Realistic wireless network model with explicit capacity evaluation

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Abstract: We consider a realistic model of wireless network where nodes are dispatched in an infinite map with uniform distribution. Signal decays with distance according to attenuation factor α. At any time we assume that the distribution of emitters is λ per square unit area. From the explicit formula of the laplace transform of received signal we derive the explicit formula for the information rate received by a random node which is $\frac{\alpha}{2}(\log 2)^{-1}$ per Hertz. We generalize to any-dimension network maps.

Key-words: wireless, networks, information theory, complex analysis, MIMO, fractal object

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Modèle réaliste de réseaux sans fil donnant lieu à des formules explicites de capacité

Résumé : Nous considérons un modèle réaliste de réseau sans fil où les nœuds sont distribués sur une carte infinie avec une distribution uniforme. Les signaux décroissent avec la distance avec un facteur d'atténuation α. A tout moment nous supposons que la distribution des emetteurs instantanés est de λ par carré unité. De l'expression explicite de la transformée de Laplace de la distribution du signal reçu nous dérivons une formule explicite du débit d'information reçue par un nœud arbitraire qui est $\frac{\alpha}{2}(\log 2)^{-1}$. Nous généralisons pour des cartes dans des espaces vectoriels de dimension quelconque.

Mots-clés : sans fil, réseau, théorie de l'information, analyse complexe, MIMO, objet fractal

1 introduction

Wireless networks are expected to be deployed extensively in densely populated areas, urban or semi-urban areas. The question is how the wireless networks can fit the increasing demand in capacity that is expected in the future. In [2] it is shown that the per node information rate is finite whatever the network density. However this is done under restrictive hypotheses where a node can only receive from a single neighbor node at a time. The problem is that a correct assessment of the capacity of a wireless network in its most general definition is at the crossing of

- physics for the wave propagation and attenuation in medium;
- geometry for the positioning of the nodes
- IT for the extraction of information from signal

This paper addresses the analytical evaluation of the wireless network capacity in a realistic model which is surprisingly tractable. This model involves the three aspects above mentionned. Attenuation is function of distance in $\frac{1}{r^\alpha}$ and of random fadings, nodes are randomly dispatched and have any given nominal power, signals superpose, information is extracted in parallel flows as in Multiple-Input-Multiple-Output (MIMO) technology, nodes transmit independent information flows. The model was primarily introduced in [1], partially developped in [3, 4].

Our main finding is that in the absence of noisy sources, and in the presence of any fading and nominal power distribution, each node can receive an average information rate in bit per Hertz:

$$I(\alpha) = \frac{\alpha}{D} \times \frac{1}{\log 2} \tag{1}$$

where α is the attenuation coefficient, D is the dimension of network map (for instance $D = 2$ for a planar map). The formula is remarkable in the sense that it is the simplest formula that collects the three main aspects of the problem: α is for the physics of the wireless communication, D is for the geometry the network map, and $\log 2$ is for information theory.

The paper is organized as follow. In a next section we present the general model and our main results in the framework of two dimensional network maps. We introduce general fading but we restrict to uniform unit nomial power. In a next section we generalize to general nominal power and show that it does not change information rate estimate. We also investigate the case where attenuation factors are not uniform, but this does not lead to closed formula for information rate). The section after is devoted to noisy conditions and in particular we analyse the case where noise comes from noisy sources randomly dispatched. This leads to a corrected estimate of information rate. Finally a last section generalizes these results to the general dimension D network maps. We set a conjecture on fractal maps.

2 Model presentation and main results

The model is an infinite plan with nodes randomly distributed. At each time the nodes which simultaneously transmit is given by a uniform Poisson distribution

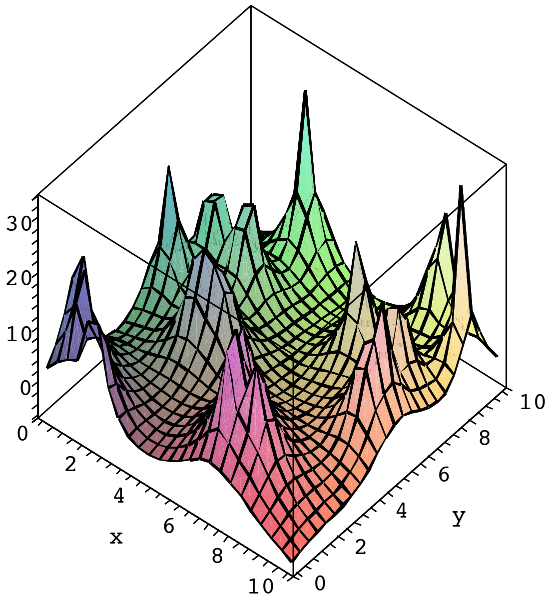

Figure 1: Signal levels landscape (in dB) for a random network with $\alpha = 2.5$

of mean λ transmitter per square area unit. Figure 3 displays an example of 400 nodes on a 1×1 map shunk.

We assume that the attenuation coefficient is $\alpha > 2$, for example $\alpha = 4$: the signal level of a transmission received at distance r is $W = \frac{\exp(F)}{r^\alpha}$ where F is a random fading of mean 0. The question is to give an estimate of the probability that a signal emitted by a transmitter is received by a given receiver with an signal-to-noise ratio (SNR) at least equal to a given value K. By noise, we mean the sum of the signal of the other messages transmitted simultaneously. We will discuss noisy conditions in a dedicated section.

It is to say that if $\mathcal{S}$ is the set of the locations z_i of the nodes transmitting during this slot, and $z_0 \in \mathcal{S}$ is the transmitter:

$$e^{F_0(z)}|z - z_0|^{-\alpha} > K \sum_{z_i \in \mathcal{S} - \{z_0\}} e^{F_i(z)}|z - z_i|^{-\alpha}$$

where the $F_i(z)$ are the respective fading experienced at point z of the messages transmitted by the elements of $\mathcal{S}$. or $W(z, \{z_0\}) > KW(z, \mathcal{S} - \{z_0\})$ where $W(z, \mathcal{S}) = \sum_{z_i \in \mathcal{S}} e^{F_i(z)}|z - z_i|^{-\alpha}$. Figure 1 shows the function $W(z, \mathcal{S})$ for z varying in the plan with $\mathcal{S}$ an arbitrary random set of transmitter. We take $\alpha = 2.5$. It is clear that the closer the receiver is to the emitter then the larger is the SNR. Figure 2 shows reception areas for various value of K. Notice the areas never overlap when $K > 1$ since there is alway only one dominant signal. For each value of K we can draw around each emitter the area where the signal is received with SNR greater or equal to K. The aim is to find the average size of this area and how it is function of parameters K and λ. This is the aim of the next section. Figure 1 displays areas of reception around transmitter for $K = 1, 4, 10$ and $\alpha = 2.5$.

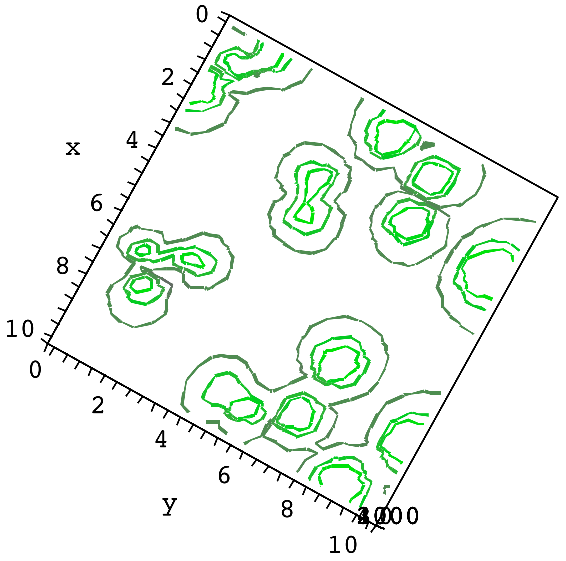

Figure 2: distribution of reception areas for various SNR parameters $K = 1, 4, 10$ for the situation of figure 1.

2.1 Distribution of signal levels

We know [3] and [1] that the Laplace transform of the signal level $W(z, \mathcal{S}(\lambda))$ (assuming all transmitters tuned at one unit nominal power) can be exactly calculated when $\mathcal{S}(\lambda)$ is given by a 2D Poisson process with intensity of λ transmitter per slot and per square area unit. The random variable $W(z, \mathcal{S}(\lambda))$ is invariant by translation and does not depend on z. We denote $W(\lambda) \equiv W(z, \lambda)$.

Theorem 1. *The Laplace transform* $\tilde{w}(\theta, \lambda) = E(e^{-W(\lambda)\theta})$*:*

$$\tilde{w}(\theta, \lambda) = \exp(-\lambda\pi\Gamma(1 - \frac{2}{\alpha})E(e^{\frac{2}{\alpha}F})\theta^{\frac{2}{\alpha}}) \tag{2}$$

Proof. If we split the map in small sub-areas of size $dx \times dy$, the contribution of every sub-areas are independent. The Laplace transform is equal to the product of the Laplace transform of the contribution of each sub-areas. The contribution of a sub-areas at distance r of z is

$$1 - \lambda dxdy + \lambda dxdy \int \phi(u) e^{-\theta r^{-alpha} e^u} du$$

with $\phi(u)$ the density probability of fading F. Therefore we have

$$\begin{array}{rcl}\log\tilde{w}(\theta,\lambda) & = & (\int\int\int(e^{-\theta r^{-\alpha}e^u}-1)\phi(u)du\lambda dxdy \\ & = & 2\pi\lambda\int\int rdr(e^{-\theta r^{-\alpha}e^u}-1)\phi(u)du \\ & = & \frac{1}{\alpha}2\pi\lambda\int\int x^{-\frac{2}{\alpha}-1}((e^{-\theta xe^u}-1)dx\phi(u)du \\ & = & \frac{2}{\alpha}\pi\lambda\Gamma(-\frac{2}{\alpha})\theta^{\frac{2}{\alpha}}\int e^{\frac{2}{\alpha}u}du \\ & = & -\pi\lambda\Gamma(1-\frac{2}{\alpha})\theta^{\frac{2}{\alpha}}E(e^{\frac{2}{\alpha}F})\end{array}$$

□

Remark: In all case $\tilde{w}(\theta,\lambda)$ is of the form $\exp(-\lambda C\theta^{\frac{2}{\alpha}})$.

From now we denote $\gamma=\frac{2}{\alpha}$. From the above formula, we can deduce the probability $w(x)=P(W(\lambda)<x)$. In [3] we show the expansion in the following theorem.

Theorem 2. *Asuming* $E(e^{-\theta W(\lambda)})=\tilde{w}(\theta,\lambda)=e^{-\lambda C\theta^{\gamma}}$ *we get*

$$P(W(\lambda)<x)=\sum_{n\geq 0}(-C\lambda)^n\frac{\sin(\pi n\gamma)}{\pi}\frac{\Gamma(n\gamma)}{n!}x^{-n\gamma}.$$

Proof. Without loss of generality we set $\lambda=1$. Denoting $\tilde{w}(\theta)=\tilde{w}(1,\theta)$, by application of the reverse Laplace transformation we get:

$$P(W(1)<x)=\frac{1}{2i\pi}\int_{-i\infty}^{+i\infty}\frac{\tilde{w}(\theta)}{\theta}e^{\theta x}d\theta$$

Expanding $\tilde{w}(\theta)=\sum_{n\geq 0}\frac{(-C)^n}{n!}\theta^{n\gamma}$, it comes

$$P(W(1)<x)=\frac{1}{2i\pi}\sum_n\frac{(-C)^n}{n!}\int_{-i\infty}^{+i\infty}\theta^{n\gamma-1}e^{\theta x}d\theta$$

Then by bending the integration path toward the negative axis we get for each term in the right handside of the above:

$$\begin{array}{rcl}\frac{1}{2i\pi}\int_{-i\infty}^{+i\infty}\theta^{n\gamma-1}e^{\theta x}d\theta & = & \frac{e^{i\pi n\gamma}-e^{-i\pi n\gamma}}{2i\pi} \\ & & \times\int_0^{\infty}\theta^{n\gamma-1}e^{-\theta x}d\theta \\ & = & \frac{\sin(\pi n\gamma)}{\pi}\Gamma(n\gamma)x^{-n\gamma}\ ,\end{array}$$

with the convention that $\frac{\sin(\pi n\gamma)}{\pi}\Gamma(n\gamma)=1$ when $n=0$. □

Remark The distribution of signal levels has a Parreto tail since $P(W>x)=O(x^{-\gamma})$ when $x\to\infty$. It means that the received signal level does not have a mean.

For completeness we shall investigate small values of x and we also have the theorem

Theorem 3. *When $x \to 0$ we have, assuming $\tilde{w}(\lambda, \theta) = \exp(-\lambda C\theta^\gamma)$:*

$$P(W(1) < x) = \frac{\exp(-c' x^{\frac{\gamma}{1-\gamma}})}{\sqrt{2\pi\theta(x)^\gamma(1-\gamma)\gamma}}(1 + O(\sqrt{x^{\frac{\gamma}{1-\gamma}}}))$$

with $c' = \frac{\frac{1}{\gamma}-1}{(C\gamma)^{\frac{1}{\gamma-1}}}$ *and* $\theta(x) = \left(\frac{x}{C\gamma}\right)^{\frac{1}{\gamma-1}}$.

This theorem is not central to our analysis and the proof is deferred in appendix.

2.2 Reception areas

Let $p(r, K, \lambda)$ be the probability to receive a signal sent at distance r with SNR at least equal to K: We have $p(r, K) = \int \phi(u) P\left(W(\lambda) < r^{-\alpha}\frac{e^u}{K}\right) du$. We define $\sigma(\lambda, K) = 2\pi \int p(r, K) r dr$ as the average size of the reception area with SNR at least equal to K around an arbitray transmitter.

Theorem 4. *the average size of the reception area with SNR at least equal to K around an arbitray transmitter satisfies the identity*

$$\sigma(\lambda, K) = \frac{1}{\lambda}\frac{\sin(\frac{2}{\alpha}\pi)}{\frac{2}{\alpha}\pi}K^{-\frac{2}{\alpha}}.$$

Remark: The fading distribution has no impact on reception area average size.

Proof. First of all we have $p(r, \lambda, K) = p(r\sqrt{\lambda}, 1, K)$ for obvious homothetic invariant. Therefore $\sigma(\lambda, K) = \frac{1}{\lambda}\sigma(1, K)$. Let $\sigma_1(K) = 2\pi \int P(W(1) < r^{-\alpha}K^{-1}) r dr$, assuming $\tilde{w}(\theta, 1) = \exp(-C\theta^\gamma)$ we have shown in [refxy] via integration by part that

$$\sigma_1(K) = \pi\alpha \int_0^\infty P(W = \frac{1}{Kr^\alpha})\frac{r}{Kr^\alpha} dr$$

We use the fact that $P(W = x) = \frac{1}{2i\pi}\int_C w(\theta) e^{\theta x} d\theta$, where C is an integration path in the definition domain of $w(\theta)$, *i.e.* with $Re(\theta) > 0$ parallel to the imaginary axis. We get by changing variable $x = (Kr^\alpha)^{-1}$ and inverting integrations.

$$\begin{aligned} \sigma_1(K) &= \frac{1}{2i}\int_C w(\theta) \int_0^\infty e^{\theta x}(Kx)^{-\gamma} dx \\ &= \frac{1}{2i}K^{-\gamma}\Gamma(1-\gamma)\int_C w(\theta)(-\theta)^{\gamma-1} d\theta \end{aligned}$$

We use $\tilde{w}(\theta, 1) = \exp(-C\theta^\gamma)$ and now deforming the integration path to stick to the negative axis:

$$\begin{aligned} \sigma_1(K) &= \frac{e^{i\pi\gamma} - e^{-i\pi\gamma}}{2i}K^{-\gamma}\Gamma(1-\gamma) \\ &\quad \times \int_0^\infty \exp(-C\theta^\gamma)\theta^{\gamma-1} d\theta \\ &= \sin(\pi\gamma)K^{-\gamma}\frac{\Gamma(1-\gamma)}{C\gamma} \end{aligned}$$

Using the expression of C: $\sigma_1(K) = \frac{\sin(\pi\gamma)}{\pi\gamma} K^{-\gamma} \frac{1}{E(e^{\gamma F})}$.

Therefore

$$\begin{aligned}
\sigma(\lambda, K) &= \frac{1}{\lambda} \int \sigma_1(Ke^{-u}) \phi(u) du \\
&= \frac{1}{\lambda E(e^{\gamma F})} \frac{\sin(\gamma\pi)}{\gamma\pi} \int K^{-\gamma} e^{\gamma u} \phi(u) du \\
&= \frac{1}{\lambda} \frac{\sin(\gamma\pi)}{\gamma\pi} K^{-\gamma}
\end{aligned}$$

□

Remark This result brings several remarks. First of all we notice that when $\alpha \to \infty$ then $\sigma(\lambda, K) \to \frac{1}{\lambda}$. This is due to the fact that when α is large the closest transmitter gives the far largest estimate, and consequently the area of reception turns to be the Voronoi cell around each transmitter, and this whatever K. The average size of the Voronoi cell being equal to the inverse density of the transmitter, say $\frac{1}{\lambda}$, we get the asymptotic result. Notice that when K grows as $\exp(O(\alpha))$ we have $\sigma(\lambda.K) \sim \frac{1}{\lambda} \exp(-\frac{2}{\alpha} \log(K))$ which suggest that the typical SNR when $\alpha \to \infty$ is of order $\exp(O(\alpha))$

On the other side when $\lambda \to 2$, we have $\sigma(\lambda, K) \to 0$ because $\sin(\frac{2}{\alpha}\pi) \to 0$. Indeed, the contribution of remote nodes tends to diverge and makes the SNR to tend to zero. This explains why $\sigma(\lambda, K) \to 0$ for any fixed value of K. Notice that when $K = O(\alpha - 2)$, then $\sigma(\lambda, K) \sim \frac{1}{\lambda} \frac{\alpha-2}{K}$ suggesting that the typical SNR is $O(alpha - 2)$.

2.3 Consequence on wireless capacity

Now we can attack the central point of our paper, that is the evaluation of the maximum information rate that can receive an arbitrary node in the network. Information flows from simultaneous transmitter are independent. We make use of Shannon law: the maximum capacity rate of an information flow receive on a SNR of K over a bandwidth f is $f \log(1 + K)$. If W_i is the signal received from node $z_i \in \mathcal{S}$, the total energy received is $W = \sum_{z_i \in \mathcal{S}} W_i$ and the total information rate received The quantity of information received by a node at location z is $\sum_{z_i \in \mathcal{S}} f \log(\frac{W}{W - W_i})$. In the sequel we want to estimate the number of bits per Hertz, therefore we assume $f = 1$ and logarithm in base 2.

Theorem 5. *The number of bit per Hertz received by an arbitrary node is in average* $\frac{\alpha}{2}(\log 2)^{-1}$

Proof. For a given transmitter the average area where its signal is received with a SNR above K is $\sigma(\lambda, K) = \frac{1}{\lambda} \frac{\sin(\pi\gamma)}{\pi\gamma} K^{-\gamma}$. Therefore the average density of node which receives signal with SNR greater than K is $\lambda\sigma(\lambda, K) = \sigma(1, K)$. Consequently the total information rate received by an arbitrary node is $I(\alpha) = \int_0^\infty \log_2(1 + K) \frac{-\partial}{\partial K} \sigma(1, K) dK$. We have by integrating by part and using the fact that $\int_0^\infty \frac{x^{-s}}{1+x} dx = \frac{\pi}{\sin(\pi s)}$

$$\begin{aligned} I(\alpha) &= \frac{1}{\log 2}\int_0^\infty \frac{1}{1+K}\sigma(1,K)dK \\ &= \frac{\sin(\pi\gamma)}{\pi\gamma\log 2}\int_0^\infty \frac{1}{1+K}K^{-\gamma}dK \\ &= \frac{1}{\gamma\log 2} \end{aligned}$$

□

Remark we notice that $I(\alpha) \to \frac{1}{\log 2} > 0$ when $\alpha \to 2$. This may look surprising since when $\alpha \to 2$ we know the individual SNR's tend to zero and we would expect the total capcity to collapse. In fact the instantaneous information rate in presence of a set $\mathcal{S}$ of simultaneous transmitters is

$$\begin{aligned} I &= \sum_{z_i\in\mathcal{S}} \log_2(\frac{W}{W-W_i}) \\ &= \sum_{z_i\in\mathcal{S}} -\log_2(1-\frac{W_i}{W}) \end{aligned}$$

Since the W_i are expected to be small in front of W, that is $-\log_2(1-\frac{W_i}{W}) \approx \frac{1}{\log 2}\frac{W_i}{W}$, we get $I \approx \sum_{z_i\in\mathcal{S}}\frac{1}{\log 2}\frac{W_i}{W} = \frac{1}{\log 2}$.

We also notice that $I(\alpha) \to \infty$ when $\alpha \to \infty$, but this is a direct consequence of the fact that the closest transmitter is received with a SNR that tends to infinity when $\alpha \to \infty$.

3 Variable power, variable attenuation

We can easily extend the above model to the case where the nominal power of the transmitters differ. Let call Q the nominal power of a transmitter; following a similar reasoning as with random fading we get

$$\tilde{w}(\theta,\lambda) = \exp\left(-\lambda\pi E(Q^\gamma)E(e^{\gamma F})\Gamma(1-\gamma)\theta^\gamma\right)$$

Let $\sigma(\lambda, K, Q)$ be the average size of the area of reception with SNR above K for a transmitter with nominal power Q, we also get

$$\sigma(\lambda,K,Q) = \frac{1}{\lambda E(Q^\gamma)}\frac{\sin(\gamma\pi)}{\gamma\pi}\left(\frac{Q}{K}\right)^\gamma$$

Summing contributions from all transmitters in the expression of $I(\alpha)$, the Q^γ's disappear and $I(\alpha)$ value remains unchanged at $\frac{1}{\gamma\log 2}$.

We also consider the case where the attenuation factor α depends on the transmitter (assume an airborne radio versus a ground device). In this case we still have a close formula

$$\tilde{w}(\lambda,\theta) = \exp\left(-\lambda\pi E(\Gamma(1-\gamma)\theta^\gamma)\right)$$

but to our best knowledge it does not provide a close formula for capacity I, since α is varying. Anyhow a numerical analysis is possible since all functions are precisely determined.

4 External noise sources

In this section we investigate the introduction of noise in our analysis. Noise was omitter in the previous section. A random noise N would affect quantity $\tilde{w}(\lambda,\theta)$ by a multicative factor $E(e^{\theta N})$. A close formula for (α) may not be possible in general but numerical analysis is tractable.

Anyhow if we consider that noise is made of external source such as microwave-owens or from other sources, artifical or naturals, then the analysis gives tractable closed formulas. Indeed

$$E(e^{\theta N}) = \exp\left(-\lambda_N E(Q_N^\gamma)\pi\Gamma(1-\gamma)\theta^\gamma\right)$$

where λ_N is the density of externam sources and Q_N their power. In this case

$$\sigma(\lambda,K,Q) = \frac{1}{\lambda_N E(Q_N^\gamma) + \lambda E(Q^\gamma)} \frac{\sin(\gamma\pi)}{\gamma\pi}\left(\frac{Q}{K}\right)^\gamma$$

Computing $I(\alpha)$ provides

$$I(\alpha) = \frac{\lambda E(Q^\gamma)}{\lambda_N E(Q_N^\gamma) + \lambda E(Q^\gamma)} \frac{1}{\gamma \log 2}$$

5 Other dimensions and fractal map conjecture

The previous sections was restricted to plain network maps of dimension 2. Now we consider map of dimension 1 (road networks), or dimension 3 (airborne, submarine or space network), or more generaly to diemsion D. Dimension 4 couls be made by the conjonction of a space network of dimension 3 and a frequency plane for radio devices.

Adapting the reasoning used for dimension 2, we get the signal level Laplace transform in dimension 1 $\tilde{w}_1(\theta,\lambda) = \exp(-\lambda\Gamma(1-\frac{1}{\alpha})\theta^{\frac{1}{\alpha}})$ under the condition that $\alpha > 1$. In dimension 3 we have $\tilde{w}_3(\theta,\lambda) = \exp(-\lambda\frac{4}{3}\pi\Gamma(1-\frac{3}{\alpha})\theta^{\frac{3}{\alpha}})$ under the condition that $\alpha > 3$. More generally we have for dimension D $\tilde{w}_D(\theta,\lambda) = \exp(-\lambda V_D\Gamma(1-\frac{D}{\alpha})\theta^{\frac{D}{\alpha}})$ under the condition that $\alpha > D$. Quantity V_D denotes the volume of the hyper-sphere of radius one in dimension D, we have $V_D = \frac{\pi^{D/2}}{D\Gamma(D/2)}$.

We also get $\sigma(\lambda,K) = \frac{\sin(\gamma\pi)}{\gamma\pi}K^{-\gamma}$ with $\gamma = \frac{D}{\alpha}$. It turns out that the information rate is unchanged $I_D(\alpha) = \frac{\alpha}{D\log 2}$.

There are set whose dimension is not an integer number, that are called *fractal* objects [ref mandelbrot] or self-similar sets. It is very tempting to generalize the above identity to such object where D is no longer an integer. Our conjecture is that the information rate identity generalizes to fractal or self-similar networks, *i.e.* where the network map is supported by a fractal set. Figure 5 displays an example of such emitter distribution of the Cantor family: a fractal tartan of dimension $\frac{4}{3}$. The fractal tartan is of the form $K \times K$ where K is a Cantor set of dimension $\frac{2}{3}$. We define $K = \frac{1}{8}K + \{0, \frac{7}{24}, \frac{7}{12}, \frac{7}{8}\}$, so that multiplying K by 8 increases the area of K by 4, leading to a dimension $\frac{\log 4}{\log 8} = \frac{2}{3}$. Figure 4 displays the set K.

The following table summarizes the simulation we have done so far. We have uniformly dispatched 400 transmitters and 400 receivers on a unit map

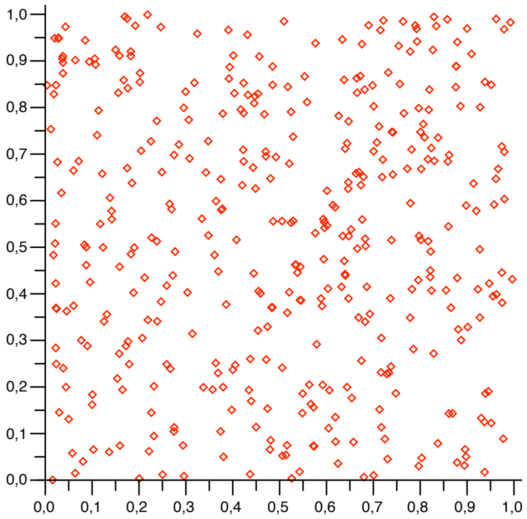

Figure 3: uniform location distribution of 400 emitters on a 1×1 square map.

and computed the information rate on any of the 400 receivers. We display the average value and compare to the theoretical value for infinite map, for various map dimensions D and attenuation coefficients α. We don't simulate fading. The factor $\log 2$ has been skipped for convenience

	$\alpha = 2$	$\alpha = 2.5$	$\alpha = 3$	$\alpha = 4$
$D = 1$	2.140	2.642	3.251	4.129
$D = \frac{4}{3}$	1.481	1.831	2.201	2.959
$D = 2$	1.157	1.303	1.562	2.049
$D = 3$	1.02	1.066	1.167	1.418

Notice the value obtained for $\alpha < D$ are equal to 1 because the simulated finite map converges when the theoretical infinite map diverges. The following table displayes the theoretical values $\frac{\alpha}{D}$:

	$\alpha = 2$	$\alpha = 2.5$	$\alpha = 3$	$\alpha = 4$
$D = 1$	2	2.5	3	4
$D = \frac{4}{3}$	1.5	1.875	2.25	3
$D = 2$	1	1.25	1.5	2
$D = 3$	-	-	1	1.333

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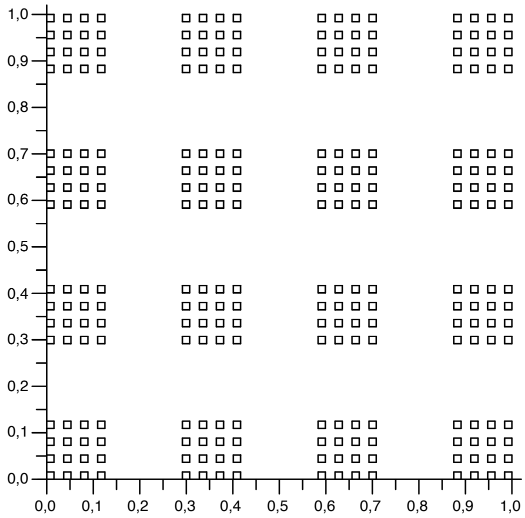

Figure 4: Fractal map of dimension $\frac{4}{3}$.

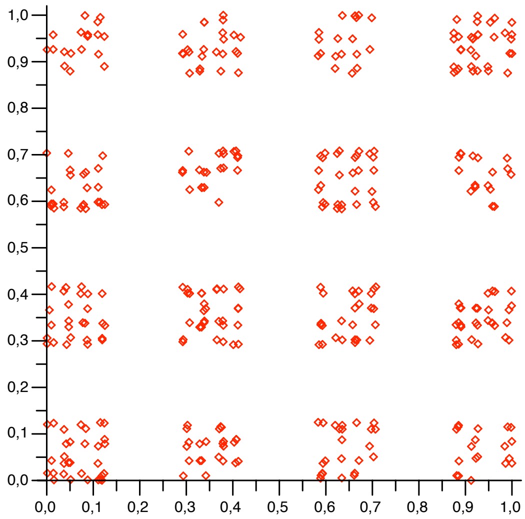

Figure 5: uniform location distribution of 400 emitters on a 1×1 fractal map of dimension $\frac{4}{3}$.

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Appendix

5.1 Proof of theorem 3

Proof. We still have

$$P(W(1) < x) = \frac{1}{2i\pi} \int_{\theta_0 - i\infty}^{\theta_0 + i\infty} \frac{\tilde{w}(\theta)}{\theta} e^{\theta x} d\theta$$

which is true for all $\theta_0 \geq 0$. Or equivalently

$$P(W(1) < x) = \frac{1}{2i\pi} \int_{\theta_0 - i\infty}^{\theta_0 + i\infty} \exp(-C\theta^\gamma + \theta x) \frac{d\theta}{\theta}$$

The function in the exponential attains its maximum value on the positive real axis at $\theta(x) = \left(\frac{x}{C\gamma}\right)^{\frac{1}{\gamma-1}}$ and the maximum value of the integrand is $\exp(C' x^{\frac{\gamma}{1-\gamma}})$. If we set $\theta_0 = \theta(x)$, then we have a Saddle point.

Using second order estimate:

$$\begin{aligned} -(\theta(x) + it)^\gamma C + (\theta(x) + it)x &= -C\theta(x)^\gamma \\ &\quad -\frac{1}{2}(1-\gamma)\gamma\theta(x)^{\gamma-2} \\ &\quad +O(\theta(x)^{\gamma-3} t^3) \end{aligned}$$

and

$$P(W(1) > x) = \frac{e^{-c' x^{\frac{\gamma}{1-\gamma}}}}{2\pi} \times \int_{-\infty}^{+\infty} \exp(-\frac{(1-\gamma)\gamma}{2}\theta(x)^{\gamma-2}t^2 + O(\theta(x)^{\gamma-3}t^3) \frac{dt}{\theta(x)+it}$$

By the change of variable $\theta(x)^{\frac{\gamma-2}{2}} = y$ we get

$$P(W(1) > x) = \frac{\exp(-c' x^{\frac{\gamma}{1-\gamma}})}{2\pi\theta(x)^{\frac{\gamma-2}{2}}} \int_{-\infty}^{+\infty} \exp(-\frac{(1-\gamma)\gamma}{2} y^2 + O(\theta(x)^{-\frac{\gamma}{2}} y^3) \frac{dt}{\theta(x) + i\theta(x)^{\frac{\gamma-2}{2}} y}$$

Using $\int_{-\infty}^{+\infty} e^{-\frac{(1-\gamma)\gamma}{2} y^2} dy = \sqrt{\frac{2\pi}{(1-\gamma)\gamma}}$ we obtain the estimate

$$P(W(1) < x) = \frac{\exp(-c' x^{\frac{\gamma}{1-\gamma}})}{\sqrt{2\pi\theta(x)^{\gamma-2}(1-\gamma)\gamma}} (1 + O(\sqrt{x^{\frac{\gamma}{\gamma-1}}}))$$

□

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