FINITENESS PROPERTIES FOR PISOT S-ADIC TILINGS

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In this paper, we will first formulate and prove some equivalent sufficient conditions to obtain the tiling property for a Pisot unimodular substitution. We will then apply these condition to the more general framework of adic systems, to extend to this more general (and non algebraic) case results already known for the substitutive case.

1. SETTING

Let $\mathcal{A}$ be a finite alphabet and $\mathcal{A}^*$ the set of finite words in $\mathcal{A}$. A *substitution* σ is an endomorphism of the free monoïd $\mathcal{A}^*$ such that the image of each letter of $\mathcal{A}$ is nonempty. A substitution can be extended to the set of bi-infinite words $\mathcal{A}^{\mathbb{Z}}$. Substitutions occur in many fields of mathematics or computer science such as illustrated in [12].

A geometric generalization of the notion of substitution has been introduced in [5]. These so-called generalized substitutions have been applied in word combinatorics [4], in dynamical systems [6], or in discrete geometry [3]. The *incidence matrix* $\mathbf{M}_\sigma$ of the substitution σ counts the number of occurrence of letters in the images of letters of $\mathcal{A}$ by σ. A substitution σ is *unimodular* if det $\mathbf{M}_\sigma = \pm 1$. Given a usual unimodular substitution, one can define a generalized substitution $E_1(\sigma)^*$ which acts on translates of faces of the unit cube by vectors with integer coordinates, and maps such a translate to a union of translates of faces (see Fig. 1.1).

We recall that an algebraic integer is a *Pisot number* if all its algebraic conjugates have modulus less than one. A substitution is said *Pisot unimodular* if its incidence matrix admits as characteristic polynomial the minimal polynomial of a Pisot unit, or equivalently if all its eigenvalues are simple, and all except one of modulus strictly smaller than 1. Let σ be a Pisot unimodular substitution over the n-letter alphabet $\mathcal{A}$. Since σ is assumed to be Pisot, then $\mathbf{M}_\sigma$ admits an expanding eigendirection and a contracting eigenspace of codimension 1.

The image by the iterates of $E_1(\sigma)^*$ of the upper half of the unit cube (that we denote by $\mathcal{U}$) are proved to belong to a discrete approximation of the contracting plane of $\mathcal{M}_\sigma$. Such an approximation is called a *stepped surface* and corresponds to the notion of standard arithmetic discrete plane, such as defined in [16]. We denote it by $\mathfrak{P}_\sigma$.

Let π denote the projection onto the contracting plane of the incidence matrix $\mathbf{M}_\sigma$ along its expanding direction. Then the sequence of renormalization by $\mathbf{M}_\sigma^n$ of the projection by π of the image by the n-th iterate of $E_1(\sigma)^*$ of the upper half of the unit cube $\mathcal{U}$, that is, $(\mathbf{M}_\sigma^n(\pi \circ (E_1(\sigma)^*)^n(\mathcal{U}))_{n\in\mathbb{N}}$, is proved in [5] to converge. Its image is called the *Rauzy fractal* $\mathcal{R}_\sigma$ of the substitution σ, according to the seminal paper [15]. For more details, see e.g., the surveys [12, 7]. It is widely conjectured that the Rauzy fractal of a Pisot substitution generates a lattice tiling of the contracting plane of $\mathbf{M}_\sigma$. This conjecture is known as the *Pisot conjecture.* It is equivalent to the fact that the associated substitutive dynamical system is measure-theoretically isomorphic to a toral translation. This tiling property is

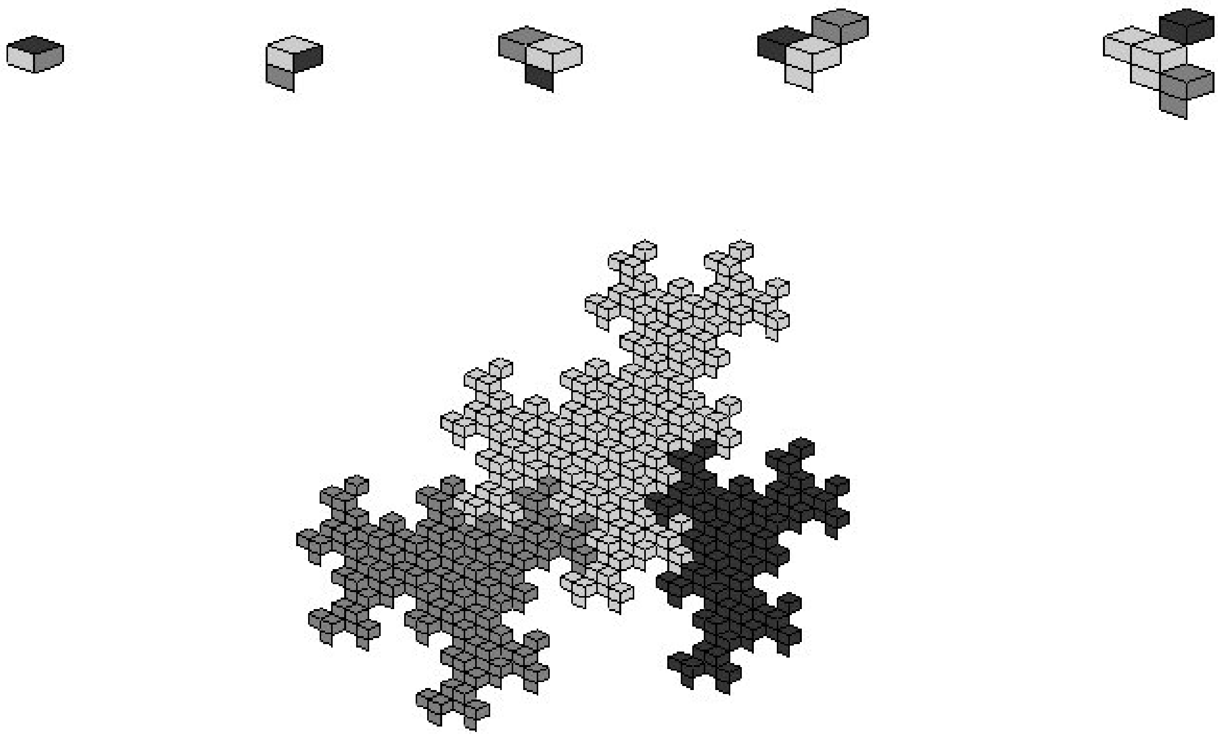

FIGURE 1.1. First iterates of a generalized substitution.

equivalent to the fact that $\pi \circ (E_1(\sigma)^*)^n(\mathcal{U})$ contains arbitrarily large balls, according e.g., to [14]. Furthermore, if the iterates $(E_1(\sigma)^*)^n(\mathcal{U})$ cover the whole stepped surface $\mathfrak{P}_\sigma$, then 0 is proved to be an inner point of $\mathcal{R}_\sigma$.

Rauzy fractals $\mathcal{R}_\beta$ can also be associated with β-numerations with β Pisot unit, according [1, 2, 17]. A classical sufficient condition for tiling is the the (F) condition, the so-called *finiteness condition* introduced in [13]: all nonnegative elements of $\mathbb{Z}[1/\beta]$ are assumed to have a finite β-expansion. The (F) property is a useful sufficient tiling condition that implies that 0 is an inner point of the Rauzy fractal $\mathcal{R}_\beta$ [1].

A substitutive counter-part to the (F) property has been introduced in [7], based on Dumont-Thomas numeration [9, 8]. It is equivalent to the fact that the iterates $(E_1(\sigma)^*)^n(\mathcal{U})$ cover the whole stepped surface $\mathfrak{P}_\sigma$ [7].

2. Results

The aim of this lecture is first to formulate several equivalent statements as well as sufficient conditions for the tiling property. In particular, we will introduce and focus on the following sufficient condition, the so-called *ring condition*: if the faces of the unit cube are located sufficiently "inside" their image under the action of $E_1(\sigma)^*$, then the finiteness property is proved to hold.

In a second part, we discuss the S-adic case, which is one of our main motivations for the present study. A sequence is said S*-adic* (according to a variation of the Vershik terminology), if there exists a finite set of substitutions $\mathcal{S}$ over an alphabet $\mathcal{D} = \{0, ..., d-1\}$, a morphism φ from $\mathcal{D}^\star$ to $\mathcal{A}^\star$, and an infinite sequence of substitutions $(\sigma_n)_{n\geqslant 1}$ with values in $\mathcal{S}$ such that $|\sigma_1\sigma_2...\sigma_n(r)| \to +\infty$ when $n \to +\infty$, for any letter $r \in \mathcal{D}$, and any word of the language of the system is a factor of $\varphi(\sigma_1\sigma_2...\sigma_n)(0)$ for some n. Let us recall that uniformy recurrent sequences with at most linear complexity are proved to be S-adic [10].

One can associate substitutions with multidimensional continued fraction algorithms such as Jacobi-Perron's algorithm or Brun's algorithm, as described in Chap. 8 of [12] (see also [11]). Let us note that one of the aims of the extension of the construction of Rauzy fractals to the $S-$adic case is to be able to handle toral translations with nonalgebraic parameters with substitutive methods.

We will thus discuss the extension of the substitutive (F) property and of the ring condition to the S-adic case.

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