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Alexandre Seuret, Thierry Floquet, Jean-Pierre Richard, Sarah Spurgeon

OBSERVER DESIGN FOR SYSTEMS WITH NON SMALL AND UNKNOWN TIME-VARYING DELAY

Alexandre Seuret * **Thierry Floquet** **
Jean-Pierre Richard ** **Sarah K. Spurgeon** *

* *Control and Instrumentation Research Group, University of Leicester, University Road, Leicester LE1 7RH, UK.*
as389@leicester.ac.uk, eon@le.ac.uk
** *LAGIS, CNRS UMR 8146, Ecole Centrale de Lille, BP 48, 59651 Villeneuve d'Ascq Cedex, France.*
Équipe Projet ALIEN, Centre de Recherche INRIA Futurs.
thierry.floquet@ec-lille.fr, jean-pierre.richard@ec-lille.fr

Abstract: This paper deals with the design of observers for linear systems with unknown, time-varying, but bounded delays (on the state and on the input). In this work, the problem is solved for a class of systems by combining the unknown input observer approach with an adequate choice of a Lyapunov-Krasovskii functional for non small delay systems. This result provides workable conditions in terms of rank assumptions and LMI conditions. The dynamic properties of the observer are also analyzed. A 4th-order example is used to demonstrate the feasibility of the proposed solution.

Keywords: Sliding Mode Observer, Time-Delay Systems, Unknown Delay, Non Small Delay, Linear Matrix Inequalities.

1. INTRODUCTION

State observation is an important issue for both linear and nonlinear systems. This work considers the observation problem for the case of linear systems with non small and unknown delay. Several authors proposed observers for delay systems (see, e.g., (Sename, 2001; Richard, 2003)). Most of the literature, as pointed out in (Richard, 2003), considers that the value of the mainly constant delay can be used in the observer realization. This means that the delay is known or measured. Likewise, what is defined as "observers without internal delay" (Darouach, 2001; Darouach *et al.*, 1999; Fairmar *et al.*, 1999) involves the output knowledge at the present and delayed instants.

There are presently very few results in which the observer does not assume knowledge of the delay (Choi and Chung, 1997; de Souza *et al.*, 1999; Fridman *et al.*, 2003*b*; Seuret *et al.*, 2007; Wang *et al.*, 1999). These interesting approaches consider linear systems and guarantee an H_∞ performance. They are based on stability techniques independent of the delay and lead to the minimization of the state observation error. It is interesting to reduce the probable conservatism of such results by taking into account information on a delay upper-bound and derive an asymptotically stable observer.

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In (Seuret *et al.*, 2007), the authors design an observer using a computational delay which can be assimilated to an estimation of the delay $\hat{h}$. The conditions which guarantee the convergence of the error dynamics developed for this observer do not take into account the value of $\hat{h}$. This means that the estimation of the state is guaranteed whatever the delay estimation $\hat{h}$ is. The property is not related to the conservatism of the conditions. The errors between the real and computational delays are controlled by the discontinuous sliding function. The greater the error between the real and computational delays, the greater the gain of the discontinuous function will be. It is thus straightforward to conclude that a worse estimate of the delay can lead to a high gain in the sliding injection.

In this paper, another method is proposed to solve the problem of the observation of linear systems with unknown time-varying delays which are assumed to be "non small" i.e. the delay function lies in an interval excluding 0. The result is based on a combination of some results on sliding mode observers (see, e.g., (Barbot *et al.*, 1996; Edwards and Spurgeon, 1998; Floquet *et al.*, 2004; Perruquetti and Barbot, 2002)) with an appropriate choice of a Lyapunov-Krasovskii functional. The dynamical properties of the observer will also be discussed. For the sake of simplicity, the unknown time delay $\tau(t)$ is assumed to be the same for the state and the input. In order to reduce the conservatism of the developed conditions, the existence of known real numbers d, τ_1 and τ_2 is assumed such that $\forall t \in \mathbb{R}_+$:

$$\begin{array}{c} \tau_1 \leq \tau(t) \leq \tau_2 \\ \dot{\tau}(t) \leq d < 1. \end{array} \tag{1}$$

Here the delay used in the observer is the average of the delay $(\tau_2 + \tau_1)/2$. Then the design of the observer does not require the definition or the computation of a delay estimate and the stability conditions only depend on the parameters of the studied system.

Throughout the article, the notation $P > 0$ for $P \in \mathbb{R}^{n\times n}$ means that P is a symmetric and positive definite matrix. $[A_1|A_2|...|A_n]$ is the concatenated matrix with matrices A_i. I_n represents the $n \times n$ identity matrix.

2. PROBLEM STATEMENT

Consider the linear time-invariant system with state and input delay:

$$\begin{cases} \dot{x}(t) = & Ax(t) + A_\tau x(t-\tau(t)) \\ & +Bu(t) + B_\tau u(t-\tau(t)) + D\zeta(t) \\ y(t) = & Cx(t) \\ x(s) = & \phi(s), \quad \forall\, s \in [-\tau_2, 0] \end{cases} \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^q$ are the state vector, the input vector and the measurement vector, respectively. $\zeta \in \mathbb{R}^r$ is an unknown and bounded perturbation that satisfies:

$$\|\zeta(t)\| \leq \alpha_1(t, y, u), \tag{3}$$

where α_1 is a known scalar function. $\phi \in C^0([-\tau_2, 0], \mathbb{R}^n)$ is the vector of initial conditions. It is assumed that A, A_τ, B, B_τ, C and D are constant known matrices of appropriate dimensions. The following structural assumptions are required for the design of the observer:

A1. $\mathrm{rank}(C[A_\tau|B_\tau|D]) = \mathrm{rank}([A_\tau|B_\tau|D]) \triangleq p$,
A2. $p < q \leq n$,
A3. The invariant zeros of $(A, [A_\tau|B_\tau|D], C)$ lie in $\mathbb{C}^-$.

Under these assumptions and using the same linear change of coordinates as in (Edwards and Spurgeon, 1998), Chapter 6, the system can be transformed into:

$$\begin{cases} \dot{x}_1(t) = & A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t), \\ \dot{x}_2(t) = & A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t) \\ & +G_1x_1(t-\tau(t)) + G_2x_2(t-\tau(t)) \\ & +G_uu(t-\tau(t)) + D_1\zeta(t), \\ y(t) = & Tx_2(t), \end{cases} \tag{4}$$

where $x_1 \in \mathbb{R}^{n-q}$, $x_2 \in \mathbb{R}^q$ and where G_1, G_2, G_u, D_1 and A_{21} are defined by:

$$G_1 = \begin{bmatrix} 0 \\ \bar{G}_1 \end{bmatrix}, G_2 = \begin{bmatrix} 0 \\ \bar{G}_2 \end{bmatrix}, G_u = \begin{bmatrix} 0 \\ \bar{G}_u \end{bmatrix},$$
$$D_1 = \begin{bmatrix} 0 \\ \bar{D}_1 \end{bmatrix},\ A_{21} = \begin{bmatrix} A_{211} \\ A_{212} \end{bmatrix},$$

with $\bar{G}_1 \in \mathbb{R}^{p\times(n-q)}$, $\bar{G}_2 \in \mathbb{R}^{p\times q}$, $\bar{G}_u \in \mathbb{R}^{p\times m}$, $\bar{D}_1 \in \mathbb{R}^{p\times r}$, $A_{211} \in \mathbb{R}^{(q-p)\times(n-q)}$, $A_{212} \in \mathbb{R}^{p\times(n-q)}$ and T an orthogonal matrix involved in the change of coordinates given in (Edwards and Spurgeon, 1998).

Under these conditions, the system can be decomposed into two subsystems. $A1$ implies that the unmeasurable state x_1 is not affected by the delayed terms and the perturbations. $A3$ ensures that the pair (A_{11}, A_{211}) is at least detectable.

In this article, the following lemma will be used:

Lemma 1. (Hu *et al.*, 2004) For any matrices A, $P_0 > 0$ and $P_1 > 0$, the inequality

$$A^TP_1A - P_0 < 0,$$

is equivalent to the existence of a matrix Y such that:

$$\begin{bmatrix} -P_0 & A^TY^T \\ YA & -Y - Y^T + P_1 \end{bmatrix} < 0.$$

3. OBSERVER DESIGN

Define the following sliding mode observer:

$$\begin{cases} \dot{\hat{x}}_1(t) = & A_{11}\hat{x}_1(t) + A_{12}x_2(t) + B_1u(t) \\ & +(LT^TG_lT - A_{11}L)(x_2(t) - \hat{x}_2(t)) \\ & +LT^T\nu(t), \\ \dot{\hat{x}}_2(t) = & A_{21}\hat{x}_1(t) + A_{22}x_2(t) + B_2u(t) \\ & +G_1\hat{x}_1(t-h) + G_2x_2(t-h) \\ & +G_uu(t-h) - T^T\nu(t), \\ & -(A_{21}L + T^TG_lT)(x_2(t) - \hat{x}_2(t)) \\ \hat{y}(t) = & T\hat{x}_2(t), \end{cases} \tag{5}$$

where the linear gain G_l is a Hurwitz matrix and L has the form $\begin{bmatrix} \bar{L} & 0 \end{bmatrix}$ with $\bar{L} \in \mathbb{R}^{(n-q)\times(q-p)}$. The computed delay $h = (\tau_2+\tau_1)/2$ is an implemented value that is chosen according to the delay definition. It corresponds to the delay average. The discontinuous injection term ν is given by:

$$\nu(t) = \begin{cases} -\rho(t,y,u)\dfrac{P_y\,(y(t)-\hat{y}(t))}{\|P_y\,(y(t)-\hat{y}(t))\,\|} \\ \text{if} \quad y(t)-\hat{y}(t) \neq 0, \\ \\ 0 \\ \text{otherwise.} \end{cases} \tag{6}$$

where $P_y > 0$, $P_y \in \mathbb{R}^{p\times p}$ and where ρ is a nonlinear positive gain yet to be defined. Note that the non delayed terms depending on x_2 are known because $x_2(t) = T^Ty(t)$. Define $\mu = (\tau_2 - \tau_1)/2$.

Remark 1. Compared to (Seuret *et al.*, 2007), this observer does not required an artificial delay $\hat{h}$. It only needs to have an average value of the delay. Contrary to the observer proposed in (Seuret *et al.*, 2007), the implemented delay will appear in the conditions which guarantee stability.

Defining the state estimation errors as $e_1 = x_1(t) - \hat{x}_1(t)$ and $e_2 = x_2(t) - \hat{x}_2(t)$, one obtains:

$$\begin{cases} \dot{e}_1(t) = & A_{11}e_1(t) - LT^T\,(G_lTe_2(t) + \nu(t)) \\ & +A_{11}Le_2(t), \\ \dot{e}_2(t) = & A_{21}e_1(t) + G_1e_1(t-\tau(t)) \\ & +(T^TG_lT + A_{21}L)e_2(t) \\ & +T^T\nu + \xi(t) + D_1\zeta(t), \end{cases} \tag{7}$$

where $\xi_0 : \mathbb{R} \longmapsto \mathbb{R}^p$ is given by:

$$\begin{aligned} \xi(t) = & \; G_1(\hat{x}_1(t-\tau(t)) - \hat{x}_1(t-h)) \\ & +G_2(x_2(t-\tau(t)) - x_2(t-h)) \\ & +G_u(u(t-\tau(t)) - u(t-h)). \end{aligned}$$

which can be rewritten as:

$$\xi(t) = \begin{bmatrix} G_1 & G_2 & G_u \end{bmatrix} \int_{t-h}^{t-\tau(t)} \begin{bmatrix} \dot{\hat{x}}_1(s) \\ \dot{x}_2(s) \\ \dot{u}(s) \end{bmatrix} ds.$$

The function ξ only depends on the known variables $\hat{x}_1$, x_2 and u and on the unknown delay $\tau(t)$. One can then assume that there exists a known scalar function α_2 such that:

$$\|\xi(t)\| \leq \alpha_2(t, \hat{x}_1, x_2, u). \tag{8}$$

Let us define an expression for ρ in (6) by using results introduced in the case of control law design (Fridman *et al.*, 2003*a*). Define γ, a real positive number and ρ such that:

$$\rho(t,y,u) = \|D_1\|\alpha_1(t,y,u) + \alpha_2(t,\hat{x}_1,x_2,u) + \gamma, \tag{9}$$

Introduce the change of coordinates $\begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \end{bmatrix} = T_L \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ with $T_L = \begin{bmatrix} I_{n-q} & L \\ 0 & T \end{bmatrix}$. Using the fact that $LG_1 = LG_2 = LG_u = LD_1 = 0$, one obtains:

$$\begin{cases} \dot{\bar{e}}_1(t) = & (A_{11} + LA_{21})\bar{e}_1(t), \\ \dot{\bar{e}}_2(t) = & TA_{21}\bar{e}_1(t) + TG_1\bar{e}_1(t-\tau(t)) \\ & +G_l\bar{e}_2(t) - TG_1L\bar{e}_2(t-\tau(t)) \\ & +T\xi(t) + TD_1\zeta(t) + \nu, \end{cases} \tag{10}$$

Theorem 1. Under assumptions $A1 - A3$ and (8) and for all Hurwitz matrices G_l, system (10) is asymptotically stable for any delay $\tau(t)$ in $[\tau_1 \quad \tau_2]$ if there exist symmetric definite positive matrices P_1 and $R_1 \in \mathbb{R}^{(n-q)\times(n-q)}$, $P_2 \in \mathbb{R}^{q\times q}$, a symmetric matrix $Z_2 \in \mathbb{R}^{q\times q}$ and a matrix $W \in \mathbb{R}^{(n-q)\times(q-p)}$ such that the following linear matrix inequalities hold:

$$\begin{bmatrix} \Psi_0 & \Psi_1 & \Psi_1 & \Psi_2 & 0 \\ * & -2P_1 + hR_1 & 0 & 0 & 0 \\ * & * & -2P_1 + hR_{1a} & 0 & 0 \\ * & * & * & \Psi_3 & -P_2TG_1 \\ * & * & * & * & -P_1 \end{bmatrix} < 0. \tag{11}$$

where

$$\begin{aligned} \Psi_0 &= A_{11}^TP_1 + P_1A_{11} + A_{211}^TW^T + WA_{211}, \\ \Psi_1 &= A_{11}^TP_1 + A_{211}^TW^T, \\ \Psi_2 &= (A_{21} + G_1)^TT^TP_2, \\ \Psi_3 &= G_l^TP_2 + P_2G_l + hZ_2 + 2\mu Z_{2a} + R_2, \end{aligned}$$

and

$$\begin{aligned} &\begin{bmatrix} -(1-d)R_2 & [W \;\; 0]^T \\ [W \;\; 0] & -P_1 \end{bmatrix} < 0, \\ &\begin{bmatrix} R_1 & (TG_1)^TP_2 \\ P_2TG_1 & Z_2 \end{bmatrix} \geq 0, \\ &\begin{bmatrix} R_{1a} & (TG_1)^TP_2 \\ P_2TG_1 & Z_{2a} \end{bmatrix} \geq 0. \end{aligned} \tag{12}$$

The gain $\bar{L}$ is given by $\bar{L} = P_1^{-1}W$.

Proof. Consider the Lyapunov-Krasovskii functional:

$$\begin{aligned} V(t) = & \; \bar{e}_1^T(t)P_1\bar{e}_1(t) + \int_{-h}^{0}\int_{t+\theta}^{t} \dot{\bar{e}}_1^T(s)R_1\dot{\bar{e}}_1(s)dsd\theta \\ & + \int_{-\mu}^{\mu}\int_{t+\theta}^{t} \dot{\bar{e}}_1^T(s)R_{1a}\dot{\bar{e}}_1(s)dsd\theta \\ & +\bar{e}_2^T(t)P_2\bar{e}_2(t) + \int_{t-\tau(t)}^{t} \bar{e}_2^T(s)R_2\bar{e}_2(s)ds. \end{aligned} \tag{13}$$

The functional V can be divided into three parts. The first line of (13) is designed to control the errors $e_1(t)$ subject to the constant delay h. The second line presents a functional which takes into account the delay variation around the average delay h. The last part which appears in the last line of (13) controls the error $e_2(t)$.

Using the following transformation $\bar{e}_i(t-\tau(t)) = \bar{e}_i(t) - \int_{t-h}^{t} \dot{\bar{e}}_i(s)ds - \int_{t-\tau(t)}^{t-h} \dot{\bar{e}}_i(s)ds$, one has:

$$\begin{array}{l}\dot{V}(t) = \\ \bar{e}_1^T(t)[(A_{11}+LA_{21})^T P_1 + P_1(A_{11}+LA_{21})]\bar{e}_1(t) \\ +2\bar{e}_2^T(t)P_2T(A_{21}+G_1)\bar{e}_1(t) + \bar{e}_2^T(t)[G_l^T P_2 \\ +P_2G_l + R_2]\bar{e}_2(t) - 2\bar{e}_2^T(t)P_2TG_1L\bar{e}_2(t-\tau(t)) \\ -(1-\dot{\tau}(t))\bar{e}_2^T(t-\tau(t))R_2\bar{e}_2(t-\tau(t)) \\ +h\dot{\bar{e}}_1^T(t)R_1\dot{\bar{e}}_1(t) - \displaystyle\int_{t-h}^{t} \dot{\bar{e}}_1^T(s)R_1\dot{\bar{e}}_1(s)ds \\ +2\mu\dot{\bar{e}}_1^T(t)R_{1a}\dot{\bar{e}}_1(t) - \displaystyle\int_{t-\tau_1}^{t-\tau_2} \dot{\bar{e}}_1^T(s)R_{1a}\dot{\bar{e}}_1(s)ds \\ +\eta_1(t)+\eta_2(t)+\eta_3(t) - 2\rho(t,y,u)\|P_2\bar{e}_2(t)\|,\end{array}$$

where

$$\begin{aligned}\eta_1(t) &= -2\bar{e}_2^T(t)P_2TG_1\int_{t-h}^{t}\dot{\bar{e}}_1(s)ds,\\ \eta_2(t) &= -2\bar{e}_2^T(t)P_2TG_1\int_{t-\tau(t)}^{t-h}\dot{\bar{e}}_1(s)ds,\\ \eta_3(t) &= 2\bar{e}_2^T(t)P_2\left[TD_1\zeta(t)+T\xi(t)\right].\end{aligned}$$

The LMI condition (12) implies that for any vector X:

$$X^T\begin{bmatrix} R_1 & (TG_1)^T P_2 \\ P_2TG_1 & Z_2\end{bmatrix}X \geq 0.$$

Developing this relation for $X = \begin{bmatrix}\dot{\bar{e}}_1(s)\\ \bar{e}_2(t)\end{bmatrix}$, the following inequality holds:

$$\begin{aligned}-2\bar{e}_2(t)P_2G_1\dot{\bar{e}}_1(s) \leq\ & \bar{e}_2(t)^T Z_2\bar{e}_2(t)\\ &+\dot{\bar{e}}_1^T(s)R_1\dot{\bar{e}}_1(s).\end{aligned}$$

Then, an integration with respect to s of the previous inequality leads to an upper bound of $\eta_1(t)$:

$$\begin{aligned}\eta_1(t) &\leq \int_{t-h}^{t}\bar{e}_2^T(t)Z_2\bar{e}_2(t)ds\\ &\quad+\int_{t-h}^{t}\dot{\bar{e}}_1^T(s)R_1\dot{\bar{e}}_1(s)ds,\\ \eta_1(t) &\leq h\bar{e}_2^T(t)Z_2\bar{e}_2(t)\\ &\quad+\int_{t-h}^{t}\dot{\bar{e}}_1^T(s)R_1\dot{\bar{e}}_1(s)ds.\end{aligned} \tag{14}$$

By using the same techniques, an upper bound of η_2 is found:

$$\begin{aligned}\eta_2(t) \leq\ & 2\mu\bar{e}_2^T(t)Z_{2a}\bar{e}_2(t)\\ &+\int_{t-h-\mu}^{t-h+\mu}\dot{\bar{e}}_1^T(s)R_{1a}\dot{\bar{e}}_1(s)ds.\end{aligned} \tag{15}$$

From (9) and from the orthogonality of the matrix T, the following inequality holds:

$$\eta_3(t) - 2\rho(t,y,u)\|P_2\bar{e}_2(t)\| \leq -2\gamma\|P_2\bar{e}_2(t)\|. \tag{16}$$

Taking into account (14), (15), (16) and the fact that $\dot{\bar{e}}_1(t) = (A_{11}+\bar{L}A_{211})\bar{e}_1(t)$, $\dot{V}$ can be upperbounded as follows:

$$\begin{aligned}\dot{V}(t) \leq\ & \bar{e}_1^T(t)(P_1\tilde{A}_{11} + \tilde{A}_{11}^T P_1 + h\tilde{A}_{11}^T R_1\tilde{A}_{11}\\ &+2\mu\tilde{A}_{11}^T R_{1a}\tilde{A}_{11})\bar{e}_1(t) - 2\gamma\|P_2e_2(t)\|\\ &+\bar{e}_2^T(t)(P_2G_l + G_l^T P_2 + R_2 + hZ_2\\ &+2\mu Z_{2a})\bar{e}_2(t) + 2\bar{e}_2^T(t)P_2(A_{21}+G_1)\bar{e}_1(t)\\ &-(1-\dot{\tau}(t))\bar{e}_2^T(t-\tau(t))R_2\bar{e}_2(t-\tau(t))\\ &-2\bar{e}_2^T(t)P_2TG_1L\bar{e}_2(t-\tau(t))\end{aligned}$$

where $\tilde{A}_{11} = (A_{11}+\bar{L}A_{211})$.

Then, the last term of this inequality can be upperbounded by noting that:

$$\begin{aligned}&-2\bar{e}_2^T(t)P_2TG_1L\bar{e}_2(t-\tau(t))\\ &\quad\leq \bar{e}_2^T(t)P_2TG_1P_1^{-1}(TG_1)^T P_2\bar{e}_2(t)\\ &\qquad+\bar{e}_2^T(t-\tau(t))L^T P_1L\bar{e}_2(t-\tau(t))\\ &\quad\leq \bar{e}_2^T(t)P_2TG_1P_1^{-1}(TG_1)^T P_2\bar{e}_2(t)\\ &\qquad+\bar{e}_2^T(t-\tau(t))(P_1L)^T P_1^{-1}(P_1L)\bar{e}_2(t-\tau(t)),\end{aligned}$$

which leads to the following upperbound:

$$\begin{aligned}\dot{V}(t) \leq\ & \begin{bmatrix}\bar{e}_1(t)\\ \bar{e}_2(t)\end{bmatrix}^T\Psi\begin{bmatrix}\bar{e}_1(t)\\ \bar{e}_2(t)\end{bmatrix} - 2\gamma\|P_2\bar{e}_2(t)\|\\ &+\bar{e}_2^T(t-\tau(t))\psi_3\bar{e}_2(t-\tau(t)),\end{aligned} \tag{17}$$

where $\Psi = \begin{bmatrix}\psi_{10} & \Psi_2\\ * & \psi_{20}\end{bmatrix}$ and :

$$\begin{aligned}\psi_{10} =\ & (A_{11}+\bar{L}A_{211})^T P_1 + P_1(A_{11}+\bar{L}A_{211})\\ &+h(A_{11}+\bar{L}A_{211})^T R_1(A_{11}+\bar{L}A_{211})\\ &+2\mu(A_{11}+\bar{L}A_{211})^T R_{1a}(A_{11}+\bar{L}A_{211}),\\ \psi_{20} =\ & G_l^T P_2 + P_2G_l + R_2 + hZ_2 + 2\mu Z_{2a}\\ &+P_2TG_1P_1^{-1}(TG_1)^T P_2,\\ \psi_{30} =\ & (1-d)R_2 + (P_1\begin{bmatrix}\bar{L} & 0\end{bmatrix})^T P_1^{-1}(P_1\begin{bmatrix}\bar{L} & 0\end{bmatrix}).\end{aligned} \tag{18}$$

This matrix inequality is not an LMI because of the multiplication of matrix variables. Considering ψ_{20} and ψ_{30}, the Schur complement can remove these nonlinearities but for ψ_{10} Lemma 1 is required. As ψ_{10} must be negative definite to have a solution to the problem (17), the use of Lemma 1 is possible. Applying it twice to ψ_{10}, the nonlinear condition can be expressed as:

$$\begin{bmatrix}\Psi_0 & \bar{A}_{11}^T Y^T & \bar{A}_{11}^T Y_a^T & \Psi_2 & 0\\ * & \psi_2 & 0 & 0 & 0\\ * & * & \psi_3 & 0 & 0\\ * & * & * & \Psi_3 & -P_2TG_1\\ * & * & * & * & -P_1\end{bmatrix} < 0. \tag{19}$$

$$\begin{bmatrix}-(1-d)R_2 & \begin{bmatrix}\bar{L}^T P_1\\ 0\end{bmatrix}\\ * & -P_1\end{bmatrix} < 0. \tag{20}$$

where

$$\begin{aligned}\bar{A}_{11} &= A_{11}+\bar{L}A_{211}\\ \psi_2 &= -Y - Y^T + hR_1\\ \psi_3 &= -Y_a - Y_a^T + hR_{1a}\end{aligned}$$

Choosing $Y = P_1$, $Y_a = P_1$ and defining $W = P_1\bar{L}$, the LMI condition from the Theorem appears. Then, if (11) and (12) are satisfied, (19) and (20) are also satisfied. Finally the error dynamics are asymptotically stable and converge to the solution $e(t) = 0$. ∎

4. DYNAMIC PROPERTIES OF THE OBSERVER

Corollary 1. With the observer design in Theorem 1, an ideal sliding motion takes place on $S_0 = \{\bar{e}_2 = 0\}$ in finite time.

Proof. Consider the Lyapunov function:

$$V_2(t) = \bar{e}_2^T(t) P_2 \bar{e}_2(t) \tag{21}$$

Differentiating (21) along the trajectories of (10) yields:

$$\begin{aligned}\dot{V}_2(t) = {} & \bar{e}_2^T(t)(G_l^T P_2 + P_2 G_l)\bar{e}_2(t) \\ & +2\bar{e}_2^T(t) P_2 T \left[T^T \nu + A_{21}\bar{e}_1(t) \right. \\ & +G_1\bar{e}_1(t-\tau(t)) + G_1 L \bar{e}_2(t-\tau(t)) \\ & \left. +D_1\zeta(t) + \xi(t)\right].\end{aligned}$$

Noting that G_l is Hurwitz and (6), the following inequality holds:

$$\begin{aligned}\dot{V}_2(t) \leq {} & 2\|P_2\bar{e}_2(t)\| \left[\|A_{21}\bar{e}_1(t) + G_1\bar{e}_1(t-\tau(t)) \right. \\ & \left. +G_1 L\bar{e}_2(t-\tau(t))\| - \gamma\right].\end{aligned}$$

From Theorem 1, the errors $\bar{e}_1$ and $\bar{e}_2$ are asymptotically stable. There thus exists an instant t_0 and a real positive number δ such that $\forall t \geq t_0$, $\|A_{21}\bar{e}_1(t) + G_1\bar{e}_1(t-\tau(t)) + G_1 L\bar{e}_2(t-\tau(t))\| \leq \gamma - \delta$. This leads to:

$$\begin{aligned}\forall t \geq t_0,\ \dot{V}_2(t) & \leq -2\delta\|P_2\bar{e}_2(t)\| \\ & \leq -2\delta\sqrt{\lambda_{min}(P_2)}\sqrt{V_2(t)}.\end{aligned} \tag{22}$$

where $\lambda_{min}(P_2)$ is the lowest eigenvalue of P_2. Integrating the previous inequality shows that a sliding motion takes place on the manifold S_0 in finite time. ■

5. EXAMPLE

Consider the system with time-varying delay (4) and:

$$A_{11} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},\ A_{12} = \begin{bmatrix} -1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} 2 & 3 \\ 2 & -1 \end{bmatrix}, A_{22} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 0 & 0 \\ 0.1 & 0.21 \end{bmatrix},\ G_2 = \begin{bmatrix} 0 & 0 \\ 0.2 & 1 \end{bmatrix},$$

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, G_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ D_1 = B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

The delay is chosen as $\tau(t) = \tau_0 + \tau_1 sin(\omega_1 t)$, with $\tau_0 = 0.225s$, $\tau_1 = 0.075$ and $\omega_1 = 0.5s^{-1}$. The control law is

$$u(t) = u_0 sin(\omega_2 t)$$

with $u_0 = 2$ and $\omega_2 = 3$ and the Hurwitz matrix G_l is $\begin{bmatrix} -5 & 0 \\ 0 & -3 \end{bmatrix}$.

Using Theorem 1, the following observer gain is obtained:

$$\bar{L} = \begin{bmatrix} -0.1144 \\ 0.0280 \end{bmatrix}$$

Since the system (4) is open loop stable, its dynamics are bounded. Thus the function $\alpha_2(t, \hat{x}_1, x_2, u)$ can be chosen as a constant $K = 4$.

The simulation results are given in the following figures. Figure 1 shows the observation errors. Figures 2 and 3 show the comparison between the real and observed states.

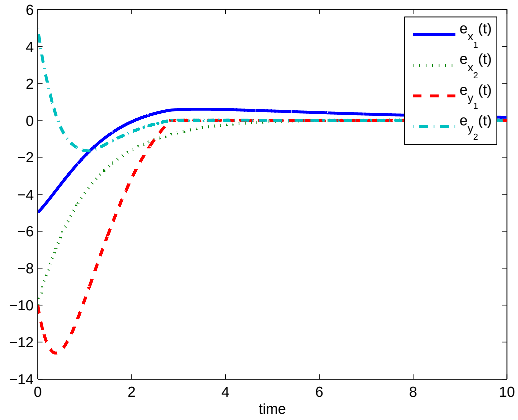

Fig. 1. Observation errors for $\tau_0 = 0.225$ and $\tau_1 = 0.075$

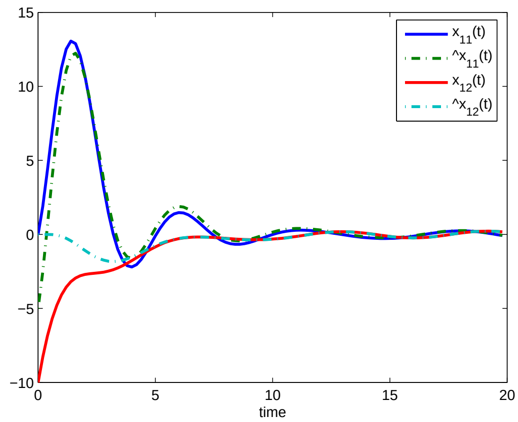

Fig. 2. x_1 and $\hat{x}_1$

Figure 1 shows the system enters a sliding motion at time $t = 2.8s$. The unmeasured variables converge asymptotically to 0.

6. CONCLUSION

The problem of designing observers for linear systems with non small and unknown variable delay on both the input and the state has been solved in this article. Delay-dependent LMI conditions have been found to guarantee asymptotic stability of the dynamical error system. The conditions only

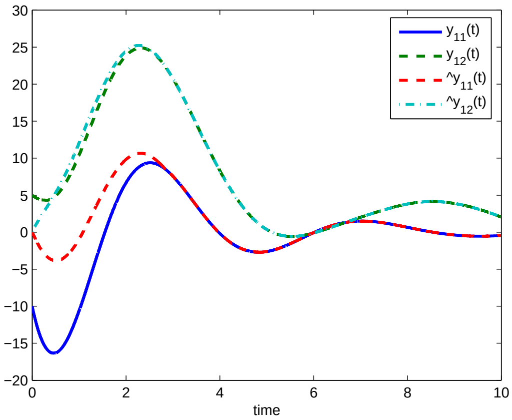

Fig. 3. x_2 and $\hat{x}_2$

depend on the real delay definition and do not require any estimated or computational delay. In addition, the dynamic properties of the proposed observer can be characterized.

REFERENCES

Barbot, J.-P., T. Boukhobza and M. Djemaï (1996). Sliding mode observer for triangular input form. *35th IEEE CDC'99, Conference on Decision and Control, Kobe, Japan* pp. 123–126.

Choi, H. H. and M. J. Chung (1997). Robust observer-based h_∞ controller design for linear uncertain time-delay systems. *Automatica* **33(9)**, 1749–1752.

Darouach, M. (2001). Linear functional observers for systems with delays in the state variables. *IEEE Trans. on Automatic Control* **46(3)**, 491–497.

Darouach, M., P. Pierrot and E. Richard (1999). Design of reduced-order observers without internal delays. *IEEE Trans. on Automatic Control* **44(9)**, 1711–1713.

de Souza, C.E., R.E. Palhares and P.L.D. Peres (1999). Robust h_∞ filtering for uncertain linear systems with multiple time-varying state: An lmi approach. *38th IEEE CDC'99, Conference on Decision and Control, Phoenix, AZ* pp. 2023–2028.

Edwards, C. and S.K. Spurgeon (1998). *Sliding Mode Control: Theory and Applications*. Taylor & Francis. London.

Fairmar, F.W., Kumar and E. Richard (1999). Design of reduced-order observers without internal delays. *IEEE Trans. on Automatic Control* **44(9)**, 1711–1713.

Floquet, T., J.-P. Barbot, W. Perruquetti and M. Djemaï (2004). On the robust fault detection via a sliding mode disturbance observer. *International Journal of control* **77(7)**, 622–629.

Fridman, E., F. Gouaisbaut, M. Dambrine and J.-P. Richard (2003*a*). Sliding mode control of systems with time-varying delays via a descriptor approach. *International Journal System and Science* **34(8-9)**, 553–559.

Fridman, E., U. Shaked and L. Xie (2003*b*). h_∞ filtering of linear systems with time-varying delay. *IEEE Trans. on Automatic Control* **48(1)**, 159–165.

Hu, L.-S., J. Huang and H.-H. Cao (2004). Robust digital model predicitve control for linear uncertain systems with saturations. *IEEE Trans. on Automatic Control* **49(5)**, 792–796.

Perruquetti, W. and J. P. Barbot (2002). *Sliding Mode Control in Engineering*. Ed. Marcel Dekker.

Richard, J.-P. (2003). Time delay systems: An overview of some recent advances and open problems. *Automatica* **39(10)**, 1667–1694.

Sename, O (2001). New trends in design of observers for time-delay systems. *Kybernetica* **37(4)**, 427–458.

Seuret, A., T. Floquet and J.P. Richard (2007). A sliding mode observer for linear systems with unknown time varying delay. *American Control Conference, New York, USA*.

Wang, Z., B. Huang and H. Unbehausen (1999). Robust h_∞ observer design for uncertain time-delay systems:(i) the continuous case. *IFAC 14th world congress, Beijing, China* pp. 231–236.