak{u}_{11}^{00} + \mathfrak{u}_{10}^{00}\,\mathfrak{w}, \\
S^1_{u_{10}} & : \mathfrak{u}_{01}^{10} - \mathfrak{u}_{11}^{00} + \mathfrak{u}_{01}^{00}\,\mathfrak{v}, & S^2_{u_{10}} & : \mathfrak{u}_{01}^{01} - \mathfrak{u}_{02}^{00} + \mathfrak{u}_{01}^{00}\,\mathfrak{w},
\end{array}$$

where

$$\mathfrak{v} = -\mathfrak{u}_{10}^{00}\,\mathfrak{u}_{20}^{00} - \mathfrak{u}_{01}^{00}\,\mathfrak{u}_{11}^{00} = \psi(v), \quad \mathfrak{w} = -\mathfrak{u}_{10}^{00}\,\mathfrak{u}_{11}^{00} - \mathfrak{u}_{01}^{00}\,\mathfrak{u}_{02}^{00}, = \psi(w).$$

$\mathfrak{T}$**,** *the relationships obtained by cross-differentiating the elements $\mathcal{I}^{s+1} \setminus \mathcal{I}^s$:*

$$\begin{array}{ll}
T^2_{u_{20}} & : \mathfrak{u}_{20}^{01} - \mathfrak{u}_{11}^{10} - 2\,\mathfrak{u}_{20}^{00}\,\mathfrak{w} + 2\,\mathfrak{u}_{11}\mathfrak{v} \\
T^2_{u_{11}} & : \mathfrak{u}_{11}^{01} - \mathfrak{u}_{02}^{10} - 2\,\mathfrak{u}_{11}^{00}\,\mathfrak{w} + 2\,\mathfrak{u}_{02}^{00}\,\mathfrak{v}.
\end{array}$$

Yet from Theorem 4.2 we know that $\{\bar{\iota}u, \bar{\iota}v, \bar{\iota}w\}$ form a generating set. As $\bar{\iota}v$ and $\bar{\iota}w$ are the coefficients of the commutation rules, we can perform a differential elimination to obtain a complete set of syzygies bearing on $\{\bar{\iota}u, \bar{\iota}v, \bar{\iota}w\}$ (Hubert, 2003, 2005b). We obtain:

$$\mathfrak{D}_1(\mathfrak{w}) - \mathfrak{D}_2(\mathfrak{v}) = 0, \qquad \mathfrak{D}_1(\mathfrak{u})^2 + \mathfrak{D}_2(\mathfrak{u})^2 = 1.$$

6 Classical examples

We treat two very classical geometries, curves and surfaces in Euclidean 3-space, in order to illustrate the general theory of this paper on well-known cases.

For surfaces we shall use the classical cross-section, show how the mean and Gauss curvature relate to the exhibited generating set of differential invariants and how the Gauss-Codazzi equation on the principal curvatures arises as the syzygy.

For curves we shall choose some non classical cross-sections that can come of use. We first illustrate Theorem 4.2 for a cross-section of minimal order that is not a coordinate cross-section and therefore not covered by Olver (2007b). The edge invariants are explicitly shown to form a generating set of differential invariants and endowed with a rewriting procedure. The syzygies there are trivial.

We then consider the cross-section introduced by Olver (2007b) to show that the minimal order condition on the cross-section is necessary for Theorem 4.2 to hold, i.e. for the edge invariants to be a generating set of differential invariants. There are then non trivial differential syzygies on the generating set of normalized invariants. Elimination on those allows to diminish the number of generators.

As should come clear from those examples, the only data we start with are the infinitesimal generators of the action and a choice of cross-section. Of course, the art of choosing the appropriate cross-section for a given application should not be underestimated.

For the benefit of a lighter notation system, we skip the Gothic notation of the formalism introduced in Section 5 when formalizing the notion of syzygies. Therefore $\bar{\iota} u_\alpha$ will in turn represent a local invariant, i.e. an element of $\mathcal{F}^{\mathcal{G}}(\mathrm{J})$, or the coordinate function $\mathfrak{u}_\alpha^0$ of $\mathfrak{A}$.

6.1 Surfaces in Euclidean geometry

We shall show how to retrieve the Codazzi equation as the syzygy between the two generators for the differential invariants.

We choose coordinate functions (x_1, x_2, u) for $\mathbb{R}^2 \times \mathbb{R}$. We consider x_1, x_2 as the independent variables and u as the dependent variable.

The infinitesimal generators of the classical action of the Euclidean group $SE(3)$ on $\mathbb{R}^3$ are:

$$\mathrm{V}_1^0 = \frac{\partial}{\partial x_1},\ \mathrm{V}_2^0 = \frac{\partial}{\partial x_2},\ \mathrm{V}_3^0 = \frac{\partial}{\partial u},$$
$$\mathrm{V}_4^0 = x_1 \frac{\partial}{\partial u} - u \frac{\partial}{\partial x_1}, \mathrm{V}_5^0 = x_2 \frac{\partial}{\partial u} - u \frac{\partial}{\partial x_2}, \mathrm{V}_6^0 = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1},$$

so that their prolongations are

$$
\mathrm{V}_1 = \frac{\partial}{\partial x_1},\ \mathrm{V}_2 = \frac{\partial}{\partial x_2},\ \mathrm{V}_3 = \frac{\partial}{\partial u},
$$
$$
\mathrm{V}_4 = \sum_\alpha \mathrm{D}^\alpha(x_1 + u_{00}u_{10})\,\frac{\partial}{\partial u_\alpha} - u_{00}\,\mathrm{D}_1, \mathrm{V}_5 = \sum_\alpha \mathrm{D}^\alpha(x_2 + u_{00}u_{01})\,\frac{\partial}{\partial u_\alpha} - u_{00}\,\mathrm{D}_2,
$$
$$
\mathrm{V}_6 = x_1\,\mathrm{D}_2 - x_2\,\mathrm{D}_1 + \sum_\alpha \mathrm{D}^\alpha(x_2u_{10} - x_1u_{01})\,\frac{\partial}{\partial u_\alpha}.
$$

Let us choose the classical cross-section defined by $P = (x_1, x_2, u_{00}, u_{10}, u_{01}, u_{11})$. The Maurer-Cartan matrix of Theorem 3.6 is

$$
K = \begin{pmatrix} 1\ 0\ 0\ \bar{\iota}u_{20} & 0 & \dfrac{\bar{\iota}u_{21}}{\bar{\iota}u_{20} - \bar{\iota}u_{02}} \\ \\ 0\ 1\ 0 \quad 0 & \bar{\iota}u_{02} & \dfrac{\bar{\iota}u_{12}}{\bar{\iota}u_{20} - \bar{\iota}u_{02}} \end{pmatrix}.
$$

Applying Proposition 3.8 we have

$$
[\mathcal{D}_2, \mathcal{D}_1] = \frac{\bar{\iota}u_{21}}{\bar{\iota}u_{20} - \bar{\iota}u_{02}}\,\mathcal{D}_1 + \frac{\bar{\iota}u_{12}}{\bar{\iota}u_{20} - \bar{\iota}u_{02}}\,\mathcal{D}_2. \tag{6.1}
$$

Given that $\bar{\iota}x_1, \bar{\iota}x_2, \bar{\iota}u_{00}, \bar{\iota}u_{10}, \bar{\iota}u_{01}, \bar{\iota}u_{11} = 0$ the non zero elements of $\mathfrak{S}$ in Theorem 5.14 are

$$
\begin{aligned}
S^1_{u_{20}} &= \mathcal{D}_1(\bar{\iota}u_{20}) - \bar{\iota}u_{30}, \qquad & S^2_{u_{20}} &= \mathcal{D}_2(\bar{\iota}u_{20}) - \bar{\iota}u_{21},\\
S^1_{u_{02}} &= \mathcal{D}_1(\bar{\iota}u_{02}) - \bar{\iota}u_{12}, \qquad & S^2_{u_{02}} &= \mathcal{D}_2(\bar{\iota}u_{02}) - \bar{\iota}u_{03},
\end{aligned}
$$

while the elements of $\mathfrak{T}$ are

$$
\begin{aligned}
T^2_{u_{12}} &= \mathcal{D}_2(\bar{\iota}u_{12}) - \mathcal{D}_1(\bar{\iota}u_{03}) - \frac{\bar{\iota}u_{12}}{\bar{\iota}u_{20} - \bar{\iota}u_{02}}\,(\bar{\iota}u_{21} + \bar{\iota}u_{03}),\\
T^2_{u_{30}} &= \mathcal{D}_2(\bar{\iota}u_{30}) - \mathcal{D}_1(\bar{\iota}u_{21}) - \frac{\bar{\iota}u_{21}}{\bar{\iota}u_{20} - \bar{\iota}u_{02}}\,(\bar{\iota}u_{12} + \bar{\iota}u_{30}),\\
T^2_{u_{21}} &= \mathcal{D}_2(\bar{\iota}u_{21}) - \mathcal{D}_1(\bar{\iota}u_{12}) - \frac{\bar{\iota}u_{21}\,\bar{\iota}u_{03} + \bar{\iota}u_{12}\,\bar{\iota}u_{30} - 2\,\bar{\iota}u_{21}{}^2 - 2\,\bar{\iota}u_{12}{}^2}{\bar{\iota}u_{20} - \bar{\iota}u_{02}} + (\bar{\iota}u_{20} - \bar{\iota}u_{02})\,\bar{\iota}u_{02}\,\bar{\iota}u_{20}.
\end{aligned}
$$

Theorem 4.2 predicts that $\{\bar{\iota}u_{20}, \bar{\iota}u_{02}, \bar{\iota}u_{21}, \bar{\iota}u_{12}\}$ form a generating set. From $\mathfrak{S}$ we see furthermore that all the third order normalized invariants can be expressed as derivatives of $\{\bar{\iota}u_{20}, \bar{\iota}u_{02}\}$. This latter set therefore already forms a generating set of invariants. Indeed, with Theorem 2.10, we can write the Gauss and mean curvatures in terms of $\{\bar{\iota}u_{20}, \bar{\iota}u_{02}\}$ (Berger and Gostiaux, 1988, (10.6.5)), (Ivey and Landsberg, 2003, (1.3))

$$\sigma = \frac{u_{20}u_{02} - u_{11}^2}{(1+u_{10}^2+u_{01}^2)^2} = \bar{\iota}u_{20}\,\bar{\iota}u_{02},$$

$$\pi = \frac{1}{2}\frac{(1+u_{01}^2)u_{20} - 2_{10}u_{01}u_{11} + (1+u_{10}^2)u_{02}}{(1+u_{10}^2+u_{01}^2)^{\frac{3}{2}}} = \frac{1}{2}(\bar{\iota}u_{20} + \bar{\iota}u_{02}).$$

Our generators $\{\bar{\iota}u_{20}, \bar{\iota}u_{02}\}$ are thus the principal curvatures. Let us write $\kappa = \bar{\iota}u_{20}$ and $\tau = \bar{\iota}u_{02}$. From $\mathfrak{S}$ we have

$$\bar{\iota}u_{30} = \mathcal{D}_1(\kappa),\ \bar{\iota}u_{21} = \mathcal{D}_2(\kappa),\ \bar{\iota}u_{12} = \mathcal{D}_1(\tau),\ \text{and}\ \bar{\iota}u_{03} = \mathcal{D}_2(\tau).$$

Making the substitution in $\mathfrak{T}$ we obtain

$$\begin{array}{l}
\mathcal{D}_2\mathcal{D}_1(\tau) - \mathcal{D}_1\mathcal{D}_2(\tau) - \dfrac{\mathcal{D}_1(\tau)}{\kappa-\tau}\left(\mathcal{D}_2(\kappa)+\mathcal{D}_2(\tau)\right)\\
\mathcal{D}_2\mathcal{D}_1(\kappa) - \mathcal{D}_1\mathcal{D}_2(\kappa) - \dfrac{\mathcal{D}_2(\kappa)}{\kappa-\tau}\left(\mathcal{D}_1(\kappa)+\mathcal{D}_1(\tau)\right)\\
\mathcal{D}_2^2(\kappa) - \mathcal{D}_1^2(\tau) - \dfrac{\mathcal{D}_1(\kappa)\,\mathcal{D}_1(\tau) + \mathcal{D}_2(\kappa)\,\mathcal{D}_2(\tau) - 2\,\mathcal{D}_2(\kappa)^2 - 2\,\mathcal{D}_1(\tau)^2}{\kappa-\tau} + (\kappa-\tau)\,\kappa\,\tau.
\end{array}$$

The two first functions vanish when one rewrites $\mathcal{D}_2\mathcal{D}_1(\tau)$ and $\mathcal{D}_2\mathcal{D}_1(\kappa)$ in terms of monotone derivatives using (6.1). The last function provides the Gauss-Codazzi equation (Ivey and Landsberg, 2003, Exercise 2.3.1).

6.2 Curves in Euclidean geometry

For this example we will first work with a cross-section of minimal order. The edge invariants are then generating and submitted to no non trivial syzygies. When we then use a cross-section that is not of minimal order, a non trivial syzygie appears on the predicted generating sets.

We consider the classical action of SE(3) on space curves. We have $\mathrm{J}^0 = \mathcal{X}^1 \times \mathcal{U}^2$ with coordinate (x, u, v). The infinitesimal generators of the action are:

$$\begin{array}{lll}
\mathrm{V}_1^0 = \dfrac{\partial}{\partial x}, & \mathrm{V}_2^0 = \dfrac{\partial}{\partial u}, & \mathrm{V}_3^0 = \dfrac{\partial}{\partial v},\\
\mathrm{V}_4^0 = v\,\dfrac{\partial}{\partial u} - u\,\dfrac{\partial}{\partial v}, & \mathrm{V}_5^0 = x\dfrac{\partial}{\partial u} - u\dfrac{\partial}{\partial x}, & \mathrm{V}_6^0 = x\dfrac{\partial}{\partial v} - v\dfrac{\partial}{\partial x}
\end{array}$$

so that their prolongations are given by

$$\begin{aligned}
&\mathrm{V}_1 = \frac{\partial}{\partial x}, \quad \mathrm{V}_2 = \frac{\partial}{\partial u}, \quad \mathrm{V}_3 = \frac{\partial}{\partial v}, \quad \mathrm{V}_4 = \sum_k v_k \frac{\partial}{\partial u_k} - u_k \frac{\partial}{\partial v_k},\\
&\mathrm{V}_5 = -u_0\,\mathrm{D} + \sum_k \mathrm{D}^k(x - u_0\, u_1)\frac{\partial}{\partial u_k} - \sum_k \mathrm{D}^k(u_0\, v_1)\frac{\partial}{\partial v_k},\\
&\mathrm{V}_6 = -v_0\,\mathrm{D} - \sum_k \mathrm{D}^k(v_0\, u_1)\frac{\partial}{\partial u_k} + \sum_k \mathrm{D}^k(x - v_0\, v_1)\frac{\partial}{\partial v_k}.
\end{aligned}$$

The action is transitive on J^1 and becomes locally free on J^2 with generic orbits of codimension 1.

Minimal order cross-section

We choose a non classical cross-section of minimal order: $P = (x, u_0, v_0, u_1, v_1, v_2 - u_2)$. Then:

$$\bar\iota(\mathrm{D}(P)) = \begin{pmatrix} 1 & 0 & 0 & \bar\iota u_2 & \bar\iota u_2 & \bar\iota(v_3 - u_3) \end{pmatrix}.$$

On one hand we know from Theorem 4.1 that $\mathcal{I}^3 = \{\bar\iota x, \bar\iota u_0, \bar\iota u_1, \bar\iota v_1, \bar\iota u_3, \bar\iota v_3\}$ is a generating set of differential invariants and rewriting any differential invariants in terms of them is a recursive process described in Section 4, or more specifically by Proposition 5.5. One can check that the complete set of syzygies on $\mathcal{I}^3$ given in Theorem 5.14 boils down to $\mathfrak{R} = \{\bar\iota x, \bar\iota u_0, \bar\iota v_0, \bar\iota u_1, \bar\iota v_1, \bar\iota v_2 - \bar\iota u_2\}$ since $\mathfrak{S} = \{0\}$ and $\mathfrak{T} = \emptyset$.

On the other hand Theorem 4.2 implies that $\mathcal{E} = \{\bar\iota u_2,\, \bar\iota w\}$, where $w = v_3 - u_3$, is a generating set of differential invariants. For the purpose of rewriting any other differential invariants in terms of them we write every element of $\mathcal{I}^3$ in terms of $\mathcal{E}$.

From Theorem 3.6 we have $\mathcal{D}(\bar\iota u_2) = \bar\iota u_3 - \frac{1}{2}\,\bar\iota w$ since

$$K = \begin{pmatrix} 1 & 0 & 0 & \dfrac{\bar\iota w}{2\,\bar\iota u_2} & \bar\iota u_2 & \bar\iota u_2 \end{pmatrix}$$

while $\bar\iota\,(\mathrm{V}(u_2)) = \begin{pmatrix} 0 & 0 & 0 & \bar\iota u_2 & 0 & 0 \end{pmatrix}^T$. Thus

$$\bar\iota v_2 = \bar\iota u_2,\ \bar\iota u_3 = \mathcal{D}(\bar\iota u_2) + \frac{\bar\iota w}{2}, \ \text{ and } \bar\iota v_3 = \mathcal{D}(\bar\iota u_2) - \frac{\bar\iota w}{2}.$$

Note that $\bar\iota u_2$ is a differential invariant of order 2 and is therefore a function of the curvature, while $\bar\iota(u_3 - v_3)$, as a differential invariant of order 3 is a function of the curvature κ and the torsion τ. There are several ways to compute the algebraic expression for $\bar\iota u_2, \bar\iota u_3$ and $\bar\iota v_3$ (Fels and Olver, 1999; Hubert and Kogan, 2007a,b). But conversely, given the analytic expression for the curvature and the torsion (Berger and Gostiaux, 1988, (8.4.13.1) and

(8.6.10.2)) it is easy to write them in terms of $\bar\iota u_2, \bar\iota u_3$ and $\bar\iota v_3$ thanks to Theorem 2.10.

$$\kappa = \sqrt{2\,\bar\iota u_2^2}, \quad \tau = \frac{\bar\iota u_3 - \bar\iota v_3}{2\,\bar\iota u_2}.$$

Non minimal cross-section

We consider now the third order cross-section $P = (x, u_0, v_0, v_1, v_2, v_3 - 1)$. Olver (2007b) introduced it to show that the minimal order condition is necessary for Theorem 4.2.

As a consequence of Theorem 4.1, $\{\bar\iota u_1,\, \bar\iota u_2,\, \bar\iota u_3,\, \bar\iota u_4, \bar\iota v_4\}$ is a generating set of differential invariants. According to Theorem 5.14 the following functions form a complete set of differential syzygies.

$$\begin{aligned}
\mathcal{D}(\bar\iota u_1) &= \bar\iota u_2 + \frac{1+\bar\iota u_1{}^2}{3\,\bar\iota u_1}\left(\frac{\bar\iota u_3}{\bar\iota u_2} - \bar\iota v_4\right)\\
\mathcal{D}(\bar\iota u_2) &= 2\,\bar\iota u_3 - \bar\iota u_2\,\bar\iota v_4\\
\mathcal{D}(\bar\iota u_3) &= \bar\iota u_4 - \left(\frac{4}{3}\,\bar\iota u_3 + \frac{\bar\iota u_2{}^2}{\bar\iota u_1}\right)\bar\iota v_4 + \frac{\bar\iota u_1{}^2+1}{\bar\iota u_2} + \frac{4}{3}\,\frac{\bar\iota u_3{}^2}{\bar\iota u_2} + \frac{\bar\iota u_2\,\bar\iota u_3}{\bar\iota u_1}
\end{aligned}$$

From the two first equations we can deduce $\bar\iota u_3$ and $\bar\iota v_4$ in terms of $\{\bar\iota u_1,\, \bar\iota u_2\}$ and their derivatives. Substituting in the last equation we can do the same for $\bar\iota u_4$ so that $\{\bar\iota u_1, \bar\iota u_2\}$ is a generating set. Concomitantly, given their explicit expressions, we can write the curvature and the torsion in terms of those through Theorem 2.10:

$$\kappa = \sqrt{\frac{\bar\iota u_2^2}{(1+\bar\iota u_1^2)^3}}, \quad \tau = \frac{1}{\bar\iota u_2(1+\bar\iota u_1^2)}.$$

7 Three independent variables

The indefinite orthogonal group $O(m_1, m_2)$ is defined as the subgroup of $GL(m_1+m_2)$ that leaves the bilinear form $x_1^2+\ldots+x_{m_1}^2-x_{m_1+1}^2-\ldots-x_{m_1+m_2}^2$ invariant. The groups $O(m_1, m_2)$ and $O(m_1, m_2) \ltimes \mathbb{R}^{m_1+m_2}$ arise as symmetries of physical differential systems. For instance, $O(m, 0) \ltimes \mathbb{R}^m$ is a group of symmetry for the Laplacian, $u_{x_1x_1} + \ldots + u_{x_mx_m}$, while $O(m-1, 1) \ltimes \mathbb{R}^m$ is a symmetry group for the D'Alembert equation $u_{x_1x_1} - u_{x_2x_2} - \ldots - u_{x_mx_m}$. Their differential invariants of all orders were determined by Xu (1998).

In the case $m = m_1 + m_2 = 3$ we offer here a classification of the generating sets in the differential sense. As far as we know very few examples dealing with three independent variables have been studied by a moving frame approach. To provide those examples we have substantially applied our symbolic computation software AIDA (Hubert, 2007b). The corresponding worksheet is available at `http://www-sop.inria.fr/cafe/Evelyne.Hubert/aida/syzygies/`

7.1 *Linear action of $O(3-l,l)$ on the independent variables*

With the help of a parameter ϵ we shall treat both the orthogonal group $O(3,0)$ and $O(2,1)$ at once. When we specialize $\epsilon = 1$ we shall retrieve the result for $O(3,0)$ and when $\epsilon = -1$ we shall retrieve the result for $O(2,1)$. The moving frame approach was applied to this action of $O(3,0)$ in (Fels and Olver, 1999, Example 15.3) which provides us with a double check while providing the analysis for $O(2,1)$.

From the knowledge of the infinitesimal generators for the action of $O(3-l,l)$ and a choice of a (minimal order) cross-section we exhibit a complete set of syzygies for the second order normalized invariants. This is a direct application of Theorem 5.14. The set of second order normalized invariants is generating but so is the much smaller set of edge invariants. This allows us to prove that an alternative set of equal cardinality is also generating. This latter makes computations easier. By differential elimination on the syzygies of the second order normalized invariants we can retrieve a complete set of syzygies for this new and smaller generating set.

Those above computations are performed without the knowledge of the moving frame nor the explicit expression to the invariants. We nonetheless provide the expressions for those latter elements for illustration.

7.1.1 *Action*

We accordingly define $O(3-l,l)$, for $l = 0$ or 1, as the subgroup of $GL(3)$ that preserves the bilinear form $x_1^2 + x_2^2 + \epsilon\, x_3^2$, where $\epsilon = (-1)^l$. We consider its linear action on the independent variables $\{x_1, x_2, x_3\}$. The dependent variable is left unchanged by those transformations.

The infinitesimal generators are then:

$$\mathrm{V}_1^0 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}, \mathrm{V}_2^0 = x_3 \frac{\partial}{\partial x_1} - \epsilon\, x_1 \frac{\partial}{\partial x_3}, \mathrm{V}_3^0 = x_3 \frac{\partial}{\partial x_2} - \epsilon\, x_2 \frac{\partial}{\partial x_3}.$$

7.1.2 *Generators*

We choose the minimal order cross-section

$$\mathcal{P} :\ x_1 = 0,\, x_2 = 0,\, u_{100} = 0.$$

The edge invariants: $\mathcal{E} = \{\bar\iota x_3, \bar\iota u_{000}, \bar\iota u_{200}, \bar\iota u_{110}, \bar\iota u_{101}\}$ form a generating set (Theorem 4.2). Furthermore, since u_{000} is invariant, and therefore $\mathrm{V}(u_{000}) = 0$, we deduce from Theorem 3.6 that $\bar\iota u_{010} = \mathcal{D}_2(\bar\iota u_{000})$ and $\bar\iota u_{001} = \mathcal{D}_3(\bar\iota u_{000})$. It will be convenient to use the following set of generators:

$$\rho = \frac{1}{\bar\iota x_3},\ \sigma = \bar\iota u_{000},\ \phi = \frac{\bar\iota u_{200}}{\bar\iota u_{010}} - \epsilon \frac{\bar\iota u_{001}}{\bar\iota u_{010}\, \bar\iota x_3},\ \zeta = \frac{\bar\iota u_{110}}{\bar\iota u_{010}},\ \psi = \frac{\bar\iota u_{101}}{\bar\iota u_{010}}$$

as we can then write the matrix K of Theorem 3.6 as

$$K = \begin{pmatrix} \phi & \rho & 0 \\ \zeta & 0 & \rho \\ \psi & 0 & 0 \end{pmatrix}.$$

Hubert (2007a) actually shows that the entries of this matrix, the Maurer-Cartan matrix, together with $\mathcal{I}^0$, always form a generating set of invariants. It is arguably a more appropriate generating set for practical purposes. Here, for instance, it saves us dealing with denominators. We then deduce from Proposition 3.8 that:

$$[\mathcal{D}_1, \mathcal{D}_2] = \phi\, \mathcal{D}_1 + \zeta\, \mathcal{D}_2,\ [\mathcal{D}_1, \mathcal{D}_3] = \rho\, \mathcal{D}_1 + \psi\, \mathcal{D}_2,\ [\mathcal{D}_3, \mathcal{D}_2] = \psi\, \mathcal{D}_1 - \rho\, \mathcal{D}_2.$$

7.1.3 Syzygies

According to Theorem 5.14 a complete set of syzygies for $\mathcal{I}^2$ is given by:

$$\mathfrak{R}\,\{\ \bar\iota x_1 = 0, \quad \bar\iota x_2 = 0, \quad \bar\iota u_{100} = 0,$$

$$\mathfrak{S} \begin{cases} \mathcal{D}_1(\bar\iota x_3) = 0, \quad \mathcal{D}_2(\bar\iota x_3) = 0, \quad \mathcal{D}_3(\bar\iota x_3) = 1, \\ \mathcal{D}_1(\bar\iota u_{000}) = 0, \quad \mathcal{D}_2(\bar\iota u_{000}) = \bar\iota u_{010}, \quad \mathcal{D}_3(\bar\iota u_{000}) = \bar\iota u_{001}, \\ \mathcal{D}_1(\bar\iota u_{010}) = \bar\iota u_{110},\ \mathcal{D}_2(\bar\iota u_{010}) = \bar\iota u_{020} - \epsilon\, \rho\, \bar\iota u_{001},\ \mathcal{D}_3(\bar\iota u_{010}) = \bar\iota u_{011}, \\ \mathcal{D}_1(\bar\iota u_{001}) = \bar\iota u_{101},\ \mathcal{D}_2(\bar\iota u_{001}) = \bar\iota u_{011} + \rho\, \bar\iota u_{010},\ \mathcal{D}_3(\bar\iota u_{001}) = \bar\iota u_{002}, \end{cases}$$

and

$$\mathfrak{T} \begin{cases} \mathcal{D}_3(\bar\iota u_{011}) - \mathcal{D}_2(\bar\iota u_{002}) = \psi\, \bar\iota u_{101} - 2\, \rho\, \bar\iota u_{011}, \\ \mathcal{D}_3(\bar\iota u_{020}) - \mathcal{D}_2(\bar\iota u_{011}) = 2\, \psi\, \bar\iota u_{110} - \zeta\, \bar\iota u_{101} - \rho\, \bar\iota u_{020} + \epsilon\, \rho\, \bar\iota u_{002}, \\ \mathcal{D}_2(\bar\iota u_{101}) - \mathcal{D}_1(\bar\iota u_{011}) = -\zeta\, \bar\iota u_{011} - \phi\, \bar\iota u_{101}, \\ \mathcal{D}_3(\bar\iota u_{101}) - \mathcal{D}_1(\bar\iota u_{002}) = -\psi\, \bar\iota u_{011} - 2\, \rho\, \bar\iota u_{101}, \\ \mathcal{D}_2(\bar\iota u_{110}) - \mathcal{D}_1(\bar\iota u_{020}) = \zeta\, \bar\iota u_{200} - \zeta\, \bar\iota u_{020} - 2\, \phi\, \bar\iota u_{110} - \epsilon\, \rho\, \bar\iota u_{101}, \\ \mathcal{D}_3(\bar\iota u_{110}) - \mathcal{D}_1(\bar\iota u_{011}) = \psi\, \bar\iota u_{200} - \psi\, \bar\iota u_{020} - \phi\, \bar\iota u_{101} - \rho\, \bar\iota u_{110}, \\ \mathcal{D}_2(\bar\iota u_{200}) - \mathcal{D}_1(\bar\iota u_{110}) = -2\, \zeta\, \bar\iota u_{110} - \phi\, \bar\iota u_{200} + \phi\, \bar\iota u_{020} + \epsilon\, \rho\, \bar\iota u_{011}, \\ \mathcal{D}_3(\bar\iota u_{200}) - \mathcal{D}_1(\bar\iota u_{101}) = -2\, \psi\, \bar\iota u_{110} + \phi\, \bar\iota u_{011} - \rho\, \bar\iota u_{200} + \epsilon\, \rho\, \bar\iota u_{002}. \end{cases}$$

By differential elimination we can rewrite any normalized invariants of order 2 and less in terms of $\{\rho, \sigma, \psi, \phi, \zeta\}$ and find a complete set of syzygies for those. The first part

$$\bar{\iota}x_1 = 0,\ \bar{\iota}x_2 = 0,\ \bar{\iota}x_3 = \frac{1}{\rho},\ \bar{\iota}u_{000} = \sigma,$$
$$\bar{\iota}u_{100} = 0,\ \bar{\iota}u_{010} = \mathcal{D}_2(\sigma),\ \bar{\iota}u_{001} = \mathcal{D}_3(\sigma),$$
$$\bar{\iota}u_{200} = \phi\,\mathcal{D}_2(\sigma) + \epsilon\,\rho\,\mathcal{D}_3(\sigma),\ \bar{\iota}u_{110} = \zeta\,\mathcal{D}_2(\sigma),\ \bar{\iota}u_{101} = \psi\,\mathcal{D}_2(\sigma),$$
$$\bar{\iota}u_{020} = \mathcal{D}_2^2(\sigma) + \epsilon\,\rho\,\mathcal{D}_3(\sigma),\ \bar{\iota}u_{011} = \mathcal{D}_2\mathcal{D}_3(\sigma) - \rho\,\mathcal{D}_2(\sigma),\ \bar{\iota}u_{002} = \mathcal{D}_3^2(\sigma)$$

allows us to rewrite any other differential invariants in terms of $\{\rho, \sigma, \psi, \phi, \zeta\}$. The second part provides a complete set of syzygies for $\{\rho, \sigma, \psi, \phi, \zeta\}$:

$$\mathcal{D}_1(\zeta) - \mathcal{D}_2(\phi) = \zeta^2 + \phi^2 + \epsilon\,\rho^2,$$
$$\mathcal{D}_1(\psi) - \mathcal{D}_3(\phi) = \phi\,\rho + \psi\,\zeta,$$
$$\mathcal{D}_1(\sigma) = 0,\ \mathcal{D}_1(\rho) = 0,\ \mathcal{D}_2(\rho) = 0,\ \mathcal{D}_3(\rho) = -\rho^2$$

We observe that ζ can actually be written in terms of $\{\phi, \psi, \rho, \sigma\}$ so that this latter is already a generating set. The first two syzygies then become:

$$\begin{aligned}\psi\,(\mathcal{D}_1^2(\psi) - \mathcal{D}_1\mathcal{D}_3(\phi)) = \mathcal{D}_3(\phi)^2 + 2\,\mathcal{D}_1(\psi)^2 - 3\,\mathcal{D}_1(\psi)\mathcal{D}_3(\phi) + \phi\rho\,(2\,\mathcal{D}_3(\phi) - 3\,\mathcal{D}_1(\psi))\\ +\psi^2\mathcal{D}_2(\phi) + \psi\rho\mathcal{D}_1(\phi) + \phi^2\,\rho^2 + \epsilon\,\psi^2\,\rho^2 + \psi^2\phi^2\end{aligned}$$

since
$$\zeta = \frac{\mathcal{D}_1(\psi) - \mathcal{D}_3(\phi) - \phi\rho}{\psi}.$$

7.1.4 Generating differential invariants

For completion on this example, let us give the explicit expressions for a set of generating differential invariants. We split that into giving the expressions for $\{\bar{\iota}x_3, \bar{\iota}u_{001}, \bar{\iota}u_{010}\}$ as we can write $\{\rho, \sigma, \phi, \psi\}$ in terms of them. To determine $\{\bar{\iota}x_3, \bar{\iota}u_{001}, \bar{\iota}u_{010}\}$ we follow Hubert and Kogan (2007a,b) so as to compute global invariants through algebraic elimination. We accordingly avoid introducing radicals by giving the algebraic combinations of $\{\bar{\iota}x_3, \bar{\iota}u_{001}, \bar{\iota}u_{010}\}$ that are global invariants. The actual expression for $\{\bar{\iota}x_3, \bar{\iota}u_{001}, \bar{\iota}u_{010}\}$, with sign determination, should be deduced from those according to the point of the cross-section in the neighborhood of which we wish to work. In this neighbourhood $\bar{\iota}f$ and f must agree on the cross-section.

$$\bar\iota x_3{}^2 = x_3{}^2 + \epsilon\, x_1{}^2 + \epsilon\, x_2{}^2,$$

$$\bar\iota x_3\, \bar\iota u_{001} = x_3 u_{001} + x_1 u_{100} + x_2 u_{010},$$

$$\bar\iota x_3{}^2 \bar\iota u_{010}{}^2 = x_1{}^2 \left(u_{001}{}^2 + \epsilon\, u_{010}{}^2\right) + x_2{}^2 \left(u_{001}{}^2 + \epsilon\, u_{100}{}^2\right) + x_3{}^2 \left(u_{100}{}^2 + u_{010}{}^2\right)$$
$$-2\,\epsilon\, x_3 x_1 u_{100} u_{001} - 2\,\epsilon\, x_3 x_2 u_{001} u_{010} - 2\,\epsilon\, x_1 x_2\, u_{100}\, u_{010},$$

$$\bar\iota x_3{}^2 \bar\iota u_{010}{}^2 \bar\iota u_{200} = x_1{}^2 \left(u_{010}{}^2 u_{002} + u_{001}{}^2 u_{020} - 2\, u_{001} u_{011} u_{010}\right)$$
$$+x_2{}^2 \left(u_{002} u_{100}{}^2 - 2\, u_{001} u_{101} u_{100} + u_{200} u_{001}{}^2\right) + x_3{}^2 \left(u_{200} u_{010}{}^2 + u_{020} u_{100}{}^2 - 2\, u_{100} u_{110} u_{010}\right)$$
$$-2\,\epsilon\, x_3 x_1 \left(u_{100} u_{001} u_{020} + u_{101} u_{010}{}^2 - u_{110} u_{001} u_{010} - u_{100} u_{010} u_{011}\right)$$
$$-2\,\epsilon\, x_3 x_2 \left(u_{011} u_{100}{}^2 - u_{010} u_{101} u_{100} + u_{200} u_{001} u_{010} - u_{100} u_{110} u_{001}\right)$$
$$+x_2 x_1 \left(2\, u_{100} u_{011} u_{001} + 2\, u_{010} u_{101} u_{001} - 2\, u_{002} u_{100} u_{010} - 2\, u_{001}{}^2 u_{110}\right),$$

$$\bar\iota x_3{}^2 \bar\iota u_{010}\, \bar\iota u_{101} = \epsilon\, x_1{}^2 \left(-u_{101} u_{010} + u_{110} u_{001}\right) - \epsilon\, x_2{}^2 \left(u_{110} u_{001} - u_{011} u_{100}\right) + x_3{}^2 \left(u_{101} u_{010} - u_{011} u_{100}\right)$$
$$+x_3\, x_1 \left(u_{200} u_{010} - \epsilon\, u_{002} u_{010} + \epsilon\, u_{011} u_{001} - u_{110} u_{100}\right)$$
$$+x_3\, x_2 \left(u_{110} u_{010} - u_{020} u_{100} - \epsilon\, u_{101} u_{001} + \epsilon\, u_{002} u_{100}\right)$$
$$-\epsilon\, x_2\, x_1 \left(u_{200} u_{001} + u_{011} u_{010} - u_{020} u_{001} - u_{101} u_{100}\right)$$

7.1.5 Invariant derivations

We can obtain an expression, depending on ϵ for the moving frame of both the action of $O(3,0)$ and $O(2,1)$. It is given by the matrix A_ϵ below

$$\begin{pmatrix} \dfrac{x_3 u_{010} - \epsilon\, x_2 u_{001}}{\bar\iota u_{010}\, \bar\iota x_3} & \dfrac{\epsilon x_1 u_{001} - x_3 u_{100}}{\bar\iota u_{010}\, \bar\iota x_3} & \dfrac{x_2\, u_{100} - x_1 u_{010}}{\bar\iota u_{010}\, \bar\iota x_3} \\ a_\epsilon & b_\epsilon & c_\epsilon \\ \dfrac{\epsilon x_1}{\bar\iota x_3} & \dfrac{\epsilon x_2}{\bar\iota x_3} & \dfrac{x_3}{\bar\iota x_3} \end{pmatrix},$$

where

$$a_\epsilon = \frac{(x_3{}^2 + \epsilon x_2{}^2) u_{100} - \epsilon x_1\, x_2\, u_{010} - \epsilon x_1\, x_3 u_{001}}{\bar\iota x_3{}^2 \bar\iota u_{010}}$$
$$b_\epsilon = \frac{(x_3{}^2 + \epsilon x_1{}^2) u_{010} - \epsilon x_2 x_3\, u_{001} - \epsilon\, x_1 x_2 u_{100}}{\bar\iota x_3{}^2 \bar\iota u_{010}}$$
$$c_\epsilon = \frac{\epsilon\, (x_2{}^2 + x_1{}^2) u_{001} - x_3\, x_1 u_{100} - x_3\, x_2\, u_{010}}{\bar\iota x_3{}^2 \bar\iota u_{010}}.$$

When $\epsilon = 1$ the matrix belongs to $O(3,0)$ so that $A_1^{-1} = A_1^T$. The invariant derivations are then given by $\mathcal{D} = A_1 \mathrm{D}$. When $\epsilon = -1$ the matrix belongs to $O(2,1)$ and the invariant derivations are then given by $\mathcal{D} = A_{-1}^{-T} \mathrm{D}$.

7.2 *Affine action of $O(3-l,l) \ltimes \mathbb{R}^3$ on the independent variables*

The action of $E(3) = O(3,0) \ltimes \mathbb{R}^3$ was considered by Mansfield (2001) in the context of the symmetry reduction for a differential elimination problem.

We show here that the differential invariants of second order are generating. We provide a complete set of syzygies for a generating set of three second order differential invariants.

7.2.1 *Action*

Compared with the action of $O(3-l,l)$ treated above, we have additionally translation. The infinitesimal generators are then:

$$\mathrm{V}_1^0 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2},\ \mathrm{V}_2^0 = x_3 \frac{\partial}{\partial x_1} - \epsilon\, x_1 \frac{\partial}{\partial x_3},\ \mathrm{V}_3^0 = x_3 \frac{\partial}{\partial x_2} - \epsilon\, x_2 \frac{\partial}{\partial x_3},$$
$$\mathrm{V}_4^0 = \frac{\partial}{\partial x_1},\ \mathrm{V}_5^0 = \frac{\partial}{\partial x_2},\ \mathrm{V}_6^0 = \frac{\partial}{\partial x_3}.$$

7.2.2 *Generators*

We choose the minimal order cross-section

$$\mathcal{P}:\ x_1 = 0,\ x_2 = 0, x_3 = 0,\ u_{100} = 0,\ u_{010} = 0,\ u_{110} = 0.$$

The edge invariants $\mathcal{E} = \{\bar{\imath}u_{000}, \bar{\imath}u_{200}, \bar{\imath}u_{020}, \bar{\imath}u_{101}, \bar{\imath}u_{011},\ \bar{\imath}u_{210},\ \bar{\imath}u_{120},\ \bar{\imath}u_{111}\}$ thus form a generating set (Theorem 4.2). Furthermore, since u_{000} is invariant, and therefore $\mathrm{V}(u_{000}) = 0$, we know from Theorem 3.6 that $\bar{\imath}u_{001} = \mathcal{D}_3(\bar{\imath}u_{000})$. It will be convenient to use the following set of generators:

$$\sigma = \bar{\imath}u_{000},\ \phi = \epsilon \frac{\bar{\imath}u_{200}}{\bar{\imath}u_{001}},\ \psi = \epsilon \frac{\bar{\imath}u_{020}}{\bar{\imath}u_{001}},\ \kappa = \frac{\bar{\imath}u_{101}}{\bar{\imath}u_{001}},\ \tau = \frac{\bar{\imath}u_{011}}{\bar{\imath}u_{001}},$$

$$\Gamma = \frac{\bar{\imath}u_{200}\bar{\imath}u_{011} - \bar{\imath}u_{210}\bar{\imath}u_{001}}{\bar{\imath}u_{001}\,(\bar{\imath}u_{200} - \bar{\imath}u_{020})},\ \Lambda = \frac{\bar{\imath}u_{020}\bar{\imath}u_{101} - \bar{\imath}u_{120}\bar{\imath}u_{001}}{\bar{\imath}u_{001}\,(\bar{\imath}u_{200} - \bar{\imath}u_{020})},\ \Omega = \frac{2\,\bar{\imath}u_{011}\bar{\imath}u_{101} - \bar{\imath}u_{111}\bar{\imath}u_{001}}{\bar{\imath}u_{001}\,(\bar{\imath}u_{200} - \bar{\imath}u_{020})},$$

as we can then write the Maurer-Cartan matrix K of Theorem 3.6 as

$$K = \begin{pmatrix} \Gamma & \phi & 0 & 1 & 0 & 0 \\ \Lambda & 0 & \psi & 0 & 1 & 0 \\ \Omega & \epsilon\kappa & \epsilon\tau & 0 & 0 & 1 \end{pmatrix}.$$

We then deduce from Proposition 3.8 that:

$$[\mathcal{D}_1, \mathcal{D}_2] = \Gamma\, \mathcal{D}_1 + \Lambda\, \mathcal{D}_2, \quad [\mathcal{D}_1, \mathcal{D}_3] = \phi\, \mathcal{D}_1 + \Omega\, \mathcal{D}_2 - \kappa\, \mathcal{D}_3,$$
$$[\mathcal{D}_3, \mathcal{D}_2] = \Omega\, \mathcal{D}_1 - \psi\, \mathcal{D}_2 - \tau\, \mathcal{D}_3.$$

7.2.3 Syzygies

According to Theorem 5.14 a complete set of syzygies for $\mathcal{I}^2$ is given by:

$$\mathfrak{R}\,\{ \quad \bar{\iota}x_1 = 0, \quad \bar{\iota}x_2 = 0, \quad \bar{\iota}x_3 = 0, \quad \bar{\iota}u_{100} = 0, \quad \bar{\iota}\bar{\iota}u_{010} = 0, \quad \bar{\iota}u_{110} = 0,$$

$$\mathfrak{S}\begin{cases}
\mathcal{D}_1(\bar{\iota}u_{000}) = 0, \mathcal{D}_2(\bar{\iota}u_{000}) = 0, \mathcal{D}_3(\bar{\iota}u_{000}) = \bar{\iota}u_{00}, \\
\mathcal{D}_1(\bar{\iota}u_{001}) = \bar{\iota}u_{101}, \mathcal{D}_2(\bar{\iota}u_{001}) = \bar{\iota}u_{011}, \mathcal{D}_3(\bar{\iota}u_{001}) = \bar{\iota}u_{002}, \\
\mathcal{D}_1(\bar{\iota}u_{200}) = \bar{\iota}u_{300} - 2\,\phi\,\epsilon\,\bar{\iota}u_{101}, \mathcal{D}_2(\bar{\iota}u_{200}) = \bar{\iota}u_{210}, \mathcal{D}_3(\bar{\iota}u_{200}) = \bar{\iota}u_{201} - 2\,\kappa\,\epsilon^2\bar{\iota}u_{101}, \\
\mathcal{D}_1(\bar{\iota}u_{101}) = \bar{\iota}u_{201} - \Gamma\,\bar{\iota}u_{011} + \phi\,\bar{\iota}u_{200} - \phi\,\epsilon\,\bar{\iota}u_{002}, \mathcal{D}_2(\bar{\iota}u_{101}) = \bar{\iota}u_{111} - \Lambda\,\bar{\iota}u_{011}, \\
\qquad\qquad \mathcal{D}_3(\bar{\iota}u_{101}) = \bar{\iota}u_{102} - \Omega\,\bar{\iota}u_{011} + \kappa\,\epsilon\,\bar{\iota}u_{200} - \kappa\,\epsilon^2\bar{\iota}u_{002}, \\
\mathcal{D}_1(\bar{\iota}u_{020}) = \bar{\iota}u_{120}, \mathcal{D}_2(\bar{\iota}u_{020}) = \bar{\iota}u_{030} - 2\,\psi\,\epsilon\,\bar{\iota}u_{011}, \mathcal{D}_3(\bar{\iota}u_{020}) = +\bar{\iota}u_{021} - 2\,\tau\,\epsilon^2\bar{\iota}u_{011}, \\
\mathcal{D}_1(\bar{\iota}u_{011}) = \bar{\iota}u_{111} + \Gamma\,\bar{\iota}u_{101}, \mathcal{D}_2(\bar{\iota}u_{011}) = \bar{\iota}u_{021} + \Lambda\,\bar{\iota}u_{101} + \psi\,\bar{\iota}u_{020} - \psi\,\epsilon\,\bar{\iota}u_{002}, \\
\qquad\qquad \mathcal{D}_3(\bar{\iota}u_{011}) = \bar{\iota}u_{012} + \Omega\,\bar{\iota}u_{101} + \tau\,\epsilon\,\bar{\iota}u_{020} - \tau\,\epsilon^2\bar{\iota}u_{002}, \\
\mathcal{D}_1(\bar{\iota}u_{002}) = +\bar{\iota}u_{102} + 2\,\phi\,\bar{\iota}u_{101}, \mathcal{D}_2(\bar{\iota}u_{002}) = +\bar{\iota}u_{012} + 2\,\psi\,\bar{\iota}u_{011}, \\
\qquad\qquad \mathcal{D}_3(\bar{\iota}u_{002}) = +\bar{\iota}u_{003} + 2\,\kappa\,\epsilon\,\bar{\iota}u_{101} + 2\,\tau\,\epsilon\,\bar{\iota}u_{011},
\end{cases}$$

and

$$
\mathfrak{T}\begin{cases}
\mathcal{D}_3(\bar\iota u_{012})-\mathcal{D}_2(\bar\iota u_{003})=\Omega\,\bar\iota u_{102}+2\,\kappa\,\epsilon\,\bar\iota u_{111}+2\,\tau\,\epsilon\,\bar\iota u_{021}-\tau\,\epsilon^2\bar\iota u_{003}-3\,\psi\,\bar\iota u_{012},\\
\mathcal{D}_3(\bar\iota u_{021})-\mathcal{D}_2(\bar\iota u_{012})=2\,\Omega\,\bar\iota u_{111}+\kappa\,\epsilon\,\bar\iota u_{120}+\tau\,\epsilon\,\bar\iota u_{030}-2\,\tau\,\epsilon^2\bar\iota u_{012}-\Lambda\,\bar\iota u_{102}\\
\qquad\qquad -2\,\psi\,\bar\iota u_{021}+\psi\,\epsilon\,\bar\iota u_{003},\\
\mathcal{D}_3(\bar\iota u_{030})-\mathcal{D}_2(\bar\iota u_{021})=3\,\Omega\,\bar\iota u_{120}-3\,\tau\,\epsilon^2\bar\iota u_{021}-2\,\Lambda\,\bar\iota u_{111}-\psi\,\bar\iota u_{030}+2\,\psi\,\epsilon\,\bar\iota u_{012},\\
\mathcal{D}_2(\bar\iota u_{102})-\mathcal{D}_1(\bar\iota u_{012})=+2\,\psi\,\bar\iota u_{111}-\Lambda\,\bar\iota u_{012}-\Gamma\,\bar\iota u_{102}-2\,\phi\,\bar\iota u_{111},\\
\mathcal{D}_3(\bar\iota u_{102})-\mathcal{D}_1(\bar\iota u_{003})=-\Omega\,\bar\iota u_{012}+2\,\kappa\,\epsilon\,\bar\iota u_{201}-\kappa\,\epsilon^2\bar\iota u_{003}+2\,\tau\,\epsilon\,\bar\iota u_{111}-3\,\phi\,\bar\iota u_{102},\\
\mathcal{D}_2(\bar\iota u_{111})-\mathcal{D}_1(\bar\iota u_{021})=\Lambda\,\bar\iota u_{201}-\Lambda\,\bar\iota u_{021}+\psi\,\bar\iota u_{120}-\psi\,\epsilon\,\bar\iota u_{102}-2\,\Gamma\,\bar\iota u_{111}-\phi\,\bar\iota u_{120},\\
\mathcal{D}_3(\bar\iota u_{111})-\mathcal{D}_1(\bar\iota u_{012})=\Omega\,\bar\iota u_{201}-\Omega\,\bar\iota u_{021}+\kappa\,\epsilon\,\bar\iota u_{210}-\kappa\,\epsilon^2\bar\iota u_{012}+\tau\,\epsilon\,\bar\iota u_{120}\\
\qquad\qquad -\tau\,\epsilon^2\bar\iota u_{102}-\Gamma\,\bar\iota u_{102}-2\,\phi\,\bar\iota u_{111},\\
\mathcal{D}_2(\bar\iota u_{120})-\mathcal{D}_1(\bar\iota u_{030})=2\,\Lambda\,\bar\iota u_{210}-\Lambda\,\bar\iota u_{030}-2\,\psi\,\epsilon\,\bar\iota u_{111}-3\,\Gamma\,\bar\iota u_{120},\\
\mathcal{D}_3(\bar\iota u_{120})-\mathcal{D}_1(\bar\iota u_{021})=2\,\Omega\,\bar\iota u_{210}-\Omega\,\bar\iota u_{030}-\kappa\,\epsilon^2\bar\iota u_{021}-2\,\tau\,\epsilon^2\bar\iota u_{111}-2\,\Gamma\,\bar\iota u_{111}-\phi\,\bar\iota u_{120},\\
\mathcal{D}_2(\bar\iota u_{201})-\mathcal{D}_1(\bar\iota u_{111})=\psi\,\bar\iota u_{210}-2\,\Lambda\,\bar\iota u_{111}-\Gamma\,\bar\iota u_{201}+\Gamma\,\bar\iota u_{021}-\phi\,\bar\iota u_{210}+\phi\,\epsilon\,\bar\iota u_{012},\\
\mathcal{D}_3(\bar\iota u_{201})-\mathcal{D}_1(\bar\iota u_{102})=\kappa\,\epsilon\,\bar\iota u_{300}-2\,\Omega\,\bar\iota u_{111}-2\,\kappa\,\epsilon^2\bar\iota u_{102}+\tau\,\epsilon\,\bar\iota u_{210}+\Gamma\,\bar\iota u_{012}\\
\qquad\qquad -2\,\phi\,\bar\iota u_{201}+\phi\,\epsilon\,\bar\iota u_{003},\\
\mathcal{D}_2(\bar\iota u_{210})-\mathcal{D}_1(\bar\iota u_{120})=\Lambda\,\bar\iota u_{300}-2\,\Lambda\,\bar\iota u_{120}-\psi\,\epsilon\,\bar\iota u_{201}-2\,\Gamma\,\bar\iota u_{210}+\Gamma\,\bar\iota u_{030}+\phi\,\epsilon\,\bar\iota u_{021},\\
\mathcal{D}_3(\bar\iota u_{210})-\mathcal{D}_1(\bar\iota u_{111})=\Omega\,\bar\iota u_{300}-2\,\Omega\,\bar\iota u_{120}-2\,\kappa\,\epsilon^2\bar\iota u_{111}-\tau\,\epsilon^2\bar\iota u_{201}\\
\qquad\qquad -\Gamma\,\bar\iota u_{201}+\Gamma\,\bar\iota u_{021}-\phi\,\bar\iota u_{210}+\phi\,\epsilon\,\bar\iota u_{012},\\
\mathcal{D}_2(\bar\iota u_{300})-\mathcal{D}_1(\bar\iota u_{210})=2\,\Gamma\,\bar\iota u_{120}+2\,\phi\,\epsilon\,\bar\iota u_{111}-3\,\Lambda\,\bar\iota u_{210}-\Gamma\,\bar\iota u_{300},\\
\mathcal{D}_3(\bar\iota u_{300})-\mathcal{D}_1(\bar\iota u_{201})=2\,\phi\,\epsilon\,\bar\iota u_{102}-3\,\Omega\,\bar\iota u_{210}-3\,\kappa\,\epsilon^2\bar\iota u_{201}+2\,\Gamma\,\bar\iota u_{111}-\phi\,\bar\iota u_{300}.
\end{cases}
$$

By differential elimination we can rewrite any normalized invariants of order 2 and less in terms of $\{\Omega,\Lambda,\Gamma,\kappa,\phi,\tau,\psi,\sigma\}$ and find a complete set of syzygies for those. It is nonetheless easier to obtain the syzygies for the Maurer-Cartan invariants $\{\Omega,\Lambda,\Gamma,\kappa,\phi,\tau,\psi\}$ from the structure equations (Mansfield and van der Kamp, 2006; Hubert, 2007a). The first part

$$\bar{\iota}u_{001} = \mathcal{D}_3(\sigma)$$
$$\bar{\iota}u_{200} = \phi\epsilon\mathcal{D}_3(\sigma),\ \bar{\iota}u_{101} = \kappa\mathcal{D}_3(\sigma),\ \bar{\iota}u_{020} = \psi\epsilon\mathcal{D}_3(\sigma),\ \bar{\iota}u_{011} = \tau\mathcal{D}_3(\sigma),\ \bar{\iota}u_{002} = \mathcal{D}_3^2(\sigma),$$
$$\bar{\iota}u_{300} = \epsilon\mathcal{D}_3(\sigma)\left(\mathcal{D}_1(\phi) + 3\,\phi\kappa\right),\ \bar{\iota}u_{210} = \epsilon\mathcal{D}_3(\sigma)\left(\Gamma\psi - \Gamma\phi + \tau\phi\right),$$
$$\bar{\iota}u_{120} = \epsilon\mathcal{D}_3(\sigma)\left(\kappa\psi + \Lambda\psi - \Lambda\phi\right),\ \bar{\iota}u_{111} = \mathcal{D}_3(\sigma)\left(2\,\kappa\tau - \epsilon\Omega\phi + \epsilon\Omega\psi\right),$$
$$\bar{\iota}u_{102} = 2\,\kappa\mathcal{D}_3^2(\sigma) + \mathcal{D}_3(\kappa)\mathcal{D}_3(\sigma) + \Omega\tau\mathcal{D}_3(\sigma) - \phi\kappa\mathcal{D}_3(\sigma),$$
$$\bar{\iota}u_{201} = \phi\epsilon\mathcal{D}_3^2(\sigma) + \mathcal{D}_3(\sigma)\mathcal{D}_1(\kappa) - \epsilon\phi^2\mathcal{D}_3(\sigma) + \Gamma\tau\mathcal{D}_3(\sigma) + \kappa^2\mathcal{D}_3(\sigma),$$
$$\bar{\iota}u_{030} = \epsilon\mathcal{D}_3(\sigma)\left(\mathcal{D}_2(\psi) + 3\,\psi\tau\right),$$
$$\bar{\iota}u_{021} = \epsilon\psi\mathcal{D}_3^2(\sigma) + \mathcal{D}_3(\sigma)\mathcal{D}_2(\tau) - \epsilon\psi^2\mathcal{D}_3(\sigma) - \Lambda\kappa\mathcal{D}_3(\sigma) + \tau^2\mathcal{D}_3(\sigma),$$
$$\bar{\iota}u_{012} = 2\,\tau\mathcal{D}_3^2(\sigma) + \mathcal{D}_3(\tau)\mathcal{D}_3(\sigma) - \Omega\kappa\mathcal{D}_3(\sigma) - \psi\tau\mathcal{D}_3(\sigma),$$
$$\bar{\iota}u_{003} = \mathcal{D}_3^3(\sigma) - 2\,\epsilon\kappa^2\mathcal{D}_3(\sigma) - 2\,\epsilon\tau^2\mathcal{D}_3(\sigma),$$

allows us to rewrite any other differential invariants in terms of $\{\Omega, \Lambda, \Gamma, \kappa, \phi, \tau, \psi, \sigma\}$. The second part consist of a complete set of syzygies for $\{\Omega, \Lambda, \Gamma, \kappa, \phi, \tau, \psi, \sigma\}$:

$$\mathcal{D}_2(\Gamma) - \mathcal{D}_1(\Lambda) = -\epsilon\,\phi\,\psi - \Gamma^2 - \Lambda^2,$$
$$\mathcal{D}_1(\Omega) - \mathcal{D}_3(\Gamma) = \Omega\,\Lambda + \phi\,\Gamma + \kappa\,\Omega + \phi\,\tau,$$
$$\mathcal{D}_3(\Lambda) - \mathcal{D}_2(\Omega) = \Omega\,\Gamma - \psi\,\Lambda - \tau\,\Omega + \kappa\,\psi,$$
$$\mathcal{D}_2(\phi) = (\psi - \phi)\ \Gamma, \qquad \mathcal{D}_1(\psi) = (\psi - \phi)\ \Lambda,$$
$$\mathcal{D}_2(\kappa) = \epsilon(\psi - \phi)\,\Omega - \tau\,\Lambda + \kappa\,\tau, \quad \mathcal{D}_1(\tau) = \epsilon\,(\psi - \phi)\,\Omega + \kappa\,\Gamma + \kappa\,\tau,$$
$$\mathcal{D}_3(\phi) - \epsilon\,\mathcal{D}_1(\kappa) = \epsilon\,\tau\,\Gamma - \epsilon\,\kappa^2 - \phi^2,$$
$$\mathcal{D}_2(\tau) - \epsilon\,\mathcal{D}_3(\psi) = \kappa\,\Lambda + \tau^2 + \epsilon\,\psi^2,$$
$$\mathcal{D}_1(\sigma) = 0,\ \mathcal{D}_2(\sigma) = 0.$$

We see that we can actually write the third order differential invariants $\{\Gamma, \Omega, \Lambda\}$ in terms of the second order differential invariants $\{\phi, \psi, \kappa, \tau\}$, so that this latter is already a generating set.

$$\Gamma = \frac{\mathcal{D}_2(\phi)}{\psi - \phi}, \qquad \Lambda = \frac{\mathcal{D}_1(\psi)}{\psi - \phi},$$

$$\Omega = \epsilon\,\frac{\tau\,\mathcal{D}_1(\psi)}{(\psi - \phi)^2} + \epsilon\,\frac{\mathcal{D}_2(\kappa)}{\psi - \phi} - \epsilon\frac{\tau\,\kappa}{\psi - \phi}$$

The coefficient of the commutation rules can now be expressed in terms of the first order derivatives of $\{\phi, \psi, \kappa, \tau\}$. We can therefore still apply the differential elimination of Hubert (2005b) to obtain a complete set of syzygies on the generating set $\{\phi, \psi, \kappa, \tau\}$. We obtain:

$$\begin{aligned}
\mathcal{D}_2(\tau) - \epsilon\, \mathcal{D}_3(\psi) &= \epsilon\, \psi^2 + \tau^2 + \frac{\kappa}{\psi-\phi}\, \mathcal{D}_1(\psi),\\
\mathcal{D}_1(\kappa) - \epsilon\, \mathcal{D}_3(\phi) &= \epsilon\, \phi^2 + \kappa^2 - \frac{\tau}{\psi-\phi}\, \mathcal{D}_2(\phi),\\
\mathcal{D}_1(\tau) - \mathcal{D}_2(\kappa) &= \tau\, \frac{\mathcal{D}_1(\psi)}{\psi-\phi} + \kappa\, \frac{\mathcal{D}_2(\phi)}{\psi-\phi},
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_2^2(\phi) - \mathcal{D}_1^2(\psi) &= \frac{\mathcal{D}_1(\psi)\, \mathcal{D}_1(\phi)}{\psi-\phi} + \frac{\mathcal{D}_2(\phi)\, \mathcal{D}_2(\psi)}{\psi-\phi} - 2\, \frac{\mathcal{D}_1(\psi)^2}{\psi-\phi} - 2\, \frac{\mathcal{D}_2(\phi)^2}{\psi-\phi} - \epsilon\, (\psi-\phi)\, \phi\psi,\\
\mathcal{D}_1^2(\tau) - \epsilon \mathcal{D}_2\mathcal{D}_3(\phi) &= \kappa\, \frac{\mathcal{D}_1\mathcal{D}_2(\phi)}{\psi-\phi} + 2\, \frac{\mathcal{D}_1(\tau)\mathcal{D}_1(\psi)}{\psi-\phi} + 2\, \epsilon \frac{\mathcal{D}_3(\phi)\mathcal{D}_2(\phi)}{\psi-\phi} - \epsilon \frac{\mathcal{D}_3(\psi)\mathcal{D}_2(\phi)}{\psi-\phi}\\
&\quad + \kappa\, \frac{\mathcal{D}_1(\phi)\mathcal{D}_2(\phi)}{(\psi-\phi)^2} - 3\, \kappa\, \frac{\mathcal{D}_1(\psi)\mathcal{D}_2(\phi)}{(\psi-\phi)^2} - \tau\, \frac{\mathcal{D}_2(\phi)^2}{(\psi-\phi)^2}\\
&\quad - \tau^2 \frac{\mathcal{D}_2(\phi)}{\psi-\phi} - 2\, \tau\, \kappa\, \frac{\mathcal{D}_1(\psi)}{\psi-\phi} - \epsilon\, \psi\, (\psi - 2\, \phi)\, \frac{\mathcal{D}_2(\phi)}{\psi-\phi} + 2\, \kappa\, \mathcal{D}_1(\tau) + \epsilon\, \tau\, \phi\, \psi.
\end{aligned}$$

From the first equation we see that κ can be written in terms of $\{\phi, \psi, \tau\}$. Substituting the expression for κ in the other three equations we obtain a complete set of syzygies for those. As the expression grow considerably we do not give them explicitly here.

We can actually compute the expressions for the normalized second order differential invariants by algebraic elimination (Hubert and Kogan (2007a,b)). Alternatively Fushchich and Yegorchenko (1992); Xu (1998) provided a functionally independent set of second order differential invariants for this action. They can be easily rewritten in terms of $\mathcal{I}^2$. With additional manipulation we can then find the expression for our generating set.

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