Single output dependent quadratic observability normal form

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SINGLE OUTPUT DEPENDENT QUADRATIC OBSERVABILITY NORMAL FORM

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Abstract: In this paper, two quadratic observability normal forms dependent on output respectively correspondent to drift dominant term and forced dominant term are firstly discussed. Moreover characteristic numbers are studied in order to simplify the calculation of these two normal forms.

1. INTRODUCTION

Normal form is a powerful tool to analyze the properties of dynamical systems, such as stability (Poincaré *et al.*, 1899), controllability (Kang *et al.*, 1992). In 2001, this concept was firstly introduced in (Boutat-Baddas *et al.*, 2001) in order to analyze the observability of dynamical system, in which the quadratic observability normal form was studied and the interests of this technique was highlighted. In 2005, a new observability normal form dependent on its output was studied in (Zheng *et al.*, 2005), and the necessary and sufficient conditions were proposed. Moreover, an extension to multi-outputs case has also been analyzed in (Boutat *et al.*, 2006). Since the observability normal form dependent on its output (or outputs) were exactly linear, another reasonable extension based on the work of (Boutat-Baddas *et al.*, 2001) and (Chabraoui *et al.*, 2003) is the study of quadratic observability normal form dependent on its output (or outputs).

Therefore, in this paper, we consider the following single input single output system:

$$\begin{cases} \dot{\zeta} = f(\zeta) + g(\zeta)u \\ y = h(\zeta) \end{cases} \tag{1}$$

where $\zeta \in D \subset \mathbb{R}^n$, $u \in \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ are analytic functions, and assume that for all $\zeta \in D$, we have $rank\left[dh, dL_f h, ..., dL_f^{n-1}h\right]^T = n$.

As an extension of our work in (Zheng *et al.*, 2005), if the theorem in (Zheng *et al.*, 2005) is not verified, we assume that system (1) can be transformed into the following form [1]:

$$\begin{cases} \dot{\eta} = \beta(y) + A(y)\eta + \gamma_y^{[2]}(\bar{\eta}) \\ \quad +B(y)u + \vartheta_y^{[1]}(\bar{\eta})u + O_y^{[3]}(\bar{\eta}, u) \\ y = C\eta \end{cases} \tag{2}$$

where $\eta \in \mathbb{R}^n$, $\bar{\eta} = [\eta_1, ...\eta_{n-1}]^T$,

$$\gamma_y^{[2]}(\bar{\eta}) = [\gamma_{y_1}^{[2]}(\bar{\eta}), ..., \gamma_{y_n}^{[2]}(\bar{\eta})]^T,$$
$$\vartheta_y^{[1]}(\bar{\eta}) = [\vartheta_{y_1}^{[1]}(\bar{\eta}), ..., \vartheta_{y_n}^{[1]}(\bar{\eta})]^T,$$

and for $1 \leq i \leq n$, $\gamma_{y_i}^{[2]}(\bar{\eta})$ and $\vartheta_{y_i}^{[1]}(\bar{\eta})$ are function of $\bar{\eta}$ with order 2 and 1 respectively parameterized by y, $\beta(y) = [\beta_1(y), ..., \beta_n(y)]^T$, and

$$A(y) = \begin{pmatrix} 0 & ... & 0 & 0 \\ \alpha_1(y) & ... & 0 & 0 \\ \vdots & \ddots & ... & \vdots \\ 0 & 0 & \alpha_{n-1}(y) & 0 \end{pmatrix},$$

[1] The necessary and sufficient condition for this transformation is still an open question.

and $B(y)$ is defined as follows:

$$B(y) = \begin{pmatrix} b(y), & 0, & \cdots, & 0 \end{pmatrix}^T \tag{3}$$

As it was shown in (Boutat-Baddas *et al.*, 2001) that the equivalence modulo an output injection is justified by the fact that the output injection can be canceled in the observation error dynamics. Therefore, by an output injection $\beta(y)$, system (2) is equivalent modulo an output injection to the following system

$$\begin{cases} \dot{x} = A(y)x + \gamma_y^{[2]}(\bar{x}) \\ \quad +B(y)u + \vartheta_y^{[1]}(\bar{x})u + O_y^{[3]}(\bar{x}, u) \\ y = x_n = Cx \end{cases} \tag{4}$$

where $\bar{x} = [x_1, \ldots, x_{n-1}]^T$.

Our problem is how to characterize the fact that all the quadratic terms can be canceled by a diffeomorphism? If this kind of diffeomorphism does not exist, then what's its normal form and its resonant terms?

2. QUADRATIC EQUIVALENCE MODULO AN OUTPUT INJECTION

Definition 1. System (4) is quadratically equivalent to system

$$\begin{cases} \dot{z} = A(y)z + \tilde{\gamma}_y^{[2]}(\bar{z}) \\ \quad +B(y)u + \bar{\vartheta}_y^{[1]}(\bar{z})u + O_y^{[3]}(\bar{z}, u) \\ y = Cz \end{cases} \tag{5}$$

if there exists a diffeomorphism of the following form:

$$z = x + \phi_y^{[2]}(\bar{x}) \tag{6}$$

which transforms the quadratic term $\gamma_y^{[2]}(\bar{x})$ into another quadratic term $\tilde{\gamma}_y^{[2]}(\bar{z})$, where $\phi_y^{[2]}(\bar{x}) = [\phi_{y,1}^{[2]}(\bar{x}), ..., \phi_{y,n}^{[2]}(\bar{x})]^T$ and $\phi_{y,i}^{[2]}(\bar{x})$ are the homogenous polynomials with order 2 in z.

Remark 1. i) In order to keep the output unchanged, we choose the output equal to x_n, which means the diffeomorphism $z = x + \phi_y^{[2]}(\bar{x})$ should verify $\phi_{y,n}^{[2]}(\bar{x}) = 0$.

ii) It should be noted that this choice is not obligatory. In fact, we can also choose $\phi_{y,n}^{[2]}(\bar{x}) = \phi_{y,n}^{[0]}(y)$, i.e., a function of the output.

Proposition 1. System (4) is quadratically equivalent to system (5) modulo an output injection, if and only if the following homologic equations are satisfied:

$$\begin{cases} \gamma_y^{[2]}(\bar{x}) + \Gamma_y^{[2]}(\bar{x}) = A(y)\phi_y^{[2]}(\bar{x}) + \tilde{\gamma}_y^{[2]}(\bar{x}) \\ \vartheta_y^{[1]}(\bar{x}) + \dfrac{\partial \phi_y^{[2]}(\bar{x})}{\partial x_1} b(y) = \bar{\vartheta}_y^{[1]}(\bar{x}) \end{cases} \tag{7}$$

where

$$\Gamma_y^{[2]}(\bar{x}) = \left[\frac{\partial \phi_y^{[2]}(\bar{x})}{\partial x_2}\alpha_1(y), \ \cdots, \ \frac{\partial \phi_y^{[2]}(\bar{x})}{\partial x_{n-1}}\alpha_{n-2}(y), \ 0, \ 0 \right] x.$$

Assume that $z = x + \phi_y^{[2]}(\bar{x})$, hence $\dot{z} = \dot{x} + \frac{\partial \phi_y^{[2]}(\bar{x})}{\partial x}\dot{x}$. According to equation (4) and (5), we have

$$\begin{aligned} \dot{z} &= \left[1 + \frac{\partial \phi_y^{[2]}(\bar{x})}{\partial x} \right] \left[\begin{array}{c} A(y)x + \gamma_y^{[2]}(\bar{x}) + B(y)u \\ +\vartheta_y^{[1]}(\bar{x})u + O_y^{[3]}(\bar{x}, u) \end{array} \right] \\ &= A(y)\left(x + \phi_y^{[2]}(\bar{x}) \right) + \tilde{\gamma}_y^{[2]}(\bar{x} + \phi_y^{[2]}(\bar{x})) \\ &\quad +B(y)u + \bar{\vartheta}_y^{[1]}(\bar{x})u + O_y^{[3]}(\bar{x}, u) \end{aligned}$$

So we obtain

$$\begin{cases} \gamma_y^{[2]}(\bar{x}) + \dfrac{\partial \phi_y^{[2]}(\bar{x})}{\partial x} A(y)x = A(y)\phi_y^{[2]}(\bar{x}) + \tilde{\gamma}_y^{[2]}(\bar{x}) \\ \vartheta_y^{[1]}(\bar{x}) + \dfrac{\partial \phi_y^{[2]}(\bar{x})}{\partial x} B(y) = \bar{\vartheta}_y^{[1]}(\bar{x}) \end{cases} \tag{8}$$

where

$$\begin{aligned} &\frac{\partial \phi_y^{[2]}(\bar{x})}{\partial x} A(y)x \\ &= \left[\frac{\partial \phi_y^{[2]}(\bar{x})}{\partial x_2}\alpha_1(y), \ \cdots, \ \frac{\partial \phi_y^{[2]}(\bar{x})}{\partial y}\alpha_{n-1}(y), \ 0, \ 0 \right] x \\ &= \Gamma_y^{[2]}(\bar{x}) + O_y^{[3]}(\bar{x}) \end{aligned}$$

and finally equation (8) becomes:

$$\begin{cases} \gamma_y^{[2]}(\bar{x}) + \Gamma_y^{[2]}(\bar{x}) = A(y)\phi_y^{[2]}(\bar{x}) + \tilde{\gamma}_y^{[2]}(\bar{x}) \\ \vartheta_y^{[1]}(\bar{x}) + \dfrac{\partial \phi_y^{[2]}(\bar{x})}{\partial x_1} b(y) = \bar{\vartheta}_y^{[1]}(\bar{x}). \end{cases}$$

3. QUADRATIC OBSERVABILITY NORMAL FORM DEPENDENT ON ITS OUTPUT

Since there exist two quadratic terms in system (5): $\tilde{\gamma}_y^{[2]}(\bar{z})$ and $\bar{\vartheta}_y^{[1]}(\bar{z})u$, we will study two normal form which correspond respectively to the drift term and forced term in this section.

3.1 Normal form with drift dominant term

In this subsection, we study the normal form correspondent to drift dominant term by simplifying the quadratic term $\tilde{\gamma}_y^{[2]}(\bar{z})$.

Theorem 1. Normal form correspondent to drift dominant term system (5) by quadratically equivalent modulo an output injection is in the following normal form:

$$\begin{cases} \dot{\xi} = \begin{pmatrix} \displaystyle\sum_{j \geq i=1}^{n-1} h_{ij}(y)\xi_i\xi_j \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} \displaystyle\sum_{i=1}^{n-1} \bar{c}_i^1(y)\xi_i \\ \vdots \\ \displaystyle\sum_{i=1}^{n-1} \bar{c}_i^{n-1}(y)\xi_i \\ \displaystyle\sum_{i=1}^{n-1} \bar{c}_i^{n}(y)\xi_i \end{pmatrix} u \\ \quad +A(y)\xi + B(y)u + O_y^{[3]}(\bar{\xi}, u) \\ y = \xi_n \end{cases} \tag{9}$$

Since the objective of this normal form is to make $\tilde{\gamma}_y^{[2]}(\bar{x}) = 0$, so the first homologic equation in (7) becomes

$$\gamma_y^{[2]}(\bar{x}) + \Gamma_y^{[2]}(\bar{x}) = A(y)\phi_y^{[2]}(\bar{x})$$

Define $\begin{cases} \phi_y^{[2]}(\bar{x}) = [\phi_{y,1}^{[2]}(\bar{x}), ..., \phi_{y,n}^{[2]}(\bar{x})]^T, \\ \gamma_y^{[2]}(\bar{x}) = [\gamma_{y,1}^{[2]}(\bar{x}), ..., \gamma_{y,n}^{[2]}(\bar{x})]^T, \\ \vartheta_y^{[1]}(\bar{x}) = \left[\vartheta_{y,1}^{[1]}(\bar{x}), ..., \vartheta_{y,n}^{[1]}(\bar{x})\right]^T, \end{cases}$ where

$$\phi_{y_n}^{[2]}(\bar{x}) = 0.$$

And we obtain

$$\begin{cases} \alpha_1(y)\phi_{y,1}^{[2]}(\bar{x}) = \gamma_{y,2}^{[2]}(\bar{x}) + \sum_{i=1}^{n-2}\left[\dfrac{\partial \phi_{y,2}^{[2]}(\bar{x})}{\partial x_{i+1}}\alpha_i(y)x_i\right] \\ \vdots \\ \alpha_{n-2}(y)\phi_{y,n-2}^{[2]}(\bar{x}) = \gamma_{y,n-1}^{[2]}(\bar{x}) \\ \qquad + \sum_{i=1}^{n-2}\left[\dfrac{\partial \phi_{y,n-1}^{[2]}(\bar{x})}{\partial x_{i+1}}\alpha_i(y)x_i\right] \\ \alpha_{n-1}(y)\phi_{y,n-1}^{[2]}(\bar{x}) = \gamma_{y,n}^{[2]}(\bar{x}) \end{cases} \tag{10}$$

and the first line of equation (7) gives

$$\gamma_{y,1}^{[2]}(\bar{x}) + \sum_{i=1}^{n-2}\left[\frac{\partial \phi_{y,1}^{[2]}(\bar{x})}{\partial x_{i+1}}x_i\alpha_i(y)\right] = 0 \tag{11}$$

Equation (10) can be used to deduce $\phi_{y,i}^{[2]}(\bar{x})$ in order to cancel the quadratic terms from $\gamma_{y,2}^{[2]}(\bar{x})$ to $\gamma_{y,n}^{[2]}(\bar{x})$ respectively. Moreover, if $\gamma_{y,i}^{[2]}(\bar{x})$ and $\phi_{y,i}^{[2]}(\bar{x})$ verify also equation (11), then this system can be quadratically linearizable. Otherwise, it gives the following resonant terms:

$$\gamma_{y,1}^{[2]}(\bar{x}) + \sum_{i=1}^{n-2}\alpha_i(y)\frac{\partial \phi_{y,1}^{[2]}(\bar{x})}{\partial x_{i+1}}x_i.$$

With $\vartheta_y^{[1]}(\bar{x}) + \frac{\partial \phi_y^{[2]}(\bar{x})}{\partial x_1}b(y) = \bar{\vartheta}_y^{[1]}(\bar{x})$, we have $\bar{\vartheta}_{y,n}^{[1]}(\bar{x}) = \vartheta_{y,n}^{[1]}(\bar{x})$. Assume $\bar{\vartheta}_{y,n}^{[1]}(\bar{x}) = \sum_{i=1}^{n-1}\bar{c}_i^n(y)x_i$ and $\vartheta_{y,n}^{[1]}(\bar{x}) = \sum_{i=1}^{n-1}c_i^n(y)x_i$, then $\bar{c}_i^n(y) = c_i^n(y)$. And it is not possible to cancel other components. Hence we obtain normal form (9).

In order to highlight the proposed method, we consider the following example.

Example 1. Consider the following system:

$$\begin{cases} \dot{x}_1 = \left(x_1^2 - 2x_3^2x_1x_2 + x_3^2x_2^2\right) \\ \qquad + \left(x_3 - 4x_3^2x_1 - 2x_3^4x_2\right)u \\ \dot{x}_2 = x_3x_1 + x_3^2x_2^2 - \left(x_3x_2 + 2x_3^2x_1\right)u \\ \dot{x}_3 = x_3x_2 + x_3^2x_1u \\ y = x_3 \end{cases} \tag{12}$$

where $y \neq 0$. According to equation (10), we have

$$\begin{cases} \phi_{y,3}^{[2]}(\bar{x}) = 0 \\ \alpha_2(y)\phi_{y,2}^{[2]}(\bar{x}) = \gamma_{y,3}^{[2]}(\bar{x}) \\ \alpha_1(y)\phi_{y,1}^{[2]}(\bar{x}) = \gamma_{y,2}^{[2]}(\bar{x}) + \alpha_1(y)\dfrac{\partial \phi_{y,2}^{[2]}(\bar{x})}{\partial x_2}x_1 \end{cases}$$

and we obtain $\begin{Bmatrix} z_1 = x_1 + x_3x_2^2 \\ z_2 = x_2 \\ z_3 = x_3 \end{Bmatrix}$. Hence the quadratic resonant terms are:

$$\gamma_{y,1}^{[2]}(\bar{x}) + \alpha_1(y)\frac{\partial \phi_{y,1}^{[2]}(\bar{x})}{\partial x_2}x_1 = x_1^2 + x_3^2x_2^2$$

Finally we have the following normal form:

$$\begin{cases} \dot{z}_1 = z_1^2 + z_3^2z_2^2 + \left(z_3 - 4z_3^2z_1 - 2z_3^4z_2\right)u \\ \qquad + O_y^{[3]}(z_1, z_2, u) \\ \dot{z}_2 = z_3z_1 - \left(z_3z_2 + 2z_3^2z_1\right)u + O_y^{[3]}(z_1, z_2, u) \\ \dot{z}_3 = z_3z_2 + z_3^2z_1u + O_y^{[3]}(z_1, z_2, u) \\ y = z_3 \end{cases} \tag{13}$$

3.2 Normal form with forced dominant term

This subsection is devoted to study another normal form by simplifying quadratic terms: $\bar{\vartheta}_y^{[1]}(\bar{z})u$, in system (5).

Theorem 2. Normal form forced dominant term of system (5) by quadratically equivalent modulo an output injection is in the following form:

$$\begin{cases} \dot{\xi} = \begin{pmatrix} \sum_{j \geq i=1}^{n-1} d_i^1(y)\xi_i\xi_j \\ \xi_1\sum_{i=1}^{n-1} d_i^2(y)\xi_i \\ \vdots \\ \xi_1\sum_{i=1}^{n-1} d_i^n(y)\xi_i \end{pmatrix} + u\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sum_{i=1}^{n-1}\bar{c}_i^n(y)\xi_i \end{pmatrix} \\ \qquad + A(y)\xi + B(y)u + O_y^{[3]}(\bar{\xi}, u) \\ y = \xi_n \end{cases} \tag{14}$$

Assuming $\phi_{y,n}^{[2]}(\bar{x}) = 0$, we have

$$\vartheta_{y,n}^{[1]}(\bar{x}) = \bar{\vartheta}_{y,n}^{[1]}(\bar{x})$$

and if we set

$$\vartheta_y^{[1]}(\bar{x}) + \frac{\partial \phi_y^{[2]}(\bar{x})}{\partial x_1}b(y) = 0 \tag{15}$$

then we obtain $\bar{\vartheta}_{y,1}^{[1]}(\bar{x}) = ... = \bar{\vartheta}_{y,n-1}^{[1]}(\bar{x}) = 0$.

According to the first homologic equation in (7),

$$\gamma_y^{[2]}(\bar{x}) + \Gamma_y^{[2]}(\bar{x}) = A(y)\phi_y^{[2]}(\bar{x}) + \tilde{\gamma}_y^{[2]}(\bar{x}) \tag{16}$$

we obtain

$$\begin{cases} \tilde{\gamma}_{y,1}^{[2]}(\bar{x}) = \gamma_{y,1}^{[2]}(\bar{x}) + \sum_{i=1}^{n-2}\left[\dfrac{\partial \phi_{y,1}^{[2]}(\bar{x})}{\partial x_{i+1}}x_i\alpha_i(y)\right] \\ \tilde{\gamma}_{y,2}^{[2]}(\bar{x}) + \alpha_1(y)\phi_{y,1}^{[2]}(\bar{x}) = \gamma_{y,2}^{[2]}(\bar{x}) \\ \qquad + \sum_{i=1}^{n-2}\left[\dfrac{\partial \phi_{y,2}^{[2]}(\bar{x})}{\partial x_{i+1}}\alpha_i(y)x_i\right] \\ \vdots \\ \tilde{\gamma}_{y,n-1}^{[2]}(\bar{x}) + \alpha_{n-2}(y)\phi_{y,n-2}^{[2]}(\bar{x}) = \gamma_{y,n-1}^{[2]}(\bar{x}) \\ \qquad + \sum_{i=1}^{n-2}\left[\dfrac{\partial \phi_{y,n-1}^{[2]}(\bar{x})}{\partial x_{i+1}}\alpha_i(y)x_i\right] \\ \tilde{\gamma}_{y,n}^{[2]}(\bar{x}) + \alpha_{n-1}(y)\phi_{y,n-1}^{[2]}(\bar{x}) = \gamma_{y,n}^{[2]}(\bar{x}) \end{cases}$$

From equation (15), the term $\phi_y^{[2]}(\bar{x})$ can be used to cancel all the quadratic terms form the second line to the last one, except for the terms $x_1 \sum_{i=1}^{n-1} d_i^j x_i,\ j \in [1, n]$, hence we obtain:

$$\tilde{\gamma}_y^{[2]}(\bar{x}) = \left(\sum_{j \geq i=1}^{n-1} d_i^1(y) x_i x_j,\ x_1 \sum_{i=1}^{n-1} d_i^2(y) x_i,\ \cdots,\ x_1 \sum_{i=1}^{n-1} d_i^n(y) x_i \right)^T$$

Finally we have the normal form (14).

The following example is to illustrate the proposed normal form.

Example 2. (Example 1 continue) According to the above method, by simple calculation, we have

$$\begin{cases} z_1 = x_1 + 2x_3 x_1^2 + 2x_3^3 x_1 x_2 + x_3 x_2^2 \\ z_2 = x_2 + x_1 x_2 + x_3 x_1^2 \\ z_3 = x_3 \end{cases} \tag{17}$$

with this diffeomorphism we have the following normal form:

$$\begin{cases} \dot{z}_1 = \left(1 + 2z_3^4\right) z_1^2 + z_3^2 z_2^2 + z_3 u \\ \qquad +O_y^{[3]}(z_1, z_2, u) \\ \dot{z}_2 = z_3 z_1 + \left(z_3 - 2z_3^2\right) z_1^2 - 2z_3^4 z_1 z_2 \\ \qquad +O_y^{[3]}(z_1, z_2, u) \\ \dot{z}_3 = z_3 z_2 - z_3 z_1 z_2 - z_3^2 z_1^2 + z_3^2 z_1 u \\ \qquad +O_y^{[3]}(z_1, z_2, u) \\ y = z_3 \end{cases} \tag{18}$$

4. CHARACTERISTIC NUMBERS

In order to simplify the calculation of diffeomorphism proposed before, a new method will be proposed in this section which permit us to determine the diffeomorphism (6) in a easer way, with which we need not to solve the homologic equation (7).

4.1 Characteristic numbers for normal form with drift dominant term

In order to simplify equation (7), we assume:

$$\begin{cases} \phi_{y,i}^{[2]}(\bar{x}) = x^T \phi_{y,i} x, \\ \tilde{\gamma}_{y,i}^{[2]}(\bar{x}) = x^T \tilde{\gamma}_{y,i} x, \\ \gamma_{y,i}^{[2]}(\bar{x}) = x^T \gamma_{y,i} x, \end{cases}$$

and

$$\begin{cases} \phi_{y,i} := \begin{pmatrix} \phi_{11}^i(y) & \cdots & \phi_{1,n-1}^i(y) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \phi_{1,n-1}^i(y) & \cdots & \phi_{n-1,n-1}^i(y) & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}, \\ \tilde{\gamma}_{y,i} := \begin{pmatrix} \tilde{\gamma}_{11}^i(y) & \cdots & \tilde{\gamma}_{1,n-1}^i(y) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \tilde{\gamma}_{1,n-1}^i(y) & \cdots & \tilde{\gamma}_{n-1,n-1}^i(y) & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}, \\ \gamma_{y,i} := \begin{pmatrix} \gamma_{11}^i(y) & \cdots & \gamma_{1,n-1}^i(y) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \gamma_{1,n-1}^i(y) & \cdots & \gamma_{n-1,n-1}^i(y) & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}. \end{cases}$$

We obtain $\Gamma_y^{[2]}(\bar{x}) = [\Gamma_{y,1}^{[2]}(\bar{x}), ..., \Gamma_{y,n}^{[2]}(\bar{x})]^T$ where $\Gamma_{y,i}^{[2]}(\bar{x}) = x^T \Gamma_{y,i} x$ and

$$\Gamma_{y,i} = \bar{A}^T(y) \phi_{y,i} + \phi_{y,i} \bar{A}(y),$$

where

$$\bar{A}(y) := \begin{pmatrix} 0 & \ldots & 0 & 0 & 0 \\ \alpha_1(y) & \ldots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \ldots & \alpha_{n-2}(y) & 0 & 0 \\ 0 & \ldots & 0 & 0 & 0 \end{pmatrix}.$$

Setting $A(y)\phi_y^{[2]}(\bar{x}) := x^T \bar{\phi}_y x$, for all x, homologic equation (7) can be written as follows:

$$x^T \gamma_y x + x^T \Gamma_y x = x^T \bar{\phi}_y x + x^T \tilde{\gamma}_y x$$

And we have

$$\gamma_y + \Gamma_y = \bar{\phi}_y + \tilde{\gamma}_y \tag{19}$$

if $\tilde{\gamma}_y = 0$, we can cancel all the quadratic terms in system (4).

Because $A(y)\phi_y^{[2]}(\bar{x}) = \begin{pmatrix} 0 \\ \alpha_1(y)\phi_{y,1}^{[2]}(\bar{x}) \\ \vdots \\ \alpha_{n-1}(y)\phi_{y,n-1}^{[2]}(\bar{x}) \end{pmatrix}$

if $\tilde{\gamma}_y^{[2]} = 0$, then

$$\begin{pmatrix} \gamma_{y,1} \\ \gamma_{y,2} \\ \vdots \\ \gamma_{y,n} \end{pmatrix} + \bar{A}^T(y) \begin{pmatrix} \phi_{y,1} \\ \phi_{y,2} \\ \vdots \\ \phi_{y,n} \end{pmatrix} + \begin{pmatrix} \phi_{y,1} \\ \phi_{y,2} \\ \vdots \\ \phi_{y,n} \end{pmatrix} \bar{A}(y) = \left(0,\ \alpha_1(y)\phi_{y,1},\ \cdots,\ \alpha_{n-1}(y)\phi_{y,n-1} \right)^T$$

we have

$$\begin{cases} \gamma_{y,1} + \bar{A}^T(y)\phi_{y,1} + \phi_{y,1}\bar{A}(y) = 0 \\ \alpha_i(y)\phi_{y,i} = \gamma_{y,i+1} + \bar{A}^T(y)\phi_{y,i+1} + \phi_{y,i+1}\bar{A}(y), \end{cases}$$

for $1 \leq i \leq n-1$.

Finally, by recurrence, we get

$$\phi_{y,i} = \sum_{k=0}^{n-1-i} \left\{ \frac{\sum_{j=0}^{k} \left[C_j^k \left(\bar{A}^T(y)\right)^{k-j} \gamma_{y,i+k+1} \bar{A}^j(y) \right]}{\prod_{m=0}^{k} \alpha_{i+m}} \right\} \tag{20}$$

for $1 \leq i \leq n-1$, where C_j^k denotes the combinatorial coefficient.

With this diffeomorphism, according to the following equality: $\vartheta_y^{[1]}(\bar{x}) + \frac{\partial \phi_y^{[2]}(\bar{x})}{\partial x_1} b(y) = \bar{\vartheta}_y^{[1]}(\bar{x})$ we have $\vartheta_{y,n}^{[1]}(\bar{x}) = \bar{\vartheta}_{y,n}^{[1]}(\bar{x})$. Setting

$$\begin{cases} \vartheta_{y,n}^{[1]}(\bar{x}) = \sum_{j=1}^{n-1} c_j^n(y) x_j, \\ \bar{\vartheta}_{y,n}^{[1]}(\bar{x}) = \sum_{j=1}^{n-1} \bar{c}_j^n(y) x_j, \end{cases}$$

then $\bar{c}_j^n(y) = c_j^n(y)$.

Because $\vartheta_y^{[1]}(\bar{x}) + \frac{\partial \phi_y^{[2]}(\bar{x})}{\partial x_1} b(y) = \bar{\vartheta}_y^{[1]}(\bar{x})$, assuming

$$\begin{cases} \vartheta_{y,i}^{[1]}(\bar{x}) = \sum_{j=1}^{n-1} c_j^i(y) x_j, \\ \bar{\vartheta}_{y,i}^{[1]}(\bar{x}) = \sum_{j=1}^{n-1} \bar{c}_j^i(y) x_j. \end{cases}$$

then, since $\phi_{y,i}^{[2]}(\bar{x}) = x^T \phi_{y,i} x$, we obtain:

$$\begin{cases} \bar{c}_j^i(y) = c_j^i(y) + 2b(y)\phi_{1,j}^i(y), \\ \bar{c}_j^n(y) = c_j^n(y) \end{cases} \tag{21}$$

for $1 \le i, j \le n-1$.

Definition 2. We define the characteristic matrix for system (4) as follows:

$$M_y = \gamma_{y,1} + \bar{A}^T(y)\phi_{y,1} + \phi_{y,1}\bar{A}(y) \tag{22}$$

and

$$C_y = \begin{pmatrix} \bar{c}_1^1(y) & \cdots & \bar{c}_{n-1}^1(y) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \bar{c}_1^n(y) & \cdots & \bar{c}_{n-1}^n(y) & 0 \end{pmatrix} \tag{23}$$

where M_y and C_y are only function of y.

We are now able to set the following theorem.

Theorem 3. Normal form correspondent to drift dominant term of system (5) is as follows:

$$\begin{cases} \dot{\xi} = \xi^T \begin{pmatrix} M_y \\ 0 \\ \vdots \\ 0 \end{pmatrix} \xi + A(y)\xi + B(y)u \\ \qquad +C_y \xi u + O_y^{[3]}(\bar{\xi}, u) \\ y = \xi_n \end{cases} \tag{24}$$

where M_y is defined in (22).

Remark 2. i) $M_y(i,j)$ in equation (24) depends on coefficients $h_{ij}(y)$ of equation (9) as follows:

$$\begin{cases} M_y(i,j) = M_y(j,i) = \frac{1}{2} h_{ij}(y), \ i < j \\ M_y(i,i) = h_{ii}(y) \end{cases}$$

ii) there are $3n(n-1)/2$ characteristic numbers in the normal form.

Example 3. (Example 1 continue) With the guide of the computation process proposed above, we have

$$M_y = \gamma_{y,1} + \bar{A}^T(y)\phi_{y,1} + \phi_{y,1}\bar{A}(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_3^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $C_y = \begin{pmatrix} -4x_3^2 & -2x_3^4 & 0 \\ -2x_3^2 & -x_3 & 0 \\ x_3^2 & 0 & 0 \end{pmatrix}$. Finally we get the same normal form correspondent to drift dominant term in (13).

4.2 Characteristic numbers for normal form with forced dominant term

With the same argument, in order to simplify equation (7), setting

$$\begin{cases} \vartheta_{y,n}^{[1]}(\bar{x}) = \sum_{j=1}^{n-1} c_j^n(y) x_j, \\ \bar{\vartheta}_{y,n}^{[1]}(\bar{x}) = \sum_{j=1}^{n-1} \bar{c}_j^n(y) x_j, \end{cases}$$

and

$$\bar{c}_j^n(y) = c_j^n(y). \tag{25}$$

Since $\vartheta_y^{[1]}(\bar{x}) + \frac{\partial \phi_y^{[2]}(\bar{x})}{\partial x_1} b(y) = \bar{\vartheta}_y^{[1]}(\bar{x})$, and

$$\phi_{y,i}^{[2]}(\bar{x}) = x^T \begin{pmatrix} \phi_{11}^i(y) & \cdots & \phi_{1,n-1}^i(y) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \phi_{1,n-1}^i(y) & \cdots & \phi_{n-1,n-1}^i(y) & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} x,$$

if we set $\phi_{1,j}^i(y) = -\frac{c_j^i(y)}{2b(y)} 1$, we obtain $\bar{\vartheta}_{y,1}^{[1]}(\bar{x}) = ... = \bar{\vartheta}_{y,n-1}^{[1]}(\bar{x}) = 0$. Then we have $\gamma_y + \Gamma_y = \bar{\phi}_y + \tilde{\gamma}_y$, which gives

$$\begin{cases} \gamma_{y,1} + \bar{A}(y)\phi_{y,1} + \phi_{y,1}\bar{A}(y) = \bar{\gamma}_{y,1} \\ \bar{\gamma}_{y,i+1} + \alpha_i(y)\phi_{y,i} = \gamma_{y,i+1} + \bar{A}^T(y)\phi_{y,i+1} + \phi_{y,i+1}\bar{A}(y) \end{cases}$$

for $1 \le i \le n-1$.

Defining

$$\Upsilon_{y,i} = \begin{pmatrix} \Upsilon_{11}^i(y) & \cdots & \Upsilon_{1,n-1}^i(y) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \Upsilon_{1,n-1}^i(y) & \cdots & \Upsilon_{n-1,n-1}^i(y) & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} = \sum_{k=0}^{n-1-i} \left\{ \frac{\sum_{j=0}^{k} \left[C_j^k \left(\bar{A}^T(y)\right)^{k-j} \begin{bmatrix} \gamma_{y,i+k+1} \\ -\tilde{\gamma}_{y,i+k+1} \end{bmatrix} \bar{A}^j(y) \right]}{\prod_{m=0}^{k} \alpha_{i+m}} \right\} \tag{26}$$

and

$$\bar{\Upsilon}_{y,i} = \begin{pmatrix} \bar{\Upsilon}_{11}^i(y) & \cdots & \bar{\Upsilon}_{1,n-1}^i(y) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \bar{\Upsilon}_{1,n-1}^i(y) & \cdots & \bar{\Upsilon}_{n-1,n-1}^i(y) & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \tag{27}$$

$$= \sum_{k=0}^{n-1-i} \left\{ \frac{\sum_{j=0}^{k} \left[C_j^k \left(\bar{A}^T(y)\right)^{k-j} \gamma_{y,i+k+1} \bar{A}^j(y) \right]}{\prod_{m=0}^{k} \alpha_{i+m}} \right\}$$

Because $\phi_{1,j}^i(y) = -\frac{c_j^i(y)}{2b(y)}$, $i, j \in [1, n-1]$, if we set

$$\phi_{l,s}^i(y) = \bar{\Upsilon}_{l,s}^i(y), i \in [1, n-1], \ l, s \in [2, n-1]$$

we have $\tilde{\gamma}_{l,s}^i = 0, i \in [2, n], \ l, s \in [2, n-1]$.

Hence we get the following diffeomorphism:

$$\phi_{y,i} : \begin{cases} \phi_{1,j}^i(y) = -\frac{c_j^i(y)}{2b(y)}, \ i, j \in [1, n-1] \\ \phi_{l,s}^i(y) = \bar{\Upsilon}_{l,s}^i(y), i \in [1, n-1], \ l, s \in [2, n-1] \end{cases}$$

Because we have $\tilde{\gamma}_{l,s}^{i} = 0, i \in [2,n],\ l,s \in [2, n-1]$, then $\tilde{\gamma}_{1,j}^{i}, i \in [2,n], j \in [1, n-1]$ can be determined by the following equation:

$$\phi_{1,j}^{i}(y) = -\frac{c_j^i(y)}{2b(y)} = \Upsilon_{1,j}^{i}(y)$$

and we note

$$M_{y_k}(i,j) = M_{y_k}(j,i) = \begin{cases} \tilde{\gamma}_{1,j}^{k},\ i = 1 \\ 0,\ i \neq 1, n \end{cases}, \quad (28)$$

for $1 \leq j \leq n-1$ and $2 \leq k \leq n$.

Finally because $\gamma_{y,1} + \bar{A}(y)\phi_{y,1} + \phi_{y,1}\bar{A}(y) = \tilde{\gamma}_{y,1}$, we obtain:

$$M_{y,1}(i,j) = M_{y,1}(j,i) = \tilde{\gamma}_{i,j}^{1},\ i < j \quad (29)$$

Then we can give the following theorem.

Theorem 4. Normal form correspondent to forced dominant term of system (5) is as follows:

$$\begin{cases} \dot{\xi} = \xi^T \begin{pmatrix} M_{y,1} \\ M_{y,2} \\ \vdots \\ M_{y,n} \end{pmatrix} \xi + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sum_{i=1}^{n-1} \bar{c}_i^n(y)\xi_i \end{pmatrix} u \\ \quad + A(y)\xi + B(y)u + O_y^{[3]}(\bar{\xi}, u) \\ y = \xi_n \end{cases}$$

where $\bar{c}_i^n$ and $M_{y,i}$ are defined by equations (25), (28) and (29).

Remark 3. The number of the free coefficients is $3n(n-1)/2$.

Example 4. (Example 1 continue) Following the proposed calculation procedure, we can obtain

$$\phi_1(y) = \begin{bmatrix} 2x_3 & x_3^3 & 0 \\ x_3^3 & x_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \phi_2(y) = \begin{bmatrix} x_3 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \phi_3(y) = 0$$

and the following matrix M_y:

$$\begin{cases} M_{y,1} = \begin{pmatrix} 1+2x_3^4 & 0 & 0 \\ 0 & x_3^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, M_{y,2} = \begin{pmatrix} x_3 - 2x_3^2 & -x_3^4 & 0 \\ -x_3^4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ M_{y,3} = \begin{pmatrix} -x_3^2 & -\frac{1}{2}x_3 & 0 \\ -\frac{1}{2}x_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{cases}$$

which yields the same normal form (18).

5. CONCLUSION

This paper is devoted to study the quadratic observability normal form parameterized by its output. Above all, two homologic equations were given in order to guarantee the equivalence of quadratic transformation. After that, we have studied two normal forms respectively correspondent to the drift dominant term and forced dominant term. Both representations are equivalent. In order to simplify the calculation, we proposed to apply quadratic terms' characteristic matrix for these two normal forms.

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