The 3D Primitive Equations in the absence of viscosity: boundary conditions and well-posedness in the linearized case

Antoine Rousseau, Roger Temam, Joe Tribbia

THE 3D PRIMITIVE EQUATIONS IN THE ABSENCE OF VISCOSITY: BOUNDARY CONDITIONS AND WELL-POSEDNESS IN THE LINEARIZED CASE

A. ROUSSEAU[♭], R. TEMAM[*], AND J. TRIBBIA[♯]

Dedicated to the memory of Jacques-Louis Lions

[♭]Institut National de Recherches en Informatique et Automatique
Team MOISE, LJK, 51 rue des Mathématiques, B.P. 53, 38041 Grenoble Cedex 9, FRANCE

[*]Corresponding Author : temam@indiana.edu
The Institute for Scientific Computing and Applied Mathematics,
Indiana University, 831 East Third Street, Bloomington, Indiana 47405, U.S.A.

[♯]National Center for Atmospheric Research, Boulder, Colorado, USA.

Abstract

In this article we consider the 3D Primitive Equations (PEs) of the ocean, without viscosity and linearized around a stratified flow. As recalled in the Introduction, the PEs without viscosity ought to be supplemented with boundary conditions of a totally new type which must be *nonlocal*. In this article a set of boundary conditions is proposed for which we show that the linearized PEs are well-posed. The proposed boundary conditions are based on a suitable spectral decomposition of the unknown functions. Noteworthy is the rich structure of the Primitive Equations without viscosity. Our study is based on a modal decomposition in the vertical direction; in this decomposition, the first mode is essentially a (linearized) Euler flow, then a few modes correspond to a stationary problem partly elliptic and partly hyperbolic; finally all the other modes correspond to a stationary problem fully hyperbolic.

Résumé

Dans cet article, nous considérons les équations primitives (EP) tridimensionnelles de l'océan, sans viscosité, linéarisées autour d'un écoulement stratifié. Comme nous le rappelons dans l'introduction, les conditions aux limites qui accompagnent les EP sans viscosité doivent être d'un type totalement nouveau ; plus précisément, elles doivent nécessairement être *non locales*. Ici, nous proposons un jeu de conditions aux limites qui rendent les EP linéarisées bien posées. Elles s'appuient sur une décomposition spectrale adaptée des inconnues, et prennent en compte la structure très particulière des équations primitives sans viscosité. Notre étude est basée sur une décomposition modale dans la direction verticale ; dans cette décomposition, le premier mode se comporte quasiment comme un écoulement d'Euler (linéarisé). Quelques-uns des modes supérieurs correspondent à un problème stationnaire à la fois elliptique et hyperbolique. Enfin, tous les autres modes sont régis par un problème stationnaire purement hyperbolique.

2000 *Mathematics Subject Classification.* 35Q35, 35L50, 76D03.
Key words and phrases. Inviscid Primitive Equations, Open Boundary Conditions, Well-posedness.

1. Introduction

The theory of the Primitive Equations with viscosity has developed parallel to that of the Navier Stokes equations of incompressible fluids, up to a certain point; see e.g. [6], [7], or the review articles [15], [9]. However the theory of the Primitive Equations of the ocean and the atmosphere in the absence viscosity is not expected to be similar to that of the Euler equations (of incompressible fluids) and we know, since the article of Oliger and Sundström [8], that there does not exist any set of local boundary conditions for which these equations are well-posed; hence the need to determine (nonlocal) boundary conditions for which the PEs are well-posed. We arrive, in this way, to boundary value problems which are totally new to the best of our knowledge, and the difficulty and the novelty occur already in the linear (linearized) context. For the primitive equations, a related problem appears also in the context of numerical simulations; this issue has been (and will be) addressed elsewhere, see e.g. [16] and [11].

In this article we focus on the linearized Primitive Equations for which the boundary condition difficulty is already fully present [8], [14]. In earlier works we have considered the PEs in space dimensions 2 and 2.5, [10], [1]. In this article we study the PEs in space dimension 3.

This article is organized as follows: in the rest of this section we recall the PEs and their linearized form. We also recall the normal modes expansion of the unknowns and their decomposition into the subcritical and supercritical modes. These two sets of modes necessitate different treatments and, unlike in dimensions 2 or 2.5, the study of the supercritical modes is not straightforward. This Section 1 also contains (Section 1.3) a study of the associated stationary operator $\mathcal{A}$, a trace theorem adapted to this stationary operator which shows that if $U = (u, v, \psi)$ and $\mathcal{A}U$ are square integrable, then the traces of v and ψ are defined on the whole boundary and the trace of u is defined on part of the boundary (Section 1.4); finally Section 1 finishes with the study of the zero mode -in the modal decomposition (Section 1.5). Section 2 is devoted to the study of the subcritical modes for which the stationary problem, partly elliptic and partly hyperbolic, possesses a regularity result. Section 3 is devoted to the study of the supercritical modes handled in a different manner; the stationary problem is then fully hyperbolic, and it does not produce any regularity. Finally in Section 4 we consider the full Primitive Equations containing both the subcritical and the supercritical modes and we prove our main existence and uniqueness results for homogeneous and nonhomogeneous boundary conditions.

Note that the boundary conditions proposed here for the subcritical modes are different than those studied in [10] and [1] in dimensions 2 and 2.5; this change is of no importance in view of the computational objectives [11]. The related open problem is the determination of *all* the sets of boundary conditions making the nonviscous primitive equation well-posed. The full nonlinear PEs with boundary conditions similar to those proposed here, will be studied in a separate work.

The article is dedicated to the memory of Jacques-Louis Lions with whom one of the authors initiated the mathematical theory of the Primitive Equations with viscosity in [6], [7].

1.1. **The Primitive Equations.**

We now recall the Primitive Equations (PEs); the emphasis will be on the case of the ocean. The case of the atmosphere can be studied similarly with minor changes, as well as the coupled atmosphere and ocean; see e.g. [15]. The equations are derived from the Boussinesq equations by making the hydrostatic assumption which amounts to replacing the conservation of momentum in the vertical direction by the hydrostatic equation. Hence the equations

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + w\dfrac{\partial \mathbf{v}}{\partial z} + k\times\mathbf{v} + \nabla\phi = 0, \\ \dfrac{\partial \phi}{\partial z} = -\rho g, \\ \nabla\cdot\mathbf{v} + \dfrac{\partial w}{\partial z} = 0, \\ \dfrac{\partial T}{\partial t} + (\mathbf{v}\cdot\nabla)T + w\dfrac{\partial T}{\partial z} = 0, \rho = \rho(T). \end{cases} \tag{1.1}$$

The notations are as follows: $\mathbf{u} = (u, v, w)$ is the velocity of the water, $\mathbf{v}$ the horizontal velocity, ρ is the density, ϕ the pressure, T the temperature; $\rho = \rho(T)$ is the equation of state. The salinity equation is not present in (1.1), but this would raise little additional difficulty to take into account the salinity S. As indicated before, the viscosity is not present in the equations (1.1), this is a crucial point in this study. Equations (1.1) correspond to the β-plane approximation of the PEs near the latitude $\theta = \theta_0$, and $f = f_0 + \beta y$, $f_0 = \Omega \sin\theta_0$ where Ω is the angular velocity of the earth, and $\beta = (df/dy)$ at $\theta = \theta_0$, that is $\beta = f_0/a$ at midlatitudes, $(\theta_0 = \pi/4)$; k is the unit vector along the south to north poles; g is the gravitational constant. The domain occupied by the water is $\mathcal{M} = (0, L_1)\times(0, L_2)\times(-L_3, 0)$ in the $Oxyz$ system of coordinates.

Equations (1.1) are linearized around the simple uniform stratified flow (1.2)

$$\overline{u} = \overline{U}_0,\ \overline{v} = 0,\ \overline{T} = \overline{T}(z),\ \overline{\rho} = \rho_0(1 - \alpha(\overline{T} - T_0)), \tag{1.2}$$

where $\overline{U}_0 > 0,\ \rho_0 > 0$ and $T_0 > 0$ are reference average values of the density and the temperature, $\alpha > 0$ is a constant and $\overline{T}$ and $\overline{\rho}$ are linear in z. We introduce the Brunt-Väisälä (buoyancy) frequency

$$N^2 = -\frac{g}{\rho_0}\frac{d\overline{\rho}}{dz}\ ,$$

and we assume that N does not depend on z. We set $u = \overline{u} + u'$, etc., we linearize the equations and drop the primes. We obtain the following system

$$
\left\{
\begin{array}{l}
u_t + \overline{U}_0 u_x - fv + \phi_x = 0, \\
v_t + \overline{U}_0 v_x + fu + \phi_y = 0, \\
T_t + \overline{U}_0 T_x + N^2 \dfrac{T_0}{g} w = 0, \\
u_x + v_y + w_z = 0 \\
\phi_z = \dfrac{gT}{T_0},
\end{array}
\right.
\tag{1.3}
$$

where $u_t = \partial u/\partial t, u_x = \partial u/\partial x$, etc. Setting

$$
\phi_z = \psi = \frac{gT}{T_0},
\tag{1.4}
$$

we can also write

$$
\left\{
\begin{array}{l}
u_t + \overline{U}_0 u_x - fv + \phi_x = 0, \\
v_t + \overline{U}_0 v_x + fu + \phi_y = 0, \\
\psi_t + \overline{U}_0 \psi_x + N^2 w = 0, \\
u_x + v_y + w_z = 0, \\
\phi_z = \psi.
\end{array}
\right.
\tag{1.5}
$$

1.2. **Normal modes expansion.** As indicated in [14], the first step of the analysis of (1.5) consists, by separation of variables, in looking for solutions of the form

$$
\left\{
\begin{array}{l}
u(x,y,z,t) = \mathcal{U}(z)\ \hat{u}(x,y,t),\ \ v(x,y,z,t) = \mathcal{V}(z)\ \hat{v}(x,y,t), \\
\psi(x,y,z,t) = \Psi(z)\ \hat{\psi}(x,y,t), \\
w(x,y,z,t) = \mathcal{W}(z)\ \hat{w}(x,y,t),\ \ \phi(x,y,z,t) = \Phi(z)\ \hat{\phi}(x,y,t).
\end{array}
\right.
\tag{1.6}
$$

Substituting these expressions into (1.5), we find that $\mathcal{U}, \mathcal{V}, \Phi$ must be proportional and $\mathcal{W}$ proportional to Ψ. So we just take $\mathcal{V} = \Phi = \mathcal{U}$, and $\Psi = \mathcal{W}$. Indeed the third equation (1.5) implies that

$$
-\frac{\hat{\psi}_t + \overline{U}_0 \hat{\psi}_x}{N^2 \hat{w}} = \frac{\mathcal{W}}{\Psi}\ (= c_1'),
$$

and these quantities are constant since the left-hand side of the last equation depends on x, y and t and the right-hand side depends on z only. For the sake of simplicity we can take this constant c_1' equal to one, that is $\mathcal{W} = \Psi$. Similarly, applying the operator $\partial/\partial t + \overline{U}_0\ \partial/\partial x$ to the first and second equations (1.5) we obtain that $\mathcal{U}, \mathcal{V}$ and Φ must be proportional, and so we can take $\mathcal{U} = \mathcal{V} = \Phi$. Finally the fourth and fifth equations (1.5) imply that

$$-\frac{\hat{u}_x+\hat{v}_y}{\hat{w}}=\frac{\mathcal{W}'}{\mathcal{U}}=c_2',\quad \frac{\hat{\phi}}{\hat{\psi}}=\frac{\Psi}{\Phi'}=c_3',$$

where c_2', c_3' are constant; hence $\mathcal{W}=c_2'\,\mathcal{U}'$ and

$$\mathcal{U}''+\lambda^2\mathcal{U}=0,\quad \mathcal{W}''+\lambda^2\mathcal{W}=0, \tag{1.7}$$

with $\lambda^2=-c_2'/c_3'$. The natural boundary conditions for w and $\mathcal{W}$ are $\mathcal{W}=0$ at $z=0$ and $-L_3$; thus $\mathcal{U}$ and $\mathcal{W}$ are solutions of the two-point boundary value problems consisting of (1.7) and

$$\mathcal{U}'(0)=\mathcal{U}'(-L_3)=\mathcal{W}(0)=\mathcal{W}(-L_3)=0. \tag{1.8}$$

We denote by λ_n^2 the corresponding eigenvalues and write

$$\begin{cases} \lambda_n=\dfrac{n\pi}{L_3},\;\; \lambda_n^2=\dfrac{1}{gH_n},\;\; i.e.\;\; H_n=\dfrac{L_3^2}{gn^2\pi^2}, \\[2ex] \mathcal{W}_n=\sqrt{\dfrac{2}{L_3}}\sin(\lambda_n z),\;\; \mathcal{U}_n=\sqrt{\dfrac{2}{L_3}}\cos(\lambda_n z),\;\; n\geq 1,\;\; \mathcal{U}_0=\dfrac{1}{\sqrt{L_3}}. \end{cases} \tag{1.9}$$

As usual the functions $\mathcal{U}_n, \mathcal{W}_n$ have been chosen to form an orthonormal set in $L^2(-L_3,0)$.

The equations satisfied by $\hat{u}, \hat{v}$, etc., will appear below. Indeed having found these special solutions to equation (1.5), we now look for the general solution in the form

$$\begin{cases} (u,v,\phi)=\displaystyle\sum_{n\geq 0}\mathcal{U}_n(z)(u_n,v_n,\phi_n)(x,y,t), \\ (w,\psi)=\displaystyle\sum_{n\geq 1}\mathcal{W}_n(z)(w_n,\psi_n)(x,y,t). \end{cases} \tag{1.10}$$

Substituting these expressions in (1.5), we arrive at the following systems, for $n\geq 1$,

$$\begin{cases} \dfrac{\partial u_n}{\partial t}+\bar{U}_0\dfrac{\partial u_n}{\partial x}-fv_n+\dfrac{\partial \phi_n}{\partial x}=0, \\[2ex] \dfrac{\partial v_n}{\partial t}+\bar{U}_0\dfrac{\partial v_n}{\partial x}+fu_n+\dfrac{\partial \phi_n}{\partial y}=0, \\[2ex] \dfrac{\partial \psi_n}{\partial t}+\bar{U}_0\dfrac{\partial \psi_n}{\partial x}+N^2w_n=0, \\[2ex] \phi_n=-\dfrac{1}{\lambda_n}\psi_n,\;\; w_n=-\dfrac{1}{\lambda_n}\left(\dfrac{\partial u_n}{\partial x}+\dfrac{\partial v_n}{\partial y}\right). \end{cases} \tag{1.11}$$

And, for $n = 0, w_0 = \psi_0 = 0$ and there remains

$$(1.12) \qquad \begin{cases} \dfrac{\partial u_0}{\partial t} + \bar{U}_0 \dfrac{\partial u_0}{\partial x} - f v_0 + \dfrac{\partial \phi_0}{\partial x} = 0, \\[2ex] \dfrac{\partial v_0}{\partial t} + \bar{U}_0 \dfrac{\partial v_0}{\partial x} + f u_0 + \dfrac{\partial \phi_0}{\partial y} = 0, \\[2ex] \dfrac{\partial u_0}{\partial x} + \dfrac{\partial v_0}{\partial y} = 0. \end{cases}$$

Note that, since the considered problem is linear, there is no coupling between the different modes; see *e.g.* [11] for the nonlinear case which introduces these couplings. We will study the zero mode separately (see Section(1.5)), and, for $n \geq 1$, we use the last two equations (1.11) and rewrite the first three in the form

$$(1.13) \qquad \begin{cases} \dfrac{\partial u_n}{\partial t} + \bar{U}_0 \dfrac{\partial u_n}{\partial x} - f v_n - \dfrac{1}{\lambda_n} \dfrac{\partial \psi_n}{\partial x} = 0; \\[2ex] \dfrac{\partial v_n}{\partial t} + \bar{U}_0 \dfrac{\partial v_n}{\partial x} + f u_n - \dfrac{1}{\lambda_n} \dfrac{\partial \psi_n}{\partial y} = 0, \\[2ex] \dfrac{\partial \psi_n}{\partial t} + \bar{U}_0 \dfrac{\partial \psi_n}{\partial x} - \dfrac{N^2}{\lambda_n} \left(\dfrac{\partial u_n}{\partial x} + \dfrac{\partial v_n}{\partial y} \right) = 0. \end{cases}$$

As indicated before, our aim is to propose boundary conditions for (1.11)-(1.13) which make these equations well–posed and consequently the equations (1.5) also. As we shall see (see also [10]), the boundary conditions are different depending on whether

$$1 \leq n \leq n_c, \quad \text{ or } \quad n > n_c,$$

where n_c, λ_{n_c} are such that

$$(1.14) \qquad \frac{n_c \pi}{L_3} = \lambda_{n_c} < \frac{N}{\bar{U}_0} < \lambda_{n_c+1} = \frac{(n_c+1)\pi}{L_3}.$$

We will not study the non generic case where $L_3 N / \pi \bar{U}_0$ is an *integer*.

The modes $0 \leq n \leq n_c$ are called *subcritical*, and the modes $n > n_c$ are called *supercritical*. It is convenient to introduce the sub and supercritical components of the functions defined by:

$$(1.15) \qquad u^0 = P_0 u = \mathcal{U}_0 u_0, u^I = P_I u = \sum_{n=1}^{n_c} \mathcal{U}_n u_n, \;\; u^{II} = P_{II} u = \sum_{n > n_c}^{\infty} \mathcal{U}_n u_n,$$

and similarly for all the other functions; of course the zero mode u^0 is a subcritical mode, but, as we will see, we need to treat it separately. With these notations, the equations (1.5), (1.11), (1.13) are equivalent to the following system:

$$\begin{cases} u^0_t + \bar{U}_0 u^0_x - f v^0 + \phi^0_x = 0, \\ v^0_t + \bar{U}_0 v^0_x + f u^0 + \phi^0_y = 0, \\ u^0_x + v^0_y = 0, \end{cases} \tag{1.16}$$

$$\begin{cases} u^I_t + \bar{U}_0 u^I_x - f v^I + \phi^I_x & = 0, \\ v^I_t + \bar{U}_0 v^I_x + f u^I + \phi^I_y & = 0, \\ \psi^I_t + \bar{U}_0 \psi^I_x + N^2 w^I & = 0, \end{cases} \tag{1.17}$$

$$\begin{cases} u^{II}_t + \bar{U}_0 u^{II}_x - f v^{II} + \phi^{II}_x & = 0, \\ v^{II}_t + \bar{U}_0 u^{II}_x + f u^{II} + \phi^{II}_y & = 0, \\ \psi^{II}_t + \bar{U}_0 \psi^{II}_x + N^2 w^{II} & = 0, \end{cases} \tag{1.18}$$

with the additional relations $\phi = \phi(\psi), w = w(u, v)$:

$$\begin{cases} \phi^I = -\displaystyle\sum_{n=1}^{n_c} \frac{1}{\lambda_n} \psi_n \mathcal{U}_n, \;\; w^I = -\displaystyle\sum_{n=1}^{n_c} \frac{1}{\lambda_n} (u_{nx} + v_{ny}) \mathcal{W}_n, \\ \phi^{II} = -\displaystyle\sum_{n>n_c} \frac{1}{\lambda_n} \psi_n \mathcal{U}_n, \;\; w^{II} = -\displaystyle\sum_{n>n_c} \frac{1}{\lambda_n} (u_{nx} + v_{ny}) \mathcal{W}_n. \end{cases} \tag{1.19}$$

We will also set $U = (u, v, \psi), U^0 = P_0 U, U^I = P_I U, U^{II} = P_{II} U$.

Hereafter, our aim will be to study separately the subscritical and supercritical modes, proposing suitable boundary conditions for them, and to combine them and obtain existence, uniqueness and regularity of the solution U. In each case we will study one (subcritical/supercritical) mode separately and then combine them for the whole subcritical and supercritical components. We now conclude this section with some remarks concerning the stationary (time independent) equations associated with (1.12), (1.13), and by a trace theorem which will be used repeatedly in the sequel.

1.3. **The stationary equations associated with (1.12)-(1.13).** The (physical) spatial domain under consideration will be $\mathcal{M} = \mathcal{M}' \times (-L_3, 0)$, where $\mathcal{M}'$ is the interface atmosphere/ocean, $\mathcal{M}' = (0, L_1) \times (0, L_2)$.

We introduce, componentwise, the differential operators $\mathcal{A}_n = (\mathcal{A}_{n1}, \mathcal{A}_{n2}, \mathcal{A}_{n3})$ operating on $U_n = (u_n, v_n, \psi_n)$,

$$\mathcal{A}_n U_n = \begin{cases} \bar{U}_0 u_{nx} - \dfrac{1}{\lambda_n} \psi_{nx}, \\ \\ \bar{U}_0 v_{nx} - \dfrac{1}{\lambda_n} \psi_{ny}, \\ \\ \bar{U}_0 \psi_{nx} - \dfrac{N^2}{\lambda_n} (u_{nx} + v_{ny}), \end{cases} \tag{1.20}$$

with $\bar{U}_0, N$ and $\lambda_n > 0$ as above.

Our object here is to study (recall) the nature of the stationary (time independent) equations in $\mathcal{M}'$:

$$\mathcal{A}_n U_n = F_n = (F_{un}, F_{vn}, F_{\psi n}), \quad n \geq 1. \tag{1.21}$$

We momentarily drop the indices n for the sake of simplicity and although this is not of direct use in the sequel, it is useful to look for the characteristics of the differential system $\mathcal{A}U = F$. We write this system in the matrix form

$$\mathcal{E}U_x + \mathcal{G}U_y = F, \tag{1.22}$$

with

$$\mathcal{E} = \begin{pmatrix} \bar{U}_0 & 0 & -\frac{1}{\lambda} \\ 0 & \bar{U}_0 & 0 \\ -\frac{N^2}{\lambda} & 0 & \bar{U}_0 \end{pmatrix}, \quad \mathcal{G} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\lambda} \\ 0 & -\frac{N^2}{\lambda} & 0 \end{pmatrix}$$

and the equation of the characteristics [4] is given by

$$\det\left(\mathcal{E}dx - \mathcal{G}dy\right) = 0,$$

that is

$$\det \begin{vmatrix} \bar{U}_0\mu & 0 & -\frac{\mu}{\lambda} \\ 0 & \bar{U}_0\mu & \frac{1}{\lambda} \\ -\frac{N^2\mu}{\lambda} & \frac{N^2}{\lambda} & \bar{U}_0\mu \end{vmatrix} = 0,$$

with $\mu = dx/dy$. Hence the equation for μ:

$$\bar{U}_0\mu\left[\left(\bar{U}_0^2 - \frac{N^2}{\lambda^2}\right)\mu^2 - \frac{N^2}{\lambda^2}\right] = 0. \tag{1.23}$$

The (real) solution $\mu_0 = 0$ exists in all cases, producing the characteristics $x =$ constant (parallel to the background flow $\bar{U}_0 e_x$). This corresponds to the first equation:

$$\frac{\partial}{\partial x}\left(\bar{U}_0 u - \frac{\psi}{\lambda}\right) = F_u + fv.$$

Then in the supercritical case, $\bar{U}_0^2 - N^2\lambda^{-2} > 0$ and we have two more real characteristics

$$\frac{dx_\pm}{dy} = \mu_\pm = \pm\frac{N}{\lambda}\left(\bar{U}_0^2 - \frac{N^2}{\lambda^2}\right)^{-1/2}, \tag{1.24}$$

whereas, in the subcritical case, these two characteristics are imaginary.

For the stationary zero mode, we obtain from (1.12) after dropping the Coriolis term:

$$\begin{aligned} &\bar{U}_0 u_x + \phi_x = F_u, \\ &\bar{U}_0 v_x + \phi_y = F_v, \\ &u_x + v_y = 0. \end{aligned} \tag{1.25}$$

By elimination of ϕ we find

$$\bar{U}_0\left(u_{xy} - v_{xx}\right) = F_{u,y} - F_{v,x}$$

and hence we find the fully elliptic equation

$$v_{xx} + v_{yy} = \frac{1}{\bar{U}_0}\left(F_{v,x} - F_{u,y}\right). \tag{1.26}$$

We infer from this remark that the stationary system $\mathcal{A}_n U_n = F_n$ is fully elliptic for the zero mode, partly hyperbolic and partly elliptic for the other subcritical modes (one real characteristic) and fully hyperbolic in the supercritical case (three real characteristics). This remark will be underlying the studies in Sections 2 and 3, although, as we said, we do not use it directly.

1.4. **A trace theorem.** We consider the same differential operator $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$, as in (1.20) operating on $U = (u, v, \psi)$, but the indices n are dropped for the sake of simplicity:

$$\mathcal{A}U = \begin{cases} \bar{U}_0 u_x - \dfrac{1}{\lambda}\psi_x, \\[2ex] \bar{U}_0 v_x - \dfrac{1}{\lambda}\psi_y, \\[2ex] \bar{U}_0 \psi_x - \dfrac{N^2}{\lambda}(u_x + v_y), \end{cases} \tag{1.27}$$

with $\bar{U}_0, N, \lambda = \lambda_n > 0$ as above, and we consider the space[1]

$$\mathcal{X} = \left\{U \in L^2(\mathcal{M}')^3, \mathcal{A}U \in L^2(\mathcal{M}')^3\right\}, \tag{1.28}$$

endowed with its natural Hilbert norm $(|U|^2_{L^2(\Gamma_i)^3} + |\mathcal{A}U|^2_{L^2(\Gamma_i)^3})^{\frac{1}{2}}$. We have

Theorem 1.1. *If $U = (u, v, \psi) \in \mathcal{X}$, the traces of v and ψ are defined on all of $\partial\mathcal{M}'$, the trace of u is defined at $x = 0$ and L_1, and they belong to the respective spaces $H_x^{-1}(0, L_1)$ and $H_y^{-1}(0, L_2)$. Furthermore the trace operators are linear continuous in the corresponding spaces, e.g. $U \in \mathcal{X} \to u|_{x=0}$ is continuous from $\mathcal{X}$ into $H_y^{-1}(0, L_2)$.*

Proof. Let us write $\mathcal{A}U = F = (f_1, f_2, f_3)$. Since $U = (u, v, \psi) \in L^2(\mathcal{M}')^3 = L_x^2(0, L_1; L_y^2(0, L_2)^3)$, we see that $U_y = \partial U/\partial y$ belongs to $L_x^2(0, L_1; H_y^{-1}(0, L_2)^3)$. From $\mathcal{A}_2 U = \bar{U}_0 v_x - \lambda^{-1}\psi_y = f_2 \in L^2(\mathcal{M}')$, we conclude that $v_x \in L_x^2(0, L_1; H_y^{-1}(0, L_2))$, and $v \in \mathcal{C}([0, L_1]; H_y^{-1}(0, L_2))$, so that its traces at $x = 0$ and L_1 are defined and belong to $H_y^{-1}(0, L_2)$. We then have $\bar{U}_0 u_x - \lambda^{-1}\psi_x = f_1 \in L_x^2(0, L_1; L_y^2(0, L_2))$, $\bar{U}_0\psi_x - (N^2/\lambda)u_x = f_2 - (N^2/\lambda)v_y \in L_x^2(0, L_1; H_y^{-1}(0, L_2))$ so that both u_x and ψ_x belong to the last space and $u, \psi \in \mathcal{C}([0, L_1]; H_y^{-1}(0, L_2))$; their traces are defined as well at $x = 0$ and L_1. Finally we write

$$\begin{gathered} \bar{U}_0 v_x - \lambda^{-1}\psi_y = f_2, \\ (\bar{U}_0 - N^2/\lambda^2\bar{U}_0)\psi_x - (N^2/\lambda)v_y = f_3 + N^2 f_1/\lambda\bar{U}_0, \end{gathered}$$

from which we conclude that v_y and $\psi_y \in L_y^2(0, L_2; H_x^{-1}(0, L_1))$ and thus v and $\psi \in \mathcal{C}_y([0, L_2]; H_x^{-1}(0, L_1))$ and their traces are both defined at $y = 0$ and L_2. Finally all the mappings above are continuous, and the theorem is proved. □

[1]We will write $\mathcal{A}_n, \mathcal{X}_n$ when it is necessary to emphasize the dependence on n through λ $(\lambda = \lambda_n)$.

Remark 1.1. Although the values of $\bar{U}_0, N, \lambda = \lambda_n$ are intended to be those above, Theorem 1.1 extends to operators $\mathcal{A}$ with the same structure and more general constant coefficients, and it will be used in this way at times.

1.5. **The zero mode.** The equations for this mode appear in (1.12) but, for the convenience of the notations, the subscripts are now changed to superscripts. Due to the form of the third equation, we proceed by analogy with the incompressible Navier Stokes equations and we determine first $\mathbf{u}^0 = (u^0, v^0)$ and then ϕ^0 by solving a Neumann problem. The natural function space for $\mathbf{u}^0$ is

$$H^0 = \left\{\mathbf{u}^0 = (u^0, v^0) \in L^2(\mathcal{M}')^2, u^0_x + v^0_y = 0, \mathbf{u}^0 \cdot \mathbf{n} = 0 \text{ on } \partial\mathcal{M}'\right\}, \tag{1.29}$$

where $\mathbf{n} = (n_x, n_y)$ is the unit outward normal on $\partial\mathcal{M}'$. Recall (see e.g. [13]) that the trace of $\mathbf{u}^0 \cdot \mathbf{n}$ on $\partial\mathcal{M}'$ makes sense for $\mathbf{u}^0 \in L^2(\mathcal{M}')^2$ with div $\mathbf{u}^0 = u^0_x + v^0_y \in L^2(\mathcal{M}')\cdot$ If the test function $\tilde{\mathbf{u}}^0 = (\tilde{u}^0, \tilde{v}^0) \in H^0$ is smooth, we classically see that (1.12) implies that

$$\frac{d}{dt}(\mathbf{u}^0, \tilde{\mathbf{u}}^0)_{H^0} + \bar{U}_0(\mathbf{u}^0_x, \tilde{\mathbf{u}}^0) + f(e_z \wedge \mathbf{u}^0, \tilde{\mathbf{u}}^0) = 0, \tag{1.30}$$

where $e_z = (0, 0, 1)$. Conversely if there exists u^0 such that (1.30) is satisfied for all such $\tilde{\mathbf{u}}^0$, then there exists ϕ^0 such that equations (1.12) are satisfied.

We then introduce the linear unbounded operator A^0 in H,

$$A^0\mathbf{u}^0 = P_{H^0}\left(\bar{U}_0\frac{\partial \mathbf{u}^0}{\partial x} + f e_z \wedge \mathbf{u}^0\right), \tag{1.31}$$

with domain

$$D(A^0) = \left\{\mathbf{u}^0 \in H^0, \mathbf{u}^0_x \in L^2(\mathcal{M}')^2\right\}, \tag{1.32}$$

where P_{H^0} is the orthogonal projector in $L^2(\mathcal{M}')^2$ onto H^0. Equation (1.30) is then equivalent to the evolution equation

$$\frac{d\mathbf{u}^0}{dt} + A^0\mathbf{u}^0 = 0. \tag{1.33}$$

Using the Hille-Phillips-Yoshida theorem, it is easy to see that equation (1.33) with initial condition $\mathbf{u}^0(0)$ given in H^0 or $D(A^0)$ produces a well-posed initial value problem. For that purpose it is sufficient to show that $-A^0$ is the infinitesimal generator of a contraction semi-group in H^0. Since the operator $\mathbf{u}^0 \longrightarrow P_{H^0}(f e_z \wedge \mathbf{u}^0)$ is continuous in H^0, it suffices to show that $\bar{A}^0\mathbf{u}^0 = P_{H^0}\bar{U}^0\mathbf{u}_{0x}$ with domain $D(\bar{A}^0)$ is dense in H^0 and $\bar{A}^0$ is closed which is easy; also $\bar{A}^0 \geq 0$ as

$$\begin{aligned}(\bar{A}^0\mathbf{u}^0, \mathbf{u}^0)_{H^0} &= \bar{U}_0\int_{\mathcal{M}'} \mathbf{u}^0_x \cdot \mathbf{u}^0 d\mathcal{M}' \\ &= \frac{\bar{U}_0}{2}\int_0^{L_2} \left[|\mathbf{u}^0|^2(L_1, y) - |\mathbf{u}^0|^2(0, y)\right] dy \\ &= 0,\end{aligned} \tag{1.34}$$

the integration in x being justified for $\mathbf{u}^0 \in D(A^0)$. We need also to show that $\bar{A}^{0*}$ is positive, but this results from the fact that $\bar{A}^{0*} = -\bar{A}^0$, with the same domain. [2] We

[2]Note that $A^{0*} = -A^0$ as well, and of course $D(A^{0*}) = D(A^0) = D(\bar{A}^0)$.

refrain from giving all the details of the proof for this partial result and refer the reader to Section 4 for the complete analysis.

We now proceed and study the subcritical modes $1 \le n \le n_c$.

2. Subcritical Modes

2.1. **One subcritical mode** $(1 \le n \le n_c)$**.** We temporarily drop the indices n and first want to set and study an initial value problem for (1.13) when the mode is subcritical, that is (see (1.14))

$$\lambda = \lambda_n < \frac{N}{\bar{U}_0} \quad (0 \le n \le n_c).$$

There are several possible choices of suitable boundary conditions; see e.g. different ones in [10] for a related situation. Here, for a simple subcritical mode $1 \le n \le n_c$, we choose the following boundary conditions:

$$\begin{cases} \psi = 0 \text{ at } x = L_1, \quad \text{and } y = 0, L_2, \\ v = 0 \text{ and } u = \psi/\lambda\bar{U}_0 \text{ at } x = 0, \end{cases} \tag{2.1}$$

and we introduce the space

$$D(A) = \left\{U \in L^2(\mathcal{M}')^3, \mathcal{A}U \in L^2(\mathcal{M}')^3, U \text{ satisfies (2.1)}\right\}, \tag{2.2}$$

and the operator[3]

$$AU = \mathcal{A}U.$$

In view of Theorem 1.1, the traces appearing in (2.1) are well defined when $U \in L^2(\mathcal{M}')^3$ and $\mathcal{A}U \in L^2(\mathcal{M}')^3$, so that the definition of $D(A_n)$ in (2.2) makes sense.

Remark 2.1. As indicated above the boundary conditions (2.1) are different than those in [10] (after neglecting the dependence on y).

We proceed with a regularity result for U in $D(A)$ (see the comments in Section 1.3).

Theorem 2.1. *If $U = (u, v, \psi) \in D(A)$, then v and ψ belong to $H^1(\mathcal{M}')$ and u_x belongs to $L^2(\mathcal{M}')$*

Proof. For $U \in D(A)$, we set $AU = F = (f_1, f_2, f_3)$. Then, in the distribution sense,

$$\begin{cases} \bar{U}_0 u_x - \dfrac{1}{\lambda}\psi_x = f_1, \\[2ex] \bar{U}_0 v_x - \dfrac{1}{\lambda}\psi_y = f_2, \\[2ex] \bar{U}_0 \psi_x - \dfrac{N^2}{\lambda}(u_x + v_y) = f_3. \end{cases} \tag{2.3}$$

[3]When needed we will write also $\mathcal{A}_n, A_n, D(A_n)$ to emphasize the dependence on n $(\lambda = \lambda_n)$.

Combining the first and third equations we find

$$\bar{U}_0 \left(1 - \frac{N^2}{\lambda^2 \bar{U}_0^2}\right) \psi_x - \frac{N^2}{\lambda} v_y = \frac{N^2}{\lambda \bar{U}_0} f_1 + f_3. \tag{2.4}$$

Combining this equation with the second equation (2.3), we obtain

$$\begin{aligned}(\bar{U}_0^2 - \frac{N^2}{\lambda^2})\psi_{xx} - \frac{N^2}{\lambda^2}\psi_{yy} = \\ = \frac{N^2}{\lambda} f_{1x} + \frac{N^2}{\lambda} f_{2y} + \bar{U}_0 f_{3x}.\end{aligned} \tag{2.5}$$

Note that this equation is elliptic in the subcritical case; of course a similar elliptic equation can be derived for v, but we will not use it. We associate to this equation the boundary condition $\psi = 0$ at $y = 0, L_2$ and $x = L_1$ contained in (2.1). Then for the side $x = 0$ of $\mathcal{M}'$, a suitable boundary condition is given by (2.4) in which $v_y = 0$ since $v = 0$ at $x = 0$; hence

$$-\frac{\partial \psi}{\partial n} = \psi_x = \bar{U}_0^{-1} \left(1 - \frac{N^2}{\lambda^2 \bar{U}_0^2}\right)^{-1} \left(\frac{N^2}{\lambda \bar{U}_0} f_1 + f_2\right), \text{at } x = 0. \tag{2.6}$$

The right-hand side of (2.6) does not make sense on $x = 0$ for $F \in L^2(\mathcal{M}')^3$. So we proceed as follows: we approximate F in $L^2(\mathcal{M}')^3$ by a sequence of smooth functions $F_m \in \mathcal{C}^\infty(\mathcal{M}')^3$. For each m, the right-hand side of (2.6) makes sense and we find a unique solution ψ_m of (2.4), (2.6) and $\psi_m = 0$ on the other sides of $\mathcal{M}'$. Of course ψ_m is $\mathcal{C}^\infty$ on $\mathcal{M}'$ away from the corners and $\psi_m \in H^1(\mathcal{M}')$ (at least), see [2]. Then from ψ_m, we determine the corresponding $v = v_m$ up to an additive constant: v_{mx} and v_{my} are given by (2.4) and the second equation (2.3), and these equations are compatible (i.e. $v_{mxy} = v_{myx}$), because of (2.5). Note that v_m belongs to $H^1(\mathcal{M}')$ at least, its trace on the side $x = 0$ of $\mathcal{M}'$ is defined, $v_{my} = 0$ on this side because of (2.6). Hence $v_m = 0$ on $x = 0$ by choosing properly the constant. Finally u_m is determined by the first equation (2.3) and the boundary condition $U_m = \psi_m / \lambda \bar{U}_0$ at $x = 0$. In conclusion $U_m = (u_m, v_m, \psi_m)$ that we just constructed belongs to $D(A)$ and satisfies $AU_m = F_m$.

To pass to the limit $m \to \infty$, we obtain the suitable a priori estimates as follows: we multiply the second equation (2.3) by $-(N^2/\lambda)\psi_{my}$, equation (2.4) by $-\bar{U}_0 \psi_{mx}$, integrate over $\mathcal{M}'$ and add these equations. We find

$$\begin{aligned}&\bar{U}_0^2 \left(\frac{N^2}{\lambda^2 \bar{U}_0^2} - 1\right) \int_{\Gamma_i} \psi_{mx}^2 d\mathcal{M}' + \frac{N^2}{\lambda^2} \int_{\mathcal{M}'} \psi_{my}^2 d\mathcal{M}' \\ &+ \frac{\bar{U}_0 N^2}{\lambda} \int_{\Gamma i} (v_{my}\psi_{mx} - v_{mx}\psi_{my})\, d\mathcal{M}' = \\ &= -\int_{\mathcal{M}'} \left(\frac{N^2}{\lambda} f_{m2}\psi_{my} + \frac{N^2}{\lambda} f_{m1}\psi_{mx} + \bar{U}_0 f_{m2}\psi_{mx}\right) d\mathcal{M}'.\end{aligned} \tag{2.7}$$

The integrals involving $v\psi$ cancel each other because it is legitimate to integrate by parts (enough regularity) and, by integration by parts, taking into account the boundary conditions (2.1) for U_m, we find

$$\int_{\mathcal{M}'} v_{my}\psi_{mx} d\mathcal{M}' = \int_{\Gamma_i} v_{mx}\psi_{my} d\mathcal{M}'. \tag{2.8}$$

Since $N^2 > \lambda^2 \bar{U}_0^2$, we then easily infer from (2.7) that

$$|\nabla\psi_m|_{L^2(\mathcal{M}')^2} \leq c|F_m|_{L^2(\Gamma_i)^2} \leq \text{ const.} \tag{2.9}$$

Thanks to the boundary conditions on ψ_m we have a Poincaré inequality which guarantees that

$$|\psi_m|_{L^2(\Gamma_i)} \leq \text{ const.}, \tag{2.10}$$

and ψ_m is bounded in $H^1(\mathcal{M}')$. As for the construction of v_m, the second equation (2.3), (2.4) and $v_m = 0$ on $x = 0$ then show that v_m is bounded in $H^1(\mathcal{M}')$. Finally $\bar{U}_0 u_m - \psi_m/\lambda$ and its x derivative are bounded in $L^2(\mathcal{M}')$ so that u_m and u_{mx} are bounded in $L^2(\mathcal{M}')$ as well.

Passing to the limit $m \to \infty$, we obtain $U_m \to \bar{U}$, with $\bar{U} \in D(A)$ and $A\bar{U} = F, \bar{U}$ satisfying the desired regularity properties. To conclude, we need to show that $\bar{U} = U$, that is A is one-to-one.

We thus consider $U \in D(A)$, such that $AU = 0$. Then U satisfies (2.3) with $f_1 = f_2 = f_3 = 0$ and the boundary conditions (2.1): ψ, v also satisfy (2.4), (2.5) and (2.6) with $F = 0$. The mixed Neumann-Dirichlet problem of which ψ is solution shows that $\psi = 0$; then $v = 0$ because of (2.4), the second equation (2.3) and $v = 0$ at $x = 0$. Finally $u = 0$ because of the first equation (2.3) and the boundary condition $\bar{U}_0 u - \psi\lambda = 0$ at $x = 0$.

Theorem 2.1 is thus proved. □

2.2. **Positivity of A and A^*.** We endow the space $H = L^2(\mathcal{M}')^3$ with the Hilbert scalar product and norm

$$(U, \tilde{U})_H = \int_{\Gamma_i} \left(u\tilde{u} + v\tilde{v} + \frac{1}{N^2}\psi\tilde{\psi} \right) d\mathcal{M}', \quad |U|_H = \{(U,U)_H\}^{1/2}.$$

Our aim is now to prove that A and its adjoint A^* defined below are positive in the sense

$$\begin{cases} (AU, U)_H \geq 0, & \forall U \in D(A), \\ (A^*U, U)_H \geq 0, & \forall U \in D(A^*). \end{cases} \tag{2.11}$$

These properties are needed to apply the Hille-Phillips-Yoshida theorem (see Section 4). The result for U is now easy thanks to Theorem 2.1. Indeed the following easy calculations are now legitimate, $\forall\ U \in D(A)$:

$$\begin{aligned} (AU, U)_H = & \int_{\Gamma_i} \left[\left(\bar{U}_0 u_x - \frac{1}{\lambda}\psi_x \right) u + \left(\bar{U}_0 v_x - \frac{1}{\lambda}\psi_y \right) v \right. \\ & \left. + \frac{\bar{U}_0}{N^2}\psi_x\psi - \frac{1}{\lambda}(u_x + v_y)\psi \right] d\mathcal{M}' \\ = & \frac{\bar{U}_0}{2}\int_0^{L_2} (u^2 + v^2)(L_1, y)dy \\ & - \int_0^{L_2} \left[\frac{\bar{U}_0}{2}(u^2 + \frac{1}{N^2}\psi^2)(0, y) - \frac{1}{\lambda}(u\psi)(0, y) \right] dy \\ \geq & \frac{\bar{U}_0}{2}\int_0^{L_2} \left(\left(\lambda\bar{U}_0\right)^{-2} - N^{-2} \right) \psi^2(0, y)dy \geq 0. \end{aligned} \tag{2.12}$$

All the integrations by parts above are easy to justify for functions in $H^1(\mathcal{M}')$. We just want to emphasize those involving u. If u and $\tilde{u}$ belong to $L^2_y(0, L_2; H^1_x(0, L_1))$, then $u, \tilde{u} \in L^2_y(0, u_2; \mathcal{C}_x([0, L_1]))$ and for a.e. $y \in (0, L_2)$:

$$\int_0^{L_1} (u_x \tilde{u} + u\tilde{u}_x)(x, y)dx = (u\tilde{u})(1, y) - (u\tilde{u})(0, y)$$

and, integrating in y,

$$\begin{aligned} &\int_{\Gamma_i} (u_x \tilde{u} + u\tilde{u}_x)dxdy = \\ &= \int_0^{L_2} [(u, \tilde{u})(1, y) - (u\tilde{u})(0, y)]dy. \end{aligned} \tag{2.13}$$

To prove (2.12), we apply (2.13) with $\tilde{u} = u, \psi$, and v.

We now turn to the definition of the formal adjoint A^* of A and its domain $D(A^*)$, in the sense of the adjoint of a linear unbounded operator (see [12]). For that purpose we first compute $(AU, \tilde{U})_H$ for U and $\tilde{U}$ smooth. By integration by parts, using Stokes formula, we find:

$$\begin{aligned} (AU, \tilde{U})_H = &\int_{\Gamma_i} \left[\left(\bar{U}_0 u_x - \frac{1}{\lambda}\psi_x \right) \tilde{u} + \left(\bar{U}_0 v_x - \frac{1}{\lambda}\psi_y \right) \tilde{v} \right. \\ &\qquad \left. + \frac{\bar{U}_0}{N^2}\psi_x\tilde{\psi} - \frac{1}{\lambda}(u_x + v_y)\tilde{\psi} \right] d\mathcal{M}' \\ = &\, I_0 + I_1, \end{aligned} \tag{2.14}$$

where I_0 stands for the integrals in $\mathcal{M}'$ and I_1 for the integrals on $\partial\mathcal{M}'$. For I_0 we have

$$I_0 = \int_{\Gamma_i} \left(\mathcal{A}_1^* \tilde{U} u + \mathcal{A}_2^* \tilde{U} v + N^{-2} \mathcal{A}_3^* \tilde{U} \psi \right) d\mathcal{M}', \tag{2.15}$$

with $\mathcal{A}^*\tilde{U} = (\mathcal{A}_1^*\tilde{U}, \mathcal{A}_2^*, \tilde{U}, \mathcal{A}_3^*\tilde{U})$, and

$$\mathcal{A}^*\tilde{U} = \begin{cases} -\bar{U}_0\tilde{u}_x + \dfrac{1}{\lambda}\tilde{\psi}_x, \\ \\ -\bar{U}_0\tilde{v}_x + \dfrac{1}{\lambda}\tilde{\psi}_y, \\ \\ -\bar{U}_0\tilde{\psi}_x + \dfrac{N^2}{\lambda}(\tilde{u}_x + \tilde{v}_y). \end{cases} \tag{2.16}$$

For I_1, taking into account the boundary conditions (2.1), there remains:

$$\begin{aligned} I_1 = & \int_0^{L_2} \left[\bar{U}_0 (u\tilde{u})(L_1, y) + \bar{U}_0 (v\tilde{v})(L_1, y) - \frac{1}{\lambda}(u\tilde{\psi})(L_1, y) \right] dy \\ & + \int_0^{L_2} \left(-\frac{\bar{U}_0}{N^2} + \frac{1}{\lambda^2 \bar{U}_0} \right) (\psi\tilde{\psi})(0, y) dy \\ & - \lambda^{-1} \int_0^{L_1} \left[(v\tilde{\psi})(x, L_2) + (v\tilde{\psi})(x, 0) \right] dx. \end{aligned}$$

According to [12], $D(A^*)$ consists of the $\tilde{U}$ in H such that $U \to (AU, \tilde{U})_H$ is continuous on $D(A)$ for the topology (norm) of H. If U is restricted to the class of $\mathcal{C}^\infty$ functions with compact support in $\mathcal{M}'$ (endowed with the norm of H), then $I_1 = 0$, and $U \to I_0$ can only be continuous if $\mathcal{A}^*\tilde{U}$ as defined in (2.16) belongs to $L^2(\mathcal{M}')^3$. We then observe that Theorem 1.1 applies to $\mathcal{A}^*$ as well and to more general constant coefficients operators. Hence if $\tilde{U} \in D(A^*)$ then $\tilde{U} \in L^2(\mathcal{M}')^3$ with $\mathcal{A}^*\tilde{U} \in L^2(\mathcal{M}')^3$, and the traces of $\tilde{U}$ are defined as in Theorem 1.1. We now restrict U to the class of $\mathcal{C}^\infty$ functions on $\bar{\mathcal{M}}'$ which belong to $D(A)$. Then the expressions above of I_0 and I_1 show that $U \to (AU, \tilde{U})_H$ can only be continuous in U for the topology (norm) of H if the following boundary conditions are satisfied:

$$\left\{ \begin{array}{l} \tilde{\psi} = 0 \text{ at } y = 0, L_2 \text{ and } x = 0, \\ \tilde{v} = 0 \text{ and } \tilde{u} = \tilde{\psi}/\lambda\bar{U}_0 \text{ at } x = L_1. \end{array} \right. \tag{2.17}$$

Hence we conclude that[4]

$$D(A^*) = \left\{ \tilde{U} \in L^2(\mathcal{M}')^3, \mathcal{A}^*\tilde{U} \in L^2(\mathcal{M}')^3, \text{ and } \tilde{U} \text{ satisfies (2.17)} \right\}. \tag{2.18}$$

We have shown indeed that $D(A^*)$ is included in the right-hand side of (2.18). Now, with exactly the same reasoning as in Theorem 2.1, we can show that
(2.19)
If $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\psi}) \in D(A^*)$, *then* $\tilde{v}$ *and* $\tilde{\psi}$ *belong to* $H^1(\mathcal{M}')$ *and* $\tilde{u}_x$ *belongs to* $L^2(\mathcal{M}')$.

Thus using again (2.13), we see that for every U in $D(A)$ and $\tilde{U}$ in $D(A^*)$ (not necessarily $\mathcal{C}^\infty$), then $(AU, \tilde{U})_H = I_0 + I_1$ as above, with $I_1 = 0$ and I_0 as in (2.15), so that $U \to (AU, \tilde{U})_H$ is continuous on $D(A)$ for the norm of H. The opposite inclusion is proven and (2.18) is established. This reasoning also shows that, for every $\tilde{U} \in D(A^*)$, $A^*\tilde{U} = \mathcal{A}^*\tilde{U}$, $\mathcal{A}^*$ as in (2.16).

It is now easy to prove the positivity of A^*, that is the second statement in (2.11). We proceed as in (2.12), using (2.13):

[4] Similarly we write $\mathcal{A}_n^*, A_n^*, D(A_n^*)$ when the dependence on n needs to be emphasized ($\lambda = \lambda_n$).

$$
\begin{aligned}
(A^*U,U)_H = &\int_{\Gamma_i} \Bigg[\left(-\bar{U}_0 u_x + \frac{1}{\lambda}\psi_x \right) u + \\
&\left(-\bar{U}_0 v_x + \frac{1}{\lambda}\psi_y \right) v + \left(-\frac{\bar{U}_0}{N^2}\psi_x + \frac{1}{\lambda}(u_x+v_y) \right) \psi \,\Bigg]\, d\mathcal{M}' \\
&= \text{(using the boundary conditions (2.17))} \\
&= \frac{\bar{U}_0}{2}\int_0^{L_2} (u^2+v^2)(0,y)dy \\
&- \int_0^{L_2} \left[\frac{\bar{U}_0}{2}(u^2+\frac{1}{N^2}\psi^2)(L_1,y) - \frac{1}{\lambda}(u\psi)(L_1,y) \right] dy \\
&\geq \frac{\bar{U}_0}{2}\int_0^{L_2} \left((\lambda\bar{U}_0)^{-2} - N^{-2} \right) \psi^2(L_1,y)dy \geq 0.
\end{aligned}
$$

Hence the positivity. Note that we cannot just write $(A^*U,U)_H = (AU,U)_H \geq 0$, because U in $D(A^*)$ may not belong to $D(A)$. In summary we have proven the following theorem:

Theorem 2.2. *For every $U \in D(A_n)$, as defined in (2.2), we have $(A_nU,U)_{L^2(\mathcal{M}')^3} \geq 0$. Similarly, for every $U \in D(A_n^*)$ defined in (2.18), we have $(A_n^*U,U)_{L^2(\mathcal{M}')^3} \geq 0$.*

Remark 2.2. Based on the previous results we can show that, for each $n, 1 \leq n \leq n_c, -A = -A_n$ is the infinitesimal generator of a contraction semi-group. Then by application of the Hille-Yoshida theorem we can solve the initial and boundary value problem associated with equations (1.13) for every such n. We refrain from developing this and will instead establish a well-posedness result for all modes together, see Section 4.

3. Supercritical Modes

We now consider the initial and boundary value problem for one single supercritical mode, that is equations (1.11) or equivalently (1.13) when $n > n_c$. We temporarily drop the indices n, and write e.g.

$$
\lambda = \lambda_n > \frac{N}{\bar{U}_0} \tag{3.1}
$$

3.1. **The operator A and its adjoint A^*.** Here, for one supercritical mode we choose the following boundary conditions:

$$
\begin{cases} u, v, \text{ and } \psi = 0 \text{ at } x = 0, \\ \text{and } \psi = 0 \text{ at } y = 0 \text{ and } L_2. \end{cases} \tag{3.2}
$$

In this case the operator $A = A_n$, is defined by $AU = \mathcal{A}U$ as in (1.27), and

$$
D(A) = \left\{ U \in H = L^2(\mathcal{M}')^2, \mathcal{A}U \in L^2(\mathcal{M}'), U \text{ satisfies (3.2)} \right\}. \tag{3.3}
$$

Note that, according to Theorem 1.1, the traces of u, v, ψ appearing in (3.2) and (3.3) are well-defined when $U \in L^2(\mathcal{M}')^3$ and $\mathcal{A}U \in L^2(\mathcal{M}')^3$.

In view of proving that $-A = -A_n$ is the infinitesimal generator of a contraction semigroup, our main task is now to show that

$$\left\{ \begin{array}{l} (AU,U)_H \geq 0, \forall U \in D(A), \text{ and} \\ (A^*U,U)_H \geq 0, \forall U \in D(A^*)^5, \end{array} \right. \tag{3.4}$$

where A^* is defined below. Our approach for (3.4) is however different from the subcritical case which was based on the regularity result Theorem 2.1. In the supercritical case the equations are hyperbolic and there are no similar regularity results. Instead we are going to prove that $(AU,U)_H \geq 0$ when U is sufficiently regular; then we define A^* and prove that $(A^*U,U)_H \geq 0$ for every U, sufficiently regular, in the domain of A^*; and finally, by passage to the limit, we prove (3.4) for all functions in $D(A)$ and $D(A^*)$ respectively.

Positivity of A

We prove that $(AU,U)_H \geq 0$ when U belongs to $D(A)$ and is sufficiently regular (say in $\mathcal{C}^2(\bar{\mathcal{M}}')^3$):

$$\begin{aligned}
(AU,U)_H &= \int_{\mathcal{M}'} [(\bar{U}_0 u_x - \frac{1}{\lambda}\psi_x)u + (\bar{U}_0 v_x - \frac{1}{\lambda}\psi_y)v \\
&\quad + \frac{1}{N^2}(\bar{U}_0\psi_x - \frac{N^2}{\lambda}(u_x+v_y))\psi] d\mathcal{M}' \\
&= \text{ (using (3.2) and (3.1))} \\
&= \frac{\bar{U}_0}{2}\int_0^{L_2} (u^2+v^2+\frac{1}{N^2}\psi^2)(L_1,y)dy \\
&\quad - \frac{1}{\lambda}\int_0^{L_2} (u\psi)(L_1,y)dy \\
&= \frac{\bar{U}_0}{2}\int_0^{L_2} v^2(L_1,y)dy + \\
&\quad + \frac{\bar{U}_0}{2}\int_0^{L_2} (u^2+\frac{1}{N^2}\psi^2 - \frac{2}{\lambda\bar{U}_0}u\psi)(L_1,y)dy \\
&\geq 0.
\end{aligned} \tag{3.5}$$

The adjoint A^*

Assume that $U \in D(A)$ and $\tilde{U} \in H$ are smooth functions; then, as in (2.14):

(3.6)

$$\begin{aligned}
(AU,\tilde{U})_H &= \int_{\mathcal{M}'} \left[(\bar{U}_0 u_x - \frac{1}{\lambda}\psi_x)\tilde{u} + (\bar{U}_0 v_x - \frac{1}{\lambda}\psi_y)\tilde{v} + \frac{1}{N^2}(\bar{U}_0\psi_x - \frac{N^2}{\lambda}(u_x+v_y))\tilde{\psi} \right] d\mathcal{M}' \\
&= I_0 + I_1,
\end{aligned}$$

where I_0 stands for the integrals on $\mathcal{M}'$ and I_1 for the integrals on $\partial\mathcal{M}'$. For I_0, we have

$$I_0 = \int_{\mathcal{M}'} \left(\mathcal{A}_1^*\tilde{U}u + \mathcal{A}_2^*\tilde{U}v + N^{-2}\mathcal{A}_3^*\tilde{U}\psi \right) d\mathcal{M}',$$

with $\mathcal{A}^*\tilde{U} = (\mathcal{A}_1^*\tilde{U}, \mathcal{A}_2^*\tilde{U}, \mathcal{A}_3^*\tilde{U})$ as in (2.16). For I_1, taking into account the boundary conditions (3.2), there remains:

$$\begin{aligned} I_1 = &\int_0^{L_2} \bar{U}_0[(u\tilde{u}) + (v\tilde{v}) + N^{-2}(\psi\tilde{\psi})](L_1, y)dy \\ &- \int_0^{L_2} \lambda^{-1}(\psi\tilde{u} + u\tilde{\psi})(L_1, y)dy \\ &- \int_0^{L_1} \lambda^{-1}[(v\tilde{\psi})(L_2, y) - (v\tilde{\psi})(0, y)]dy. \end{aligned} \tag{3.7}$$

According to [12] , $D(A^*)$ consists of the $\tilde{U}$ in H such that $U \longrightarrow (AU, \tilde{U})_H$ is continuous on $D(A)$ for the topology (norm) of H. If U is restricted to the class of $\mathcal{C}^\infty$ functions with compact support in $\mathcal{M}'$ (endowed with the norm of H), then $I_1 = 0$ and $U \longrightarrow I_0$ can only be continuous if $\mathcal{A}^*\tilde{U}$ as defined in (2.16) belongs to $L^2(\mathcal{M}')^3$. If $\tilde{U}$ belongs to H and $\mathcal{A}^*\tilde{U}$ belongs to $L^2(\mathcal{M}')^3$, then we already observed that Theorem 1.1 applies to $\mathcal{A}^*$ as well. Consequently the traces of $\tilde{U}$ are defined as in Theorem 1.1 and the calculations in (3.6) are now valid for any such $\tilde{U}$ (and U in $D(A)$ not necessarily smooth). We now restrict U to the class of $\mathcal{C}^\infty$ function on $\bar{\mathcal{M}}'$ which belong to $D(A)$. Then the expressions above of I_0 and I_1 show that $U \longrightarrow (AU, \tilde{U})_H$ can only be continuous in U for the topology (norm) of H if the following boundary conditions are satisfied

$$\begin{cases} \tilde{u}, \tilde{v} \text{ and } \tilde{\psi} = 0 \text{ at } x = L_1, \\ \text{and } \tilde{\psi} = 0 \text{ at } y = 0 \text{ and } L_2. \end{cases} \tag{3.8}$$

Conversely if $\tilde{U} \in H, \mathcal{A}^*\tilde{U} \in L^2(\mathcal{M}')^3$ and the conditions (3.8) are satisfied, then the calculation (3.6) are valid, $I_1 = 0$, and $U \longrightarrow (AU, \tilde{U})_H$ is continuous on $D(A)$ for the norm of H. Hence $\tilde{U} \in D(A^*)$ and we conclude[6] that

$$D(A^*) = \left\{\tilde{U} \in L^2(\mathcal{M}')^3, \mathcal{A}^*\tilde{U} \in L^2(\mathcal{M}')^3, \text{ and } \tilde{U} \text{ satisfies (3.8)}\right\}; \tag{3.9}$$

and that $A^*\tilde{U} = \mathcal{A}^*\tilde{U}$ for $\tilde{U}$ in $D(A^*), \mathcal{A}^*$ as in (2.16).

Positivity of A and A^*

The proof of the positivity is not done as in the subcritical case, since the regularity result of Theorem 2.1 is not available in this case. Instead, for A, to prove that $(AU, U)_H \geq 0$, for U in $D(A)$, we will construct a sequence of smooth functions $U_n \in D(A)$ such that, as $n \longrightarrow \infty$,

$$\begin{aligned} &U_n \longrightarrow U \text{ in } H \text{ strongly}, \\ &AU_n \rightharpoonup AU \text{ in } H \text{ weakly}. \end{aligned}$$

Then $(AU_n, U_n)_H \longrightarrow (AU, U)_H$ and since $(AU_n, U_n)_H \geq 0$ by (3.5), $(AU, U)_H \geq 0$ follows. The proof for A^* would be similar.

Given $U \in D(A)$, with $F = (f_1, f_2, f_3) = AU \in H$, we observe that the calculations (2.3)-(2.5) are still valid but now, since $\lambda > N/\tilde{U}_0$ equation (2.5 is hyperbolic. In fact we are now going to treat (2.5) as a second order evolution equation in x (wave equation), in which x is the time-like variable and y is the spatial variable. For such a wave equation

[6]Remember that A, A^* depend on n through $\lambda = \lambda_n$; we write A_n, A_n^* when the dependance on n needs to be emphasized.

we need to prescribe ψ and ψ_x at $x = 0_1$, and ψ at $y = 0$ and L_2. These values of ψ are given equal to 0, and we are missing ψ_x which we infer from the first and third equations (2.3) when U is smooth, which we assume for the moment. Indeed since $v = 0$ at $x = 0, v_y = 0$ and these equations, restricted to $x = 0$, become a system

$$\bar{U}_0 u_x - \lambda^{-1}\psi_x = f_1,$$
$$\bar{U}_0 \psi_x - N^2\lambda^{-1} u_x = f_3,$$

which allows us to compute u_x and ψ_x at $x = 0$; hence for ψ_x:

$$\psi_x(0,y) = \frac{1}{\bar{U}_0^2 - N^2\lambda^{-2}} \left(\frac{N^2}{\lambda} f_1(L_1,y) + \bar{U}_0 f_3(L_1,y)\right), \quad 0 < y < L_2. \tag{3.10}$$

We continue to assume that all functions $(f_1, f_2, f_3, u, v, \psi)$ are sufficiently regular and we integrate (2.5) from 0 to x. Setting

$$\Psi(x,y) = \int_0^x \psi(x',y)dy \ , \tag{3.11}$$

we obtain:

$$\begin{aligned} &-(\bar{U}_0^2 - \frac{N^2}{\lambda^2})(\psi_x(x,y) - \psi_x(0,y)) \\ &- \frac{N^2}{\lambda^2}\Psi_{yy}(x,y) = \frac{N^2}{\lambda}(f_1(x,y) - f_1(0,y)) \\ &+ \frac{N^2}{\lambda}F_{2y}(x,y) + \bar{U}_0(f_3(x,y) - f_3(0,y)), \end{aligned}$$

where

$$F_i(x,y) = \int_0^x f_i(x',y)dx'. \tag{3.12}$$

Taking (3.10) into account, there remains

$$\begin{aligned} &(\bar{U}_0^2 - \frac{N^2}{\lambda^2})\Psi_{xx} - \frac{N^2}{\lambda^2}\Psi_{yy} = \\ &\qquad = \frac{N^2}{\lambda} f_1 + \frac{N^2}{\lambda}F_{2y} + \bar{U}_0 f_3, \end{aligned} \tag{3.13}$$

which we aim to consider for $x > 0$, with the "initial" and boundary conditions:

$$\begin{cases} \Psi = 0 \text{ and } \Psi_x = \psi = 0 \text{ at } x = 0, \\ \Psi = 0 \text{ at } y = 0 \text{ and } L_2. \end{cases} \tag{3.14}$$

We obtain a priori estimates for Ψ in a standard way by multiplying (3.13) by Ψ_x, integrating in y and integrating by parts. We find

$$\begin{aligned} &\frac{1}{2}(\bar{U}_0^2 - \frac{N^2}{\lambda^2})\frac{d}{dx}\int_0^{L_2} \Psi_x^2(x,y)dy + \frac{N^2}{2\lambda^2}\frac{d}{dx}\int_0^{L_2} \Psi_y^2(x,y)dy \\ &= -\int_0^{L_2} [(\frac{N^2}{\lambda} f_1 - \frac{N^2}{\lambda}F_{2y} + \bar{U}_0 f_3)\Psi_x](x,y)dy. \end{aligned}$$

We then integrate in x from 0 to x to obtain, using (3.14):

$$\begin{aligned}\frac{1}{2}(\bar{U}_0^2 - \frac{N^2}{\lambda^2}) \int_0^{L_2} \Psi_x^2(x,y)dy + \frac{N^2}{2\lambda^2} \int_0^{L_2} \Psi_y^2(x,y)dy \\ = - \int_0^x \int_0^{L2} [(\frac{N^2}{\lambda} f_1 - \frac{N^2}{\lambda} F_{2y} + \bar{U}_0 f_3)\Psi_x](x',y)dx'dy.\end{aligned} \tag{3.15}$$

The term involving F_{2y} can be integrated by parts, using (3.14); we find, all functions being sufficiently regular:

$$\begin{aligned}&\frac{N^2}{\lambda} \int_0^x \int_0^{L_2} (F_{2y}\Psi_x)(x',y)dx'dy \\ &\quad = \frac{N^2}{\lambda} \int_0^x \int_0^{L_2} (F_2 \Psi_{xy})(x',y)dx'dy \\ &\quad = \frac{N^2}{\lambda} \int_0^{L_2} (F_2\Psi_y)(x,y)dy - \frac{N^2}{\lambda} \int_0^x \int_0^{L_2} (F_{2x}\Psi_y)(x',y)dx'dy \\ &\quad = \frac{N^2}{\lambda} \int_0^{L_2} \Psi_y(x,y) \int_0^x f_2(x',y)dx'dy - \frac{N^2}{\lambda} \int_0^x \int_0^{L_2} (f_2\Psi_y)(x',y)dx'dy.\end{aligned}$$

We insert this expression in (3.15) and integrate (3.15) in x from 0 to L_1, which leads to:

$$\begin{aligned}&(\bar{U}_0^2 - \frac{N^2}{\lambda^2}) \int_{\mathcal{M}'} \Psi_x^2(x,y)dxdy + \frac{N^2}{\lambda^2} \int_{\mathcal{M}'} \Psi_y^2(x,y)dxdy \\ &= \frac{2N^2}{\lambda} \int_{\mathcal{M}'} \Psi_y(x,y)(\int_0^x f_2(x',y)dx')dydx \\ &- 2 \int_{\mathcal{M}'} [\frac{N^2}{\lambda} f_1 \Psi_x + \frac{N^2}{\lambda} (\int_0^x f_2(x',y)dx')\Psi_y + \bar{U}_0 f_3 \Psi_x](x,y)dxdy.\end{aligned} \tag{3.16}$$

Since $\bar{U}_0 > N/\lambda$, we easily deduce from (3.16) an estimate

$$\begin{aligned}&\int_{\mathcal{M}'} (\Psi_x^2 + \Psi_y^2)(x,y)dxdy \\ &\leq \kappa_1 (|f_1|^2_{L^2(\mathcal{M}')} + |f_2|^2_{L^2(\mathcal{M}')} + |f_3|^2_{L^2(\mathcal{M}')}),\end{aligned} \tag{3.17}$$

where κ_1 depends only on the data, namely, $L_1, L_2, \bar{U}_0, N$ and λ. Alternatively (3.17) can be written as

$$\int_{\mathcal{M}'} (\psi^2 + \Psi_y^2)(x,y)dxdy \leq \kappa_1 |F|^2_{L^2(\mathcal{M}')^3}. \tag{3.18}$$

The calculations above have been made under the assumption that $U \in D(A)$ (and $AU = F$) are sufficiently regular. The lemma below extends (3.18) to all U in $D(A)$.

Lemma 3.1. *In the supercritical case (i.e. assuming (3.1)), (3.18) is valid for every $U = (u, v, \psi)$ in $D(A)$. There also exists a constant κ_2 depending only on the data such that*

$$|U|_H \leq \kappa_2 |AU|_H, \quad \forall \; U \in D(A). \tag{3.19}$$

Proof. Given U in $D(A)$, then $AU = F = (f_1, f_2, f_3)$ belongs to $H = L^2(\mathcal{M}')^3$ and it can be approximated in $L^2(\mathcal{M}')^3$ by a sequence of smooth functions $F_m = (f_{1m}, f_{2m}, f_{3m})$ which are $\mathcal{C}^\infty$ with compact support in $\mathcal{M}'$. With these F_m, we solve equation (3.13) with boundary and initial conditions (3.14) so that we obtain the Ψ_m which satisfy (3.18).

As $n \to \infty$, the F_m converge to F in $L^2(\mathcal{M}')^3$ and the Ψ_m converge to $\bar{\Psi}$ weakly in $H^1(\mathcal{M}')$, where $\bar{\Psi}$ is the (unique) solution of (3.13), (3.14) in $H^1(\mathcal{M}')$. We then define $\bar{\psi} = \partial\bar{\Psi}/\partial x$ which satisfies equation (2.5) in the distributional sense and (3.18) is satisfied by $\bar{\psi}, \bar{\Psi}$ and F. By inspection of (3.13), we notice that $\bar{\Psi}_{yy}$ and F_{2y} belong to $L^2_x(0, L_1; H^{-1}_y(0, L_2))$ so that

$$\bar{\psi}_x = \bar{\Psi}_{xx} \in L^2_x(0, L_1; H^{-1}_y(0, L_2)),$$

and $\bar{\psi} \in \mathcal{C}_x([0, L_1]; H^{-1}_y(0, L_2))$. Hence $\bar{\psi}(0, \cdot)$ is defined and it vanishes according to (3.14).

Now, integrating in x the first and second equations (2.3) and imposing $\bar{u} = \bar{v} = 0$ at $x = 0$, we define $\bar{u}$ and $\bar{v}$ by setting

$$\begin{aligned} \bar{U}_0\bar{u} - \frac{1}{\lambda}\bar{\psi} &= \int_0^x f_1 dx', \\ \bar{U}_0\bar{v} - \frac{1}{\lambda}\bar{\psi}_y dx' &= \int_0^x f_2 dx'. \end{aligned} \tag{3.20}$$

We want to show that the third equation (2.3) is satisfied as well: differentiating the first equation (3.20) in x and the second equation (3.20) in y we find:

$$\bar{U}_0\bar{u}_x - \frac{1}{\lambda}\bar{\psi}_x = f_1, \;\; \bar{U}_0\bar{v}_y - \frac{1}{\lambda}\bar{\Psi}_{yy} = F_{2y},$$

and then

$$\begin{aligned} \bar{U}_0\bar{\psi}_x - \frac{N^2}{\lambda}(\bar{u}_x + \bar{v}_y) &= \\ &= \bar{U}_0\bar{\psi}_x - \frac{N^2}{\lambda\bar{U}_0}\Big(\frac{1}{\lambda}\bar{\psi}_x + f_1 + \frac{1}{\lambda}\bar{\Psi}_{yy} + F_{2y}\Big) \\ &= \left(\bar{U}_0 - \frac{N^2}{\lambda^2\bar{U}_0}\right)\bar{\psi}_x - \frac{N^2}{\lambda^2\bar{U}_0}\bar{\Psi}_{yy} - \frac{N^2}{\lambda\bar{U}_0}(f_1 + F_{2y}) \\ &= (\text{ by (3.13)}) \\ &= f_3, \end{aligned}$$

so that all three equations (2.3) are satisfied by $\bar{U}$. Furthermore $\bar{U}$ satisfies the boundary conditions (3.14) and we conclude that $\bar{U} \in D(A)$ and $A\bar{U} = F$. Since $AU = F$ as well, we will conclude that $\bar{U} = U$ by showing that A is one-to-one.

To show that A is one-to-one, consider $\tilde{U} \in D(A)$ such that $A\tilde{U} = 0$. Then $\tilde{\Psi}$ defined by (3.11) satisfies (3.13) and (3.14). At this point we do not know that $\tilde{\Psi} \in H^1(\mathcal{M}')$, but, at least, we infer from (3.11) that $\tilde{\Psi} \in L^2(\mathcal{M}')$ since $\tilde{\psi} \in L^2(\mathcal{M}')$. We then infer from [5] that (3.13) - (3.14) has a unique solution in $L^2(\mathcal{M}')$, so that $\tilde{\Psi} = 0$. From this we conclude that $\tilde{\psi} = 0$ and $\tilde{u}$ and $\tilde{v}$ also vanish since they satisfy equations (3.20), because of the boundary conditions at $x = 0$. Hence $\tilde{U} = 0$ and A is one-to-one.

Returning to U, we conclude at this point that ψ and Ψ satisfy (3.18) which was the first statement in this lemma.

There remains to prove (3.19); $|\psi|_{L^2(\mathcal{M}')} \leq \kappa|AU|_H$ follows from (3.18), and the analogue results for u and v follow from (3.20)(and (3.18)).

The proof of the Lemma is complete. □

We can now prove (3.4).[7]

Theorem 3.1. *In the supercritical case (i.e. assuming (3.1)), for every $U \in D(A_n), A_n$ defined in (3.3), we have $(A_n U, U)_{L^2(\mathcal{M}')^3} \geq 0$. Similarly, we have $(A_n^* U, U)_{L^2(\mathcal{M}')^3} \geq 0$, for every U in $D(A_n^*), A_n^*$ and $D(A_n^*)$ defined in (3.9).*

Proof. We prove the result for A, the proof would be similar for A^*.

Considering $U \in D(A)$, we approximate $AU = F$ by a sequence of smooth functions F_m as in Lemma 3.1. To each function F_m, we associate $U_m \in D(A)$ such that $AU_m = F_m$: each U_m is constructed exactly as we constructed $\bar{U}$ in Lemma 3.1, and U_m is smooth. We easily check that, as $m \to \infty, U_m$ weakly converges in H to U, whereas $AU_m = F_m$ strongly converges in H to $AU = F$. Hence

$$(AU_m, U_m)_H \longrightarrow (AU, U)_H,$$

and since $(AU_m, U_m)_H \geq 0$ by (3.5), U_m being sufficiently regular, we conclude that $(AU, U)_H \geq 0$. □

Remark 3.1. As indicated in Remark 2.2, and based on the previous results, we can show for each $n > n_c$ that $-A = -A_n$ is the infinitesimal generator of a contraction semi-group. Then by application of the Hille-Yoshida theorem we can solve the initial and boundary value problem associated with equations (1.13), for each such n. We refrain from developing this and we will study all subcritical and supercritical modes at once (together) in the next section.

4. The initial and boundary value problem for the full system

In this section we aim to combine the results of the previous sections and to investigate the well-posedness for equations (1.5) associated with the suitable initial and boundary conditions. We successively consider the case of homogeneous and nonhomogeneous boundary conditions.

4.1. **The homogeneous boundary condition case.** As explained in (1.15) the function U and its respective components are decomposed in the form $U = U^0 + U^I + U^{II}$. Accordingly the basic function space H will be $L^2(\mathcal{M})^3$ or

$$H = H^0 \times \dot{L}^2(\mathcal{M})^3,$$

where H^0 is the same as H_0 in (1.29), and $\dot{L}^2(\mathcal{M})$ consists of the orthogonal, in $L^2(\mathcal{M})$, of the space of functions independent of z. Like in Section 1.5 the elements of H^0 will be the vectors $\mathbf{u}^0 = (u^0, v^0)$. The elements of $\dot{L}^2(\mathcal{M})^3$ will be the triplets $U = (u, v, \psi)$; each of these functions possesses an expansion of the form (1.10) from which we can accordingly identify the functions with the product of their components, and the space

[7]We recall that A and A^* depend on n as $\lambda = \lambda_n$.

$L^2(\mathcal{M})$ with the product of an infinite sequence of spaces $L^2(\mathcal{M}')$. The space H is a subspace of $L^2(\mathcal{M})^3$, just remembering that $\psi_0 = 0$, and its natural scalar product and norms are essentially those of $L^2(\mathcal{M})^3$, more precisely,

$$\begin{aligned}(U, \tilde{U})_H &= \Big((u, v, \psi), (\tilde{u}, \tilde{v}, \tilde{\psi})\Big)_{L^2(\mathcal{M})^3} \\ &= (u, \tilde{u})_{L^2(\mathcal{M})} + (v, \tilde{v})_{L^2(\mathcal{M})} + \frac{1}{N^2}(\psi, \tilde{\psi})_{L^2(\mathcal{M})}, \\ |U|_H &= [(U, U)_H]^{1/2}.\end{aligned}$$

Each U can be seen as the sum of its three components

$$U = U^0 + U^I + U^{II}, \tag{4.1}$$

or it can be identified with the infinite sequence of its components $\{U_n\}_{n\geq 0}$, in which case[8]

$$|U|_H^2 = |\mathbf{u}^0|^2_{L^2(\mathcal{M}')^2} + \Sigma_{n=1}^{\infty} |U_n|^2_{L^2(\mathcal{M})^2}.$$

The semigroup

We now introduce the operator A and its domain $D(A)$ in H. We have $D(A) = D(A^0) \times D(A^I) \times D(A^{II})$, where the space $D(A^0)$ is the same as in (1.32),

$$D(A^0) = \left\{\mathbf{u}^0 \in H^0, \mathbf{u}^0_x \in L^2(\mathcal{M}')^2\right\}. \tag{4.2}$$

Then (compare to (2.2)):

$$\begin{aligned}D(A^I) = \{U^I = (U_1, \ldots, U_{n_c}),\ \ &U_n \in L^2(\mathcal{M}')^3, \mathcal{A}_n U_n \in L^2(\mathcal{M}')^3, \\ &n = 1, \ldots, n_c,\ \ U^I \text{ satisfies } , (4.4)\},\end{aligned} \tag{4.3}$$

$$\begin{cases} \psi^I & = 0 \text{ at } x = L_1, \text{ and } y = 0, L_2, \\ v^I & = 0 \text{ and } u^I + \phi^I/\bar{U}_0 = 0 \text{ at } x = 0. \end{cases} \tag{4.4}$$

Here we introduced for convenience the function $\phi = \psi^0 + \phi^I + \phi^{II} = \{\phi_n\}_{n\geq 0}$, with, according to (1.11),

$$\phi_n = -\frac{1}{\lambda_n}\psi_n, \quad n \geq 1. \tag{4.5}$$

Finally (compare to (3.3)):

$$\begin{aligned}D(A^{II}) = \big\{U^{II} = \{U_n\}_{n>n_c}, &U_n \in L^2(\mathcal{M}')^3, \mathcal{A}_n U_n \\ &\in L^2(\mathcal{M}')^3, n = 1, \ldots, n_c,\ \ U^{II} \text{ satisfies } (4.7)\big\},\end{aligned} \tag{4.6}$$

$$\begin{cases} u^{II} = v^{II} = \psi^{II} = 0 \quad \text{at } x = 0, \\ \psi^{II} = 0 \quad \text{at } y = 0 \text{ and } L_2. \end{cases} \tag{4.7}$$

For $U = \left(\mathbf{u}_0, U^I, U^{II}\right)$ in $D(A)$, we set $AU = (A^0\mathbf{u}^0, A^I U^I, A^{II} U^{II})$ where

[8]Remember that $\psi^0 = 0$ so that $U^0 = \mathbf{u}^0 = (u^0, v^0)$.

$$A^0\mathbf{u}^0 = P_{H^0}\left(\bar{U}_0\frac{\partial \mathbf{u}^0}{\partial x} + fe_z \wedge \mathbf{u}^0\right)$$

as in (1.31) and we define $A^I U^I$ and $A^{II}U^{II}$ componentwise by setting

$$A_n U_n = \mathcal{A}_n U_n, \quad \text{for } 1 \leq n,$$

$\mathcal{A}_n$ as in (1.27) with $\lambda = \lambda_n$.

We now need to define the adjoint A^* of A and prove that A and A^* are positive which will follow promptly from the results in the previous sections.

For the adjoint, it is easy to see that

$$D(A^*) = D(A^{0*}) \times D(A^{I*}) \times D(A^{II**}), \tag{4.8}$$

with $D(A^{0*}) = D(A^0)$ as shown in Section 1.5, $D(A^{I*})$ defined in (2.18) and $D(A^{II*})$ defined in (3.9). Indeed, according to [12], $\tilde{U} \in D(A^*)$ if and only if,

$$U \to (AU, \tilde{U})_H = (A^0\mathbf{u}^0, \tilde{\mathbf{u}}^0) + (A^I U^I, \tilde{U}^I) + (A^{II}U^{II}, \tilde{U}^{II}),$$

is continuous on $D(A)$ for the topology (norm) of H. Considering successively $U = (\mathbf{u}^0, 0, 0), U = (0, U^I, 0)$, and $U = (0, 0, U^{II})$, we obtain that $D(A)$ is included in the space in the right-hand side of (4.8). Conversely any $\tilde{U}$ in the right-hand side of (4.8) belongs to $D(A)$ and hence (4.8) is proven.

We can prove the following

Theorem 4.1. *The operator $-A$ is infinitesimal generator of a semigroup of contractions in H.*

Proof. According to [17] and [3], it suffices to show that

i) A and A^* are closed operators, and their domains $D(A)$ and $D(A^*)$ are dense in H.

ii) A and A^* are positive:

$$\begin{aligned} &(AU, U)_H \geq 0, \qquad \forall U \in D(A), \\ &(A^*U, U)_H \geq 0, \qquad \forall U \in D(A^*). \end{aligned} \tag{4.9}$$

For i) we observe, as is well-known, that $D(A^*)$ (resp. $D(A)$) dense in H implies that A (resp. A^*) is closed. We proceed componentwise for, say, $D(A) : D(A^0)$ defined in (4.2) is dense in H^0, since the $\mathcal{C}^\infty$ functions $\mathbf{u}^0 = (u^0, v^0)$ with compact support in $\mathcal{M}'$ and such that div $\mathbf{u}^0 = u^0_x + v^0_y = 0$ are dense in H^0; see e.g. [13]; and for $D(A^I)$ and $D(A^{II})$ we simply observe that the $\mathcal{C}^\infty$ functions with compact support in $\mathcal{M}'$ are dense in $L^2(\mathcal{M}')$.

Finally for (4.9) we proceed componentwise and use the results of the previous sections, e.g. for A :

$$(AU, U)_H = (A^0\mathbf{u}^0, \mathbf{u}^0)_{H^0} + (A^I U^I, U^I)_{H^I} + (A^{II}U^{II}, U^{II})_{H^{II}}. \tag{4.10}$$

The first term in the right-hand side of (4.10) has been shown to be positive ($= 0$ in fact, see (1.34)). The second term is equal to

$$\sum_{n=1}^{nc}(A_n U_n, U_n)_{L^2(\mathcal{M}')^3},$$

and each of these terms is positive as shown in (2.11). Finally the third term

$$(A^{II}U^{II}, U^{II})_{H^{II}} = \sum_{n>n_c}(A_n U_n, U_n)_{L^2(\Gamma_i)^3},$$

and each term of the series is positive according to (3.4). □

The initial and boundary value problems

We now consider the whole system of three-dimensional linearized Primitive Equations, namely (1.5) and introduce the initial and boundary conditions. We start with the homogeneous boundary conditions and treat subsequently the case of nonhomogeneous boundary conditions.

As implied by the previous sections the boundary conditions will be different for the subcritical and supercritical components of $U = (u, v, \psi) = (U^0, U^I, U^{II})$. Hence for $U^0 = \mathbf{u}^0(\psi^0 = 0)$, we set (see (1.32)):

$$\mathbf{u}^0 \cdot \mathbf{n} = 0 \text{ on } \partial\mathcal{M}'. \tag{4.11}$$

For U^I, according to (2.1), the boundary conditions read

$$\begin{cases} \psi^I = 0 \text{ at } x = L_1, \text{ and } y = 0, L_2, \\ v^I = 0 \text{ and } u_n = \psi_n/\lambda_n \bar{U}_0 \text{ at } x = 0, n = 1, \dots, n_c. \end{cases} \tag{4.12}$$

For U^{II} the boundary conditions are inferred from (3.3) and read

$$\begin{cases} u^{II} = v^{II} = \psi^{II} = 0 \text{ at } x = 0, \\ \text{and } \psi^{II} = 0 \text{ at } y = 0 \text{ and } L_2. \end{cases} \tag{4.13}$$

All these boundary conditions are taken into account in the domain $D(A)$ of A. Finally if we add the initial conditions

$$U(0) = (u(0), v(0), \psi(0)) = U_0 = (u_0, v_0, \psi_0), \tag{4.14}$$

then the initial and boundary value problem consisting of equations (1.5), and (4.11) - (4.14) is equivalent to the abstract initial value problem

$$\frac{dU}{dt} + AU = F, \tag{4.15}$$

$$U(0) = U_0. \tag{4.16}$$

Note that $F = (F_u, F_v, F_w)$ which does not appear in (1.5) is added here for mathematical generality and to study below the case of nonhomogeneous boundary conditions. By Theorem 4.1 this problem is now solved by the Hille-Yoshida theorem and we have

Theorem 4.2. *Let H, A and $D(A)$ be defined as in (4.1) - (4.7). Then the initial value problem (4.15) - (4.16) is well-posed. That is, for every $U_0 \in D(A)$, and $F \in L^1(0,T,H)$, with $F' = dF/dt$ in $L^1(0,T;H)$, (4.15) - (4.16) has a unique solution U such that*

$$U \in \mathcal{C}([0,T];H) \cap L^\infty(0,T;D(A)), \quad \frac{dU}{dt} \in L^\infty(0,T;H). \tag{4.17}$$

4.2. **The nonhomogeneous boundary conditions.** We now turn to the case of nonhomogeneous boundary conditions for (4.11) - (4.13), that is we want to solve (1.5) with (4.11) - (4.13) in which the boundary conditions are now nonhomogeneous, and with initial condition (4.14). We assume that all boundary data are inferred from a function $U^g = (\mathbf{u}^{g0}, U^{gI}, U^{gII})$ which is defined in $\mathcal{M} \times [0,T]$. We also assume that U^g is given by its normal modes expansions:

(4.18)
$$U^g(x,y,z,t) = \left(\sum_{n\geq 0} u_n^g(x,y,t)\mathcal{U}_n(z), \sum_{n\geq 0} v_n^g(x,y,t)\mathcal{U}_n(z), \sum_{n\geq 1} \psi_n^g(x,y,t)\mathcal{W}_n(z) \right).$$

We now set
$$U = U^\# + U^g,$$
and observe that $U^\# \in D(A)$ if $U^\#$ is smooth enough (homogeneous boundary conditions). Then $U^\#$ will be sought as the solution of the linear evolution equation

$$\begin{aligned} &\frac{dU^\#}{dt} + AU^\# = F^\#, \\ &U^\#(0) = U_0^\#, \end{aligned} \tag{4.19}$$

where

$$U_0^\# = U_0 - U^g|_{t=0} \tag{4.20}$$

and

$$F^\# = F - \frac{\partial U^g}{\partial t} - \mathcal{A}U^g. \tag{4.21}$$

Here $\mathcal{A}U^g$ is defined by its normal mode expansion, where each $(\mathcal{A}U^g)_n$ is equal to $\mathcal{A}_n U_n^g, \mathcal{A}_n$ as in (1.27).

Theorem 4.2 will be applicable to (4.19) and we will obtain the desired existence and uniqueness result for U, provided we assume that $U_0^\#$ and $F^\#$ satisfy the hypotheses of Theorem 4.2. It is very easy to give sufficient (non necessarily optimal) conditions on U^g which guarantee that $U_0^\# \in D(A)$ and $F^\#$ and $dF^\#/dt$ are in $L^1(0,T;H)$. We assume e.g. the following

$$\begin{aligned} &U_0, \frac{\partial U_0}{\partial x}, \frac{\partial U_0}{\partial y} \in L^2(\mathcal{M})^3, \text{ and } \operatorname{div} \mathbf{u}_0^0 = 0, \\ &F, \frac{\partial F}{\partial t} \in L^1(0,T;L^2(\mathcal{M})^3), \\ &U^g, \frac{\partial U^g}{\partial t}, \frac{\partial U^g}{\partial x}, \frac{\partial U^g}{\partial y}, \frac{\partial^2 U^g}{\partial t^2}, \frac{\partial^2 U^g}{\partial x \partial t}, \frac{\partial^2 U^g}{\partial y \partial t} \in \mathcal{C}([0,T];L^2(\mathcal{M})^3). \end{aligned} \tag{4.22}$$

In addition we require that U_0 and U^g satisfy certain compatibility conditions, for $t=0$, and $(x,y)\in\partial\mathcal{M}'$, conditions which guarantee that $U_0^{\#}\in D(A)$. Setting $U_0=(\tilde{u}_0,\tilde{v}_0,\tilde{\psi}_0)=(\tilde{U}_0^0,\tilde{U}_0^I,\tilde{U}_0^{II})$,[9] we require

$$\begin{aligned}
&\tilde{\mathbf{u}}^0\cdot\mathbf{n}=\mathbf{u}^{0g}\cdot\mathbf{n},\quad \text{on } \partial\mathcal{M}',\ \text{at } t=0\\
&\tilde{\psi}_0^I=\tilde{\psi}^{gI} \text{ at } t=0 \text{ and } x=L_1,\ \text{or } y=0 \text{ or } L_2,\\
&\tilde{v}_0^I=\tilde{v}^{gI} \text{ and } \tilde{u}_{on}=\tilde{\psi}_{on}/\lambda_n\bar{U}_0=\tilde{u}_n^g-\tilde{\psi}_n^g/\lambda_n\bar{U}_0\\
&\qquad\qquad \text{at } x=0 \text{ and } t=0, n=1,\dots,n_c,\\
&\tilde{u}_0^{II}-\tilde{u}^{gII}=\tilde{v}_0^{II}-\tilde{v}^{gII}=\tilde{\psi}_0^{II}-\tilde{\psi}^{gII},\ \text{at } x=0 \text{ and } t=0,\\
&\tilde{\psi}_0^{II}-\tilde{\psi}^{gII}=0,\ \text{at } t=0 \text{ and } y=0 \text{ or } L_2.
\end{aligned}\tag{4.23}$$

With the regularity hypotheses (4.22) and the compatibility hypotheses (4.23), we obtain U satisfying

$$\begin{aligned}
&U\in\mathcal{C}([0,T];L^2(\mathcal{M})^3),\\
&\mathcal{A}U\in L^\infty(0,T;L^2(\mathcal{M})^3),\\
&\frac{\partial U}{\partial t}\in L^\infty(0,T;L^2(\mathcal{M})^3),
\end{aligned}\tag{4.24}$$

and the boundary conditions for $0<t<T$:

$$\begin{aligned}
&\mathbf{u}^0\cdot\mathbf{n}=\mathbf{u}^g\cdot\mathbf{n} \text{ on } \partial\mathcal{M}', -L_3<z<0,\\
&\psi^I=\psi^{gI} \text{ at } x=L_1 \text{ and } y=0,L_2,\\
&v^I=v^{gI} \text{ at } x=0,\\
&u_n^I-\psi_n^I/\lambda_n\bar{U}_0=u_n^{gI}-\psi_n^{gI}/\lambda_n\bar{U}_0,\ \text{at } x=0,\ \ n=1,\dots,n_c,\\
&u^{II}=u^{gII}, v^{II}=v^{gII}, \psi^{II}=\psi^{gII} \text{ at } x=0,\\
&\psi^{II}=\psi^{gII} \text{ at } y=0 \text{ and } L_2.
\end{aligned}\tag{4.25}$$

In summary, we have proven the following theorem:

Theorem 4.3. *We assume that U_0, F and U^g are given satisfying the hypotheses (4.22) and (4.23). Then there exists a unique U solution of the Primitive Equations (1.5), satisfying the regularity properties (4.24), the boundary condition (4.25) and the initial condition (4.16).*

Acknowledgments. This work was partially supported by the National Science Foundation under the grant NSF-DMS-0604235, and by the Research Fund of Indiana University.

References

[1] Qingshan Cheng, Jacques Laminie, Antoine Rousseau, Roger Temam, and Joe Tribbia. Some 2.5d models for the equations of the ocean and the atmosphere. *Analysis and Applications*, 5(3):199–229, 2007.

[9] The tildes here on u_0, v_0, ψ_0, etc. are intended to distinguish these initial data from the zero modes of $U(t)$.

[2] Pierre Grisvard. *Elliptic problems in nonsmooth domains*. Monographs and Studies in Mathematics. Pitman, Boston, 1985.

[3] Einar Hille and Ralph S. Phillips. *Functional analysis and semi-groups*. American Mathematical Society, Providence, R. I., 1974. Third printing of the revised edition of 1957, American Mathematical Society Colloquium Publications, Vol. XXXI.

[4] Fritz John. *Partial Differential Equations*. Springer-Verlag, New-York, 1982. Fourth Edition.

[5] Jacques-Louis Lions and Enrico Magenes. *Non-homogeneous boundary value problems and applications*. Springer-Verlag, New-York-Heidelberg, 1972. Vols. I & II, Translated from the French by P. Kenneth.

[6] Jacques-Louis Lions, Roger Temam, and Shouhong Wang. New formulations of the primitive equations of atmosphere and applications. *Nonlinearity*, 5(2):237–288, 1992.

[7] Jacques-Louis Lions, Roger Temam, and Shouhong Wang. On the equations of the large-scale ocean. *Nonlinearity*, 5(5):1007–1053, 1992.

[8] Joseph Oliger and Arne Sundström. Theoretical and practical aspects of some initial boundary value problems in fluid dynamics. *SIAM J. Appl. Math.*, 35(3):419–446, 1978.

[9] Madalina Petcu, Roger Temam, and Mohammed Ziane. Mathematical problems for the primitive equations with viscosity. In R. Temam P.G. Ciarlet EDs and J. Tribbia Guest Eds, editors, *Handbook of Numerical Analysis. Special Issue on Some Mathematical Problems in Geophysical Fluid Dynamics*, Handb. Numer. Anal. Elsevier, New York, 2008.

[10] Antoine Rousseau, Roger Temam, and Joe Tribbia. Boundary conditions for the 2D linearized PEs of the ocean in the absence of viscosity. *Discrete Contin. Dyn. Syst.*, 13(5):1257–1276, 2005.

[11] Antoine Rousseau, Roger Temam, and Joe Tribbia. Numerical simulations of the inviscid primitive equations in a limited domain. In Caterina Calgaro, Jean-François Coulombel, and Thierry Goudon, editors, *Analysis and Simulation of Fluid Dynamics*, Advances in Mathematical Fluid Mechanics, pages 163–181. Birkhaäuser, 2006.

[12] Walter Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill Inc., New York, second edition, 1991.

[13] Roger Temam. *Navier-Stokes equations*. AMS Chelsea Publishing, Providence, RI, 2001. Theory and numerical analysis, Reprint of the 1984 edition.

[14] Roger Temam and Joe Tribbia. Open boundary conditions for the primitive and Boussinesq equations. *J. Atmospheric Sci.*, 60(21):2647–2660, 2003.

[15] Roger Temam and Mohammed Ziane. Some mathematical problems in geophysical fluid dynamics. In S. Friedlander and D. Serre, editors, *Handbook of mathematical fluid dynamics*. North-Holland, 2004.

[16] Thomas T. Warner, Ralph A. Peterson, and Russel E. Treadon. A tutorial on lateral boundary conditions as a basic and potentially serious limitation to regional numerical weather prediction. *Bull. Amer. Meteor. Soc.*, 78(11):2599–2617, 1997.

[17] Kōsaku Yosida. *Functional analysis*. Springer-Verlag, Berlin, 6th edition, 1980.