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Abstract: We study the well-posedness of a coupled system of PDEs and ODEs arising in the numerical simulation of electrocardiograms. It consists of a system of degenerate reaction-diffusion equations, the so-called bidomain equations, governing the electrical activity of the heart, and a diffusion equation governing the potential in the surrounding tissues. Global existence of weak solutions is proved for an abstract class of ionic models including Mitchell-Schaeffer, FitzHugh-Nagumo, Aliev-Panfilov and MacCulloch. Uniqueness is proved in the case of the FitzHugh-Nagumo ionic model. The proof is based on a regularisation argument with a Faedo-Galerkin/compactness procedure.

Key-words: Bidomain model, reaction-diffusion system, electrocardiograms modelling

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Etude d'un système couplé d'EDO et EDP intervenant dans la modélisation d'éléctrocardiogrammes

Résumé : Nous étudions le caractère bien posé d'un système d'équations intervenant dans la simulation numérique d'électrocardiogrammes. Ce système, qui couple des équations aux dérivées ordinaires et des équations aux dérivées partielles, est constitué d'équations de réaction-diffusion dégénérées, les équations bidomaine qui modélisent l'activité électrique du coeur, et d'une équation de diffusion qui modélise l'activité électrique du tissu environnant (le thorax). L'existence globale de solutions faibles est obtenue pour un ensemble de modèles ioniques, dont les modèles de Mitchell-Schaeffer, FitzHugh-Nagumo, Aliev-Panfilov et MacCulloch. L'unicité est prouvée pour le modèle de FitzHugh-Nagumo. La preuve s'appuie sur un argument de régularisation et la méthode d'approximation de Faedo-Galerkin, combinés avec des résultats de compacité.

Mots-clés : Modèle bidomaine, système de reaction-diffusion, modélisation de l'électrocardiogramme

1 Introduction

We analyze the well-posedness of a coupled system arising in the numerical simulation of electrocardiograms (ECG). It consists of two partial differential equations (PDEs) and a system of ordinary differential equations (ODEs), describing the electrical activity of the heart, coupled to a third PDE which describes the electrical potential of the surrounding tissue within the torso.

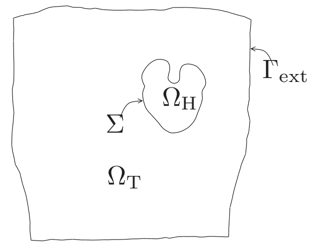

Figure 1: The heart and torso domains: $\Omega_{\rm H}$ and $\Omega_{\rm T}$

We assume the cardiac tissue to be located in a domain (an open bounded subset with locally Lipschitz continuous boundary) $\Omega_{\rm H}$ of $\mathbb{R}^3$. The surrounding tissue within the torso occupies a domain $\Omega_{\rm T}$. We denote by $\Sigma \stackrel{\rm def}{=} \overline{\Omega_{\rm H}} \cap \overline{\Omega_{\rm T}} = \partial\Omega_{\rm H}$ the interface between both domains, and by $\Gamma_{\rm ext}$ the external boundary of $\Omega_{\rm T}$, *i.e.* $\Gamma_{\rm ext} \stackrel{\rm def}{=} \partial\Omega_{\rm T} \backslash \Sigma$, see figure 1. At last, we define Ω the global domain $\overline{\Omega_{\rm H} \cup \Omega_{\rm T}}$.

A widely accepted model of the macroscopic electrical activity of the heart is the so-called *bidomain model* (see *e.g.* the monographs [19, 17, 20]). It consists of two degenerate parabolic reaction-diffusion PDEs coupled to a system of ODEs:

$$\begin{cases} C_{\rm m}\partial_t v_{\rm m} + I_{\rm ion}(v_{\rm m}, w) - \operatorname{div}(\boldsymbol{\sigma}_{\rm i} \boldsymbol{\nabla} u_{\rm i}) = I_{\rm app}, & \text{in} \quad \Omega_{\rm H} \times (0,T), \\ C_{\rm m}\partial_t v_{\rm m} + I_{\rm ion}(v_{\rm m}, w) + \operatorname{div}(\boldsymbol{\sigma}_{\rm e} \boldsymbol{\nabla} u_{\rm e}) = I_{\rm app}, & \text{in} \quad \Omega_{\rm H} \times (0,T), \\ \partial_t w + g(v_{\rm m}, w) = 0, & \text{in} \quad \Omega_{\rm H} \times (0,T). \end{cases} \tag{1.1}$$

The two PDEs describe the dynamics of the averaged intra- and extracellular potentials $u_{\rm i}$ and $u_{\rm e}$, whereas the ODE, also known as *ionic model*, is related to the electrical behavior of the myocardium cells membrane, in terms of the (vector) variable w representing the averaged ion concentrations and gating states. In (1.1), the quantity $v_{\rm m} \stackrel{\rm def}{=} u_{\rm i} - u_{\rm e}$ stands for the transmembrane potential, $C_{\rm m}$ is the membrane capacitance, $\boldsymbol{\sigma}_{\rm i}, \boldsymbol{\sigma}_{\rm e}$ are the intra- and extracellular conductivity tensors and $I_{\rm app}$ is an external applied volume current. The nonlinear reaction term $I_{\rm ion}(v_{\rm m}, w)$ and the vector-valued function $g(v_{\rm m}, w)$ depend on the ionic model under consideration (*e.g.* Mitchell-Schaeffer [14], FitzHugh-Nagumo [15] or Luo-Rudy [12, 13]).

The PDE part of (1.1) has to be completed with boundary conditions for $u_{\rm i}$ and $u_{\rm e}$. The intracellular domain is assumed to be electrically isolated, so we prescribe

$$\boldsymbol{\sigma}_{\rm i} \boldsymbol{\nabla} u_{\rm i} \cdot \boldsymbol{n} = 0, \quad \text{on} \quad \Sigma.$$

Conversely, the boundary conditions for $u_{\rm e}$ will depend on the interaction with the surrounding tissue.

The numerical simulation of the ECG signals requires a description of how the surface potential is perturbed by the electrical activity of the heart. In general, such a description is based on the coupling of (1.1) with a diffusion equation in Ω_{T}:

$$\operatorname{div}(\boldsymbol{\sigma}_{\mathrm{T}}\boldsymbol{\nabla}u_{\mathrm{T}}) = 0, \quad \text{in} \quad \Omega_{\mathrm{T}}, \tag{1.2}$$

where u_{T} stands for the torso potential and $\boldsymbol{\sigma}_{\mathrm{T}}$ for the conductivity tensor of the torso tissue. The boundary Γ_{ext} can be supposed to be insulated, which corresponds to the condition

$$\boldsymbol{\sigma}_{\mathrm{T}}\boldsymbol{\nabla}u_{\mathrm{T}} \cdot \boldsymbol{n}_{\mathrm{T}} = 0 \text{ on } \Gamma_{\mathrm{ext}},$$

where $\boldsymbol{n}_{\mathrm{T}}$ stands for the outward unit normal on Γ_{ext}.

The coupling between (1.1) and (1.2) is operated at the heart-torso interface Σ. Generally, by enforcing the continuity of potentials and currents (see *e.g* [11, 9, 16, 17, 20]):

$$\begin{cases} u_{\mathrm{e}} = u_{\mathrm{T}}, & \text{on} \quad \Sigma, \\ \boldsymbol{\sigma}_{\mathrm{e}}\boldsymbol{\nabla}u_{\mathrm{e}} \cdot \boldsymbol{n} = \boldsymbol{\sigma}_{\mathrm{T}}\boldsymbol{\nabla}u_{T} \cdot \boldsymbol{n}, & \text{on} \quad \Sigma. \end{cases} \tag{1.3}$$

These conditions represent a perfect electrical coupling between the heart and the surrounding tissue. More general coupling conditions, which take into account the impact of the pericardium (a double-walled sac which separates the heart and the surrounding tissue), have been reported by the authors in a recent work [4].

In summary, from (1.1), (1.2) and (1.3) we obtain the following coupled heart-torso model (see *e.g* [9, 16, 17, 20]):

$$\begin{cases} C_{\mathrm{m}}\partial_t v_{\mathrm{m}} + I_{\mathrm{ion}}(v_{\mathrm{m}}, w) - \operatorname{div}(\boldsymbol{\sigma}_{\mathrm{i}}\boldsymbol{\nabla}u_{\mathrm{i}}) = I_{\mathrm{app}}, & \text{in} \quad \Omega_{\mathrm{H}}, \\ C_{\mathrm{m}}\partial_t v_{\mathrm{m}} + I_{\mathrm{ion}}(v_{\mathrm{m}}, w) + \operatorname{div}(\boldsymbol{\sigma}_{\mathrm{e}}\boldsymbol{\nabla}u_{\mathrm{e}}) = I_{\mathrm{app}}, & \text{in} \quad \Omega_{\mathrm{H}}, \\ \partial_t w + g(v_{\mathrm{m}}, w) = 0, & \text{in} \quad \Omega_{\mathrm{H}}, \\ \operatorname{div}(\boldsymbol{\sigma}_{\mathrm{T}}\boldsymbol{\nabla}u_{\mathrm{T}}) = 0, & \text{in} \quad \Omega_{\mathrm{T}}, \\ \boldsymbol{\sigma}_{\mathrm{i}}\boldsymbol{\nabla}u_{\mathrm{i}} \cdot \boldsymbol{n} = 0, & \text{on} \quad \Sigma \\ \boldsymbol{\sigma}_{\mathrm{e}}\boldsymbol{\nabla}u_{\mathrm{e}} \cdot \boldsymbol{n} = \boldsymbol{\sigma}_{\mathrm{T}}\boldsymbol{\nabla}u_{\mathrm{T}} \cdot \boldsymbol{n}, & \text{on} \quad \Sigma, \\ u_{\mathrm{e}} = u_{\mathrm{T}}, & \text{on} \quad \Sigma, \\ \boldsymbol{\sigma}_{\mathrm{T}}\boldsymbol{\nabla}u_{\mathrm{T}} \cdot \boldsymbol{n}_{\mathrm{T}} = 0, & \text{on} \quad \Gamma_{\mathrm{ext}}. \end{cases} \tag{1.4}$$

Problem (1.4) is completed with initial conditions:

$$v_{\mathrm{m}}(0, x) = v_0(x) \text{ and } w(0, x) = w_0(x) \quad \forall x \in \Omega_{\mathrm{H}}, \tag{1.5}$$

and the identity

$$v_{\mathrm{m}} \stackrel{\text{def}}{=} u_{\mathrm{i}} - u_{\mathrm{e}}, \quad \text{in} \quad \Omega_{\mathrm{H}}. \tag{1.6}$$

Finally, let us notice that u_{e} and u_{T} are defined up to the same constant. This constant can be fixed, for instance, by enforcing the following condition

$$\int_{\Omega_{\mathrm{H}}} u_{\mathrm{e}} = 0,$$

on the extra-cellular potential.

Introduced in the late 70's [21], the system of equations (1.1) can be derived mathematically using homogenization techniques. Typically, by assuming that the myocardium has periodic structure at the cell scale [10] (see also [6]). A first well-posedness analysis of (1.1), with $I_{\text{ion}}(v_{\text{m}}, w)$ and $g(v_{\text{m}}, w)$ given by the FitzHugh-Nagumo ionic model [15], has been reported in [6]. The proof is based on a reformulation of (1.1) in terms of an abstract evolutionary variational inequality. The analysis for a simplified ionic model, namely $I_{\text{ion}}(v_{\text{m}}, w) \stackrel{\text{def}}{=} I_{\text{ion}}(v_{\text{m}})$, has been addressed in [2]. In the recent work [5], existence of solution is proved for a wide class of ionic models (including Aliev-Panfilov [1] and MacCulloch [18]). Uniqueness, however, is achieved only for the FitzHugh-Nagumo ionic model. Finally, in [22], existence, uniqueness and some regularity results are proved with a generalized phase-I Luo-Rudy ionic model [12].

None of the above mentioned works consider the coupled bidomain-torso problem (1.4). The aim of this paper is to provide a well-posedness analysis of this coupled problem. Our main result states the existence of global weak solutions for (1.4) with an abstract class of ionic models, including: FitzHugh-Nagumo [8, 15], Aliev-Panfilov [1], Roger-McCulloch [18] and Mitchell-Schaeffer [14]. For the sake of completeness, we give here the expressions of I_{ion} and g for these models.

- FitzHugh-Nagumo model:

$$I_{\text{ion}}(v, w) = kv(v-a)(v-1) + w, \quad g(v, w) = -\epsilon(\gamma v - w). \tag{1.7}$$

- Aliev-Panfilov model:

$$I_{\text{ion}}(v, w) = kv(v-a)(v-1) + vw, \quad g(v, w) = \epsilon(\gamma v(v-1-a) + w). \tag{1.8}$$

- Roger-McCulloch model:

$$I_{\text{ion}}(v, w) = kv(v-a)(v-1) + vw, \quad g(v, w) = -\epsilon(\gamma v - w). \tag{1.9}$$

- Mitchell-Schaeffer model:

$$\begin{aligned} I_{\text{ion}}(v, w) &= \frac{w}{\tau_{\text{in}}} v^2 (v-1) - \frac{v}{\tau_{\text{out}}}, \\ g(v, w) &= \begin{cases} \dfrac{1-w}{\tau_{\text{open}}} & \text{if } v \le v_{\text{gate}}, \\ \dfrac{-w}{\tau_{\text{close}}} & \text{if } v > v_{\text{gate}}. \end{cases} \end{aligned} \tag{1.10}$$

Here $0 < a < 1$, k, ϵ, γ, τ_{in}, τ_{out}, τ_{open}, τ_{close} and $0 < v_{\text{gate}} < 1$ are given positive constants.

To the best of our knowledge, the ionic model (1.10) has not yet been considered within a well-posedness study of the bidomain equations (1.1). Compared to models (1.7)-(1.9), the Mitchell-Schaeffer model has different structure that makes the proof of our results slightly more involved. As far as the ECG modeling is concerned, in [4, 3], the authors point out that realistic ECG signals can be obtained with this model, whereas it seems to be not the case for standard FitzHugh-Nagumo type models (1.7).

The remainder of the paper is organized as follows. In the next section we state our main existence result for problem (1.4), under general assumptions on the ionic model. In Section 3 we provide the proof of this result. We use a regularization argument and a standard Faedo-Galerkin/compactness procedure based on a specific spectral basis in Ω. Uniqueness is proved for the FitzHugh-Nagumo ionic model.

2 Main result

We assume that the conductivities of the intracellular, extracellular and thoracic media $\boldsymbol{\sigma}_{\mathrm{i}}, \boldsymbol{\sigma}_{\mathrm{e}}, \boldsymbol{\sigma}_{\mathrm{T}} \in [L^\infty(\Omega_{\mathrm{H}})]^{3\times 3}$ are symmetric and uniformly positive definite, *i.e.* there exist $\alpha_{\mathrm{i}} > 0$, $\alpha_{\mathrm{e}} > 0$ and $\alpha_{\mathrm{T}} > 0$ such that, $\forall \boldsymbol{x} \in \mathbb{R}^3$, $\forall \boldsymbol{\xi} \in \mathbb{R}^3$,

$$\boldsymbol{\xi}^T \boldsymbol{\sigma}_{\mathrm{i}}(\boldsymbol{x})\boldsymbol{\xi} \geq \alpha_{\mathrm{i}}|\boldsymbol{\xi}|^2, \quad \boldsymbol{\xi}^T \boldsymbol{\sigma}_{\mathrm{e}}(\boldsymbol{x})\boldsymbol{\xi} \geq \alpha_{\mathrm{e}}|\boldsymbol{\xi}|^2, \quad \boldsymbol{\xi}^T \boldsymbol{\sigma}_{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\xi} \geq \alpha_{\mathrm{T}}|\boldsymbol{\xi}|^2. \tag{2.11}$$

Moreover, we shall use the notation $\alpha \stackrel{\text{def}}{=} \min\{\alpha_{\mathrm{e}}, \alpha_{\mathrm{T}}\}$.

For the reaction terms we consider two kinds of (two-variable) ionic models:

- **I1**: Generalized FitzHugh-Nagumo models, where functions I_{ion} and g are given by
$$\begin{aligned} I_{\mathrm{ion}}(v,w) &= f_1(v) + f_2(v)w, \\ g(v,w) &= g_1(v) + c_1 w. \end{aligned} \tag{2.12}$$
Here, f_1, f_2 and g_1 are given real functions and c_1 is a real constant.

- **I2**: A regularized version of the Mitchell-Schaeffer model (see *e.g.* [7]), for which the functions I_{ion} and g are given by:
$$\begin{aligned} I_{\mathrm{ion}}(v,w) &= \frac{w}{\tau_{\mathrm{in}}} f_1(v) - \frac{v}{\tau_{\mathrm{out}}}, \\ g(v,w) &= \Big(\frac{1}{\tau_{\mathrm{close}}} + \frac{\tau_{\mathrm{close}} - \tau_{\mathrm{open}}}{\tau_{\mathrm{close}}\tau_{\mathrm{open}}} h_\infty(v)\Big)(w - h_\infty(v)), \end{aligned} \tag{2.13}$$
where f_1 is a real function and h_∞ is given by
$$h_\infty(v) = \frac{1}{2}\left[1 - \tanh\left(\frac{v - v_{\mathrm{gate}}}{\eta_{\mathrm{gate}}}\right)\right], \tag{2.14}$$
and τ_{in}, τ_{out}, $\tau_{\mathrm{open}} < \tau_{\mathrm{close}}$, v_{gate}, η_{gate} are positive constants.

In what follows we will consider the following two problems:

- **P1**: System (1.4) with the ionic model (**I1**) given by (2.12).
- **P2**: System (1.4) with the ionic model (**I2)** given by (2.13)-(2.14).

In order to analyze the well-posedness of these problems, we shall make use of the following assumptions on the behavior of the reaction terms:

- **A1**: We assume that f_1, f_2 and g_1 belong to $C^1(\mathbb{R})$ and that, $\forall v \in \mathbb{R}$,
$$\begin{aligned} |f_1(v)| &\leq c_2 + c_3|v|^3, \\ f_2(v) &= c_4 + c_5 v, \\ |g_1(v)| &\leq c_6 + c_7|v|^2, \end{aligned} \tag{2.15}$$
with $\{c_i\}_{i=2}^7$ given real constants and c_2, c_3, c_6, c_7 are positives.

- **A2**: For any $v \in \mathbb{R}$,
$$f_1(v)v \geq a|v|^4 - b|v|^2, \tag{2.16}$$
with $a > 0$ and $b \geq 0$ given constants.

The next assumption will be also used in order to prove uniqueness of the solution of problem (**P1**).

- **A3**: For all $\mu > 0$, we introduce F_μ as
$$\begin{aligned} F_\mu \quad : \quad & \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ & (v, w) \mapsto (\mu I_{\mathrm{ion}}(v, w), g(v, w)), \end{aligned}$$
and Q_μ as:
$$Q_\mu(z) \stackrel{\mathrm{def}}{=} \frac{1}{2}\left(\boldsymbol{\nabla} F_\mu(z) + \boldsymbol{\nabla} F_\mu(z)^{\mathrm{T}}\right), \forall\, z \in \mathbb{R}^2.$$
In addition, we assume that there exist $\mu_0 > 0$ and a constant $C_{\mathrm{ion}} \leq 0$ such that the eigenvalues $\lambda_{1,\mu_0}(z) \leq \lambda_{2,\mu_0}(z)$ of $Q_{\mu_0}(z)$, satisfy
$$C_{\mathrm{ion}} \leq \lambda_{1,\mu_0}(z) \leq \lambda_{2,\mu_0}(z), \quad \forall z \in \mathbb{R}^2. \tag{2.17}$$

Remark 3 *One can check that models* (1.7)-(1.9) *enter the general framework* (2.12) *and satisfy the assumptions* **A1** *and* **A2**. *In addition,* **A3** *holds true for the FitzHugh-Nagumo model. We refer to [5], for the details.*

In what follows, we shall make use of the following function spaces
$$\begin{aligned} V_{\mathrm{i}} &\stackrel{\mathrm{def}}{=} H^1(\Omega_{\mathrm{H}}), \\ V_{\mathrm{e}} &\stackrel{\mathrm{def}}{=} \left\{\phi \in H^1(\Omega_{\mathrm{H}}) \,:\, \int_{\Omega_{\mathrm{H}}} \phi = 0\right\}, \\ V_{\mathrm{HT}} &\stackrel{\mathrm{def}}{=} \left\{\phi \in H^1(\Omega_{\mathrm{T}}) \,:\, \phi_{|\Sigma} = 0\right\}, \\ V &\stackrel{\mathrm{def}}{=} \left\{\phi \in H^1(\Omega) \,:\, \int_{\Omega_{\mathrm{H}}} \phi = 0\right\}. \end{aligned}$$

We introduce the cylindrical time-space domain $Q_T \stackrel{\mathrm{def}}{=} (0, T) \times \Omega_{\mathrm{H}}$, and we define u as the extracellular cardiac potential in Ω_{H}, and the thoracic potential in Ω_{T}, *i.e.*:
$$u \stackrel{\mathrm{def}}{=} \begin{cases} u_{\mathrm{e}} \text{ in } \Omega_{\mathrm{H}}, \\ u_{\mathrm{T}} \text{ in } \Omega_{\mathrm{T}}. \end{cases}$$
From the first coupling condition in (1.3), it follows that $u \in H^1(\Omega)$ provided that $u_{\mathrm{e}} \in H^1(\Omega_{\mathrm{H}})$ and $u_{\mathrm{T}} \in H^1(\Omega_{\mathrm{T}})$. Similarly, we define the global conductivity tensor $\boldsymbol{\sigma} \in [L^\infty(\Omega)]^{3\times 3}$ as
$$\boldsymbol{\sigma} \stackrel{\mathrm{def}}{=} \begin{cases} \boldsymbol{\sigma}_{\mathrm{e}} \text{ in } \Omega_{\mathrm{H}}, \\ \boldsymbol{\sigma}_{\mathrm{T}} \text{ in } \Omega_{\mathrm{T}}. \end{cases}$$

Definition 3.1 *A weak solution of problem (***P1***) is a quadruplet of functions* $(v_{\mathrm{m}}, u_{\mathrm{i}}, u, w)$ *with the regularity*
$$\begin{aligned} & v_{\mathrm{m}} \in L^\infty(0, T; H^1(\Omega_H)) \cap H^1(0, T; L^2(\Omega_{\mathrm{H}})), \\ & u \in L^\infty(0, T; V), \quad w \in H^1(0, T; L^2(\Omega_{\mathrm{H}})), \end{aligned} \tag{3.18}$$

and satisfying (1.5), (1.6) *and*

$$C_{\mathrm{m}} \int_{\Omega_{\mathrm{H}}} \partial_t v_{\mathrm{m}} \phi_{\mathrm{i}} + \int_{\Omega_{\mathrm{H}}} \boldsymbol{\sigma}_{\mathrm{i}} \boldsymbol{\nabla} u_{\mathrm{i}} \cdot \boldsymbol{\nabla} \phi_{\mathrm{i}} + \int_{\Omega_{\mathrm{H}}} I_{\mathrm{ion}}(v_{\mathrm{m}}, w) \phi_{\mathrm{i}} = \int_{\Omega_{\mathrm{H}}} I_{\mathrm{app}} \phi_{\mathrm{i}}, \quad (3.19)$$

$$C_{\mathrm{m}} \int_{\Omega_{\mathrm{H}}} \partial_t v_{\mathrm{m}} \psi - \int_{\Omega} \boldsymbol{\sigma} \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} \psi + \int_{\Omega_{\mathrm{H}}} I_{\mathrm{ion}}(v_{\mathrm{m}}, w) \psi = \int_{\Omega_{\mathrm{H}}} I_{\mathrm{app}} \psi, \quad (3.20)$$

$$\partial_t w + g(v_{\mathrm{m}}, w) = 0. \quad (3.21)$$

for all $(\phi_{\mathrm{i}}, \psi, \theta) \in H^1(\Omega_{\mathrm{H}}) \times V \times L^2(\Omega_{\mathrm{H}})$. *Equations* (3.19) *and* (3.20) *holds in* $\mathcal{D}'(0,T)$ *and equation* (3.21) *holds almost everywhere. On the other hand, a weak solution of problem (***P2***) is a quadruplet* $(u_{\mathrm{i}}, u, v_{\mathrm{m}}, w)$ *satisfying* (3.18) (1.5), (1.6), (3.19)-(3.20) *and*

$$w \in W^{1,\infty}(0,T, L^\infty(\Omega_{\mathrm{H}})), \quad \partial_t w + g(v_{\mathrm{m}}, w) = 0, \, a.e. \text{ on } Q_T.$$

Remark 4 *Since* $w \in H^1(0,T; L^2(\Omega_{\mathrm{H}}))$ *it follows that* $w \in C^0(0,T; L^2(\Omega_{\mathrm{H}}))$, *which gives a sense to the initial data of* w. *In the same manner, the initial condition on* v_{m} *makes sense.*

The next theorem provides the main result of this paper, it states the existence of solution for problems (**P1**) and (**P2**).

Theorem 4.1 *Let* $T > 0$, $I_{\mathrm{app}} \in L^2(Q_T)$, $\boldsymbol{\sigma}_{\mathrm{i}}, \boldsymbol{\sigma}_{\mathrm{e}} \in [L^\infty(\Omega_{\mathrm{H}})]^{3\times 3}$ *symmetric and satisfying* (2.11), $w_0 \in L^2(\Omega_{\mathrm{H}})$ *and* $v_0 \in H^1(\Omega_{\mathrm{H}})$ *be given data. Assume that (***A1***) and (***A2***) hold. Then:*

- *Problem (***P1***) has a weak solution in the sense of Definition 3.1. Moreover, if assumption (***A3***) holds true, the solution is unique.*
- *If, in addition,* $w_0 \in L^\infty(\Omega_{\mathrm{H}})$ *with a positive lower bound* $r > 0$, *such that*

$$r < w_0 \leq 1 \quad \text{in} \quad \Omega_{\mathrm{H}}, \quad (4.22)$$

  *problem (***P2***) has a weak solution in the sense of Definition 3.1.*

The next section is fully devoted to the proof of this theorem.

5 Proof of the main result

Two main issues arise in the analysis of problem (1.4). Firstly, the non-linear reaction-diffusion equations $(1.4)_{1,2}$ are degenerated in time. And secondly, we have a coupling with a diffusion equation through the interface Σ. The first issue is overcome here by adding a couple of regularization terms, making bidomain equations parabolic. The method we propose simplifies the approach used in [2] by merging regularization and approximation of the solution . Then, the resulting regularized system can be analyzed by standard arguments, namely, through a Faedo-Galerkin/compactness procedure and a specific treatment of the non-linear terms. On the other hand, the second matter can be handled through a specific definition of the Galerkin basis.

In paragraph 5.1, regularization and Faedo-Galerkin techniques are merged by introducing a regularized problem in finite dimension n. In the next paragraph, existence of solution for this problem is proved. In paragraph 6.1, energy estimates are derived, independent of the regularization parameter $\frac{1}{n}$. Paragraph 6.2 is devoted to the proof of global existence of discrete solution. Existence of solution for the continuous problem is addressed in section 6.3 whereas, in 6.4, uniqueness is proved for problem (**P1**), under the additional assumption (**A3**).

5.1 A regularized problem in finite dimension

Let $\{h_k\}_{k\in\mathbb{N}^*}$ be a Hilbert basis of V_i, $\{f_k\}_{k\in\mathbb{N}^*}$ be a Hilbert basis of V_e and $\{g_k\}_{k\in\mathbb{N}^*}$ a Hilbert basis of V_HT. Without loss of generality, we assume that these basis functions are (sufficiently) smooth and that $\{h_k\}_{k\in\mathbb{N}^*}$ is an orthogonal basis in $L^2(\Omega_\mathrm{H})$. We introduce a Galerkin basis of V by defining, for all $k\in\mathbb{N}$, $\tilde{f}_k\in H^1(\Omega)$ as an extension of f_k in $H^1(\Omega)$, given by an arbitrary continuous extension operator. We also extend, for all $k\in\mathbb{N}$, g_k by $\tilde{g_k}\in H^1(\Omega)$ such that $\tilde{g_k}=0$ in Ω_H. One can check straighforwardly that $\{e_k\}_{k\in\mathbb{N}^*}$, defined as, $e_{2k-1}=\tilde{f}_k,\quad e_{2k}=\tilde{g_k},\quad \forall k\in\mathbb{N}^*$, is a Galerkin basis of V.

Finally, for all $n\in\mathbb{N}^*$, we can define the finite dimensional spaces $V_{\mathrm{i},n}$, $V_{\mathrm{e},n}$, $V_{\mathrm{T},n}$ and V_n generated, respectively, by $\{h_k\}_{k=1}^n$,$\{f_k\}_{k=1}^n$, $\{g_k\}_{k=1}^n$ and $\{e_k\}_{k=1}^{2n}$, *i.e*

$$V_{\mathrm{i},n}\stackrel{\mathrm{def}}{=}<\{h_k\}_{k=1}^n>,\quad V_{\mathrm{e},n}\stackrel{\mathrm{def}}{=}<\{f_k\}_{k=1}^n>,$$
$$V_{\mathrm{T},n}\stackrel{\mathrm{def}}{=}<\{g_k\}_{k=1}^n>,\quad V_n\stackrel{\mathrm{def}}{=}<\{e_k\}_{k=1}^{2n}>.$$

Hence, we can introduce, for each $n\in\mathbb{N}^*$, the following two discrete problems $\mathbf{P1}_n$ and $\mathbf{P2}_n$ associated to problems **P1** and **P2**, respectively:

- $\mathbf{P1}_n$: Find $(v_n,u_{\mathrm{i},n},u_n)\in C^0(0,T;V_\mathrm{i}^2\times V)$, $w_n\in C^1(0,T;L^2(\Omega_\mathrm{H}))$ with $(\partial_t v_n,\partial_t u_{\mathrm{i},n},\partial_t u_n)\in[L^2(Q_T)]^3$ satisfying, for all $(h,e,\theta)\in V_{\mathrm{i},n}\times V_\mathrm{n}\times V_{\mathrm{i},n}$,

$$\begin{aligned}
&C_\mathrm{m}\int_{\Omega_\mathrm{H}}\partial_t v_n h+\frac{1}{n}\int_{\Omega_\mathrm{H}}\partial_t u_{\mathrm{i},n}h+\int_{\Omega_H}\boldsymbol{\sigma}_\mathrm{i}\boldsymbol{\nabla}u_{\mathrm{i},n}\cdot\boldsymbol{\nabla}h\\
&\qquad\qquad+\int_{\Omega_H}I_\mathrm{ion}(v_n,w_n)h=\int_{\Omega_H}I_\mathrm{app}h,\\
&C_\mathrm{m}\int_{\Omega_\mathrm{H}}\partial_t v_n e-\frac{1}{n}\int_{\Omega}\partial_t u_n e-\int_{\Omega}\boldsymbol{\sigma}\boldsymbol{\nabla}u_n\cdot\boldsymbol{\nabla}e\\
&\qquad\qquad+\int_{\Omega_H}I_\mathrm{ion}(v_n,w_n)e=\int_{\Omega_H}I_\mathrm{app}e,\\
&\int_{\Omega_\mathrm{H}}\partial_t w_n\theta+\int_{\Omega_\mathrm{H}}g(v_n,w_n)\theta=0,
\end{aligned}\tag{5.23}$$

with $v_n\stackrel{\mathrm{def}}{=}u_{\mathrm{i},n}-u_{n|\Omega_\mathrm{H}}$ and verifying the initial conditions

$$\begin{aligned}
&v_n(0)=v_{0,n},\quad u_{\mathrm{i},n}(0)=u_{\mathrm{i},0,n},\quad \text{in}\quad \Omega_\mathrm{H};\quad u_n(0)=u_{0,n}\quad \text{in}\quad \Omega,\\
&w_n(0)=w_{0,n},\quad \text{in}\quad \Omega_\mathrm{H},
\end{aligned}\tag{5.24}$$

Here, $v_{0,n},w_{0,n}$ are suitable approximations of v_0 and w_0 in $V_{\mathrm{i},n}$, and $u_{\mathrm{i},0,n},u_{0,n}$ are *auxiliary* initial to be specified later on.

- $\mathbf{P2}_n$: Find $(v_n, u_{\mathrm{i},n}, u_n) \in C^0(0,T; V_{\mathrm{i}}^2 \times V)$ and $w_n \in C^1(0,T,L^\infty(\Omega_{\mathrm{H}}))$ with $(\partial_t v_n, \partial_t u_{\mathrm{i},n}, \partial_t u_n) \in [L^2(Q_T)]^3$ satisfying $(5.23)_{1,2}$-$(5.24)_1$ and

$$\begin{aligned} \partial_t w_n + g(v_n, w_n) = 0, &\quad \text{a.e. in} \quad Q_T, \\ w_n(0) = w_0, &\quad \text{a.e. in} \quad \Omega_{\mathrm{H}}. \end{aligned} \tag{5.25}$$

The (auxiliary) initial conditions for $u_{\mathrm{i},n}$ and u_n, needed by the two problems below, are defined by introducing two arbitrary functions $u_{\mathrm{i},0} \in H^1(\Omega_{\mathrm{H}})$ and $u_0 \in V$ such that $v_0 = u_{\mathrm{i},0} - u_0$ in Ω_{H}. Then, for $n \in \mathbb{N}^*$, we define $(v_{0,n}, u_{\mathrm{i},0,n}, u_{0,n}, w_{0,n})$ as the orthogonal projections, on $V_{\mathrm{i},n}^2 \times V_{\mathrm{n}} \times V_{\mathrm{i},n}$, of $(v_0, u_{\mathrm{i},0}, u_0, w_0)$. Clearly, by construction of these sequences, we have

$$(v_{0,n}, u_{\mathrm{i},0,n}, u_{0,n}, w_{0,n}) \longrightarrow (v_0, u_{\mathrm{i},0}, u_0, w_0), \tag{5.26}$$

in $V_{\mathrm{i}}^2 \times V \times L^2(\Omega_{\mathrm{H}})$.

5.2 Local existence of the discretized solution

Lemma 5.1 *For all $n \in \mathbb{N}^*$ there exists a positive time $0 < t_n \le T$ such that problems $\mathbf{P1}_n$ and $\mathbf{P2}_n$ admit a unique solution over the time interval $(0, t_n)$.*

Proof. For the sake of conciseness we only give here the details of the proof for problem $\mathbf{P1}_n$, the proof for problem $\mathbf{P2}_n$ follows with minor modifications.

Since $\{h_l\}_{1\le l\le n}$ and $\{e_l\}_{1\le l\le 2n}$ are basis of $V_{\mathrm{i},n}$ and V_{n}, respectively, we can write

$$\begin{aligned} &u_{\mathrm{i},n}(t) = \sum_{l=1}^{n} c_{\mathrm{i},l}(t) h_l, \quad u_n(t) = \sum_{l=1}^{2n} c_l(t) e_l, \quad w_n(t) = \sum_{l=1}^{n} c_{\mathrm{w},l}(t) h_l, \\ &u_{\mathrm{i},0,n} = \sum_{l=1}^{n} c_{\mathrm{i},l}^0 h_l, \quad u_{0,n} = \sum_{l=1}^{2n} c_l^0 e_l, \quad w_{0,n} = \sum_{l=1}^{n} c_{\mathrm{w},l}^0 h_l. \end{aligned} \tag{5.27}$$

Thus, by introducing the notations

$$\begin{aligned} &\boldsymbol{c}_{\mathrm{i}} \stackrel{\mathrm{def}}{=} \{c_{\mathrm{i},l}\}_{l=1}^n, \quad \boldsymbol{c} \stackrel{\mathrm{def}}{=} \{c_l\}_{l=1}^{2n}, \quad \boldsymbol{c}_{\mathrm{w}} \stackrel{\mathrm{def}}{=} \{c_{\mathrm{w},l}\}_{l=1}^n, \\ &\boldsymbol{c}_{\mathrm{i}}^0 \stackrel{\mathrm{def}}{=} \{c_{\mathrm{i},l}^0\}_{l=1}^n, \quad \boldsymbol{c}^0 \stackrel{\mathrm{def}}{=} \{c_l^0\}_{l=1}^{2n}, \quad \boldsymbol{c}_{\mathrm{w}}^0 \stackrel{\mathrm{def}}{=} \{c_{\mathrm{w},l}^0\}_{l=1}^n, \end{aligned}$$

it follows that problem $\mathbf{P1}_n$ is equivalent to the following non-linear system of ordinary differential equations (ODE)

$$\boldsymbol{M} \begin{bmatrix} \boldsymbol{c}_{\mathrm{i}}' \\ \boldsymbol{c}' \\ \boldsymbol{c}_{\mathrm{w}}' \end{bmatrix} = \begin{bmatrix} \boldsymbol{G}_{\mathrm{i}}\big(t, \boldsymbol{c}_{\mathrm{i}}, \boldsymbol{c}, \boldsymbol{c}_{\mathrm{w}}\big) \\ \boldsymbol{G}\big(t, \boldsymbol{c}_{\mathrm{i}}, \boldsymbol{c}, \boldsymbol{c}_{\mathrm{w}}\big) \\ \boldsymbol{G}_{\mathrm{w}}\big(t, \boldsymbol{c}_{\mathrm{i}}, \boldsymbol{c}, \boldsymbol{c}_{\mathrm{w}}\big) \end{bmatrix}, \quad \begin{bmatrix} \boldsymbol{c}_{\mathrm{i}}(0) \\ \boldsymbol{c}(0) \\ \boldsymbol{c}_{\mathrm{w}}(0) \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}_{\mathrm{i}}^0 \\ \boldsymbol{c}^0 \\ \boldsymbol{c}_{\mathrm{w}}^0 \end{bmatrix}. \tag{5.28}$$

Here, thematrix $\boldsymbol{M} \in \mathbb{R}^{4n\times 4n}$ is given by

$$\boldsymbol{M} \stackrel{\text{def}}{=} \begin{bmatrix} (C_\text{m} + \frac{1}{n})\boldsymbol{M}_{V_\text{i}} & \vdots & -C_\text{m}\boldsymbol{M}_{V_\text{ie}} & \vdots & 0 \\ \cdots\cdots & \vdots & \cdots\cdots & \vdots & \cdots\cdots \\ & \vdots & & \vdots & \\ -C_\text{m}\boldsymbol{M}_{V_\text{ie}}^T & \vdots & C_\text{m}\boldsymbol{M}_{V_\text{e}} + \frac{1}{n}\boldsymbol{M}_{V_\text{HT}} & \vdots & 0 \\ & \vdots & & \vdots & \\ \cdots\cdots & \vdots & \cdots\cdots & \vdots & \cdots\cdots \\ 0 & \vdots & 0 & \vdots & \boldsymbol{M}_{V_\text{i}} \end{bmatrix},$$

with $\boldsymbol{M}_{V_\text{i}} \in \mathbb{R}^{n\times n}$, $\boldsymbol{M}_{V_\text{ie}} \in \mathbb{R}^{n\times 2n}$ and $\boldsymbol{M}_{V_\text{e}}, \boldsymbol{M}_{V_\text{HT}} \in \mathbb{R}^{2n\times 2n}$

$$\boldsymbol{M}_{V_\text{i}} \stackrel{\text{def}}{=} \Big(\int_{\Omega_\text{H}} h_k h_l\Big)_{1\le k,l\le n}, \quad \boldsymbol{M}_{V_\text{ie}} \stackrel{\text{def}}{=} \Big(\int_{\Omega_\text{H}} h_k e_l\Big)_{1\le k\le n,\, 1\le l\le 2n},$$
$$\boldsymbol{M}_{V_\text{e}} \stackrel{\text{def}}{=} \Big(\int_{\Omega_\text{H}} e_k e_l\Big)_{1\le k,l\le 2n}, \quad \boldsymbol{M}_{V_\text{HT}} \stackrel{\text{def}}{=} \Big(\int_{\Omega} e_k e_l\Big)_{1\le k,l\le 2n}.$$

On the other hand, from the notations

$$\boldsymbol{G}_\text{i} \stackrel{\text{def}}{=} \{G_{\text{i},k}\}_{k=1}^n, \quad \boldsymbol{G} \stackrel{\text{def}}{=} \{G_k\}_{k=1}^{2n}, \quad \boldsymbol{G}_\text{w} \stackrel{\text{def}}{=} \{G_{\text{w},k}\}_{k=1}^n,$$

the right-hand side of (5.28) is given by

$$G_{\text{i},k}(t, \boldsymbol{c}_\text{i}, \boldsymbol{c}, \boldsymbol{c}_\text{w}) \stackrel{\text{def}}{=} -\int_{\Omega_H} \boldsymbol{\sigma}_\text{i}\boldsymbol{\nabla} u_{\text{i},n}\cdot\boldsymbol{\nabla} h_k - \int_{\Omega_H} I_\text{ion}(v_n, w_n)h_k + \int_{\Omega_H} I_\text{app}h_k,$$

for all $1 \le k \le n$,

$$G_k(t, \boldsymbol{c}_\text{i}, \boldsymbol{c}, \boldsymbol{c}_\text{w}) \stackrel{\text{def}}{=} -\int_{\Omega} \boldsymbol{\sigma}\boldsymbol{\nabla} u_n\cdot\boldsymbol{\nabla} e_k + \int_{\Omega_H} I_\text{ion}(v_n, w_n)e_k - \int_{\Omega_H} I_\text{app}e_k,$$

for all $1 \le k \le 2n$, and finally,

$$G_{\text{w},k}(t, \boldsymbol{c}_\text{i}, \boldsymbol{c}, \boldsymbol{c}_\text{w}) \stackrel{\text{def}}{=} -\int_{\Omega_H} g(v_n, w_n)h_k,$$

for all $1 \le k \le n$.

Existence of a local solution for the ODE system (5.28) follows by using the Cauchy-Lipschitz theorem. Indeed, according to Lemma 5.2, given below, the mass matrix $\boldsymbol{M}$ is positive definite and hence invertible and, on the other hand, the right-hand side of (5.28) is a C^1 function with respect to the arguments $\boldsymbol{c}_\text{i}$, $\boldsymbol{c}$ and $\boldsymbol{c}_\text{w}$. This completes the proof. □

Lemma 5.2 *For all $n \in \mathbb{N}^*$, the matrix $\boldsymbol{M}$ is positive definite.*

Proof. We can decompose $\boldsymbol{M}$ as $\boldsymbol{M} = C_{\mathrm{m}}\boldsymbol{N} + \dfrac{1}{n}\boldsymbol{D}$, with

$$\boldsymbol{D} \stackrel{\text{def}}{=} \begin{bmatrix} \boldsymbol{M}_{V_{\mathrm{i}}} & \vdots & 0 & \vdots & 0 \\ \cdots\cdots & \vdots & \cdots\cdots\cdots & \vdots & \cdots\cdots \\ & \vdots & & \vdots & \\ 0 & \vdots & \boldsymbol{M}_{V_{\mathrm{HT}}} & \vdots & 0 \\ & \vdots & & \vdots & \\ \cdots\cdots & \vdots & \cdots\cdots\cdots & \vdots & \cdots\cdots \\ 0 & \vdots & 0 & \vdots & n\boldsymbol{M}_{V_{\mathrm{i}}} \end{bmatrix},$$

and

$$\boldsymbol{N} \stackrel{\text{def}}{=} \begin{bmatrix} \boldsymbol{M}_{V_{\mathrm{i}}} & \vdots & -\boldsymbol{M}_{V_{\mathrm{ie}}} & \vdots & 0 \\ \cdots\cdots & \vdots & \cdots\cdots\cdots & \vdots & \cdots\cdots \\ & \vdots & & \vdots & \\ -\boldsymbol{M}_{V_{\mathrm{ie}}}{}^{T} & \vdots & \boldsymbol{M}_{V_{\mathrm{e}}} & \vdots & 0 \\ & \vdots & & \vdots & \\ \cdots\cdots & \vdots & \cdots\cdots\cdots & \vdots & \cdots\cdots \\ 0 & \vdots & 0 & \vdots & 0 \end{bmatrix}.$$

Since the matrices $\boldsymbol{M}_{V_{\mathrm{i}}}$, $\boldsymbol{M}_{V_{\mathrm{HT}}}$ and $\boldsymbol{M}_{V_{\mathrm{i}}}$ are mass matrices, we obtain that the block-diagonal matrix D is definite positive. On the other hand, for each $\begin{bmatrix}\boldsymbol{c}_{\mathrm{i}} & \boldsymbol{c} & \boldsymbol{c}_{\mathrm{w}}\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{4n}$ we have

$$\begin{aligned} \begin{bmatrix}\boldsymbol{c}_{\mathrm{i}} \\ \boldsymbol{c} \\ \boldsymbol{c}_{\mathrm{w}}\end{bmatrix}^{\mathrm{T}} \boldsymbol{N} \begin{bmatrix}\boldsymbol{c}_{\mathrm{i}} \\ \boldsymbol{c} \\ \boldsymbol{c}_{\mathrm{w}}\end{bmatrix} &= \sum_{l,k=1}^{n} \left(\int_{\Omega_{\mathrm{H}}} c_{\mathrm{i},l} c_{\mathrm{i},k} h_l h_k - 2\int_{\Omega_{\mathrm{H}}} c_{\mathrm{i},l} c_{2k-1} h_l f_k + \int_{\Omega_{\mathrm{H}}} c_{2l-1} c_{2k-1} f_l f_k \right) \\ &= \left\| \sum_{l,1}^{n} (c_{\mathrm{i},l} h_l - c_{2l-1} f_l) \right\|^2_{L^2(\Omega_{\mathrm{H}})} \\ &\geq 0, \end{aligned}$$

so that $\boldsymbol{N}$ is positive. It then follows that $\boldsymbol{M}$ is positive definite. □

Remark 6 *The above lemma points out the role of the regularization term* $\frac{1}{n}\boldsymbol{D}$*. It allows to obtain a matrix* $\boldsymbol{M}$ *in* (5.28) *which is non-singular, so that the resulting system of ODE is non-degenerated.*

6.1 Energy estimates

In the next Lemma, we state some uniform estimates (with respect to n) of the solution of problems $\mathbf{P1}_n$ and $\mathbf{P2}_n$. We also provide similar estimates for the time derivative, which will be useful for the passage to the limit. For the sake of clarity, in what follows, $c > 0$ stands for a generic constant that depends on T and the physical parameters, but which is independent of n.

Lemma 6.1 *Let $u_{\mathrm{i},0} \in H^1(\Omega_{\mathrm{H}})$, $u_0 \in V$, $w_0 \in L^2(\Omega_{\mathrm{H}})$ and $I_{\mathrm{app}} \in L^2(Q_T)$ be given data and $(v_n, u_{\mathrm{i},n}, u_n, w_n)$ the solution of $\mathbf{P1}_n$ over $(0, t_n)$. Assume that* **A1** *and* **A2** *hold true. Then, for all $n \in \mathbb{N}^*$ and $t \in (0, t_n)$, we have*

$$\begin{aligned}
&\|v_n\|_{L^\infty(0,t;L^2(\Omega_{\mathrm{H}}))} + \|v_n\|_{L^4(Q_t)} + \frac{1}{\sqrt{n}}\left(\|u_{\mathrm{i},n}\|_{L^\infty(0,t;L^2(\Omega_{\mathrm{H}}))} + \|u_n\|_{L^\infty(0,t;L^2(\Omega))}\right)\\
&\qquad + \|\boldsymbol{\nabla} u_{\mathrm{i},n}\|_{L^2(Q_t)} + \|\boldsymbol{\nabla} u_n\|_{L^2((0,t)\times\Omega)} \le c,\\
&\|\partial_t v_n\|_{L^2(Q_t)} + \|v_n\|_{L^\infty(0,t;H^1(\Omega_{\mathrm{H}}))} + \frac{1}{\sqrt{n}}\left(\|\partial_t u_{\mathrm{i},n}\|_{L^2(Q_t)} + \|\partial_t u_n\|_{L^2((0,t)\times\Omega)}\right)\\
&\qquad + \|\boldsymbol{\nabla} u_{\mathrm{i},n}\|_{L^\infty(0,t;L^2(\Omega_{\mathrm{H}}))} + \|\boldsymbol{\nabla} u_n\|_{L^\infty(0,t;L^2(\Omega))} \le c,
\end{aligned} \tag{6.29}$$

and

$$\|w_n\|_{L^\infty(0,t;L^2(\Omega_{\mathrm{H}}))} \le c, \quad \|\partial_t w_n\|_{L^2(Q_t)} \le c. \tag{6.30}$$

If, in addition, $w_0 \in L^\infty(\Omega_{\mathrm{H}})$ with (4.22)*, there exists a positive constant $w_{\min}$ (independent of t_n) such that the solution $(v_n, u_{\mathrm{i},n}, u_n, w_n)$ of $\mathbf{P2}_n$ over $(0, t_n)$ satisfies* (6.29) *and*

$$\|w_n\|_{W^{1,\infty}(0,t,L^\infty(\Omega_{\mathrm{H}}))} \le c, \quad w_{\min} \le w_n \le 1, \quad \text{in} \quad Q_{t_n}. \tag{6.31}$$

Proof. We start by proving the estimates for problem $\mathbf{P1}_n$. By taking $h = u_{\mathrm{i},n}$, $e = -u_n$, $\theta = w_n$ in (5.23) and using the uniform coercivity of the conductivity tensors (2.11), we obtain:

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}&\left[\|w_n\|^2_{L^2(\Omega_{\mathrm{H}})} + C_{\mathrm{m}}\|v_n\|^2_{L^2(\Omega_{\mathrm{H}})} + \frac{1}{n}\left(\|u_{\mathrm{i},n}\|^2_{L^2(\Omega_{\mathrm{H}})} + \|u_n\|^2_{L^2(\Omega)}\right)\right]\\
&\qquad + \alpha_{\mathrm{i}}\|\boldsymbol{\nabla} u_{\mathrm{i},n}\|^2_{L^2(\Omega_{\mathrm{H}})} + \alpha\|\boldsymbol{\nabla} u_n\|^2_{L^2(\Omega)} + \int_{\Omega_{\mathrm{H}}} I_{\mathrm{ion}}(v_n, w_n)v_n\\
&\qquad\qquad + \int_{\Omega_{\mathrm{H}}} g(v_n, w_n)w_n \le \int_{\Omega_{\mathrm{H}}} I_{\mathrm{app}} v_n.
\end{aligned} \tag{6.32}$$

From assumption **A2**, we get

$$I_{\mathrm{ion}}(v, w)v + g(v, w)w \ge a|v|^4 - (c_8|v|^2 + c_9|w|^2) - c_{10},$$

with $c_8, c_9, c_{10} > 0$. Thus, inserting this expression in (6.32) and using the Cauchy-Schwarz's inequality, it follows that

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}&\left[\|w_n\|^2_{L^2(\Omega_{\mathrm{H}})} + C_{\mathrm{m}}\|v_n\|^2_{L^2(\Omega_{\mathrm{H}})} + \frac{1}{n}\left(\|u_{\mathrm{i},n}\|^2_{L^2(\Omega_{\mathrm{H}})} + \|u_n\|^2_{L^2(\Omega)}\right)\right]\\
&\qquad + \alpha_{\mathrm{i}}\|\boldsymbol{\nabla} u_{\mathrm{i},n}\|^2_{L^2(\Omega_{\mathrm{H}})} + \alpha\|\boldsymbol{\nabla} u_n\|^2_{L^2(\Omega)} + a\|v_n\|^4_{L^4(\Omega_{\mathrm{H}})}\\
&\le \left(c_8 + \frac{1}{2}\right)\|v_n\|^2_{L^2(\Omega_{\mathrm{H}})} + c_9\|w_n\|^2_{L^2(\Omega_{\mathrm{H}})} + c_{10}|\Omega_{\mathrm{H}}| + \frac{1}{2}\|I_{\mathrm{app}}\|^2_{L^2(\Omega_{\mathrm{H}})}.
\end{aligned}$$

Therefore, by integrating over $(0,t)$, with $t \in (0,t_n)$, we have

$$\begin{aligned}
&\|w_n\|^2_{L^2(\Omega_{\rm H})} + C_{\rm m}\|v_n\|^2_{L^2(\Omega_{\rm H})} + \frac{1}{n}\left(\|u_{{\rm i},n}\|^2_{L^2(\Omega_{\rm H})} + \|u_n\|^2_{L^2(\Omega)}\right)\\
&\qquad + \alpha_{\rm i}\|\boldsymbol{\nabla} u_{{\rm i},n}\|^2_{L^2(Q_t)} + \alpha\|\boldsymbol{\nabla} u_n\|^2_{L^2(\Omega\times(0,t))} + a\|v_n\|^4_{L^4(Q_t)}\\
&\quad \leq c\int_0^t \left(\|v_n\|^2_{L^2(\Omega_{\rm H})} + \|w_n\|^2_{L^2(\Omega_{\rm H})}\right) + c_{10}|\Omega_{\rm H}|T + \frac{1}{2}\|I_{\rm app}\|^2_{L^2(Q_T)}\\
&\quad + \|w_{0,n}\|^2_{L^2(\Omega_{\rm H})} + C_{\rm m}\|v_{0,n}\|^2_{L^2(\Omega_{\rm H})} + \frac{1}{n}\left(\|u_{{\rm i},0,n}\|^2_{L^2(\Omega_{\rm H})} + \|u_{0,n}\|^2_{L^2(\Omega)}\right),
\end{aligned}$$

for all $t \in (0,t_n)$. Estimates $(6.29)_1$ and $(6.30)_1$ follow by applying Gronwall lemma and using the fact that, from (5.26),

$$\|w_{0,n}\|^2_{L^2(\Omega_{\rm H})} + C_{\rm m}\|v_{0,n}\|^2_{L^2(\Omega_{\rm H})} + \frac{1}{n}\left(\|u_{{\rm i},0,n}\|^2_{L^2(\Omega_{\rm H})} + \|u_{0,n}\|^2_{L^2(\Omega)}\right),$$

is uniformly bounded with respect to n.

For the estimate of the time derivative, following [2], we notice that

$$\int_{\Omega_{\rm H}} f_1(v)\partial_t v = \frac{d}{dt}\int_{\Omega_{\rm H}} H(v), \quad H(v) \stackrel{\rm def}{=} \int_0^v f_1. \tag{6.33}$$

On the other hand, taking $h = \partial_t u_{{\rm i},n}$, $e = \partial_t u_n$ and $\theta = \partial_t w_n$ in (5.23) and integrating over $(0,t)$, with $t \in (0,t_n)$, yields

$$\begin{aligned}
&\|\partial_t w_n\|^2_{L^2(Q_t)} + C_{\rm m}\|\partial_t v_n\|^2_{L^2(Q_t)} + \frac{1}{n}\left(\|\partial_t u_{{\rm i},n}\|^2_{L^2(Q_t)} + \|\partial_t u_n\|^2_{L^2(0,t;L^2(\Omega))}\right)\\
&+ \frac{\alpha_{\rm i}}{2}\|\boldsymbol{\nabla} u_{{\rm i},n}\|_{L^2(\Omega_{\rm H})} + \frac{\alpha}{2}\|\boldsymbol{\nabla} u_n\|_{L^2(\Omega)}\\
&\quad \leq \frac{1}{2}\int_{\Omega_{\rm H}} \boldsymbol{\sigma}_{\rm i}\boldsymbol{\nabla} u_{{\rm i},0,n}\cdot\boldsymbol{\nabla} u_{{\rm i},0,n} + \frac{1}{2}\int_{\Omega} \boldsymbol{\sigma}\boldsymbol{\nabla} u_{0,n}\cdot\boldsymbol{\nabla} u_{0,n} + \int_{\Omega_{\rm H}} H(v_{0,n})\\
&- \int_{\Omega_{\rm H}} H(v_n) + \int_0^t\int_{\Omega_{\rm H}} I_{\rm app}\partial_t v_n - \int_0^t\int_{\Omega_{\rm H}} (f_2(v_n)w_n\partial_t v_n + g(v_n,w_n)\partial_t w_n).
\end{aligned} \tag{6.34}$$

It remains now to estimate the right-hand side of this expression. The first two terms can be bounded using (5.26). For the third term, we use **A1**, the continuous embedding of $H^1(\Omega_{\rm H})$ into $L^4(\Omega_{\rm H})$ and (5.26) to obtain

$$|H(v_{0,n})| = \left|\int_0^{v_{0,n}} f_1(s)ds\right| \leq c(v_{0,n}^4 + 1) \leq c.$$

For the fourth term, according to assumption **A2**, we have $f_1(v)v + bv^2 \geq 0$. In other words, $f_1(v) + bv \geq 0$ for $v \geq 0$, and $f_1(v) + bv \leq 0$ for $v \leq 0$. As a result, integrating over $(0,v)$ yields

$$-H(v) \leq \frac{b}{2}v^2. \tag{6.35}$$

On the other hand, the fifth term can be controlled using the Cauchy-Schwarz inequality.

In summary, from (6.34) and (2.12), we get

$$
\begin{aligned}
&\|\partial_t w_n\|^2_{L^2(Q_t)} + \frac{C_\mathrm{m}}{2}\|\partial_t v_n\|^2_{L^2(Q_t)} + \frac{1}{n}\|\partial_t u_{\mathrm{i},n}\|^2_{L^2(Q_t)} \\
&+ \frac{1}{n}\|\partial_t u_n\|^2_{L^2(0,t;L^2(\Omega))} + \frac{\alpha_\mathrm{i}}{2}\|\boldsymbol{\nabla} u_{\mathrm{i},n}\|_{L^2(\Omega_\mathrm{H})} + \frac{\alpha}{2}\|\boldsymbol{\nabla} u_n\|_{L^2(\Omega)} \\
&\quad \le c + \frac{1}{2C_\mathrm{m}}\|I_\mathrm{app}\|^2_{L^2(Q_t)} + \frac{b}{2}\|v_n\|^2_{L^2(\Omega_\mathrm{H})} \\
&- \int_0^t\int_{\Omega_\mathrm{H}} f_2(v_n) w_n \partial_t v_n - \int_0^t\int_{\Omega_\mathrm{H}} g_1(v_n)\partial_t w_n - \int_0^t\int_{\Omega_\mathrm{H}} \frac{c_1}{2}\partial_t w_n^2.
\end{aligned}
\tag{6.36}
$$

For the last three terms of the right-hand side we proceed as follows. First, using **A1** and Young's inequality we notice that

$$
\begin{aligned}
\Big|\int_0^t\int_{\Omega_\mathrm{H}} f_2(v_n) w_n\partial_t v_n\Big| &= \Big|\int_0^t\int_{\Omega_\mathrm{H}} c_4\partial_t v_n w_n + c_5 v_n \partial_t v_n w_n\Big| \\
&\le \frac{C_\mathrm{m}}{4}\|\partial_t v_n\|^2_{L^2(Q_t)} + c\|w_n\|^2_{L^2(Q_t)} + \left|\frac{c_5}{2}\int_0^t\int_{\Omega_\mathrm{H}} w_n\partial_t v_n^2\right|.
\end{aligned}
$$

In addition, integration by parts in the last term with the Cauchy-Schwarz inequality yields

$$
\begin{aligned}
\left|\frac{c_5}{2}\int_0^t\int_{\Omega_\mathrm{H}} w_n\partial_t v_n^2\right| &\le \frac{|c_5|}{2}\Big|\int_0^t\int_{\Omega_\mathrm{H}}\partial_t w_n v_n^2\Big| + \frac{|c_5|}{2}\int_{\Omega_\mathrm{H}}\left|w_n(t)v_n^2(t) - w_{0,n}v_{0,n}^2\right| \\
&\le c\|v_n\|^4_{L^4(Q_t)} + \frac{1}{4}\|\partial_t w_n\|^2_{L^2(Q_t)} + c\left(\|v_{0,n}\|^4_{L^4(\Omega_\mathrm{H})} + \|w_{0,n}\|^2_{L^2(\Omega_\mathrm{H})}\right) \\
&\qquad + c\|w_n(t)\|_{L^2(\Omega_\mathrm{H})}\|v_n(t)\|^2_{L^4(\Omega_\mathrm{H})},
\end{aligned}
$$

where the last term can be estimated by combining Hölder's inequality and the continuous embedding of $H^1(\Omega_\mathrm{H})$ in $L^6(\Omega_\mathrm{H})$, namely,

$$
\|v_n(t)\|^2_{L^4(\Omega_\mathrm{H})} \le \|v_n(t)\|^{\frac{1}{2}}_{L^2(\Omega_\mathrm{H})}\|v_n(t)\|^{\frac{3}{2}}_{L^6(\Omega_\mathrm{H})} \le c\|v_n(t)\|^{\frac{1}{2}}_{L^2(\Omega_\mathrm{H})}\|v_n(t)\|^{\frac{3}{2}}_{H^1(\Omega_\mathrm{H})}.
$$

Finally, using again **A1** we have,

$$
\left|\int_0^t\int_{\Omega_\mathrm{H}} g_1(v_n)\partial_t w_n\right| \le c(|\Omega_\mathrm{H}|t + \|v_n\|^4_{L^4(Q_t)}) + \frac{1}{4}\|\partial_t w_n\|^2_{L^2(Q_t)},
$$

and

$$
\left|\int_0^t\int_{\Omega_\mathrm{H}}\frac{c_1}{2}\partial_t w_n^2\right| \le \frac{1}{4}\|w_n(t)\|^2_{L^2(\Omega_\mathrm{H})} + c\|w_{0,n}\|^2_{L^2(\Omega_\mathrm{H})}.
$$

As a result, by inserting this last estimates in (6.36), we obtain

$$
\begin{aligned}
&\frac{1}{2}\|\partial_t w_n\|^2_{L^2(Q_t)} + \frac{C_\mathrm{m}}{4}\|\partial_t v_n\|^2_{L^2(Q_t)} + \frac{1}{n}\|\partial_t u_{\mathrm{i},n}\|^2_{L^2(Q_t)} + \frac{1}{n}\|\partial_t u_n\|^2_{L^2(0,t;L^2(\Omega))} \\
&+ \frac{\alpha_\mathrm{i}}{2}\|\boldsymbol{\nabla} u_{\mathrm{i},n}\|_{L^2(\Omega_\mathrm{H})} + \frac{\alpha}{2}\|\boldsymbol{\nabla} u_n\|_{L^2(\Omega)} \le c + \frac{1}{2C_\mathrm{m}}\|I_\mathrm{app}\|^2_{L^2(Q_t)} + c\|v_n(t)\|^2_{L^2(\Omega_\mathrm{H})} \\
&\qquad + c\|w_n\|^2_{L^2(Q_t)} + c\|v_n\|^4_{L^4(Q_t)} + c\left(\|v_{0,n}\|^4_{L^4(\Omega_\mathrm{H})} + \|w_{0,n}\|^2_{L^2(\Omega_\mathrm{H})}\right) \\
&\qquad\qquad + c\|w_n(t)\|_{L^2(\Omega_\mathrm{H})}\|v_n(t)\|^{\frac{1}{2}}_{L^2(\Omega_\mathrm{H})}\|v_n(t)\|^{\frac{3}{2}}_{H^1(\Omega_\mathrm{H})} + c|\Omega_\mathrm{H}|t,
\end{aligned}
\tag{6.37}
$$

for all $t \in (0, t_n)$.

Therefore, using (5.26), the previous estimates $(6.29)_1$, $(6.30)_1$, and since $t_n \leq T$, inequality (6.37) reduces to

$$\frac{1}{2}\|\partial_t w_n\|^2_{L^2(Q_t)} + \frac{C_\mathrm{m}}{4}\|\partial_t v_n\|^2_{L^2(Q_t)} + \frac{1}{n}\|\partial_t u_{\mathrm{i},n}\|^2_{L^2(Q_t)} + \frac{1}{n}\|\partial_t u_n\|^2_{L^2(0,t;L^2(\Omega))} + \frac{\alpha_\mathrm{i}}{2}\|\boldsymbol{\nabla} u_{\mathrm{i},n}\|_{L^2(\Omega_\mathrm{H})} + \frac{\alpha}{2}\|\boldsymbol{\nabla} u_n\|_{L^2(\Omega)} \leq c\left(1 + \|v_n(t)\|^{\frac{3}{2}}_{H^1(\Omega_\mathrm{H})}\right),$$

for all $t \in (0, t_n)$. In particular, using Poincaré inequality, this implies that

$$\frac{1}{2}\min(\alpha, \alpha_i)\|v_n(t)\|^2_{H^1(\Omega_\mathrm{H})} \leq c\left(1 + \|v_n(t)\|^{\frac{3}{2}}_{H^1(\Omega_\mathrm{H})}\right),$$

so that v_n is uniformly bounded in $L^\infty(0, t_n; H^1(\Omega_\mathrm{H}))$. Hence, we obtain the desired estimates $(6.29)_2$ and $(6.30)_2$.

Now, we consider problem $\mathbf{P2}_n$, by proving the estimate (6.31). From $(5.25)_1$ it follows that $\partial_t w_n = -g(v_n, w_n)$ and, on the other hand, according to (2.14), we have $0 \leq h_\infty \leq 1$. Thus, from $(2.13)_2$ we have, *a.e.* in $(0, t_n)$,

$$\begin{aligned} \partial_t w_n &\geq -\, w_n\Big(\frac{1}{\tau_\mathrm{close}} + \frac{\tau_\mathrm{close} - \tau_\mathrm{open}}{\tau_\mathrm{close}\tau_\mathrm{open}} h_\infty(v_n)\Big), \\ \partial_t w_n &\leq (1 - w_n)\Big(\frac{1}{\tau_\mathrm{close}} + \frac{\tau_\mathrm{close} - \tau_\mathrm{open}}{\tau_\mathrm{close}\tau_\mathrm{open}} h_\infty(v_n)\Big), \end{aligned} \tag{6.38}$$

which combined with Gronwall lemma yields

$$\begin{aligned} w_n &\geq w_0 \exp\left[-\int_0^t \left(\frac{1}{\tau_\mathrm{close}} + \frac{\tau_\mathrm{close} - \tau_\mathrm{open}}{\tau_\mathrm{close}\tau_\mathrm{open}} h_\infty(v_n)\right)\right], \\ w_n &\leq 1 - (1 - w_0)\exp\left[-\int_0^t \left(\frac{1}{\tau_\mathrm{close}} + \frac{\tau_\mathrm{close} - \tau_\mathrm{open}}{\tau_\mathrm{close}\tau_\mathrm{open}} h_\infty(v_n)\right)\right]. \end{aligned}$$

Using (4.22), we then obtain that

$$w_\mathrm{min} \stackrel{\mathrm{def}}{=} r\exp\left(\frac{-T}{\tau_\mathrm{open}}\right) \leq w_n \leq 1, \quad a.e. \text{ in } \quad Q_{t_n}.$$

On the other hand, by combining this estimate with (6.38), we get

$$\frac{-1}{\tau_\mathrm{open}} \leq \partial_t w_n \leq \frac{1}{\tau_\mathrm{open}}, \quad a.e. \text{ in } \quad Q_{t_n}.$$

which completes the proof of (6.31).

Finally, the energy estimates $(6.29)_1$ are obtained in a standard fashion by taking $h = u_{\mathrm{i},n}$ and $e = -u_n$ in $(5.23)_{1,2}$, which yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left[C_\mathrm{m}\|v_n\|^2_{L^2(\Omega_\mathrm{H})} + \frac{1}{n}\left(\|u_{\mathrm{i},n}\|^2_{L^2(\Omega_\mathrm{H})} + \|u_n\|^2_{L^2(\Omega)}\right)\right] + \alpha_\mathrm{i}\|\boldsymbol{\nabla} u_{\mathrm{i},n}\|^2_{L^2(\Omega_\mathrm{H})} + \alpha\|\boldsymbol{\nabla} u_n\|^2_{L^2(\Omega)} + \int_{\Omega_\mathrm{H}} I_\mathrm{ion}(v_n, w_n)v_n \leq \int_{\Omega_\mathrm{H}} I_\mathrm{app} v_n. \tag{6.39}$$

Conversely, assumption **A2** and the estimate (6.31) lead to

$$I_\mathrm{ion}(v, w)v \geq \frac{a}{\tau_\mathrm{in}} w_\mathrm{min}|v|^4 - \left(\frac{b}{\tau_\mathrm{in}} + \frac{1}{\tau_\mathrm{out}}\right)|v|^2,$$

so that, from (6.39), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left[C_{\mathrm{m}}\|v_n\|^2_{L^2(\Omega_{\mathrm{H}})}+\frac{1}{n}\left(\|u_{\mathrm{i},n}\|^2_{L^2(\Omega_{\mathrm{H}})}+\|u_n\|^2_{L^2(\Omega)}\right)\right]$$
$$+\alpha_{\mathrm{i}}\|\boldsymbol{\nabla}u_{\mathrm{i},n}\|^2_{L^2(\Omega_{\mathrm{H}})}+\alpha\|\boldsymbol{\nabla}u_n\|^2_{L^2(\Omega)}+\frac{a}{\tau_{in}}w_{\min}\|v_n\|^4_{L^4(\Omega_{\mathrm{H}})}$$
$$\leq\left(\frac{b}{\tau_{\mathrm{in}}}+\frac{1}{\tau_{\mathrm{out}}}+\frac{1}{2}\right)\|v_n\|^2_{L^2(\Omega_{\mathrm{H}})}+\frac{1}{2}\|I_{\mathrm{app}}\|^2_{L^2(\Omega_{\mathrm{H}})}.$$

We obtain the energy estimate $(6.29))_1$ by applying Gronwall lemma.

For the estimate on the time derivatives, we take $h=\partial_t u_{\mathrm{i},n}$ and $e=\partial_t u_n$ in (5.23) and we integrate over $(0,t)$, with $t\in(0,t_n)$. By using Cauchy-Schwarz and Young's inequalities, this yields

$$\frac{C_{\mathrm{m}}}{4}\|\partial_t v_n\|^2_{L^2(Q_t)}+\frac{1}{n}\left(\|\partial_t u_{\mathrm{i},n}\|^2_{L^2(Q_t)}+\|\partial_t u_n\|^2_{L^2(0,t;L^2(\Omega))}\right)+\frac{\alpha_{\mathrm{i}}}{2}\|\boldsymbol{\nabla}u_{\mathrm{i},n}\|_{L^2(\Omega_{\mathrm{H}})}$$
$$+\frac{\alpha}{2}\|\boldsymbol{\nabla}u_n\|_{L^2(\Omega)}\leq c\left(\|\boldsymbol{\nabla}u_{\mathrm{i},0,n}\|^2_{L^2(\Omega_{\mathrm{H}})}+\|\boldsymbol{\nabla}u_{0,n}\|^2_{L^2(\Omega)}\right)+\frac{1}{2C_{\mathrm{m}}}\|I_{\mathrm{app}}\|^2_{L^2(Q_t)}$$
$$+\frac{1}{\tau^2_{\mathrm{out}}C_{\mathrm{m}}}\|v_n\|^2_{L^2(Q_t)}-\frac{1}{\tau_{\mathrm{in}}}\int_0^t\int_{\Omega_{\mathrm{H}}}w_n f_1(v_n)\partial_t v_n. \quad (6.40)$$

On the other hand, using (6.35), (6.33), assumption **A1** and integration by parts, for the last term, we have

$$-\int_0^t\int_{\Omega_{\mathrm{H}}}w_n f_1(v_n)\partial_t v_n=-\int_0^t\int_{\Omega_{\mathrm{H}}}w_n\partial_t H(v_n)$$
$$=-\int_{\Omega_{\mathrm{H}}}w_n H(v_n)+\int_{\Omega_{\mathrm{H}}}w_0 H(v_{0,n})+\int_0^t\int_{\Omega_{\mathrm{H}}}\partial_t w_n H(v_n)$$
$$\leq c\|w_n(t)\|_{L^\infty(\Omega_{\mathrm{H}})}\|v_n(t)\|^2_{L^2(\Omega_{\mathrm{H}})}$$
$$+c\|w_0\|_{L^\infty(\Omega_{\mathrm{H}})}\left(1+\|v_{0,n}\|^4_{L^4(\Omega_{\mathrm{H}})}\right)$$
$$+c\|\partial_t w_n\|_{L^\infty(Q_t)}\left(1+\|v_n\|^4_{L^4(Q_t)}\right).$$

Therefore, by inserting this estimate in (6.40), using (5.26) and the previous estimates $(6.29)_1$ and (6.31), we obtain $(6.29)_2$, which completes the proof. □

6.2 Global existence of the discretized solution

For the time being, the solution of the approximated differential system $\mathbf{P1}_n$ is defined on the time interval $[0,t_n]$. In this paragraph, we prove that we can extend this solution over the whole interval $[0,T]$.

In the proof of Lemma 5.1, Cauchy-Lipschitz theorem provides an existence time t_n which depends on n and the initial conditions $\boldsymbol{c}^0_{\mathrm{i}},\boldsymbol{c}^0_{\mathrm{w}}\in\mathbb{R}^n$ and $\boldsymbol{c}^0\in\mathbb{R}^{2n}$. Now, for a fixed $n\in\mathbb{N}^*$, we can consider the problem of solving (5.28) with initial data $\boldsymbol{c}^0_{\mathrm{i}}\stackrel{\mathrm{def}}{=}\{c_{\mathrm{i},l}(t_n)\}_{l=1}^n$, $\boldsymbol{c}^0_{\mathrm{w}}\stackrel{\mathrm{def}}{=}\{c_{\mathrm{w},l}(t_n)\}_{l=1}^n$ and $\boldsymbol{c}^0\stackrel{\mathrm{def}}{=}\{c_l(t_n)\}_{l=1}^{2n}$ at $t=t_n$, where

$$u_{i,n}(t_n)=\sum_{l=1}^n c_{i,l}(t_n)h_l,\quad u_n(t_n)=\sum_{l=1}^{2n}c_l(t_n)e_l,\quad w_n(t_n)=\sum_{l=1}^n c_{\mathrm{w},l}(t_n)h_l.$$

According to Lemma 6.1, the new initial conditions satisfy

$$\frac{1}{n}\sum_{l=1}^{n}|c_{i,l}(t_n)|^2+\frac{1}{n}\sum_{l=1}^{2n}|c_l(t_n)|^2+\sum_{l=1}^{n}|c_{\mathrm{w},l}(t_n)|^2\|h_l\|^2_{L^2(\Omega_{\mathrm{H}})}\leq c, \tag{6.41}$$

with c is independent of t_n. Thus, using the Cauchy-Lipschitz theorem, we can define a solution on the time interval $[t_n, t_n+\rho_n]$ where $\rho_n>0$ depends only on n and c. Thanks to the energy estimates of Lemma 6.1, which now hold true over $[0, t_n+\rho_n]$, $c_{i,l}(t_n+\rho_n)$, $c_l(t_n+\rho_n)$ and $c_{\mathrm{w},l}(t_n+\rho_n)$ still satisfy the estimate (6.41), with the same constant c. Therefore, by iterating this argument, we obtain the existence of solution on time intervals of fixed length ρ_n, which allows to reach any arbitrary time $T>0$. For problem $\mathbf{P2}_n$, the proof follows with minor modifications by noticing that, from (6.31), w_n is bounded in $L^\infty(Q_T)$.

Finally, we note that the estimates provid by Lemma 6.1 can be extended to the whole time interval $[0,T]$.

6.3 Weak solution of the bidomain-torso problem

We first consider problem **P1**. Let us multiply (5.23) by a function $\alpha\in\mathcal{D}(0,T)$ and integrate between 0 and T. For all $k\leq n$, we have

$$\begin{aligned}C_{\mathrm{m}}\int_0^T\int_{\Omega_{\mathrm{H}}}\alpha\,\partial_t v_n h_k+\frac{1}{n}\int_0^T\int_{\Omega_{\mathrm{H}}}\alpha\,\partial_t u_{\mathrm{i},n}h_k+\int_0^T\int_{\Omega_{\mathrm{H}}}\alpha\,\boldsymbol{\sigma}_{\mathrm{i}}\boldsymbol{\nabla}u_{\mathrm{i},n}\cdot\boldsymbol{\nabla}h_k\\ +\int_0^T\int_{\Omega_{\mathrm{H}}}\alpha\,I_{\mathrm{ion}}(v_n,w_n)h_k=\int_0^T\int_{\Omega_{\mathrm{H}}}\alpha\,I_{\mathrm{app}}h_k,\end{aligned} \tag{6.42}$$

$$\begin{aligned}C_{\mathrm{m}}\int_0^T\int_{\Omega_{\mathrm{H}}}\alpha\,\partial_t v_n e_k-\frac{1}{n}\int_0^T\int_{\Omega}\alpha\,\partial_t u_n e_k-\int_0^T\int_{\Omega}\alpha\,\boldsymbol{\sigma}\boldsymbol{\nabla}u_n\cdot\boldsymbol{\nabla}e_k\\ +\int_0^T\int_{\Omega_{\mathrm{H}}}\alpha\,I_{\mathrm{ion}}(v_n,w_n)e_k=\int_0^T\int_{\Omega_{\mathrm{H}}}\alpha\,I_{\mathrm{app}}e_k\end{aligned} \tag{6.43}$$

$$\int_0^T\int_{\Omega_{\mathrm{H}}}\alpha\,\partial_t w_n h_k+\int_0^T\int_{\Omega_{\mathrm{H}}}\alpha\,g(v_n,w_n)h_k=0. \tag{6.44}$$

From Lemma 6.1, it follows that there exists four functions $u\in L^\infty(0,T;V)$, $v_{\mathrm{m}}\in L^\infty(0,T;H^1(\Omega_{\mathrm{H}}))\cap L^4(Q_T)\cap H^1(0,T;L^2(\Omega_{\mathrm{H}}))$, $u_{\mathrm{i}}\in L^\infty(0,T;H^1(\Omega_{\mathrm{H}}))$ and $w\in H^1(0,T;L^2(\Omega_{\mathrm{H}}))$ such that, up to extracted subsequences, we have:

$$\begin{cases} u_n\to u \text{ in } L^\infty(0,T;V) \text{ weak}\star,\\ v_n\to v_{\mathrm{m}} \text{ in } L^\infty(0,T;H^1(\Omega_{\mathrm{H}})) \text{ weak}\star,\\ v_n\to v_{\mathrm{m}} \text{ weakly in } L^4(Q_T),\\ v_n\to v_{\mathrm{m}} \text{ weakly in } H^1(0,T;L^2(\Omega_{\mathrm{H}})),\\ u_{\mathrm{i},n}\to u_{\mathrm{i}} \text{ in } L^\infty(0,T;H^1(\Omega_{\mathrm{H}})) \text{ weak}\star,\\ w_n\to w \text{ weakly in } H^1(0,T;L^2(\Omega_{\mathrm{H}})).\end{cases} \tag{6.45}$$

Moreover, according to lemma 6.1, we also notice that $\dfrac{1}{\sqrt{n}} u_{\mathrm{i},n}$ and $\dfrac{1}{\sqrt{n}} u_n$ are bounded in $L^\infty(0,T;L^2(\Omega_{\mathrm{H}}))$ and $L^\infty(0,T;L^2(\Omega))$, respectivley. Thus, for all $k \le n$ and $\alpha \in \mathcal{D}(0,T)$, we have

$$\lim_{n\to+\infty} \frac{1}{n} \int_0^T \int_{\Omega_{\mathrm{H}}} \alpha\, \partial_t u_{\mathrm{i},n} h_k = 0, \qquad \lim_{n\to+\infty} \frac{1}{n} \int_0^T \int_{\Omega} \alpha\, \partial_t u_n e_k = 0.$$

Let us consider now the nonlinear terms in (6.42)-(6.44). Since $\{v_n\}$ is bounded in $L^2(0,T;H^1(\Omega_{\mathrm{H}}))\cap H^1(0,T;L^2(\Omega_{\mathrm{H}}))$, we have that $\{v_n\}$ is bounded in $H^1(Q_T)$. Hence, thanks to the compact embedding of $H^1(Q_T)$ in $L^3(Q_T)$, the sequence $\{v_n\}$ strongly converges to v_{m} in $L^3(Q_T)$. In addition, using the Lebesgue's dominated convergence Theorem, we deduce that there exists a positive function $\mathcal{V} \in L^1(Q_T)$ such that, up to extraction, $v_n^3 \le \mathcal{V}$ and that $v_n \to v_{\mathrm{m}}$ *a.e.* in Q_T. Thus, from **A1** and using once again the Lebesgue's dominated convergence Theorem, it follows that $\{f_1(v_n)\}$ strongly converges to $f_1(v_{\mathrm{m}})$ in $L^1(Q_T)$. As a result,

$$\lim_{n\to+\infty} \int_0^T \int_{\Omega_{\mathrm{H}}} \alpha\, f_1(v_n) h_k = \int_0^T \int_{\Omega_{\mathrm{H}}} \alpha\, f_1(v_{\mathrm{m}}) h_k.$$

On the other hand, since $\{w_n\}$ is bounded in $L^2(Q_T)$ and $\{v_n\}$ strongly converges to v_{m} in $L^2(Q_T)$, we have

$$\lim_{n\to+\infty} \int_0^T \int_{\Omega_{\mathrm{H}}} \alpha\, f_2(v_n) w_n h_k = \int_0^T \int_{\Omega_{\mathrm{H}}} \alpha\, f_2(v_{\mathrm{m}}) w h_k.$$

Thus, in summary,

$$\lim_{n\to+\infty} \int_0^T \int_{\Omega_{\mathrm{H}}} \alpha\, I_{\mathrm{ion}}(v_n, w_n) h_k = \int_0^T \int_{\Omega_{\mathrm{H}}} \alpha\, I_{\mathrm{ion}}(v_{\mathrm{m}}, w) h_k.$$

Similar arguments allow to prove that

$$\lim_{n\to+\infty} \int_0^T \int_{\Omega_{\mathrm{H}}} \alpha\, g(v_n) h_k = \int_0^T \int_{\Omega_{\mathrm{H}}} \alpha\, g(v_{\mathrm{m}}) h_k.$$

We can then pass to the limit in n in (6.42)-(6.44), yielding

$$\begin{aligned} C_{\mathrm{m}} \int_0^T \int_{\Omega_{\mathrm{H}}} \alpha\, \partial_t v_{\mathrm{m}} h_k + \int_0^T \int_{\Omega_H} \alpha\, \boldsymbol{\sigma}_{\mathrm{i}} \boldsymbol{\nabla} u_{\mathrm{i}} \cdot \boldsymbol{\nabla} h_k \\ + \int_0^T \int_{\Omega_H} \alpha\, I_{\mathrm{ion}}(v_{\mathrm{m}}, w) h_k = \int_0^T \int_{\Omega_H} \alpha\, I_{\mathrm{app}} h_k, \end{aligned} \tag{6.46}$$

$$\begin{aligned} C_{\mathrm{m}} \int_0^T \int_{\Omega_{\mathrm{H}}} \alpha\, \partial_t v_{\mathrm{m}} e_k - \int_0^T \int_{\Omega} \alpha\, \boldsymbol{\sigma} \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} e_k \\ + \int_0^T \int_{\Omega_H} \alpha\, I_{\mathrm{ion}}(v_{\mathrm{m}}, w) e_k = \int_0^T \int_{\Omega_H} \alpha\, I_{\mathrm{app}} e_k, \end{aligned} \tag{6.47}$$

$$\int_0^T \int_{\Omega_{\mathrm{H}}} \alpha\, \partial_t w h_k + \alpha\, g(v_{\mathrm{m}}, w) h_k = 0, \tag{6.48}$$

for all $k \in \mathbb{N}^*$ and $\alpha \in \mathcal{D}(0,T)$. We obtain (3.19)-(3.21) from the density properties of the spaces spanned by $\{h_k\}_{k\in\mathbb{N}^*}$ and $\{e_k\}_{k\in\mathbb{N}^*}$.

Finally, it only remains to prove that v_{m} and w satisfy the initial conditions (1.5). Since (v_n) weakly converges to v_{m} in $H^1(0,T;L^2(\Omega_{\mathrm{H}}))$, (v_n) strongly converges to v_{m} in $C(0,T;H^{-1}(\Omega_{\mathrm{H}}))$ for instance. This allows to assert that $v_{\mathrm{m}}(0) = v_0$ in Ω_{H} since, by construction, $v_n(0) \to v_0$ in $L^2(\Omega_{\mathrm{H}})$. The same argument holds for w.

For problem **P2**, the arguments of passing to the limit can be adapted without major modifications. For the nonlinear terms, we can (as previously) prove that $\{v_n\}$ strongly converges to v_{m} in $L^3(Q_T)$. Thus $f_1(v_n)$ strongly converges to $f_1(v_{\mathrm{m}})$ in $L^1(Q_T)$. Since

$$w_n \to w \text{ in } L^\infty(Q_T) \text{ weak-star},$$

this allows to prove that

$$\lim_{n\to+\infty} \int_0^T \int_{\Omega_{\mathrm{H}}} \alpha\, I_{\mathrm{ion}}(v_n, w_n) h_k = \int_0^T \int_{\Omega_{\mathrm{H}}} \alpha\, I_{\mathrm{ion}}(v_{\mathrm{m}}, w) h_k.$$

Moreover, since $h_\infty(v_n) \to h_\infty(v_{\mathrm{m}})$ a.e. in Q_T and $\{h_\infty(v_n)\}$ is bounded in $L^\infty(Q_T)$, $\{h_\infty(v_n)\}$ strongly converges in $L^2(Q_T)$ to $h_\infty(v_{\mathrm{m}})$. Thus we can also pass to the limit in equation (5.25). This allows to obtain a weak solution of **P2** as defined by Definition 3.1.

6.4 Uniqueness of the weak solution

In this paragraph we prove the uniqueness of weak solution for problem **P1**, under the additional assumption **A3**. This is a direct consequence of the following *comparison Lemma.*

Lemma 6.2 *Asssume that assumption* **A3** *holds and that*

$$(v_{\mathrm{m},1}, u_{\mathrm{i},1}, u_1, w_1), \quad (v_{\mathrm{m},2}, u_{\mathrm{i},2}, u_2, , w_2),$$

are two weak solutions of problem **P1** *corresponding, respectively, to the initial data* $(v_{1,0}, w_{1,0})$ *and* $(v_{2,0}, w_{2,0})$*, and right-hand sides* $I_{\mathrm{app},1}$ *and* $I_{\mathrm{app},2}$*. For all* $t \in (0,T)$ *there holds*

$$\begin{aligned} &\|v_1(t) - v_2(t)\|^2_{L^2(\Omega_{\mathrm{H}})} + \|w_1(t) - w_2(t)\|^2_{L^2(\Omega_{\mathrm{H}})} \\ &\le \exp(K_1 t)\, K_2 \Big(\|v_{1,0} - v_{2,0}\|^2_{L^2(\Omega_{\mathrm{H}})} + |w_{1,0} - w_{2,0}|^2_{L^2(\Omega_{\mathrm{H}})} + \|I_{\mathrm{app},1} - I_{\mathrm{app},2}\|^2_{L^2(Q_t)} \Big), \end{aligned}$$

for all $t \in (0,T)$*, with* $K_1, K_2 > 0$ *positive constants depending only on* C_{m}*,* μ_0 *and* C_{ion}*.*

Proof. The proof follows the argument provided in [5] for the isolated bidomain equations. According to Definition 3.1, we have, for all $\phi_{\mathrm{i}} \in L^2(0,T;H^1(\Omega_{\mathrm{H}}))$,

$\psi \in L^2(0,T;V)$ and $\theta \in L^2(0,T;L^2(\Omega_{\mathrm{H}})$,

$$\begin{aligned}
&C_{\mathrm{m}} \int_0^t \int_{\Omega_{\mathrm{H}}} \partial_t (v_1 - v_2)\phi_{\mathrm{i}} + \int_0^t \int_{\Omega_{\mathrm{H}}} \boldsymbol{\sigma}_{\mathrm{i}} (\boldsymbol{\nabla} u_{\mathrm{i},1} - \boldsymbol{\nabla} u_{\mathrm{i},2}) \cdot \boldsymbol{\nabla} \phi_{\mathrm{i}} \\
&\qquad + \int_0^t \int_{\Omega_{\mathrm{H}}} (I_{\mathrm{ion}}(v_1, w_1) - I_{\mathrm{ion}}(v_2, w_2))\phi_{\mathrm{i}} = \int_0^t \int_{\Omega_{\mathrm{H}}} (I_{\mathrm{app},1} - I_{\mathrm{app},2})\phi_{\mathrm{i}}, \\
&C_{\mathrm{m}} \int_0^t \int_{\Omega_{\mathrm{H}}} \partial_t (v_1 - v_2)\psi - \int_0^t \int_{\Omega} \boldsymbol{\sigma} (\boldsymbol{\nabla} u_1 - \boldsymbol{\nabla} u_2) \cdot \boldsymbol{\nabla} \psi \\
&\qquad + \int_0^t \int_{\Omega_{\mathrm{H}}} (I_{\mathrm{ion}}(v_1, w_1) - I_{\mathrm{ion}}(v_2, w_2))\psi = \int_0^t \int_{\Omega_{\mathrm{H}}} (I_{\mathrm{app},1} - I_{\mathrm{app},2})\psi, \\
&\int_0^t \int_{\Omega_{\mathrm{H}}} \partial_t (w_1 - w_2)\theta + \int_0^t \int_{\Omega_{\mathrm{H}}} (g(v_1, w_1) - g(v_2, w_2))\theta = 0.
\end{aligned}$$

For $\mu > 0$, we take in this expresion $\phi_{\mathrm{i}} = \mu(u_{\mathrm{i},1} - u_{\mathrm{i},2})$, $\psi = -\mu(u_1 - u_2)$ and $\theta = w_1 - w_2$. Thus, by adding the resulting equalities, we have

$$\begin{aligned}
&\frac{\mu C_{\mathrm{m}}}{2} \|v_1(t) - v_2(t)\|^2_{L^2(\Omega_{\mathrm{H}})} + \frac{1}{2} \|w_1(t) - w_2(t)\|^2_{L^2(\Omega_{\mathrm{H}})} \\
&\qquad + \mu \left(\alpha_{\mathrm{i}} \|\boldsymbol{\nabla}(u_{\mathrm{i},1} - u_{\mathrm{i},2})\|^2_{L^2(Q_t)} + \alpha \|\boldsymbol{\nabla}(u_1 - u_2)\|^2_{L^2(\Omega\times(0,t))} \right) \\
&\qquad + \mu \int_0^t \int_{\Omega_{\mathrm{H}}} (I_{\mathrm{ion}}(v_1, w_1) - I_{\mathrm{ion}}(v_2, w_2))(v_1 - v_2) \\
&\qquad + \int_0^t \int_{\Omega_{\mathrm{H}}} (g(v_1, w_1) - g(v_2, w_2))(w_1 - w_2) \\
&\leq \frac{\mu C_{\mathrm{m}}}{2} \|v_{1,0} - v_{2,0}\|^2_{L^2(\Omega_{\mathrm{H}})} + \frac{1}{2} \|w_{1,0} - w_{2,0}\|^2_{L^2(\Omega_{\mathrm{H}})} \\
&\qquad + \frac{\mu^2}{2} \|I_{\mathrm{app},1} - I_{\mathrm{app},2}\|^2_{L^2(Q_t)} + \frac{1}{2} \|v_1 - v_2\|^2_{L^2(Q_t)}.
\end{aligned} \tag{6.49}$$

Let $\mu_0 > 0$ the parameter provided by assumption **A3**. We define

$$\begin{aligned}
\Phi(v_1, w_1, v_2, w_2) \stackrel{\mathrm{def}}{=} &\int_{\Omega_{\mathrm{H}}} \mu_0 \big(I_{\mathrm{ion}}(v_1, w_1) - I_{\mathrm{ion}}(v_2, w_2) \big)(v_1 - v_2) \\
&+ \int_{\Omega_{\mathrm{H}}} \big(g(v_1, w_1) - g(v_2, w_2) \big)(w_1 - w_2),
\end{aligned} \tag{6.50}$$

By denoting $z \stackrel{\mathrm{def}}{=} (v, w)$ and using (2), we have

$$\Phi(v_1, w_1, v_2, w_2) = \Phi(z_1, z_2) = \int_{\Omega_{\mathrm{H}}} \big(F_{\mu_0}(z_1) - F_{\mu_0}(z_2) \big) \cdot (z_1 - z_2).$$

Since F_{μ_0} is continuously differentiable, a Taylor expansion with integral remainder yields

$$F_{\mu_0}(z_1) - F_{\mu_0}(z_2) = \int_0^1 \boldsymbol{\nabla} F_{\mu_0} \big(\xi z_1 + (1 - \xi) z_2 \big) \cdot (z_1 - z_2) \mathrm{d}\xi, \quad \forall\, z_1, z_2 \in \mathbb{R}^2.$$

By inserting this expression in (6.50) and using the assumed spectral bound (2.17), there follows

$$\begin{aligned}\Phi(z_1,z_2) &= \int_0^1 \int_{\Omega_{\mathrm{H}}} (z_1-z_2)\cdot \boldsymbol{\nabla} F_{\mu_0}(\xi z_1 + (1-\xi)z_2)\cdot(z_1-z_2)\mathrm{d}\xi \\ &\geq C_{\mathrm{ion}} \int_0^1 \|z_1-z_2\|^2_{L^2(\Omega_{\mathrm{H}})}\mathrm{d}\xi \\ &= C_{\mathrm{ion}}(\|v_1-v_2\|^2_{L^2(\Omega_{\mathrm{H}})} + \|w_1-w_2\|^2_{L^2(\Omega_{\mathrm{H}})}).\end{aligned}$$

Therefore, from (6.49) with $\mu=\mu_0$, we have

$$\begin{aligned}&\frac{\mu_0 C_{\mathrm{m}}}{2}\|v_1(t)-v_2(t)\|^2_{L^2(\Omega_{\mathrm{H}})} + \frac{1}{2}\|w_1(t)-w_2(t)\|^2_{L^2(\Omega_{\mathrm{H}})} \\ &\leq \frac{\mu C_{\mathrm{m}}}{2}\|v_{1,0}-v_{2,0}\|^2_{L^2(\Omega_{\mathrm{H}})} + \frac{1}{2}\|w_{1,0}-w_{2,0}\|^2_{L^2(\Omega_{\mathrm{H}})} + \frac{\mu^2}{2}\|I_{\mathrm{app},1}-I_{\mathrm{app},2}\|^2_{L^2(Q_t)} \\ &\qquad + \left|\frac{1}{2}-C_{\mathrm{ion}}\right| \|v_1-v_2\|^2_{L^2(Q_t)} + |C_{\mathrm{ion}}|\|w_1-w_2\|^2_{L^2(Q_t)}.\end{aligned} \tag{6.51}$$

We conclude the proof by using Gronwall Lemma. □

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