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Abstract: We prove that the expected size of the 3D Delaunay triangulation of n points evenly distributed on a cylinder is $\Theta(n \log n)$. This shows that the $n\sqrt{n}$ behavior of the cylinder-example of Erickson [9] is pathological.

Key-words: Delaunay triangulation, random distribution, random sample, surface reconstruction

Ce travail préliminaire a été joint avec un travail parallèle de Jeff Erickson et sera publié à SODA 2008.

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L'échantillonnage aléatoire du cylindre aboutit à une triangulation de Delaunay raisonnable

Résumé : Nous montrons que la taille moyenne de la triangulation de Delaunay 3D de n points distribués aléatoirement sur un cylindre a une taille $\Theta(n \log n)$. L'exemple d'Erickson [9] de taille $n\sqrt{n}$ est donc très pathologique.

Mots-clés : Delaunay triangulation, échatinnonage aléatoire, reconstruction de surface

1 Introduction

Context. Delaunay triangulations and Voronoi diagrams are widely used in application areas, for instance in mesh generation [5], surface reconstruction [6] or molecular modeling [8] and, as such, are among the most extensively studied structures in computational geometry. Recently, considerable effort was devoted to understanding the discrepancy between the (tight) quadratic worst-case complexity of the Delaunay triangulation of n points in $\mathbb{R}^3$ and the near-linear behavior observed in most practical situation; we refer to the paper by Erickson [10] for an overview of these results.

Previous work. Among the various "realistic input models" considered, the case where the point set $\mathcal{P}$ samples a 2-dimensional surface Σ received considerable attention in the recent years; this model is essentially motivated by recent surface reconstruction algorithms [6] that start by computing a Delaunay triangulation of the set of sample points. Previous results of this flavor essentially discuss two types of samples: random samples, where the points are distributed uniformly on the surface, and (ϵ, κ)-samples, where the points need only satisfy that any ball of radius ϵ centered on the surface contains between 1 and κ points. Since a random sample is a $\left(\sqrt{\frac{\log n}{n}}, O(\log n)\right)$-sample with high probability [10], bounds for (ϵ, κ)-samples yield similar bounds for random samples.

For polyhedral surfaces, the complexity of the Delaunay triangulation is $O(n \text{ polylog } n)$ for random samples [11, 12] and $O(n)$ for (ϵ, κ)-samples [2, 3]. When the surface is smooth and generic, this complexity becomes $O(n \log^3 n)$ for random samples and $O(n \log n)$ for (ϵ, κ)-samples [4], where a surface is generic if it meets any medial ball in a constant number of points. In particular, surfaces containing a portion of a cylinder are not generic. Whether the complexity of the Delaunay triangulation of a sample of a smooth surface remains $O(n \log n)$ when the genericity assumption is dropped remains an open question. For "reasonable" surfaces of finite area, including cylinders of constant height, Erickson [10] proved that the complexity of the Delaunay triangulation of a random sample is $O(n\sqrt{n \log n})$. In an earlier paper [9], he also gave an example of a $(\sqrt{\frac{1}{n}}, 4)$ sample of a right circular cylinder of constant radius and height with Delaunay triangulation of complexity order $\Omega(n\sqrt{n})$.

Our results. In this paper, we prove:

Theorem 1. *The average complexity of the 3D Delaunay triangulation of n points uniformly distributed on a right circular cylinder of constant radius and height is $\Theta(n \log n)$.*

This indicates that it should be possible to relax the genericity hypotheses required by Attali et al. [4] and shows that the behavior of Erickson's example [9] is pathological among random samples of a cylinder. We also present experimental evidence that (i) the asymptotic behavior is reached for a small number of points, (ii) the constant hidden in the $\Theta()$ is actually small and (iii) a similar behavior is observed for points randomly distributed on an ellipsoid with two equal radii, another non-generic surface.

2 Preliminaries

We equip $\mathbb{R}^3$ with a cylindrical coordinate system (ρ, θ, z). We denote by $\mathcal{C}_\infty$ the cylinder $\rho = 1$ and by $\mathcal{C}$ its portion in-between the planes $z = -H$ and $z = H$, for a given constant H. We consider a set $\mathcal{P}$ of points independently and identically distributed uniformly on $\mathcal{C}$. The number of edges, faces and simplices of a 3D Delaunay triangulation depend linearly on one-another so the number of any of these objects can be used to measure the size of the triangulation. Recall that pq is a Delaunay edge of $\mathcal{P}$ if there exists a sphere S containing p and q and no other point from S. We denote by $\mathcal{D}(\mathcal{P})$ the graph whose vertices are the points in $\mathcal{P}$ and whose edges are the Delaunay edges of $\mathcal{P}$.

In a cylindrical coordinate system, the equation of a sphere S of radius r centered in $(R, 0, 0)$ is:

$$(\rho \cos\theta - R)^2 + \rho^2 \sin^2\theta + z^2 = r^2.$$

When $R - 1 < r$ this sphere intersects $\mathcal{C}_\infty$ in the curve (see Figure 1) of equation:

$$\rho = 1 \; ; \; z = \pm\sqrt{2R\cos\theta + r^2 - R^2 - 1}. \tag{1}$$

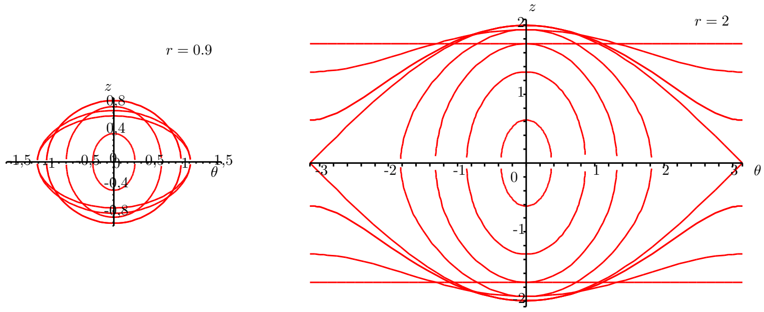

Figure 1: Intersection $\mathcal{C}_\infty \cap S$ (left) for $r < 1$ and (right) for $r > 1$ for different values of R..

We will use the derivatives of the upper part of $z(\theta)$ with respect to θ:

$$\frac{dz}{d\theta}(\theta) = -\frac{R\sin\theta}{\sqrt{2R\cos\theta + r^2 - R^2 - 1}} = -\frac{R\sin\theta}{z(\theta)}, \tag{2}$$

and

$$\frac{d^2z}{d\theta^2}(\theta) = -\frac{R^2\sin^2\theta}{(2\,R\cos\theta - 1 - R^2 + r^2)^{3/2}} - \frac{R\cos\theta}{(2\,R\cos\theta - 1 - R^2 + r^2)^{1/2}} = -\frac{R^2\sin^2\theta}{z(\theta)^3} - \frac{R\cos\theta}{z(\theta)}. \tag{3}$$

Lemma 2. *If S is a sphere centered in $(R,0,0)$ and with radius $r < R+1$ then $S \cap \mathcal{C}_\infty$ is a closed curve that is convex in the (θ, z) plane.*

Proof. We first prove that $\gamma = S \cap \mathcal{C}_\infty$ is a simple closed curve. The topology of the intersection of a sphere and a cylinder only changes as the sphere becomes tangent to the cylinder, so it is sufficient to prove this for $R = 1/2$ and $r = 1$. The symmetric matrices associated to the cylinder and the sphere S are then

$$A_C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad A_S = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{4} & 0 & 0 & -\frac{3}{4} \end{pmatrix},$$

and the characteristic polynomial of their pencil is:

$$\det(A_C + xA_S) = -\frac{1}{4}x(x+1)(4x^2 + 7x + 4).$$

Since this polynomial has exactly two real roots, the intersection is a smooth closed curve [15].

To prove the convexity of γ it is now sufficient to show that its second derivative does not vanish. By symmetry, we need only argue about the convexity of the upper branch, which is given by

$$z(\theta) = \sqrt{2R\cos\theta + r^2 - R^2 - 1}.$$

Since $z(\theta) \geq 0$ and $R > 0$, Equation (3) yields that

$$\frac{d^2z}{d\theta^2}(\theta) < 0 \quad \Leftrightarrow \quad -z(\theta)^2\cos\theta < R\sin^2\theta \quad \Leftrightarrow \quad P(\cos\theta) > 0,$$

where $P(t) = t^2 + 2\left(\frac{R^2+1-r^2}{2R}\right)t + 1$. As $r < R+1$ we have that $P(t) > (t-1)^2$ and $P(\cos\theta) > 0$. □

3 Triangulated slab graph and Rhombus graph

We now introduce two graphs on $\mathcal{P}$ which we use to prove Theorem 1.

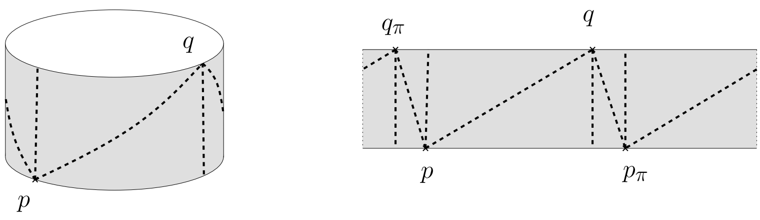

Figure 2: The triangulated slab of two points p and q of $\mathcal{C}$.

Triangulated slab graph. Given a point $p = (\theta_p, z_p, r_p) \in \mathbb{R}^3$ we denote by p_π its image by a symmetry with respect to the z axis, i.e. $p_\pi = (\theta_p + \pi, z_p, r_p)$. Let p and q be two points from $\mathcal{C}$, q being above p. The *slab* of p and q is the region of the cylinder in-between the horizontal planes through p and q, seen as a subset of $\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$. The vertical lines through p, q, p_π and q_π and the segments connecting pq, pq_π, $p_\pi q$ and $p_\pi q_\pi$ decompose the slab of p and q into eight triangles (see Figure 2). We call this partition the *triangulated slab of p and q*.
The *triangulated slab graph* of $\mathcal{P}$ is the graph on $\mathcal{P}$ where an edge connects two points if and only if some triangle in their triangulated slab contains no point from $\mathcal{P}$ other than p or q. We use the triangulated slab graph $\mathcal{T}(\mathcal{P})$ of $\mathcal{P}$ to obtain an upper bound on the size of its Delaunay graph:

Lemma 3. *For any point set $\mathcal{P}$ on $\mathcal{C}$, $\mathcal{D}(\mathcal{P}) \subset \mathcal{T}(\mathcal{P})$.*

Proof. Let (p, q) be a Delaunay edge. Let S be a sphere containing p and q and not containing any other point in $\mathcal{P}$. We denote by r the radius of S and assume, by symmetry of revolution, that its center is $(R, 0, 0)$. By symmetry between p and q, we also assume that $z_p \leq z_q$. From Equation (1), the upper part of $S \cap \mathcal{C}_\infty$ in the range $[-\pi, \pi]$ is given by:

$$z(\theta) = \sqrt{2R\cos\theta + r^2 - R^2 - 1}.$$

Its first derivative, given by Equation (2), has the sign of $-\theta$ and thus the curve is increasing on $[-\pi, 0]$ then decreasing on $[0, \pi]$. Also, since the second derivative, given by Equation (3), is negative if $\cos\theta \geq 0$, the function $z(\theta)$ is convex on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Let Γ be the set of points of $\mathcal{C}$ that have θ-coordinates in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and are inside S. This region is bounded by convex arcs of $\mathcal{C} \cap S$ and vertical segments and is thus a convex subset of $\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$. If $p \notin \Gamma$, then p_π has its θ-coordinate in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and, by monotonicity of the curve, $z_p \leq z(\theta_p + \pi)$. Thus, either p or p_π belongs to Γ. The same argument yields that exactly two points from $\{p, q, p_\pi, q_\pi\}$ are in Γ. Since Γ is convex, the segment joining these points is inside Γ. By symmetry, one of the triangles of the slab of p and q incident to this segment is also inside Γ. Therefore, (p, q) is an edge of the triangulated slab graph of $\mathcal{P}$. $\square$

Rhombus graph. Let p and q be two points in $\mathcal{P}$ such that $\theta_p \leq \theta_q \leq \theta_p + \pi$. The rhombus *associated with* p and q is the rhombus drawn on $\mathcal{C}$ centered at p, passing through q and such that the slopes of its sides have absolute value $\left|\frac{z_q - z_p}{\theta_q - \theta_p}\right|$ (see Figure 3).

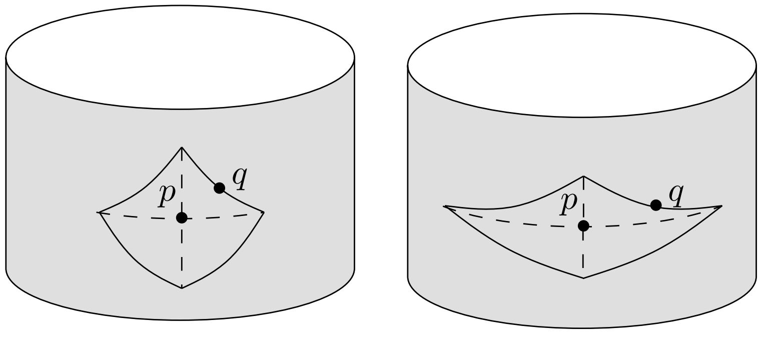

Figure 3: Two possible rhombus centered at p.

The *rhombus graph* $\mathcal{R}(\mathcal{P})$ of $\mathcal{P}$ is the graph on $\mathcal{P}$ where an edge connects two points p and q if and only if the rhombus associated with p and q does not contain any point from $\mathcal{P}$. We use the rhombus graph of $\mathcal{P}$ to obtain a lower bound on the size of its Delaunay graph:

Lemma 4. *For any point set $\mathcal{P}$ on $\mathcal{C}$, $\mathcal{R}(\mathcal{P}) \subset \mathcal{D}(\mathcal{P})$.*

Proof. Let pq be an edge of $\mathcal{R}(\mathcal{P})$ and let us denote by δ their associated rhombus. Without loss of generality we set the origin of the (θ, z) frame in the midpoint of pq. Notice that this implies that $\theta_q \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Let S be the sphere whose intersection γ with $\mathcal{C}_\infty$ is centered at this origin and that is tangent to δ in q. By symmetry around the θ-axis and the z-axis, the four points with coordinates $(\pm\theta_q, \pm z_q)$ belong to γ. Also, the tangents to γ in these points have slopes $\pm\frac{z_q}{\theta_q}$ and define a rhombus $\delta' \subset \delta$ (see Figure 4).

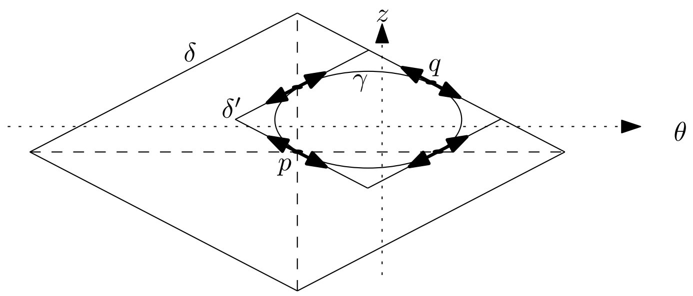

Figure 4: The rhombus δ' is inside δ.

From

$$z(\theta_q) = z_q \quad \text{and} \quad \frac{dz}{d\theta}(\theta_q) = -\frac{z_q}{\theta_q}$$

we deduce, using Equations (1) and (2):

$$R = \frac{z_q^2}{\theta_q \sin\theta_q} \quad \text{and} \quad r = \sqrt{(\cos\theta_q - R)^2 + \sin^2\theta_q + z_q^2}.$$

Since

$$r^2 = R^2 + 1 + \left(z_q^2 - 2R\cos\theta_q\right)$$

we get:

$$r < R+1 \quad \Leftrightarrow \quad R > \frac{z_q^2}{2(1+\cos\theta_q)} \quad \Leftrightarrow \quad 2(1+\cos\theta_q) > \theta_q \sin\theta_q.$$

The function $t \mapsto 2(1+\cos t) - t\sin t$ is positive on $[0, \frac{\pi}{2}]$ since it is decreasing on this interval and positive in $\frac{\pi}{2}$; by symmetry, it is thus positive on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and the inequality

$$2(1+\cos\theta_q) > \theta_q \sin\theta_q$$

holds. It follows that $r < R+1$ and Lemma 2 yields that γ is a a convex closed curve. This implies that γ is contained in δ' and therefore in δ. It follows that γ does not contain any point from $\mathcal{P}$ and that pq is an edge of $\mathcal{D}(\mathcal{P})$. □

4 Proof of Theorem 1

We first bound from below the complexity of the rhombus graph. The idea of the proof is based on the remark that in the set

$$\mathcal{P}' = \{(0,0)\} \cup \{(|\theta|, |z|); (\theta, z) \in \mathcal{P}\},$$

the neighbors of the origin in $\mathcal{R}(\mathcal{P}')$ are among the vertices of the convex hull of $\mathcal{P}' \setminus \{(0,0)\}$. It is well known that the expected complexity of the convex hull of n random points in a square is $\Theta(\log n)$ [14]. We show that the same holds for the expected degree of a point p in $\mathcal{R}(\mathcal{P})$.

Lemma 5. *The expected complexity of $\mathcal{R}(\mathcal{P})$ is $\Omega(n \log n)$.*

Proof. Two points $p, q \in \mathcal{P}$ are neighbors in $\mathcal{R}(\mathcal{P})$ only if the rhombus δ associated to these two points does not contain any point from $\mathcal{P}$:

$$Pr(pq \in \mathcal{R}(\mathcal{P})) \geq Pr\left(\delta \cap \mathcal{C} \text{ is empty}\right).$$

We consider the point p fixed and, without loss of generality, assume that it is at the origin of the θ-axis. Integrating over all possible values of (θ_q, z_q) we obtain:

$$Pr\left(pq \in \mathcal{R}(\mathcal{P})\right) \geq \int_{z=-H}^{z=H} \int_{\theta=-\pi}^{\theta=\pi} \left(1 - \frac{\text{area of } \delta \cap \mathcal{C}}{\text{area of } \mathcal{C}}\right)^{n-2} Pr\left((z_q, \theta_q) \in [z, z+dz] \times [\theta, \theta + d\theta]\right)$$

Since the area of $\delta \cap \mathcal{C}$ is always bounded from above by the area of δ, which is $8|\theta(z - z_p)|$, we get:

$$Pr\left(pq \in \mathcal{R}(\mathcal{P})\right) \geq \int_{z=-H-z_p}^{z=H-z_p} \int_{\theta=-\pi}^{\theta=\pi} \max\left(0, \left(1 - \frac{2|\theta z|}{\pi H}\right)^{n-2}\right) d\theta dz.$$

The expression $\left(1 - \frac{2|\theta z|}{\pi H}\right)^{n-2}$ is even with respect to z and for any value of $z_p \in [-H, H]$ the interval $[-H - z_p, H - z_p]$ contains $[-H, 0]$ or $[0, H]$. We can thus bound from below the previous probability as follows:

$$Pr\left(pq \in \mathcal{R}(\mathcal{P})\right) \geq \int_{z=0}^{z=H} \int_{\theta=-\pi}^{\theta=\pi} \max\left(0, \left(1 - \frac{2z|\theta|}{\pi H}\right)^{n-2}\right) d\theta dz \geq \int_{z=0}^{z=H} \int_{\theta=-\pi}^{\theta=\pi} \left(1 - \frac{2z|\theta|}{\pi H}\right)^{n-2} d\theta dz.$$

Then, integrating with respect to θ yields:

$$Pr\left(pq \in \mathcal{R}(\mathcal{P})\right) \geq \frac{2\pi H}{n-1} \int_{z=0}^{z=H} \left(1 - \left(1 - \frac{2z}{H}\right)^{n-1}\right) \frac{dz}{z}.$$

Integrating from $z = \frac{1}{n-1}$ instead of $z = 0$ and using the inequality $1 - t \leq e^{-t}$, we obtain

$$Pr\left(pq \in \mathcal{R}(\mathcal{P})\right) \geq \frac{2\pi H}{n-1} \int_{z=\frac{1}{n-1}}^{z=H} \left(1 - e^{-\frac{2z(n-1)}{H}}\right) \frac{dz}{z}.$$

and since the term $1 - e^{-\frac{2z(n-1)}{H}}$ is decreasing for $z \geq \frac{1}{n-1}$, we finally get:

$$Pr\left(pq \in \mathcal{R}(\mathcal{P})\right) \geq \frac{2\pi H}{n-1} \left(1 - e^{-\frac{2}{H}}\right) \int_{z=\frac{1}{n-1}}^{z=H} \frac{dz}{z} = \frac{2\pi H}{n-1} \left(1 - e^{-\frac{2}{H}}\right) \left(\log H + \log(n-1)\right).$$

Summing over all points q, we get that the expected degree of p in $\mathcal{R}(\mathcal{P})$ is $\Omega(\log n)$. Summing over all points p we get that the expected number of edges in $\mathcal{R}(\mathcal{P})$ is $\Omega(n \log n)$. □

We now bound from above the complexity of the triangulated slab graph:

Lemma 6. *The expected complexity of $\mathcal{T}(\mathcal{P})$ is $O(n \log n)$.*

Proof. We first bound the average degree of the triangulated slab graph by bounding the number of vertices $q \in \mathcal{P}$ *above* a given vertex p such that $pq \in \mathcal{T}(\mathcal{P})$. By symmetry of rotation, we assume that the θ-coordinate of p is zero. We have:

$$Pr(pq \in \mathcal{T}(\mathcal{P})) = \int_{z_p}^{H} \int_{-\pi}^{\pi} \quad Pr\left(pq \in \mathcal{T} \,|(z_q, \theta_q) = (z, \theta)\right)$$
$$*Pr\left((z_q, \theta_q) \in [z, z+dz] \times [\theta, \theta + d\theta]\right).$$

By symmetry, we can restrict ourselves to $\theta \geq 0$:

$$Pr(pq \in \mathcal{T}(\mathcal{P})) = 2 \int_{z_p}^{H} \int_{0}^{\pi} Pr\left(pq \in \mathcal{T} \,|(z_q, \theta_q) = (z, \theta)\right) d\theta dz.$$

We can bound from above

$$Pr\left(pq \in \mathcal{T}(\mathcal{P})\,|(z_q,\theta_q) = (z,\theta)\,\right) \leq \sum_{i=1}^{8} Pr(T_i(z,\theta) \text{ is empty})$$

where the T_i are the triangles of the slab of p and q. Each such triangle has height $z - z_p$ and width at least $\min(\theta, \pi - \theta)$. The area of T_i is thus at least $\frac{1}{2}\min(\theta,\pi-\theta)|z-z_p|$ and

$$Pr(pq \in \mathcal{T}(\mathcal{P})) \leq 16 \int_{z_p}^{H}\int_{0}^{\pi} Pr(\text{Triangle of area } \tfrac{1}{2}\min(\theta,\pi-\theta)|z-z_p| \text{ is empty}) d\theta dz$$

which rewrites as

$$Pr(pq \in \mathcal{T}(\mathcal{P})) \leq 32 \int_{z_p}^{H}\int_{0}^{\frac{\pi}{2}} \left(1 - \frac{\theta(z-z_p)}{8\pi H}\right)^{n-2} d\theta dz.$$

Since for $0 \leq t \leq 1$ we have $0 \leq (1-t) \leq e^{-t}$, we finally get

$$Pr(pq \in \mathcal{T}(\mathcal{P})) \leq 32 \int_{0}^{H-z_p}\int_{0}^{\frac{\pi}{2}} e^{-(n-2)\frac{\theta z}{8\pi H}} d\theta dz$$

Since $H - z_p \leq 2H$, the previous expression is bounded from above by

$$Pr(pq \in \mathcal{T}(\mathcal{P})) \leq 32 \int_{0}^{2H}\int_{0}^{\frac{\pi}{2}} e^{-(n-2)\frac{\theta z}{8\pi H}} d\theta dz$$

This integrates, e.g. using Maple [13], into

$$Pr(pq \in \mathcal{T}(\mathcal{P})) \leq \frac{2^8 \pi H}{n-2}\left(\ln(n-2) + \varphi(n)\right)$$

where:

$$\varphi(n) = \gamma + \ln(\pi) + Ei(1, \frac{n-2}{8}),$$

with the convention that γ denotes Euler's constant and $Ei(1,A)$ is $\int_0^{+\infty} \frac{e^{-At}}{t} dt$. The function $Ei(1,A)$ is bounded from above by (see [1, Chapter 5.1])

$$Ei(1,A) \leq e^{-A} \ln\left(1 + \frac{1}{A}\right)$$

so $\varphi(n)$ is $O(1)$ and we get that

$$Pr(pq \in \mathcal{T}(\mathcal{P})) = O\left(\frac{\log n}{n}\right).$$

Summing over all points $q \in \mathcal{P}$ above p, we get that the expected degree of p if $O(\log n)$. The expected size of the triangulated slab graph of $\mathcal{P}$ is thus $O(n \log n)$. □

We can now prove the main result of this paper: Theorem 1 follows from Lemmas 4 and 5 for the lower bound and Lemmas 3 and 6 for the upper bound.

5 Experiments

We now present experimental measures of the complexity of the 3D Delaunay triangulation of points randomly distributed on surfaces that are not generic in the sense of Attali et al. [4].

We consider two different surfaces: a right cylinder of radius 1 and height 1 and an ellipsoid with radii 1, 1 and 2. We distribute on each surface up to 2^{20} points uniformly and independently and construct their Delaunay triangulation incrementally. For $k = 1, \ldots, 20$, we measure the complexity of the triangulation after inserting 2^k points. The complexity is measured by the number of 3D cells of the triangulation. For each surface, we run 20 experiments. The computations were performed using CGAL [7].

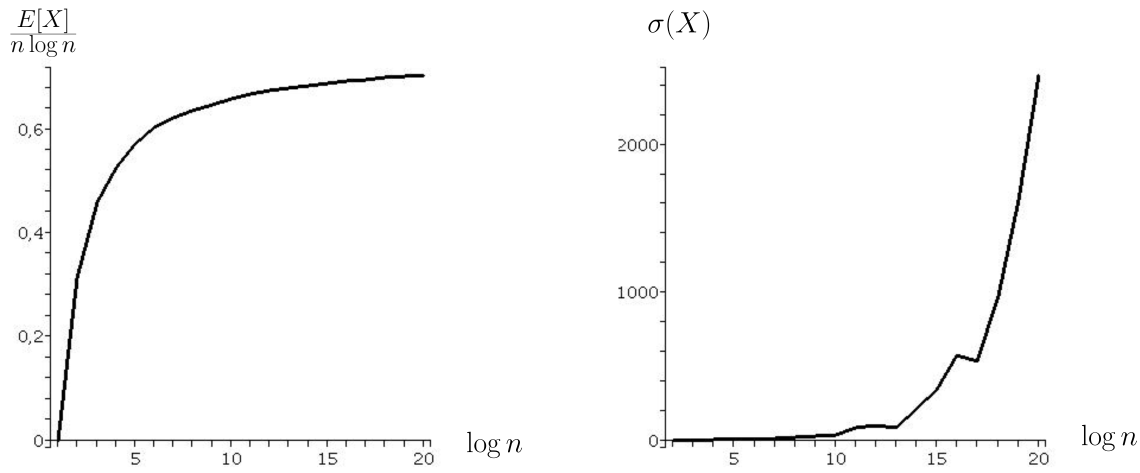

Figure 5: Size X of the Delaunay triangulation of n points distributed on a cylinder for $n = 2, \ldots, 2^{20}$: (left) ratio of the average size over $n \log n$ and (right) standard deviation. .

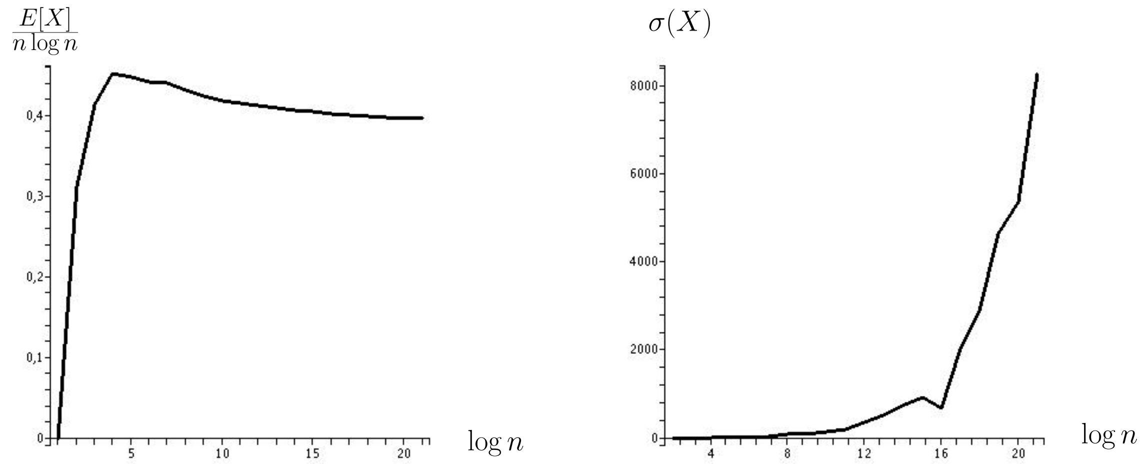

Figure 6: Size X of the Delaunay triangulation of n points distributed on an ellipsoid of radii 1, 1 and 2 for $n = 2, \ldots, 2^{21}$: (left) ratio of the average size over $n \log n$ and (right) standard deviation. .

The results are summarized in Figures 5 and 6.

In both Figure 5 and 6 the asymptotic behavior seems to be reached quickly, for less than a hundred points. Also, the constant hidden in the bound of Theorem 1 seems small. On Figure 6 we can observe that the behavior for another typical non-generic surface is similar to that of the cylinder. In both cases, the relatively small standard deviation suggests that the average behavior may even be attained with high probability.

6 Conclusion and open problems

We prove that for a specified 3D surface whose skeleton is one dimensional, namely a right cylinder, the Delaunay triangulation of n uniformly distributed points has complexity $\Theta(n \log n)$. Experimental evidence that this bound holds for other non-generic surfaces is given. This result suggests that the genericity hypotheses made on surfaces such as the one by Attali *et al.* [4] may possibly be relaxed and that Erickson's $O(n^{\frac{3}{2}})$ example [9] is pathological.

Open problems include proving that the complexity of the Delaunay triangulation of n points uniformly distributed on a smooth surface is $\Theta(n \log n)$. For generic smooth surfaces, the exact bound remains open and we conjecture $\Theta(n)$.

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