Helly-type Theorems for Line transversals to Disjoint Unit Balls (Extended abstract)

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Abstract

We prove Helly-type theorems for line transversals to disjoint unit balls in $\mathbb{R}^d$. In particular, we show that a family of $n \geqslant 2d$ disjoint unit balls in $\mathbb{R}^d$ has a line transversal if, for some ordering $\prec$ of the balls, every subfamily of $2d$ balls admits a line transversal consistent with $\prec$. We also prove that a family of $n \geqslant 4d-1$ disjoint unit balls in $\mathbb{R}^d$ admits a line transversal if every subfamily of size $4d-1$ admits a transversal.

Helly's celebrated theorem, published in 1923, states that a finite family of convex sets in $\mathbb{R}^d$ has non-empty intersection if and only if every subfamily of size at most $d+1$ has non-empty intersection. Subsequent results of similar flavor (that is, if every subset of size k of a set $\mathcal{S}$ has property $\mathcal{P}$ then $\mathcal{S}$ has property $\mathcal{P}$) have been called *Helly-type theorems* and the minimal such k is known as the associated *Helly number*. Helly-type theorems and tight bounds on Helly numbers have been the object of active research in combinatorial geometry. In this paper, we investigate Helly-type theorems for the existence of line transversals to a family of objects, i.e. lines that intersect every member of the family.

History. The earliest Helly-type theorems in geometric transversal theory appeared about five decades ago. In 1957, Hadwiger [10] showed that an ordered family $\mathcal{S}$ of compact convex figures in the plane admits a line transversal if every triple admits a line transversal compatible with the ordering. (Note that a line transversal to $\mathcal{S}$ may not respect the ordering on $\mathcal{S}$; to prove the existence of a line transversal that respects the ordering on $\mathcal{S}$ one needs the assumption that any *four* admits an order-respecting line transversal.) In what follows, we shall talk about a Hadwiger-type theorem when the family of objects under consideration is ordered.

The same year, L. Danzer [4] proved the following result concerning families of pairwise disjoint unit discs in the plane: if such a family consists of at least 5 discs, and if any 5 of these discs are met by some line, then there exists a line meeting all the discs of the family. This answered a question of Hadwiger [8], who gave an example (5 circles, almost touching and with centers forming a regular pentagon) which shows that 5 cannot be replaced by 4. Grünbaum [6] showed that the same result holds if "unit disc" is replaced by "unit square", and conjectured that the result holds for families of disjoint translates of any compact convex set in the plane. This long-standing conjecture was finally proved by Tverberg [14]. A weaker form of the conjecture which assumed 128 instead of 5 had been established earlier by Katchalski [13].

In three dimensions, neither Hadwiger nor Helly-type theorems exist for line transversals to general convex objects, not even for translates of a convex compact set [12]. However, Hadwiger [9] proved a Helly-type theorem for line transversals to "thinly distributed" disjoint balls in dimension d with Helly number d^2. A family of balls is thinly distributed if the distance between any two balls is at least the sum of their radii. Grünbaum [7] improved this Helly number to $2d-1$ using the topological Helly theorem. For the special case of unit balls in three dimensions—but without any additional assumption on their distribution—Holmsen et al. [11] showed a Hadwiger-type theorem with constant 12, and a Helly-type theorem with constant 46. These constants were later improved to 9 and 18 by Cheong et al. [3].

We refer the reader to the recent survey by Wenger [15] for a broader discussion of geometric transversal theory.

Our results. In this paper we prove Helly-type theorems for line transversals to families of *pairwise-inflatable* balls in $\mathbb{R}^d$. A family $\mathcal{F}$ of balls in $\mathbb{R}^d$ is called pairwise-inflatable if for every pair of balls $B_1, B_2 \in \mathcal{F}$ we have $\gamma^2 > 2(r_1^2 + r_2^2)$, where r_i is the radius of B_i, and γ is the distance between their centers. A family of disjoint unit balls is pairwise-inflatable, and so is a family of balls that is "thinly distributed" in Hadwiger's sense. Pairwise-inflatable families of balls are not only more general than families of disjoint congruent balls but allow to generalize most of our proofs obtained in three or four dimensions to arbitrary dimension; the key property, which we prove in this paper, is that the set of pairwise-inflatable families is closed under intersection with

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affine subspaces, unlike the set of families of disjoint congruent balls.

An order-respecting line transversal to a subset of an ordered family is a line transversal that respects the order induced by the family on that subset. An ordered family $\mathcal{F}$ of pairwise-inflatable balls is said to have property $(OR)T$ if it admits a (order-respecting) line transversal. If every k or fewer members of $\mathcal{F}$ admit a (order-respecting) line transversal then $\mathcal{F}$ is said to have property $(OR)T(k)$. Our first main result requires that the line transversals to the subfamilies induce consistent orderings:

Proposition 1 *For any ordered family of pairwise-inflatable balls in $\mathbb{R}^d$, $ORT(2d)$ implies T and $ORT(2d+1)$ implies ORT.*

We then remove the condition on the ordering at the cost of increasing the Helly number to $4d-1$ and restricting ourselves to disjoint unit balls:

Proposition 2 *For any family of disjoint unit balls in $\mathbb{R}^d$, $T(4d-1)$ implies T.*

Our results are thus both qualitative and quantitative: we generalize Danzer's result to arbitrary dimension and prove that the Helly number grows at most linearly with the dimension. We build on the work of Holmsen et al. [11] who obtained results similar to Propositions 1 and 2 for disjoint unit balls in three dimensions, albeit with larger bounds on Helly numbers (12 and 46 instead of 6 and 11, respectively). A previous version of this paper, also restricted to disjoint unit balls in three dimensions, appeared in the Symposium on Computational Geometry 2005 [1].

Approach. To prove Proposition 1, we start with a family of balls having property $ORT(2d)$ and continuously shrink them until that property no longer holds, following Hadwiger's approach [10]. Before the set of order-respecting line transversals to a $2d$-tuple of balls disappears (i) it first reduces to a single line and (ii) this line is an isolated line transversal to $2d-1$ of the balls. That line has then to be a line transversal to the whole family and Proposition 1 follows; considerations on geometric permutations yield Proposition 2.

Proving the properties (i) and (ii) mentioned above is elementary in the plane but requires considerably more work in higher dimension. For a sequence $\mathcal{F}$, let $\mathcal{K}(\mathcal{F}) \subset \mathbb{S}^{d-1}$ denote the set of directions of line transversals to $\mathcal{F}$. Our proofs rely on the following proposition:

Proposition 3 *The directions of order-respecting line transversals to a family of pairwise-inflatable balls in $\mathbb{R}^d$ form a strictly convex subset of $\mathbb{S}^{d-1}$.*

This directly implies property (i) and yields that order-respecting line transversals form a contractible set in line space. From there, a well-known topological analogue of Helly's theorem leads to a weaker version of Proposition 1 sufficient to prove property (ii), namely:

Proposition 4 *If a line ℓ is an isolated line transversal to a sequence $\mathcal{F}$ of $n \geqslant 2d$ pairwise-inflatable balls in $\mathbb{R}^d$ then there exists a subsequence $\mathcal{F}' \subset \mathcal{F}$ of size $2d-1$ such that ℓ is an isolated line transversal to $\mathcal{F}'$.*

For the proofs, omitted in this extended abstract, we refer the reader to the full version [2].

Open problems. We conclude by a few open problems suggested by our results.

Problem 1 *What is the maximum number of geometric permutations of pairwise-inflatable balls in $\mathbb{R}^d$?*

A geometric permutation of a collection of disjoint convex sets is an ordering of these sets that can be realized by a line transversal. To prove Proposition 2 we use the fact that the number of geometric permutations of n disjoint balls in $\mathbb{R}^d$ is at most 3 if the balls have equal radii [3]. If the ratio

$$\frac{\text{largest radius}}{\text{smallest radius}}$$

is not bounded independently of n then the number of geometric permutations is known to be $\Theta(n^{d-1})$ [16].

Problem 2 *For which classes of objects is the cone of directions $\mathcal{K}(A_1, \ldots, A_n)$ convex, or at least contractible?*

Our proof of convexity for the cone of directions of balls collapses for balls that are not pairwise-inflatable. In fact, the set Q_{AB}^F is not necessarily convex if B is much smaller than A but very close to it.

Problem 3 *For which classes of objects is the set of order-respecting line transversals always connected?*

Our proof of Proposition 1 follows from (i) a bounded pinning number and (ii) the fact that as the set of order-respecting line transversals to a sequence disappears it first reduces to a single line. For strictly convex objects, property (ii) follows from the connectivity of the set of order-respecting transversals. Surprisingly, it is an open question whether this set is connected for even 4 disjoint balls in $\mathbb{R}^3$, whereas it is known to be connected for any triple of disjoint convex objects [5, Lemma 74]. We conjecture that general convex sets in $\mathbb{R}^d$ have a bounded pinning number. Thus, understanding how general this connectivity property is would provide insight in how general the example of Holmsen and Matousek [12], convex sets whose translates do not admit a Hadwiger theorem, actually is. Of course, a positive answer to Problem 2 for a particular family of convex sets implies a positive answer to Problem 3 for that family as well.

Problem 4 *Is the pinning number of disjoint unit balls in $\mathbb{R}^d$ equal to $2d-1$?*

Surprisingly, the only known lower bound on the Helly number is the construction done by Hadwiger fifty years ago. Note that the bound in our Hadwiger theorem has to be higher than the pinning number of the corresponding family and one can therefore look for a lower bound on the pinning number. Intuitively, considerations on the dimension suggest that the pinning number in dimension d cannot be less than $2d-1$, the dimension of the underlying line space being $2d-2$.

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