Second order sliding mode control of underactuacted mechanical systems II: Orbital stabilization of an inverted pendulum with application to swing up / balancing control

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Second Order Sliding Mode Control of Underactuated Mechanical Systems II: Orbital Stabilization of an Inverted Pendulum with Application to Swing Up/Balancing Control

Raul Santiesteban, Thierry Floquet, Yuri Orlov, Samer Riachy, and Jean-Pierre Richard[1]

Abstract

Orbital stabilization of an underactuated cart-pendulum system is under study. The quasihomogeneous control synthesis is utilized to design a second order sliding mode controller that drives the actuated cart to a periodic reference orbit in finite time, while the non-actuated pendulum produces bounded oscillations. A modified Van der Pol oscillator is introduced into the synthesis as an asymptotic generator of the periodic motion. The resulting closed-loop system is capable of moving from one orbit to another by simply changing the parameters of the Van der Pol modification. Performance issues of the proposed synthesis are illustrated in numerical and experimental studies of the swing up/balancing control problem of moving a pendulum, located on an actuated cart, from its stable downward position to the unstable inverted position and stabilizing it about the vertical.

1 Introduction

Motivated by applications where the natural operation mode is periodic, orbital stabilization of mechanical systems has received significant attention over the last few years (see, e.g., [18] and references therein). For these systems the orbital stabilization paradigm, referred to as periodic balancing [2], differs from typical formulations of output tracking where the reference trajectory to follow is known *a priori*. The control objective for the periodic balancing, e.g., a walking rabbit [4] is to result in the closed-loop system that generates its own periodic orbit similar to that produced by a nonlinear

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oscillator. Apart from this, the closed-loop system should be capable of moving from one orbit to another by simply modifying the orbit parameters such as frequency and/or amplitude.

In the present paper, a periodic balancing problem is under study for an underactuated mechanical system. A non-actuated pendulum, located on an actuated cart, appears as a special test bed. Orbital stabilization of this system captures all the essential features of the general treatment thereby forming a basis for the extension to other underactuated systems.

We demonstrate that the quasihomogeneous synthesis, developed in the first part of the paper [17], is applicable to the cart-pendulum system to design a Second Order Sliding Mode (SOSM) controller that drives an actuated part of the system to a periodic reference orbit in finite time in spite of the presence of friction forces and external disturbances with an *a priori* known magnitude bound. The resulting controller, that presents a contribution of the paper, exhibits an infinite number of switches on a finite time interval, however, in contrast to first order sliding mode controllers, it does not rely on the generation of sliding motions on the switching manifolds but on their intersections.

A modified Van der Pol oscillator, proposed in [15], is introduced into the synthesis as a reference model. The proposed modification still possesses a stable limit cycle, governed by a standard linear oscillator equation, and therefore it constitutes an asymptotic harmonic generator as opposed to a standard Van der Pol oscillator, exhibiting a non-sinusoidal response in its limit cycle.

Another example of an asymptotic harmonic generator (nearly the only one available in the literature) is the variable structure Van der Pol oscillator from [19]. However, it is hardly possible to use that oscillator for generating a reference signal because the system response would be contaminated by high frequency oscillations (a so-called chattering effect) caused by fast switching of the structure of the Van der Pol oscillator.

In contrast to a linear oscillator, whose amplitude depends on the initial conditions, both the amplitude and frequency of the sinusoidal signal, generating by the modified Van der Pol oscillator, can readily be modified on-line by simply changing the oscillator parameters. Due to this, the modified Van der Pol oscillator is well-suited for addressing the problem in question.

Effectiveness of the orbitally stabilizing synthesis is illustrated in numerical and experimental studies of the swing up/balancing control problem for a laboratory cart-pendulum system. In our studies, the pendulum, driven by a hybrid controller to be constructed, is required to move from its stable downward position to the unstable upright position and be stabilized about the vertical while the cart is stabilized about a desired endpoint.

The proposed hybrid controller is based on the orbital transfer strategy, similar to that used in [16] to swing up a Pendubot to its upright position. First, a swinging controller is composed by an inner loop controller, partially linearizing the cart-pendulum system, and an orbitally stabilizing outer loop controller, that pumps into the system as much energy as required to approach a homoclinic orbit with the same energy level as that corresponding to the desired equilibrium point. Once the cart-pendulum system reaches this homoclinic orbit, the orbitally stabilizing controller is turned off and the system is therefore obliged to evolve along the homoclinic orbit.

Finally, turning on a locally stabilizing controller, when the homoclinic motion enters the attraction basin of the latter controller, completes a unified framework for the swing up/balancing control of the pendulum, located on the cart. The locally stabilizing controller from [17], which is also based on quasihomogeneous considerations, is involved into our hybrid controller design. Being verified experimentally, the proposed framework is another contribution of the paper and it presents an interesting addition to the energy-based approach from [1, 12] to stabilization of mechanical systems with underactuation degree one.

The paper is organized as follows. The quasihomogeneous orbital stabilization of the cart-pendulum system is developed in Section 2. Numerical and experimental results on application of the orbitally stabilizing synthesis to the swing up/balancing control problem are given in Section 3. Section 4 finalizes the paper with some conclusions.

2 Orbitally Stabilizing Synthesis

In order to facilitate exposition, the orbitally stabilizing synthesis is developed for a laboratory cart-pendulum system from our companion paper [17].

2.1 Problem Statement

The cart-pendulum system, presented in [17], is governed by

$$\begin{aligned} (M+m)\ddot{x} + ml\sin\theta\,\dot{\theta}^2 - ml\cos\theta\,\ddot{\theta} &= \tau + w_1(t) - \psi(\dot{x}), && (1) \\ \frac{4}{3}ml^2\ddot{\theta} - ml\cos\theta\,\ddot{x} - mgl\sin\theta &= w_2(t) - \varphi(\dot{\theta}) && (2) \end{aligned}$$

where x is the cart position, θ is the angular deviation of the pendulum from the vertical, M is the cart mass, m is the rod mass, l is the distance to the center of mass of the pendulum, g is the gravitational acceleration, τ is the controlled input, $w_1(t), w_2(t)$ are external disturbances, $\psi(\dot{x})$ and $\varphi(\dot{\theta})$ are friction forces, affecting the cart and the pendulum, respectively.

In order to describe the friction forces the classical model is utilized:

$$\psi(\dot{x}) = \psi_v\dot{x} + \psi_c sign(\dot{x}),\ \varphi(\dot{\theta}) = \varphi_v\dot{\theta} + \varphi_c sign(\dot{\theta}). \quad (3)$$

The above model comes with the viscous friction coefficients $\psi_v, \varphi_v > 0$, the Coulomb friction level $\psi_c, \varphi_c > 0$, and the standard notation $sign(\cdot)$ for the signum function. Subject to (3) the right-hand side of the dynamic system (1)–(2) is piece-wise continuous. Throughout, solutions of such a system are defined in the sense of Filippov [5] as that of a certain differential inclusion with a multi-valued right-hand side.

Because the phenomenon of friction is hard to model, we have introduced the uncertain terms $w_1(t), w_2(t)$ into the dynamic equations (1), (2) to account for destabilizing model discrepancies such as Stribeck effect and backlash. Due to dissipative properties of mechanical systems, upper bounds $N_i > 0,\ i = 1, 2$ for the magnitudes of these terms can normally be estimated a priori:

$$|w_i(t)| \leq N_i \quad (4)$$

for all t.

Our objective is to design a controller that causes the actuated part of the cart-pendulum system to track

$$\lim_{t\to\infty}[z(t)+x(t)]=0, \tag{5}$$

a trajectory $z(t)$ generated by the modified Van der Pol equation [15]

$$\ddot{z}+\varepsilon[(z^2+\frac{\dot{z}^2}{\nu^2})-\rho^2]\dot{z}+\nu^2 z=0, \tag{6}$$

while also attenuating the effect of the friction forces (3) and external disturbances (4).

To this end, we present several arguments to consider the modified Van der Pol equation as a good choice of a reference model. The Van der Pol modification (6) is firstly shown [15] to possess a stable limit cycle, being expressible in the explicit form

$$z^2+\frac{\dot{z}^2}{\nu^2}=\rho^2 \tag{7}$$

where the parameter ρ stands for the amplitude of the limit cycle and ν is for its frequency. By substituting the orbit equation (7) into (6) the limit cycle of the modified Van der Pol equation (6) is secondly concluded to be remarkably generated by a standard linear harmonic oscillator

$$\ddot{z}+\nu^2 z=0, \tag{8}$$

initialized on (7). This is completely opposite to the non-modified Van der Pol oscillator, whose general representation is given by the second order scalar nonlinear differential equation

$$\ddot{z}+\varepsilon[(z-z_0)^2-\rho^2]\dot{z}+\nu^2(z-z_0)=0 \tag{9}$$

with positive parameters ε,ρ,ν, and which exhibits a nonsinusoidal periodic response in its limit cycle (see, e.g., [9] for details).

Summarizing, the modified Van der Pol oscillator (6) constitutes a *nonlinear asymptotic harmonic generator* which naturally exhibits an ideal sinusoidal signal (8) in its limit cycle (7). In contrast to the linear oscillator (8), whose amplitude depends on the initial conditions of the oscillator, the amplitude and frequency of this sinusoidal signal can be varied at will by tuning the parameters ρ and ν of the asymptotic harmonic generator (6).

2.2 Control Strategy

In order to present a control strategy that allows one to achieve the above objective let us partially linearize the cart-pendulum dynamics. For this purpose, let us rewrite the state equation (2) in the form

$$\ddot{\theta}=\frac{3}{4ml^2}[ml\cos\theta\ddot{x}+mgl\sin\theta+w_2(t)-\varphi(\dot{\theta})]. \tag{10}$$

Now substituting equation (10) into (1) yields

$$[(m+M)-\frac{3}{4}m\cos^2\theta]\ddot{x}=\tau+w_1(t)-\psi(\dot{x})-ml\sin\theta\,\dot{\theta}^2+\frac{3}{4}mg\cos\theta\sin\theta+\frac{3}{4l}[w_2(t)-\varphi(\dot{\theta})]\cos\theta. \tag{11}$$

Finally, setting $J = (m+M) - \frac{3}{4}m\cos^2\theta = \frac{1}{4}m + M + \frac{3}{4}m\sin^2\theta$ and

$$\tau = Ju + ml\sin\theta\,\dot{\theta}^2 - \frac{3}{4}mg\cos\theta\sin\theta \tag{12}$$

where u is the new control input, and taking into account that the relation $J \neq 0$ holds for all values of θ, the desired linearization is obtained:

$$\ddot{x} = u + \frac{3\cos\theta}{4lJ}[w_2(t) - \varphi(\dot{\theta})] + \frac{1}{J}[w_1(t) - \psi(\dot{x})] \tag{13}$$

$$\ddot{\theta} = \frac{3}{4l}\{u\cos\theta + \frac{3m\cos^2\theta + 4J}{4mlJ}[w_2(t) - \varphi(\dot{\theta})] + \frac{\cos\theta}{J}[w_1(t) - \psi(\dot{x})] + g\sin\theta\} \tag{14}$$

Since system (13), (14) describes the linearized actuated joint model it is referred to as *collocated linearization* [21].

The control strategy is now formalized as follows. The control input (12) is composed by an inner loop controller, partially linearizing the cart-pendulum, and an outer loop controller u to be constructed. Given the system output

$$y(t) = z(t) + x(t), \tag{15}$$

that combines the actuated state $x(t)$ of the system and the reference variable $z(t)$ governed by the modified Van der Pol equation (6), the outer loop controller u is to drive the system output (15) to the surface $y = 0$ in finite time and maintain it there in spite of the friction forces $\psi(\dot{x}), \varphi(\dot{\theta})$ and external disturbances $w_1(t), w_2(t)$, affecting the system.

2.3 SOSM Control Synthesis

Due to (6), (13), (15), the output dynamics is given by

$$\ddot{y} = u + \frac{3\cos\theta}{4lJ}[w_2(t) - \varphi(\dot{\theta})] + \frac{1}{J}[w_1(t) - \psi(\dot{x})] - \varepsilon[(z^2 + \frac{\dot{z}^2}{\nu^2}) - \rho^2]\dot{z} - \nu^2 z \tag{16}$$

The following control law

$$u = \frac{3\varphi_v\cos\theta}{4lJ}\dot{\theta} + \frac{\psi_v}{J}\dot{x} + \varepsilon[(z^2 + \frac{\dot{z}^2}{\nu^2}) - \rho^2]\dot{z} + \nu^2 z - \alpha sign(y) - \beta sign(\dot{y}) - hy - p\dot{y} \tag{17}$$

with the parameters such that

$$h, p \geq 0,\ \alpha - \beta > \frac{3(\varphi_c + N_2)}{4lJ} + \frac{\psi_c + N_1}{J} \tag{18}$$

is proposed.

The closed-loop system (3), (16), (17) is then feedback transformed to the one

$$\begin{aligned} \ddot{y} = {} & \frac{3\cos\theta}{4lJ}[w_2(t) - \varphi_c sign(\dot{\theta})] + \frac{1}{J}[w_1(t) - \psi_c sign(\dot{x})] \\ & -\alpha sign(y) - \beta sign(\dot{y}) - hy - p\dot{y} \end{aligned} \tag{19}$$

with piece-wise continuous right-hand side.

Relating the quasihomogeneous synthesis from [14], the above controller has been composed of the linear viscous friction compensator

$$u_f = \frac{3\varphi_v \cos\theta}{4lJ}\dot{\theta} + \frac{\psi_v}{J}\dot{x}, \tag{20}$$

nonlinear trajectory compensator

$$u_c = \varepsilon[(z^2 + \frac{\dot{z}^2}{\nu^2}) - \rho^2]\dot{z} + \nu^2 z, \tag{21}$$

the homogeneous switching part (the so-called twisting controller from [6])

$$u_h = -\alpha sign(y) - \beta sign(\dot{y}), \tag{22}$$

and the linear remainder

$$u_l = -hy - p\dot{y} \tag{23}$$

that vanishes in the origin $y = \dot{y} = 0$. By Theorem 1 from [17] the quasihomogeneous system (19) with the parameter subordination (18) is finite time stable regardless of which friction forces (3) and uniformly bounded external disturbances subject to (4) affect the system. The control objective is thus achieved and the following result is obtained.

Theorem 1 *Let the modified Van der Pol equation (6) with positive parameters ε, ν, ρ be a reference model of the cart-pendulum dynamics (1)–(3) and let the system output be given by (15). Then the quasihomogeneous controller (12), (17), (18) drives the cart-pendulum system to the zero dynamics surface $y = 0$ in finite time, uniformly in friction forces (3) and admissible disturbances (4). After that the actuated part $x(t)$ follows the output $-z(t)$ of the modified Van der Pol equation (6) whereas the non-actuated part $\theta(t)$ is governed by the zero dynamics equation*

$$\frac{4}{3}\, l\ddot{\theta} = \cos\theta\nu^2 x + g\sin\theta + \frac{1}{ml}[w_2(t) - \varphi(\dot{\theta})]. \tag{24}$$

on any finite time interval.

Proof: By Theorem 8 of [5, p. 85] the closed-loop system (13), (14), (17) has a local solution for all initial data and admissible disturbances (4). Let us demonstrate that each solution of this system is globally continuable on the right.

Due to (4), the magnitude $|W(t)|$ of the uncertainty

$$W(t) = \frac{3\cos\theta}{4lJ}[w_2(t) - \varphi_c sign(\dot{\theta})] + \frac{1}{J}[w_1(t) - \psi_v sign(\dot{x})]$$

that appears in the right-hand side of (19) is upper estimated as follows:

$$|W(t)| \le \frac{3(\varphi_c + N_2)}{4lJ} + \frac{\psi_c + N_1}{J}. \tag{25}$$

Since this estimate, coupled to (18), ensures that

$$\alpha - \beta > |W(t)|, \tag{26}$$

Theorem 4.2 from [14] turns out to be applicable to the quasihomogeneous system (18), (19). By applying this theorem, system (19) subject to (18) is proved to be finite time stable, uniformly in admissible disturbances (4). Now employing (15), it follows that along with a solution $y(t)$ of (19), an arbitrary solution $x(t) = y(t) - z(t)$ of (13) is globally continuable on the right and uniformly bounded in t.

Moreover, due to the uniform boundedness of $y(t)$, the control signal (17) is uniformly bounded, too. Thus, an arbitrary solution $\theta(t)$ of (14) is also globally continuable on the right.

So, starting from a finite time moment the cart-pendulum system evolves in the second order sliding mode on the zero dynamics surface $y = 0$ and to complete the proof it remains to derive the sliding mode system dynamics. For this purpose let us apply the equivalent control method [23] and substitute the only solution u_{eq} of the algebraic equation

$$u + \frac{3\cos\theta}{4lJ}[w_2(t) - \varphi(\dot\theta)] + \frac{1}{J}[w_1(t) - \psi(\dot x)] - \varepsilon[(x^2 + \frac{\dot x^2}{\nu^2}) - \rho^2]\dot x - \nu^2 x = 0$$

with respect to u (i.e., the equivalent control input u_{eq} that ensures equality $\ddot y = 0$) into (14). Then, while being restricted to this surface, the system dynamics is given by

$$\begin{aligned}\frac{4}{3}\, l\ddot\theta \quad &= \quad \cos\theta\{\frac{3\cos\theta}{4lJ}[\varphi(\dot\theta) - w_2(t)] + \frac{1}{J}[\psi(\dot x) - w_1(t)] + \varepsilon[(x^2 + \frac{\dot x^2}{\nu^2}) - \rho^2]\dot x + \nu^2 x\} \\ &\quad + \frac{3m\cos^2\theta + 4J}{4mlJ}[w_2(t) - \varphi(\dot\theta)] + \frac{\cos\theta}{J}[w_1(t) - \psi(\dot x)] + g\sin\theta \\ &= \quad \cos\theta\{\varepsilon[(x^2 + \frac{\dot x^2}{\nu^2}) - \rho^2]\dot x + \nu^2 x\} + \frac{1}{ml}[w_2(t) - \varphi(\dot\theta)] + g\sin\theta\end{aligned} \tag{27}$$

where $x(t)$ is the reference trajectory governed by the modified Van der Pol equation (6). For the orbits $x(t)$ of the Van der Pol modification (6), initialized on the limit cycle (7), the zero dynamics (27) is simplified to

$$\frac{4}{3}\, l\ddot\theta = \cos\theta\nu^2 x + g\sin\theta + \frac{1}{ml}[w_2(t) - \varphi(\dot\theta)].$$

The sliding mode equation (27) on the surface $y = 0$ is thus validated for the non-actuated variable $\theta(t)$, restricted to any finite time interval. Theorem 1 is proved.

Analyzing the proof of Theorem 1 one can conclude that the actuated variable $x(t)$ remains bounded regardless of which admissible disturbances affect the closed-loop system. For practical reasons the non-actuated variable $\theta(t)$ is also required to remain bounded under arbitrary disturbances of sufficiently small magnitudes. Due to this, the zero dynamics (24) is required to be a bounded input - bounded state system, locally in (w_1, w_2). While being beyond the scope of the paper, this item is not studied here in details, but only numerical and experimental evidences, demonstrating that this is indeed the case, are presented in the next section (for analysis of bounded input - bounded state systems see, e.g., [8, 11, 22]).

In the remainder, capabilities of the synthesis procedure, constituted by Theorem 1, are tested in numerical and experimental studies of the laboratory cart-pendulum system.

3 Swing up control and stabilization

In the test swing up/balancing control problem, the pendulum, located on the cart, is required to move from its stable downward position to the unstable upright position and be stabilized about the vertical while the cart is stabilized about a desired endpoint. An orbitally stabilizing controller is designed to swing up the pendulum to make it reach a homoclinic orbit of the energy level, corresponding to the cart pendulum system at the desired equilibrium point. Once the homoclinic orbit is reached at a state where the cart velocity is infinitesimal, the orbitally stabilizing controller is turned off thereby letting the system evolve along the homoclinic orbit. A locally stabilizing controller from [17] is then turned on when the homoclinic motion enters the attraction basin of the latter controller.

The hybrid control strategy, to be tested in an experimental study, is to select the amplitude ρ and the frequency ν of the model limit cycle (7) reasonably small and the parameter ε, controlling the speed of the limit cycle transient in the modified Van der Pol equation (6), reasonably large to ensure that the cart-pendulum system reaches its homoclinic orbit of the corresponding energy level. Turning off the orbitally stabilizing controller, once the system is synchronized with the homoclinic orbit, and proper switching to the stabilizing controller when the subsequent homoclinic motion enters the attraction basin of the latter controller yields the generation of a swing up motion, asymptotically stable about the vertical.

3.1 Cart-pendulum prototype

In order to observe the performance of the proposed synthesis we made simulations on Matlab using Simulink. We considered the real parameters of the laboratory cart-pendulum system from [17]. These parameters are listed in Table 1.

Table 1: Parameters of the Cart-Pendulum.

Notation	**Value**	**Units**
M	3.4	kg
m	0.147	kg
l	0.175	m
ψ_v	8.5	$N \cdot s/m$
φ_v	0.0015	$N \cdot m \cdot s/rad$
ψ_c	6.5	N
φ_c	0.00115	$N \cdot m$

3.2 Swinging controller design

By a suitable choice of the reference parameters ε, ρ, ν, the orbitally stabilizing synthesis (6), (12), (17) can be applied to the cart-pendulum system to swing the pendulum up from the downward position to the upright position. To successfully apply this synthesis the cumulative energy, pumped by the resulting controller into the cart-pendulum system, is to be of an appropriate level. According to the energy-based approach [12], such a level in the friction-free case must ensure that the total energy

$$E(q,\dot{q}) = \frac{1}{2}(M+m)\dot{x}^2 - ml\dot{x}\dot{\theta}\cos\theta + \frac{2}{3}ml^2\dot{\theta}^2 + mgl(\cos\theta - 1) \tag{28}$$

of the closed-loop system increases from the negative value

$$E_0 = -2mgl, \tag{29}$$

at the initial time moment to the zero-value energy

$$E_0 = 0, \tag{30}$$

at a time instant when the cart velocity becomes infinitesimal and ready to change the rotation direction. Thus synchronized, the unforced friction-free system generates a homoclinic orbit converging to the desired equilibrium that has the same energy level (30).

In order to take into account friction forces, resulting in an energy loss of the laboratory cart-pendulum system, the required energy level (30) has to deliberately be increased to a positive value

$$E_0 = \delta > 0. \tag{31}$$

Under a certain δ, found experimentally, the unforced friction system also generates an orbit, converging to the desired equilibrium. This orbit is further referred to as a *quasihomoclinic orbit*.

So, tuning the energy parameter δ and the reference parameters ε, ρ, ν is crucial to a successful swing up. Appropriate values of these parameters are carried out in successive simulations. The parameter ρ, responsible for the amplitude of the cart oscillations, is simply set slightly smaller than the admissible road length of the laboratorial equipment. The parameters ε and ν are responsible for making a reasonably high convergence rate of the modified Van der Pol oscillator (6) and, respectively, for providing the desired energy level (31) of the system at a position of the cart where it possesses an infinitesimal velocity (the higher ε the faster convergence; the larger ν the greater the energy of the limit cycle and hence the greater the energy of the cart, tracking the limit cycle). These parameters are iteratively tuned to approach the quasihomoclinic orbit. Turning off the controller at the time instant, when the system attains the quasihomoclinic orbit, makes the system follow this orbit, thereby approaching the desired endpoint.

In our simulation study the controller gains were set to $\alpha = 3\ N\cdot m, \beta = 1\ N\cdot m,\ h = 0,\ p = 0$. With these parameters, condition (18) holds and the output tracking of the modified Van der Pol oscillator (6) is therefore guaranteed. The reference parameters were tuned to $\varepsilon = 40\ [rad]^{-2}s^{-1}$, $\rho = 0.5\ rad$, $\nu = 1\ s^{-1}$. Being found numerically for the above parameters, the energy parameter took the value $\delta = 0.8\ N \cdot m$. Once the system energy attained this value under $\|x\| = \rho,\ \dot{x} = 0$, the orbitally stabilizing controller was turned off and the unforced pendulum was swang up along the quasihomoclinic orbit. Detailed experimental results, supporting the proposed swing up synthesis, are presented in Subsection 3.6.

3.3 Locally stabilizing controller design

Being developed in the companion paper [17], the locally stabilizing controller

$$\tau = \frac{4lJ\cos\theta}{\left[3 + 8l\lambda_2\dot{\theta}\sin\theta\right]}\left[-\mu(\theta,\dot{\theta}) - \alpha_1 sign(\xi) - \beta_1 sign(\dot{\xi}) - h_1\xi - p_1\dot{\xi}\right],\ \alpha_1, \beta_1, h_1, p_1 > 0 \tag{32}$$

subject to

$$\begin{gathered}\mu(\theta,\dot{\theta}) = 2\frac{\tan\theta}{\cos^2\theta}\dot{\theta}^2 + \left[\frac{3+8l\lambda_2\dot{\theta}\sin\theta}{\cos\theta}\right]\left[\frac{(M+m)g - ml\cos\theta\ \dot{\theta}^2}{D}\right]\tan\theta \\ +\lambda_1\left(g + \frac{4}{3}l\frac{\dot{\theta}^2}{\cos\theta}\right)\tan\theta + \lambda_2\left(g + \frac{4}{3}l\frac{\dot{\theta}^2(1+\sin^2\theta)}{\cos\theta}\right)\frac{\dot{\theta}}{\cos^2\theta},\end{gathered} \tag{33}$$

$$\xi = \tan\theta - \lambda_1\eta - \lambda_2\dot{\eta},\ \lambda_1, \lambda_2 > 0 \tag{34}$$

$$\eta = x - \frac{4l}{3}\ln\left(\frac{1+\sin\theta}{\cos\theta}\right), \quad |\theta| < \frac{\pi}{2}, \tag{35}$$

was tested in a simulation study. Quite impressive numerical results were obtained for the local stabilization of the pendulum about the upright position with the controller parameters $\alpha_1 = 55\ m/s^2,\ \beta_1 = 35\ m/s^2,\ h_1 = 0,\ p_1 = 0,\ \lambda_1 = 0.2\ 1/m,\ \lambda_2 = 0.26\ s/m$. The results can be observed in Subsection 3.5 where the proposed controller is introduced into our hybrid synthesis of swinging the Cart-Pendulum up and balancing it about the vertical.

3.4 Hybrid Controller Design

In order to accompany swinging the pendulum up by the subsequent stabilization around the upright position the swinging controller, presented in Subsection 3.2, is turned off, once the system reaches the corresponding homoclinic orbit, and then the locally stabilizing controller from [17] is turned on once the pendulum, evolving along the quasihomoclinic orbit, enters the basin of attraction, numerically found for the latter controller. The resulting hybrid controller moves the inverted pendulum, located on the cart, from its downward position to the upright position and stabilizes it about the vertical whereas the cart is stabilized at the desired endpoint. The capability of the closed-loop system of reaching the homoclinic orbit and entering the attraction basin of the locally stabilizing controller is additionally supported by experimental results.

3.5 Numerical verification

The initial conditions of the position of the cart-pendulum system and that of the modified Van der Pol oscillator, selected for all experiments, were $x(0) = 0$, $\theta(0) = 3.14\ rad$, and $z(0) = 0.05\ rad$, whereas all the velocity initial conditions were set to zero.

To begin with, we separately studied the orbitally stabilizing controller from Subsection 3.2. For demonstrating the capability of the controller to move the pendulum from one orbit to another by modifying the orbit parameters we then introduced a random time instant t_0 (it was $t_0 \approx 10s$ in the experiment), when the amplitude ρ and frequency ν of the model limit cycle was changed from their initial values $\rho = 0.1\ rad,\ \nu = 10\ s^{-1}$ to the new one $\rho = 0.5\ rad,\ \nu = 1\ s^{-1}$.

Finally, the hybrid controller from Subsection 3.4 was implemented to swing the pendulum up and stabilize it about the vertical while the cart is stabilized around the initial position.

Numerical results for the resulting motion, enforced by the orbitally stabilizing controller are depicted in Figs. 1. This figure demonstrates that while being driven by the orbitally stabilizing controller, the closed-loop system, perturbed by the permanent external disturbances $w_1(t) \equiv 0.5\ N,\ w_2(t) \equiv 0.5\ N \cdot m$, generates a bounded, quasi-periodic motion. As predicted by theory, the desired orbital transfer is achieved by simply changing the amplitude of the orbit limit cycle. Thus, good performance of the orbitally stabilizing controller is concluded from Fig. 1.

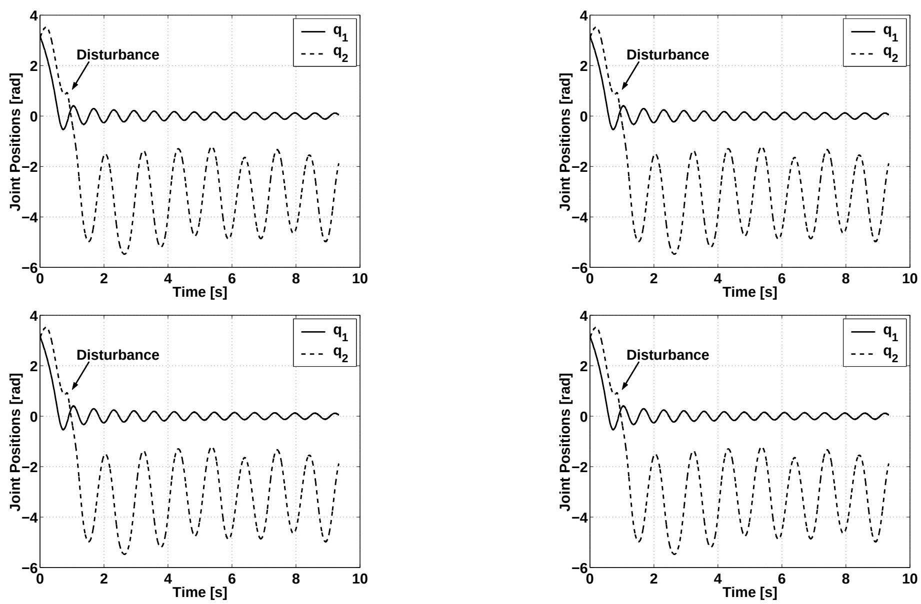

Figure 1: Orbital stabilization of the cart-pendulum system: left column for the orbital recovery under the permanent disturbances, right column for the transfer from one orbit to another.

3.6 Experimental verification

The initial conditions of the position of the cart-pendulum system and that of the modified Van der Pol oscillator, selected for the experiment, were $x(0) = -0.3\ m$, $\theta(0) = \pi\ rad$ (downward position), and $z(0) = 0.3\ m$, whereas all the velocity initial conditions were set to zero.

The hybrid controller from Subsection 3.4 was implemented to swing the pendulum up and stabilize it about the vertical while the cart is stabilized around the desired endpoint.

Experimental results for the resulting motion, enforced by the orbitally stabilizing controller and the hybrid controller, are depicted in Fig. 2.

Figure 2: Experimental results: left cart and pendulum position, right control input.

The switching moment occurs in the interval 2-3 seconds. Notice that the control effort is zero in this interval and that the cart velocity is zero. So, as stated above, the pendulum travels on its homoclinic orbit towards the unstable equilibrium and, once it is close enough, switching to the local controller occurs. The local controller is a robust nonlinear one, having a large domain of attraction (see [17]), and switching, that occurs at 10° from the upright position, is constrained by the road length of the laboratorial equipment.

In order to obtain a rough idea concerning the performance of the developed controller, the experimental results had been compared to the one in [12], where the switching from the swinging controller to a local state feedback was done after approximately 30 seconds. In the approach presented in this paper, the switching to the local quasi-homogeneous controller occurs at approximately 2.5 seconds, and the stabilization is accomplished in a shorter time compared to that of [12]. It should also be noted that while the energy-based approach from [12] is limited to models with negligible friction forces, the present approach does not suffer from this drawback.

4 Conclusions

Orbital stabilization of a cart-pendulum system, presenting a simple underactuated (two degrees-of-freedom, one actuator) manipulator, is under study. The quasihomogeneous SOSM-based control synthesis is utilized to design a variable structure controller that drives the pendulum to a desired zero dynamics manifold in finite time and maintains it there in spite of the presence of external disturbances.

Capabilities of the quasihomogeneous synthesis and its robustness are illustrated in an experimental study of the swing up/balancing control problem of moving the pendulum from its stable downward position to the unstable inverted position and stabilizing it about the vertical.

The orbitally stabilizing synthesis, being of interest in itself and as a part of the hybrid controller design of the swing up/balancing control of the cart-pendulum system, is the main contribution of the paper.

Eliminating undesirable chattering oscillations, caused by fast switching in the implemented hybrid controller, is among other problems calling for further investigation. While being non-trivial, this problem is however well-understood from the existing literature (see, e.g., [7] and references therein) and hopefully general methods of chattering reduction apply here as well.

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