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Fluctuations of Interacting Markov Chain Monte Carlo Models

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Abstract: We present a functional central limit theorem for a general class of interacting Markov chain Monte Carlo interpretations of discrete generation measure-valued equations. The path space models associated with these stochastic processes belong to the class of nonlinear Markov chains interacting with their empirical occupation measures. We develop an original theoretical analysis based on resolvent operators and semigroup techniques to analyze the fluctuation of their occupation measures around their limiting value. We also present a set of simple regularity conditions that applies to interacting Markov chain Monte Carlo models on path spaces, yielding what seems to be the first fluctuation theorems for this class of self-interacting models.

Key-words: Multivariate and functional central limit theorems, random fields, martingale limit theorems, self-interacting Markov chains, Markov chain Monte Carlo models.

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Fluctuations de Modèles de Monte Carlo par Chaînes de Markov en Interaction

Résumé : Nous présentons un théorème de la limite centrale fonctionnel pour une classe générale d'algorithmes de Monte Carlo par chaînes de Markov en interaction, utilisés dans la résolution numérique d'équations à valeurs mesures non linéaires. Les modèles trajectoriels associés à ces processus stochastiques appartiennent à la classe des modèles de chaînes de Markov non linéaires, en interaction avec leurs mesures d'occupation temporelle. Nous développons une analyse des fluctuations originale fondée sur l'étude fine d'opérateurs résolvants et sur des techniques de semigroupes sur des espaces de distributions. Cette étude dépend d'un jeu de conditions de régularité simples permettant d'analyser des modèles de Monte Carlo par chaînes de Markov en interaction sur des espaces trajectoriels. Ces résultats semblent être les premiers de ce type pour ces classes de processus en auto-interaction.

Mots-clés : Théorèmes de la limite centrale multidimensionnels, champs aléatoires, théorèmes limites de martingales, processus en auto-interaction, méthodes de Monte Carlo par chaînes de Markov

1 Introduction

1.1 Interacting Markov chain Monte Carlo models

Interacting Markov chain Monte Carlo methods (*i-MCMC*) are a new class of stochastic simulation algorithms for the numerical solving of a fairly general class of discrete generation nonlinear measure-valued equations. This class of models are defined in the following way. Let $(S^{(l)}, \mathcal{S}^{(l)})_{l\geq 0}$ be a sequence of measurable spaces, and for every $l \geq 0$ we denote by $\mathcal{P}(S^{(l)})$ the set of all probability measures on $S^{(l)}$. We consider a flow of probability measures $\pi^{(l)} \in \mathcal{P}(S^{(l)})$ that satisfies an evolution equation of the following form

$$\forall l \geq 0 \qquad \Phi_l(\pi^{(l-1)}) = \pi^{(l)} \tag{1.1}$$

for some mappings $\Phi_l : \mathcal{P}(S^{(l-1)}) \to \mathcal{P}(S^{(l)})$. In this notation system, we simplify the presentation using the convention $\Phi_0(\pi^{(-1)}) = \pi^{(0)}$, for $l = 0$. The i-MCMC model associated to (1.1) is defined in terms of a series of Markov transitions. That is, we consider a Markov transition $M^{(0)}$ from $S^{(0)}$ into itself , and a collection of Markov transitions $M^{(l)}_\mu$ from $S^{(l)}$ into itself, indexed by the parameter $l \geq 0$ and the set of probability measures $\mu \in \mathcal{P}(S^{(l-1)})$. We further assume that the fixed point measure of each operator $M^{(l)}_\mu$ is given by $\Phi_l(\mu)$; that is we have

$$\forall l \geq 0 \quad \forall \mu \in \mathcal{P}(S^{(l-1)}) \qquad \Phi_l(\mu)(dy) = \int \Phi_l(\mu)(dx)\ M^{(l)}_\mu(x,y)$$

For $l = 0$, we use the convention $M^{(0)}_\mu = M^{(0)}$ and $\Phi_0(\mu) = \pi^{(0)}$. The initial distributions of the i-MCMC algorithm will be defined in terms of a series of probability measures ν_l on $S^{(l)}$, with $l \geq 0$.

Now, we are in position to defined inductively the i-MCMC stochastic algorithm. At level $l = 0$, we let $X^{(0)} := (X^{(0)}_n)_{n\geq 0}$ be a Markov chain on $S^{(0)}$ with initial distribution ν_0 and elementary Markov transitions $M^{(0)}$. For every $k \geq 1$, given a realization of the chain $X^{(k-1)} := (X^{(k-1)}_n)_{n\geq 0}$, the k-th level chain $X^{(k)}_n$ is a Markov chain with initial distribution ν_k and with random Markov transitions $M^{(k)}_{\eta^{(k-1)}_n}$ depending on the current occupation measures $\eta^{(k-1)}_n$ of the chain at level $(k-1)$; that is we have that

$$\mathbb{P}(X^{(k)}_{n+1} \in dx \mid X^{(k-1)},\ X^{(k)}_n) = M^{(k)}_{\eta^{(k-1)}_n}(X^k_n, dx) \tag{1.2}$$

with

$$\eta^{(k-1)}_n := \frac{1}{n+1}\sum_{p=0}^{n} \delta_{X^{(k-1)}_p}$$

The rationale behind this is that the k-th level chain $X^{(k)}_n$ behaves asymptotically as a Markov chain with time homogeneous elementary transitions $M^{(k)}_{\pi^{(k-1)}}$, as long as $\eta^{(k-1)}_n$ is a good approximation of $\pi^{(k-1)}$.

For a detailed discussion on the origins and the application model areas of these i-MCMC models we refer the reader to the pair of recent articles [2, 3]. For sufficiently regular models, we proved in [2, 3] that the occupations measures $\eta^{(l)}_n$ converge to the solution $\pi^{(l)}$ of the equation (1.1), in the sense that

$\lim_{n\to\infty} \eta_n^{(l)}(f) = \pi^{(l)}$, almost surely and for any bounded measurable function f on $S^{(l)}$. The pair of articles [3, 4] also provide a collection of non asymptotic $\mathbb{L}_r$-mean error estimates and exponential deviations inequalities. The fluctuation analysis of $\eta_n^{(l)}$ around the limiting measure $\pi^{(l)}$ has been initiated in [4] in the special case where $M_\mu^{(k)}(x^k, .) = \Phi_k(\mu)$. In this situation, we observe that the k-th level chain $X^{(k)}$ is a collection of independent random variables with distributions ν_k for $n = 0$ and $\Phi_k(\eta_{n-1}^{(k-1)})$ for $n \geq 1$. The fluctuation analysis of this model is essentially based on the study of sequences of conditionally independent random variables $X^{(k)}$, whose distributions depend in a nonlinear way on the occupation measure of a similar series of random states $X^{(k-1)}$ on the lower level set, and so on for lower indexed levels.

The present article studies the fluctuations of the occupation measures associated with the abstract general class of i-MCMC models discussed above. The theoretical analysis of this type of models is much more involved than the one of traditional Markov chains since it relies on the stability properties of sophisticated Markov chains with elementary transitions that depend in a nonlinear way on the occupation measure of the chain.

The rest of the paper is organized as follows.

The main notation and convention used in this work are presented in a short preliminary section, section 1.2. The main result of the article is presented in full details in section 2. The regularity conditions are summarized in section 2.1. In section 2.2 we state a functional central limit theorem in terms of the semigroup associated with a first order expansion of the one step evolution mappings Φ_l of the limiting measure-valued equation (1.1). Section 4 addresses the fluctuation analysis of an abstract class of time inhomogeneous Markov chains. In section 4.2, we present a preliminary resolvent analysis to estimate the regularity properties of resolvent operators and invariant measure type mappings. In section 4.3, we apply these results to study the local fluctuations of a class of weighted occupation measures associated with these models. Section 5 addresses the fluctuation analysis of local interaction random fields associated with the i-MCMC model. The proof of the main theorem presented in section 2.2 is a direct consequence of a fluctuation theorem for local interaction random fields, and it is given at the end of section 5.1.

1.2 Notation and conventions

We denote respectively by $\mathcal{M}(E)$, $\mathcal{M}_0(E)$, $\mathcal{P}(E)$, and $\mathcal{B}(E)$, the set of all finite signed measures on some measurable space $(E, \mathcal{E})$, the convex subset of measures with null mass, the subset of all probability measures, and finally the Banach space of all bounded and measurable functions f on E equipped with the uniform norm $\|f\| = \sup_{x\in E} |f(x)|$. We also denote by $\mathcal{B}_1(E) \subset \mathcal{B}(E)$ the unit ball of functions $f \in \mathcal{B}(E)$ with $\|f\| \leq 1$, and by $\mathrm{Osc}_1(E)$, the convex set of $\mathcal{E}$-measurable functions f with oscillations less than one; that is,

$$\mathrm{osc}(f) = \sup \{|f(x) - f(y)| \ ; \ x, y \in E\} \leq 1$$

We let $\mu(f) = \int \ \mu(dx) \ f(x)$, be the Lebesgue integral of a function $f \in \mathcal{B}(E)$, with respect to a measure $\mu \in \mathcal{M}(E)$. We slight abuse the notation, and sometimes we denote by $\mu(A) = \mu(1_A)$ the measure of a measurable subset $A \in \mathcal{E}$.

We recall that a bounded integral operator M from a measurable space $(E, \mathcal{E})$ into an auxiliary measurable space $(F, \mathcal{F})$ into itself, is an operator $f \mapsto M(f)$ from $\mathcal{B}(F)$ into $\mathcal{B}(E)$, so that the functions

$$M(f)(x) = \int_F M(x, dy)\ f(y)\ \in \mathbb{R}$$

are $\mathcal{E}$-measurable and bounded, for any $f \in \mathcal{B}(F)$. By Fubini's theorem, we recall that a bounded integral operator M from a measurable space $(E, \mathcal{E})$ into an auxiliary measurable space $(F, \mathcal{F})$ also generates a dual operator $\mu \mapsto \mu M$ from $\mathcal{M}(E)$ into $\mathcal{M}(F)$ defined by $(\mu M)(f) := \mu(M(f))$.

We denote by $\|M\| := \sup_{f \in \mathcal{B}_1(E)} \|M(f)\|$ the norm of the operator $f \mapsto M(f)$ and we equip the Banach space $\mathcal{M}(E)$ with the corresponding total variation norm $\|\mu\| = \sup_{f \in \mathcal{B}_1(E)} |\mu(f)|$. We let $\beta(M)$ be the Dobrushin coefficient of a bounded integral operator M defined by the following formula

$$\beta(M) := \sup \{\mathrm{osc}(M(f))\ ;\ \ f \in \mathrm{Osc}_1(F)\}$$

When M has a constant mass, that is $M(1)(x) = M(1)(y)$ for any $(x, y) \in E^2$, the operator $\mu \mapsto \mu M$ maps $\mathcal{M}_0(E)$ into $\mathcal{M}_0(F)$, and $\beta(M)$ coincides with the norm of this operator. We equip the sets of distribution flows $\mathcal{M}(E)^{\mathbb{N}}$ with the uniform total variation distance defined by

$$\forall \eta = (\eta_n)_{n \geq 0}, \mu = (\mu_n)_{n \geq 0} \in \mathcal{M}(E)^{\mathbb{N}} \qquad \|\eta - \mu\| := \sup_{n \geq 0} \|\eta_n - \mu_n\|$$

We extend a given bounded integral operator $\mu \in \mathcal{M}(E) \mapsto \mu M \in \mathcal{M}(F)$ into an mapping

$$\eta = (\eta_n)_{n \geq 0}\ \in \mathcal{M}(E)^{\mathbb{N}} \mapsto \eta D = (\eta_n M)_{n \geq 0} \in \mathcal{M}(F)^{\mathbb{N}}$$

Sometimes, we slightly abuse the notation and we denote by ν instead of $(\nu)_{n \geq 0}$ the constant distribution flows equal to a given measure $\nu \in \mathcal{P}(E)$.

For any $\mathbb{R}^d$-valued function $f = (f^i)_{1 \leq i \leq d} \in \mathcal{B}(F)^d$, any integral operator M from E into F, and any $\mu \in \mathcal{M}(F)$, we will slightly abuse the notation, and we write $M(f)$ and $\mu(f)$ the $\mathbb{R}^d$-valued function and the point in $\mathbb{R}^d$ given respectively by

$$M(f) := \left(M(f^1), \ldots, M(f^d)\right) \quad \text{and} \quad \mu(f) := \left(\mu(f^1), \ldots, \mu(f^d)\right)$$

We also simplify the notation and sometimes we write

$$M[(f^1 - M(f^1))\ (f^2 - M(f^2))](x)$$

instead of

$$M[(f^1 - M(f^1)(x))\ (f^2 - M(f^2)(x))](x) = M(f^1 f^2)(x) - M(f^1)(x)\ M(f^2)(x)$$

Unless otherwise is stated, we denote by $c(k)$, $k \in \mathbb{N}$, a constant whose value may vary from line to line but only depends on the parameter k. For any pair of integers $0 \leq m \leq n$, we denote by $(n)_m := n!/(n-m)!$ the number of one to one mappings from $\{1, \ldots, m\}$ into $\{1, \ldots, n\}$. Finally, we shall use the conventions $\sum_\emptyset = 0$ and $\prod_\emptyset = 1$.

2 Statement of some results

2.1 Regularity conditions

Our first regularity condition is a first order weak regularity condition on the mappings Φ_l governing the measure-valued equation (1.1). Namely, we further assume that, for any $l \geq 0$, the mappings $\Phi_{l+1} : \mathcal{P}(S^{(l)}) \to \mathcal{P}(S^{(l+1)})$ satisfy the following first order local decomposition

$$\left[\Phi_{l+1}(\mu) - \Phi_{l+1}(\pi^{(l)})\right] = (\mu - \pi^{(l)})D_{l+1} + \Xi_l(\mu, \pi^{(l)}) \tag{2.1}$$

In the above display, $D_{l+1} \ : \ \mathcal{B}(S^{(l+1)}) \to \mathcal{B}(S^{(l)})$ stands for a bounded integral operator from $S^{(l)}$ into $S^{(l+1)}$ (that may depend on the measure $\pi^{(l)}$); and $\Xi_l(\mu, \eta)$ stands for a remainder signed measure on $S^{(l+1)}$ indexed by the set of probability measures $\mu, \eta \in \mathcal{P}(S^{(l)})$. We further require that

$$|\Xi_l(\mu, \eta)(f)| \leq \int \ \left|(\mu - \eta)^{\otimes 2}(g)\right| \ \mathit{\Xi}_l(f, dg) \tag{2.2}$$

for some integral operator $\mathit{\Xi}_l$ from $\mathcal{B}(S^{(l+1)})$ into the set $\mathcal{T}_2(S^{(l)})$ of all tensor product functions $g = \sum_{i\in I} \ a_i \ (h_i^1 \otimes h_i^2)$, with $I \subset \mathbb{N}$, $(h_i^1, h_i^2)_{i\in I} \in (\mathcal{B}(S^{(l)})^2)^I$, and a sequence of numbers $(a_i)_{i\in I} \in \mathbb{R}^I$ such that

$$|g| := \sum_{i\in I} \ |a_i| \ \|h_i^1\|\|h_i^2\| < \infty \quad \text{and} \quad \chi_l := \sup_{f\in\mathcal{B}_1(S^{(l)})} \int \ |g| \ \mathit{\Xi}_l(f, dg) < \infty$$

Our second regularity condition concerns the integral operators $M_\mu^{(l)}$. We assume the operators $M_\mu^{(l)}$ satisfy the following pair of regularity conditions

$$m_l(n_l) := \sup_{\eta\in\mathcal{P}(S^{(l-1)})} \beta((M_\eta^{(l)})^{n_l}) < 1 \tag{2.3}$$

and

$$\left|\left[M_\mu^{(l)} - M_\nu^{(l)}\right](f)\right| \leq \int \ |[\mu - \nu]\,(g)| \ \Gamma_{l,\mu}(f, dg) \tag{2.4}$$

for some collection of bounded integral operators $\Gamma_{l,\mu}$ from $\mathcal{B}(S^{(l)})$ into $\mathcal{B}(S^{(l-1)})$, and indexed by the set of measures $\mu \in \mathcal{P}(S^{(l-1)})$ with

$$\sup_{\mu\in\mathcal{P}(S^{(l-1)})} \int \ \Gamma_{l,\mu}(f, dg) \ \|g\| \leq \Lambda_l \ \|f\| \quad \text{and} \quad \Lambda_l < \infty$$

We end this section with some comments on this set of conditions.

Firstly, we mention that the regularity condition (2.1) is a first order refinement of a Lipschitz type condition we used in [2, 3] to derive a series of $\mathbb{L}_p$-mean error bounds and exponential inequalities. This condition has been introduced in [4] for studying the fluctuations of the elementary i-MCMC model when $M_\mu^{(l)}(x, .) = \Phi_l(\mu)$. The second regularity condition (2.4) is a local Lipschitz type continuity condition on the collection of operators $M_\mu^{(l)}$. This conditions is less stringent than the one we used in [3]. In the latter referenced article, we assumed that (2.4) holds for some operators $\Gamma_{l,\mu} = \Gamma_l$ that do not depend on μ.

Therefore, most of the asymptotic results presented in [3] do not apply directly in the present context. Nevertheless, it can be checked that the inductive proof of the $\mathbb{L}_p$-mean error bounds presented in theorem 5.2 hold true under the weaker condition (2.4); thus, for every $l \geq 0$ and any function $f \in \mathcal{B}(S^{(l)})$, we also know that $\eta_n^{(l)}(f)$ converge almost surely to $\pi^{(l)}(f)$ as $n \to \infty$.

The main advantage of this set of conditions is that it stable under a state space enlargement, so that the asymptotic analysis of such models, including the functional central limit theorem presented in the next section, applies directly without further work to i-MCMC models on path spaces. We shall return to this key property in section 3.

2.2 A functional central limit theorem

To describe precisely the fluctuations of the empirical measures $\eta_n^{(l)}$ around their limiting value $\pi^{(l)}$, we need a few notation. We denote by $D_{k,l}$ with $0 \leq k \leq l$ the semigroup associated with the bounded integral operators D_k introduced in (2.1). More formally, we have

$$\forall 1 \leq k \leq l \qquad D_{k,l} = D_k D_{k+1} \ldots D_l$$

For $k > l$, we use the convention $D_{k,l} = Id$, the identity operator.

We refer the reader to [4] for an explicit functional representation of these semigroups for a class of Feynman-Kac models arising in physics, biology and information engineering sciences.

In this notation, the functional central limit theorem describing the fluctuations of the i-MCMC model around the solution of the equation (1.1) is stated as follows.

Theorem 2.1 *For every $k \geq 0$, the sequence of random fields $(U_n^{(k)})_{n\geq 0}$ on $\mathcal{B}(S^{(k)})$ defined below*

$$U_n^{(k)} := \sqrt{n}\ \left[\eta_n^{(k)} - \pi^{(k)}\right]$$

converges in law, as n tends to infinity and in the sense of finite dimensional distributions, to a sequence of Gaussian random fields $U^{(k)}$ on $\mathcal{B}(S^{(k)})$ given by the following formula

$$U^{(k)} := \sum_{0\leq l\leq k} \frac{\sqrt{(2l)!}}{l!}\ V^{(k-l)} D_{(k-l)+1,k}$$

In the above display, $\left(V^{(l)}\right)_{l\geq 0}$ stands for a collection of independent and centered Gaussian fields with a variance function given by

$$\begin{aligned} &\mathbb{E}\left(V^{(l)}(f)^2\right) \\ &= \pi^{(l)}\left[(f-\pi^{(l)}(f))^2\right] + 2\textstyle\sum_{n\geq 1} \pi^{(l)}\left[(f-\pi^{(l)}(f))\ \left(M^{(l)}_{\pi^{(l-1)}}\right)^n (f-\pi^{(l)}(f))\right] \end{aligned} \tag{2.5}$$

We end this section with some comments on this functional central limit theorem.

Firstly, in the special case where $M^{(l)}_{\mu}(x^l,.) = \Phi_l(\mu)$ for all $l \geq 1$, the variance function (2.5) reduces to the one we obtained in [4]; that is we have

$$\mathbb{E}\left(V^{(l)}(f)^2\right) = \pi^{(l)}\left[(f - \pi^{(l)}(f))^2\right]$$

Loosely speaking, this special class of i-MCMC model behaves as a sequence of independent random variables with distributions $\Phi_l(\eta^{(l-1)}_n)$ given by the local invariant measures of Markov chain Monte Carlo model with general transitions $M^{(l)}_{\eta^{(l-1)}_n}(x^l,.)$.

In these interpretations, the additional terms in the right hand side (2.5) reflects the fluctuations of this MCMC model around their limiting invariant measure. Although the choice $M^{(l)}_{\mu}(x^l,.) = \Phi_l(\mu)$ seems in some sense optimal, it is difficult to compare the fluctuation variances of these models.

Our second remark is that the functional central limit theorem stated above applies directly without further work to the analysis of the fluctuations of the occupation measures of $(X^{(k)}_n)_{0\leq k\leq l}$ around the limiting tensor product measure $\otimes_{0\leq k\leq l}\pi^{(k)}$, for any time horizon $l \geq 0$. We shall discuss this stability property in the next section devoted to i-MCMC models on path spaces.

Since preparing the original version of this paper, the authors' attention have been drawn to a recent work of Yves Atchadé [1] which analyzes the local fluctuations of some related algorithms; namely the importance-resampling MCMC algorithm and a version of the equi-energy sampler. In [1], the author provides a pair of local central limit theorems for quantities associated to some random measures $\pi^{(k)}_n$ which converge almost surely to $\pi^{(k)}$ as n tends to infinity, and given by the following formula

$$\frac{1}{\sqrt{n+1}} \sum_{p=0}^{n} \left[f_k(X^{(k)}_p) - \pi^{(k)}_p(f_k)\right]$$

In our notational system, it is shown in [1] that the above quantities converge in law to a centered Gaussian random variable $V^{(k)}(f_k)$ that does not depend on the first order semigroup $D_{k,l}$, nor on the fluctuations of the i-MCMC model on lower indexed levels. As it will become transparent in the further developments of the present article, this local fluctuation result is only true if we choose the random sequence $\pi^{(k)}_p := \Phi_k(\eta^{(k-1)}_{p-1})$, and it does not reflect the complete fluctuations of the i-MCMC model around the limiting measures $\pi^{(l)}$.

3 Path space i-MCMC models

In this section, we fix a final time horizon l of the equation (1.1) and we design an i-MCMC interpretation of the path space model given by

$$\forall n \geq 0 \qquad X^{[l]}_n := (X^{(0)}_n, \ldots, X^{(l)}_n) \in S^{[l]} := \prod_{0\leq k\leq l} S^{(k)}$$

For every $0 \leq k \leq l$ we denote by $\eta^{(k)} \in \mathcal{P}(S^{(k)})$ the image measure of a measure $\eta \in \mathcal{P}(S^{[l]})$ on the k-th coordinate level set $S^{(k)}$ of the product space

$S^{[l]} := \prod_{0\leq k\leq l} S^{(k)}$. In this notation, it is readily checked that $X_n^{[l]}$ is an $S^{[l]}$-valued self-interacting Markov chain with elementary transitions defined by

$$\mathbb{P}(X_{n+1}^{[l]} \in dx \mid (X_p^{[l-1]})_{0\leq p\leq n}, X_n^{[l]}) = M^{[l]}_{\eta_n^{[l-1]}}(X_n^{[l]}, dx) \tag{3.1}$$

with the occupation measures $\eta_n^{[l-1]}$ and the collection of transitions $M^{[l]}_{\eta_n^{[l-1]}}$ defined by the following synthetic formulae

$$\eta_n^{[l-1]} := \frac{1}{n+1}\sum_{p=0}^{n} \delta_{X_p^{[l-1]}} \quad \text{and} \quad M^{[l]}_{\eta_n^{[l-1]}}(X_n^{[l]}, dx) = \prod_{0\leq k\leq l} M^{(k)}_{\eta_n^{[l],(k-1)}}(X_n^{(k)}, dx^l)$$

In the above displayed formulae we have use the convention $M^{(0)}_{\eta_n^{(-1)}} = M^{(0)}$ for $l=0$, and $dx := dx^0 \times \ldots \times dx^m$ stands for an infinitesimal neighborhood of a generic point in the product space $x := (x^0, \ldots, x^l) \in S^{[l]}$. We leave the reader to check that (3.1) coincides with the i-MCMC model associated with the limiting evolution equation

$$\pi^{[l]} = \Phi_{[l]}(\pi^{[l-1]}) \quad \text{with} \quad \pi^{[l]} := \pi^{(0)} \otimes \ldots \otimes \pi^{(l)}$$

and the invariant measure mapping

$$\Phi_{[l]} \;:\; \mu \in \mathcal{P}(S^{[l-1]}) \mapsto \Phi_{[l]}(\mu) := \pi^{(0)} \otimes \Phi_1(\mu^{(0)}) \otimes \ldots \otimes \Phi_l(\mu^{(l-1)}) \in \mathcal{P}(S^{[l]})$$

To describe the main result of this section, we need to introduce additional notation. For any $0 \leq k_1 \leq k_2$, we set

$$S^{[k_1,k_2]} := \prod_{k_1\leq k\leq k_2} S^{(k)} \quad \text{and} \quad \pi^{[k_1,k_2]} := \otimes_{k_1\leq k\leq k_2} \pi^{(k)} \in \mathcal{P}(S^{[k_1,k_2]})$$

For any $0 \leq k < l$, any pair $(\mu_1, \mu_2) \in \left(\mathcal{P}(S^{[0,k]}) \times \mathcal{P}(S^{[k+2,l+1]})\right)$ and any integral operator D from $S^{(k)}$ into $S^{(k+1)}$, we denote by $(\mu_1 \otimes D \otimes \mu_2)$ from $S^{[l]}$ into $S^{[l+1]}$

$$(\mu_1 \otimes D \otimes \mu_2)((x_1, x_2, x_3), d(y_1, y_2, y_3)) = \mu_1(dy_1)\; D(x_1, dy_2)\; \mu_2(dy_3)$$

where $d(y_1, y_2, y_3) = dy_1 \times dy_2 \times dy_3$ stands for an infinitesimal neighborhood of a generic point $(y_1, y_2, y_3) \in S^{[l+1]} = \left(S^{[0,k]} \times S^{(k+1)} \otimes S^{[k+2,l+1]}\right)$, and (x_1, x_2, x_3) a point in the set $S^{[l]} = \left(S^{[0,k-1]} \times S^{(k)} \otimes S^{[k+1,l]}\right)$.

Proposition 3.1 *For any $l \geq 0$, the mappings $\Phi_{[l]}$ and the collection of Markov transitions $M_\mu^{[l]}$ satisfy the set of regularity conditions (2.1), (2.3), and (2.4) as long as the corresponding conditions are met for the marginal mappings Φ_k and the transitions $M_\mu^{(k)}$ where $1 \leq k \leq l$. In addition, the mappings $\Phi_{[l+1]}$ satisfy the first order decomposition (2.1) with bounded integral operators $D_{[l+1]}$ from $S^{[l]}$ into $S^{[l+1]}$ given by*

$$D_{[l+1]} = \pi^{[0,l]} \otimes D_{l+1} + \sum_{0\leq k<l} \pi^{[0,k]} \otimes D_{k+1} \otimes \pi^{[k+2,l+1]}$$

Before presenting the proof of this proposition, we emphasize that this latter directly implies that the functional central limit theorem stated in section 2.2 is also valid for the path space i-MCMC model discussed above. In other words, for every $k \geq 0$, the sequence of random fields $(U_n^{[k]})_{n\geq 0}$ on $\mathcal{B}(S^{[k]})$ defined by

$$U_n^{[k]} := \sqrt{n}\ \left[\eta_n^{[k]} - \pi^{[k]}\right] = \frac{1}{\sqrt{n+1}} \sum_{p=0}^{n} \left[\delta_{(X_p^{(0}, \ldots, X_p^{(k)})} - \left(\pi^{(0)} \otimes \ldots \otimes \pi^{(k)}\right)\right]$$

converges in law, as n tends to infinity and in the sense of finite dimensional distributions, to a sequence of Gaussian random fields $U^{[k]}$ defined as $U^{(k)}$ by replacing the the semigroups D_{l_1,l_2} and the limiting measures $\pi^{(l)}$ by the corresponding objects on path spaces.

Proof of proposition 3.1 :
With some obvious notation, we have

$$\forall n \geq 1 \qquad \left(M_\mu^{[l]}\right)^n = \bigotimes_{0\leq k\leq l} \left(M^{(k)}_{\mu^{(k-1)}}\right)^n$$

Using the fact that

$$\| \bigotimes_{0\leq k\leq l} \mu_k - \bigotimes_{0\leq k\leq l} \eta_k \| \leq \sum_{0\leq k\leq l} \|\mu_k - \eta_k\|$$

for any sequence of probability measures $\mu_k, \eta_k \in \mathcal{P}(S^{(k)})$, with $0 \leq k \leq l$, we prove that

$$\beta\left(\left(M_\mu^{[l]}\right)^n\right) \leq \sum_{0\leq k\leq l} \beta\left(\left(M^{(k)}_{\mu^{(k-1)}}\right)^n\right) \leq \frac{1}{l+1} \sum_{0\leq k\leq l} m_k(n_k) < 1$$

as soon as

$$n \geq n_{[l]} := (\vee_{0\leq k\leq l} n_k) \times \left(1 + \frac{\log{(l+1)}}{\wedge_{0\leq k\leq l} \log{(1/m_k(n_k))}}\right)$$

We prove the pair of regularity conditions (2.1) and (2.4) by induction on the parameter l. We use the notation $D_{[l]}$, $\Xi_{[l]}$, $\Xi_{[l]}$ $m_{[l]}$, $n_{[l]}$ and $\Gamma_{[l],\mu}$ the corresponding objects introduced in the statement of conditions (2.1), (2.3), and (2.4). The results are clearly true for $m = 0$ with

$$\Phi_{[0]}(\mu) := \pi^{(0)} \quad \text{and} \quad M^{[0]}_{\eta_n^{[-1]}} := M^{(0)}$$

In this case, we readily find that

$$D_{[0]} = D_0 = 0,\ \Xi_{[l]} = 0,\ m_{[0]} = m_0,\ n_{[0]} = n_0 \quad \text{and} \quad \Gamma_{[0],\mu} = \Gamma_{0,\mu} = 0$$

Assume now that the result has been proved at some rank l. For any measure μ on $S^{[l+1]} = S^{[l]} \times S^{(l+1)}$, we denote by $\mu^{[l]}$ and $\mu^{(l+1)}$ its image measures on $S^{[l]}$ and $S^{(l)}$. In this notation we have

$$\Phi_{[l+1]}(\mu) = \Phi_{[l]}(\mu^{[l]}) \otimes \Phi_{l+1}(\mu^{(l)})$$

and

$$M_\mu^{[l+1]}((u,v), d(x,y)) = M^{[l]}_{\mu^{[l]}}(u, dx) \times M^{(l)}_{\mu^{(l)}}(v, dy)$$

for any $(u,v) \in S^{[l+1]} = (S^{[l]} \times S^{(l)})$, where $d(x,y) = dx \times dy$ stand for an infinitesimal neighborhood of a generic point $(x,y) \in S^{[l+1]} = (S^{[l]} \times S^{(l)})$. After some elementary computations, using the decomposition

$$\begin{aligned}
&\left[\Phi_{[l+1]}(\mu) - \Phi_{[l+1]}(\pi^{(l)})\right] \\
&= \Phi_{[l]}(\mu^{[l-1]}) \otimes \Phi_{l+1}(\mu^{(l)}) - \Phi_{[l]}(\pi^{[l-1]}) \otimes \Phi_{l+1}(\pi^{(l)}) \\
&= \Phi_{[l]}(\pi^{[l-1]}) \otimes \left[\Phi_{l+1}(\mu^{(l)}) - \Phi_{l+1}(\pi^{(l)})\right] \\
&\quad + \left[\Phi_{[l]}(\mu^{[l-1]}) - \Phi_{[l]}(\pi^{[l-1]})\right] \otimes \Phi_{l+1}(\pi^{(l)}) \\
&\qquad + \left[\Phi_{[l]}(\mu^{[l-1]}) - \Phi_{[l]}(\pi^{[l-1]})\right] \otimes \left[\Phi_{l+1}(\mu^{(l)}) - \Phi_{l+1}(\pi^{(l)})\right]
\end{aligned}$$

we find that the first order condition (2.1) is satisfied with an integral operator $D_{[l+1]}$ from $S^{[l]}$ into $S^{[l+1]}$ defined for any $f \in$ by

$$\begin{aligned}
D_{[l+1]}(u, d(x,y)) &= \Phi_{[l]}(\pi^{[l-1]})(dx)\ D_{(l+1)}(u,dy) + D_{[l]}(u,dx)\ \Phi_{l+1}(\pi^{(l)})(dy) \\
&= \pi^{[l]}(dx)\ D_{(l+1)}(u,dy) + D_{[l]}(u,dx)\ \pi^{(l+1)}(dy)
\end{aligned}$$

Condition (2.4) is proved using the same type of arguments. This ends the proof of the proposition. ∎

4 On the fluctuations of time inhomogeneous Markov chains

We discuss in this section the fluctuations of time inhomogeneous Markov chains with elementary Markov transitions that may depend on some predictable flow of distributions on some possibly different state space.

4.1 Description of the models

We consider a collection of Markov transitions M_η on some measurable space $(S,\mathcal{S})$ indexed by the set of probability measures $\eta \in \mathcal{P}(S')$, on some possibly different measurable space $(S',\mathcal{S}')$. We further assume that there exists an integer $n_0 \geq 0$ such that

$$m(n_0) := \sup_{\eta \in \mathcal{P}(S')} \beta(M_\eta^{n_0}) < 1 \quad \text{and we set} \quad p(n_0) := 2n_0/(1-m(n_0)) \tag{4.1}$$

We also assume that for any pair of measures $(\eta,\mu) \in \mathcal{P}(S')^2$ we have

$$|[M_\mu - M_\eta](f)| \leq \int \ |[\mu - \eta](g)|\ \Gamma_\mu(f,dg) \tag{4.2}$$

for some collection of bounded integral operator Γ_μ from $\mathcal{B}(S)$ into $\mathcal{B}(S')$, indexed by the set of measures $\mu \in \mathcal{P}(S')$ with

$$\sup_{\mu \in \mathcal{P}(S')} \int \Gamma_\mu(f,dg)\ \|g\| \leq \Lambda\ \|f\| \quad \text{for some finite constant} \quad \Lambda < \infty$$

We associate to the collection of transitions M_η an S-valued non homogeneous Markov chain X_n with a prescribed initial distribution $\nu \in \mathcal{P}(S)$, and some elementary transitions defined by

$$\forall n \geq 0 \qquad \mathbb{P}(X_{n+1} \in dx \mid \mathcal{F}_n) = M_{\eta_n}(X_n, dx) \tag{4.3}$$

In the above displayed formula, $\mathcal{F}_n$ is a σ-field that contains the σ-field generated by the random states X_p from the origin $p = 0$ up to the current time horizon $p = n$. In other words, the randomness included in η_n may come from a different stochastic process but the distributions η_n are measurable with respect to $\mathcal{F}_n$. We further assume that the variations of the flow η_n are controlled by some sequence of random variables $\tau(n)$ in the sense that

$$\forall n \geq 0 \qquad \|\eta_n - \eta_{n-1}\| \leq \tau(n), \quad \text{and we set} \quad \overline{\tau}(n) := \sum_{0 \leq p \leq n} \tau(p) \tag{4.4}$$

For $n = 0$ we use the convention $\eta_{-1} = 0$, the null measure on S'.

4.2 Regularity properties of resolvent operators

The main simplification of conditions (4.1) comes from the fact that M_η has an unique invariant measure

$$\Phi(\eta)M_\eta = \Phi(\eta) \in \mathcal{P}(S)$$

In addition, the resolvent operators

$$P_\eta \ : \ f \in \mathcal{B}(S) \to P_\eta(f) := \sum_{n \geq 0} \left[M_\eta^n - \Phi(\eta)\right](f) \in \mathcal{B}(S) \tag{4.5}$$

are well defined absolutely convergent series that satisfy the Poisson equation given by

$$\left\{ \begin{array}{rcl} (M_\eta - Id)P_\eta & = & (\Phi(\eta) - Id) \\ \Phi(\eta)P_\eta & = & 0 \end{array} \right.$$

In the rest of the article, we shall use the following definition.

Definition 4.1 *The integral operator P_η defined by the series (4.5) is called the resolvent operator associated to the Markov transition M_η and the fixed point measure $\Phi(\eta)$.*

Resolvent operators are classical tools for the asymptotic analysis of time inhomogeneous Markov chains. In our context the Markov chain interacts with a flow a probability measures. To analyze the situation where this flow converges to some limiting measure, it is convenient to study the regularity properties of the resolvent operators P_η as well as the ones of the invariant measure mapping $\Phi(\eta)$ associated with M_η.

Proposition 4.2 *Under the regularity conditions (4.1) and (4.2), we have*

$$\sup_{\eta\in\mathcal{P}(S')} \|P_\eta\| \leq p(n_0)$$

In addition, for any $f \in \mathcal{B}(S)$ and any $(\mu,\eta) \in \mathcal{P}(S')$ we have the following Lipschitz type inequalities

$$|[\Phi(\eta)-\Phi(\mu)](f)| \leq \int\ |[\eta-\mu](g)|\, \Upsilon_\mu(f,dg) \tag{4.6}$$

and

$$\|\,[P_\eta - P_\mu](f)\| \leq \int\ |[\eta-\mu](g)|\, \Upsilon'_\mu(f,dg) \tag{4.7}$$

where $(\Upsilon_\mu, \Upsilon'_\mu)$ is a pair of bounded integral operators from $\mathcal{B}(S)$ into $\mathcal{B}(S')$ indexed by the set of measures $\mu \in \mathcal{P}(S')$ such that

$$\int\ \|g\|\ \Upsilon_\mu(f,dg) \leq p(n_0)\ \Lambda\ \|f\|$$

and

$$\int\ \|g\|\ \Upsilon'_\mu(f,dg) \leq p(n_0)(1+p(n_0))\ \Lambda\ \|f\|$$

Proof:
The first assertion is proved in [3]. For completeness, it is only sketched here. We simply use the fact that

$$P_\eta(f)(x) = \sum_{n\geq 0} \int\ \left[M^n_\eta(f)(x) - M^n_\eta(f)(y)\right] \Phi(\eta)(dy)$$

to check that

$$\|P_\eta(f)\| \leq \sum_{n\geq 0} \mathrm{osc}(M^n_\eta(f))$$

and

$$\|P_\eta(f)\| \leq \left[\sum_{n\geq 0} \beta(M^n_\eta)\right]\ \mathrm{osc}(f) \Rightarrow \begin{array}{rcl} \|P_\eta\| & \leq & 2\ \sum_{p\geq 1}\ \sum_{r=0}^{n_0-1} \beta(M_\eta^{(p-1)n_0+r}) \\ & \leq & \frac{2n_0}{1-\beta(M_\eta^{n_0})} \end{array}$$

The end of the proof of the first assertion is now immediate. The proof of (4.6) is based on the following decomposition

$$[\Phi(\eta)-\Phi(\mu)](f) = \{[\Phi(\eta)-\Phi(\mu)]\, M_\mu + \Phi(\eta)\, [M_\eta - M_\mu]\}\, (f_\mu)$$

with $f_\mu := (f - \Phi(\mu)(f))$. Under our regularity conditions on the integral operators M_μ, we find that

$$\begin{aligned} |\,[\Phi(\eta)-\Phi(\mu)]\,(f)| &\leq |\,[\Phi(\eta)-\Phi(\mu)]\, M_\mu(f_\mu)| + \|\,[M_\eta - M_\mu]\,(f_\mu)\| \\ &\leq |\,[\Phi(\eta)-\Phi(\mu)]\, M_\mu(f_\mu)| + \int\ |[\mu-\eta]\,(g)|\ \Gamma_\mu(f_\mu,dg) \end{aligned}$$

This recursion readily implies (4.6) with the integral operator given by

$$\Upsilon_\mu(f, dg) := \sum_{n \geq 0} \Gamma_\mu(M_\mu^n(f_\mu), dg)$$

Finally we observe that

$$\int \|g\| \ \Upsilon_\mu(f, dg) \leq \sum_{n \geq 0} \int \|g\| \ \Gamma_\mu(M_\mu^n(f_\mu), dg) \leq \Lambda \ \sum_{n \geq 0} \|M_\mu^n(f_\mu)\|$$

Arguing as above, we conclude that

$$\int \|g\| \ \Upsilon_\mu(f, dg) \leq \Lambda \ \sum_{n \geq 0} \mathrm{osc}(M_\mu^n(f)) \leq p(n_0) \ \Lambda \ \|f\|$$

This ends the proof of (4.6). The proof of (4.7) follows the same type of arguments. Firstly, we observe that

$$P_\eta - P_\mu = P_\mu(M_\eta - M_\mu)P_\eta + [\Phi(\mu) - \Phi(\eta)] \, P_\eta$$

To check this formula, we first use the fact that $M_\mu P_\mu = P_\mu M_\mu$ to prove that

$$P_\mu(M_\mu - Id) = (M_\mu - Id)P_\mu = (\Phi(\mu) - Id)$$

This yields

$$P_\mu(M_\mu - Id)P_\eta = (\Phi(\mu) - Id)P_\eta$$

Using the Poisson equation and the fact that $P_\mu(1) = 0$ we also have the decomposition

$$P_\mu(M_\eta - Id)P_\eta = P_\mu(\Phi(\eta) - Id) = -P_\mu$$

Combining these two formulae, we conclude that

$$P_\mu(M_\eta - M_\mu)P_\eta = [P_\eta - P_\mu] - [\Phi(\mu) - \Phi(\eta)] \, P_\eta$$

This ends the proof of the decomposition given above. It is now easily checked that

$$\begin{aligned} \| [P_\mu - P_\eta] (f)\| &\leq \|(M_\mu - M_\eta)P_\mu(f)\| + \int \ |[\eta - \mu] (g)| \, \Upsilon_\mu(P_\mu(f), dg) \\ &\leq \int \ |[\mu - \eta] (g)| \ \ \{\Gamma_\mu(P_\mu(f), dg) + \Upsilon_\mu(P_\mu(f), dg)\} \end{aligned}$$

The end of the proof follows the same type of arguments as before. This ends the proof of the proposition. ■

4.3 Local fluctuations of weighted occupation measures

This section is concerned with the fluctuation analysis of the occupation measures of the time inhomogeneous Markov chain introduced in (4.3). In section 5, we shall use these results to analyze the fluctuations of i-MCMC models. The fluctuation analysis of this type of models is related to the fluctuations of weighted occupation measures with respect to some weight array type functions (cf. for instance [4]).

Definition 4.3 *We let $\mathcal{W}$ be the set of non negative and non increasing weight array functions $w = (w_n(p))_{0\leq p\leq n, 0\leq n}$, satisfying the following conditions*

$$\lim_{n\to\infty} \vee_{0\leq p\leq n} w_n(p) = 0 \quad \textit{with} \quad \forall \epsilon \in [0,1] \qquad \varpi(\epsilon) := \lim_{n\to\infty} \sum_{0\leq p\leq \lfloor \epsilon n\rfloor} w_n^2(p) < \infty$$

and some scaling function ϖ such that $\lim_{(\epsilon_0,\epsilon_1)\to(0-,1+)}(\varpi(\epsilon_0), \varpi(\epsilon_1)) = (0,1)$. We observe that the traditional and constant fluctuation rates sequences $w_n(p) = 1/\sqrt{n}$ belong to $\mathcal{W}$, with the identity function $\varpi(\epsilon) = \epsilon$.

Definition 4.4 *We associate to a given weight array function $w \in \mathcal{W}$ the mapping*

$$W \ : \ \eta \in \mathcal{M}(S)^{\mathbb{N}} \mapsto W(\eta) = (W_n(\eta))_{n\geq 0} \in \mathcal{M}(S)^{\mathbb{N}}$$

defined for any flow of measures $\eta = (\eta_n)_{n\geq 0} \in \mathcal{P}(S)$, and any $n \geq 0$, by the weighted measures

$$W_n(\eta) = \sum_{0\leq p\leq n} w_n(p) \ \eta_p$$

The next proposition presents a pivotal decomposition formula of the weighted occupation measures in terms of a particular martingale on fixed time horizon with a negligible remainder bias term.

Proposition 4.5 *We consider the flow of random measures $\zeta := (\zeta_n)_{n\in\mathbb{N}} \in \mathcal{M}(S)^{\mathbb{N}}$ defined for any $n \geq 0$ by the following formula*

$$\forall n \geq 0 \qquad \zeta_n = [\delta_{X_n} - \Phi(\eta_{n-1})]$$

For $n = 0$, we use the convention $\Phi(\eta_{-1}) = \nu_0$, so that $\zeta_0 = [\delta_{X_0} - \nu_0]$. For any weight array function $w \in \mathcal{W}$, the weighted measures $W_n(\zeta)$ satisfy the following decomposition

$$W_n(\zeta)(f) = \sum_{0\leq p\leq n} w_n(p) \ \Delta\mathbb{M}_{p+1}(f) + \mathbb{L}_n(f) \tag{4.8}$$

with the martingale increments

$$\Delta\mathbb{M}_{p+1}(f) = \mathbb{M}_{p+1}(f) - \mathbb{M}_p(f) := \left(P_{\eta_{p-1}}(f)(X_{p+1}) - M_{\eta_p} P_{\eta_{p-1}}(f)(X_p)\right)$$

and a remainder signed measure $\mathbb{L}_n$ such that

$$\|\mathbb{L}_n\| \leq w_n(0) \ (1 + p(n_0)\overline{\tau}(n)) \ (2 + p(n_0)) \ \Lambda$$

Proof:
We let $P_{\eta_{n-1}}$ be the integral operator solution of the Poisson equation associated with the Markov transition $M_{\eta_{n-1}}$, with a fixed point measure $\Phi(\eta_{n-1})$. By construction, we have

$$\zeta_n(f) = [f(X_n) - \Phi(\eta_{n-1})(f)] = P_{\eta_{n-1}}(f)(X_n) - M_{\eta_{n-1}}(P_{\eta_{n-1}}(f))(X_n)$$

For $n = 0$, we use the convention $P_{\eta_{-1}} = Id$ and $M_{\eta_{-1}} = \nu_0$. The proof of (4.8) is based on the following decomposition

$$\zeta_n(f) = A_n(f) + B_n(f) + C_n(f) + \Delta\mathbb{M}_{n+1}(f)$$

with the random processes $A_n(f)$, $B_n(f)$ and $C_n(f)$ defined below

$$\begin{aligned}
A_n(f) &:= \left[P_{\eta_n} - P_{\eta_{n-1}}\right](f)(X_{n+1}) \\
B_n(f) &:= \left[P_{\eta_{n-1}}(f)(X_n) - P_{\eta_n}(f)(X_{n+1})\right] \\
C_n(f) &:= \left[M_{\eta_n} - M_{\eta_{n-1}}\right] P_{\eta_{n-1}}(f)(X_n)
\end{aligned}$$

Using the Lipschitz inequality (4.7) presented in proposition 4.2, we prove that

$$\begin{aligned}
|A_n(f)| &\leq \| \left[P_{\eta_n} - P_{\eta_{n-1}}\right](f)\| \\
&\leq \int \;\; |[\eta_n - \eta_{n-1}](g)| \, \Upsilon'_{\eta_n}(f, dg) \leq \tau(n)\ p(n_0)(1 + p(n_0))\ \Lambda\ \|f\|
\end{aligned}$$

In addition, using the Lipschitz regularity condition (4.2), we also obtain

$$\begin{aligned}
|C_n(f)| &\leq \| \left[M_{\eta_n} - M_{\eta_{n-1}}\right](P_{\eta_{n-1}}(f)) \\
&\leq \| \int \;\; |[\eta_n - \eta_{n-1}](g)|\ \ \Gamma_{\eta_n}(P_{\eta_{n-1}}(f), dg) \\
&\leq \tau(n)\ \Lambda\ \|P_{\eta_{n-1}}\|\ \|f\| \leq \tau(n)\ \Lambda\ p(n_0)\ \|f\|
\end{aligned}$$

By definition of the weighted measure $W_n(\zeta)$, we have

$$\begin{aligned}
W_n(\zeta) &:= \sum_{0 \leq p \leq n} w_n(p)\zeta_p(f) \\
&= \sum_{0 \leq p \leq n} w_n(p)\ \Delta\mathbb{M}_{p+1}(f) + \sum_{0 \leq p \leq n} w_n(p)(A_p(f) + B_p(f) + C_p(f))
\end{aligned} \tag{4.9}$$

From previous calculations, we have

$$\left| \sum_{0 \leq p \leq n} w_n(p)(A_p(f) + C_p(f)) \right| \leq w_n(0)\ \overline{\tau}(n)\ \Lambda\ p(n_0)(2 + p(n_0))\ \|f\|$$

Finally, we use the following decomposition

$$\begin{aligned}
\sum_{0 \leq p \leq n} w_n(p)B_p(f) = &\sum_{0 \leq p \leq n} \left[w_n(p)\ P_{\eta_{p-1}}(f)(X_p) - w_n(p+1)P_{\eta_p}(f)(X_{p+1})\right] \\
&+ \sum_{0 \leq p \leq n} [w_n(p+1) - w_n(p)]\ \ P_{\eta_p}(f)(X_{p+1})
\end{aligned}$$

with the convention $w_n(n+1) = 0$. This readily implies that

$$\begin{aligned}
\left| \sum_{0 \leq p \leq n} w_n(p)B_p(f) \right| &\leq 2\ w_n(0)\ \|f\| + p(n_0)\ \|f\| \sum_{0 \leq p \leq n} [w_n(p) - w_n(p+1)] \\
&= (2 + p(n_0))\ \|f\|\ w_n(0)
\end{aligned}$$

The end of the proof is now a direct consequence of formula (4.9). ■

Now, we are in position to state and to prove the main result of this section.

Theorem 4.6 *We suppose there exists some measure η, and some $m \geq 1$ such that*

$$\forall f \in \mathcal{B}_1(S') \qquad \mathbb{E}(|\eta_n(f) - \eta(f)|^m) \leq \epsilon_m(n) \qquad \text{with} \quad \sum_{n\geq 0} \epsilon_m(n) < \infty$$

We let $V_n := W_n(\zeta)$ be the sequence of random fields on $\mathcal{B}(S)$ associated with a given weight array function $w \in \mathcal{W}$ and defined in (4.8). We suppose that $w \in \mathcal{W}$ is chosen so that $w_n(0)\overline{\tau}(n)$ tends to 0 as as $n \to \infty$. In this situation, V_n converges in law, as $n \to \infty$ to a Gaussian random field V on $\mathcal{B}(S)$ such that

$$\forall (f,g) \in \mathcal{B}(S)^2 \qquad \mathbb{E}(V(f)V(g)) = \Phi(\eta)\left[C_\eta(f,g)\right]$$

with the local covariance functions

$$C_\eta(f,g) := M_\eta\left[(P_\eta(f) - M_\eta P_\eta(f))\,(P_\eta(g) - M_\eta P_\eta(g))\right]$$

Proof:
Using proposition 4.5, it is clearly sufficient to prove that the random fields

$$W'_n(\zeta) := \sum_{0\leq p\leq n} w_n(p)\ \Delta\mathbb{M}_{p+1} \tag{4.10}$$

converge in law to the Gaussian random field V as $n \to \infty$. To use the Lindeberg central limit theorem for triangular arrays of $\mathbb{R}^d$-valued random variables, we let $f = (f^i)_{1\leq i\leq d} \in \mathcal{B}(S)^d$ be a collection of d-valued functions; and we consider the $\mathbb{R}^d$-valued random variables $W'_n(\zeta)(f) = (W'_n(\zeta)(f^i))_{1\leq i\leq d}$. We further denote by $\mathcal{F}_p$ the σ-field generated by the random variables X_q for any $q \leq p$. By construction, for any functions f and $g \in \mathcal{B}(S)$ and for every $0 \leq p \leq n$ we find that

$$\mathbb{E}(w_n(p)\ \Delta\mathbb{M}_{p+1}(f) \mid \mathcal{F}_p) = 0$$
$$\mathbb{E}(w_n(p)^2\ \Delta\mathbb{M}_{p+1}(f)\Delta\mathbb{M}_{p+1}(g) \mid \mathcal{F}_p) = w_n(p)^2\ C'_p(f,g)(X_p)$$

with the local covariance function

$$C'_p(f,g) := M_{\eta_p}\left[\left(P_{\eta_{p-1}}(f) - M_{\eta_p}P_{\eta_{p-1}}(f_k)\right)\left(P_{\eta_{p-1}}(g) - M_{\eta_p}P_{\eta_{p-1}}(g)\right)\right]$$

Using proposition 4.2, after some tedious but elementary calculations we find that

$$\|C'_p(f,g) - C_\eta(f,g)\| \leq c(\eta)\ \left\{\int |[\eta_{p-1} - \eta]\,(h)|\,\Upsilon^1_\eta((f,g),dh) + \int |[\eta_p - \eta]\,(h)|\,\Upsilon^2_\eta((f,g),dh)\right\}$$

with pair of bounded integral operator Υ^i_η, $i = 1,2$, from $\mathcal{B}(S)^2$ into $\mathcal{B}(S')$ such that

$$\int\ \|h\|\ \Upsilon^i_\eta((f,g),dh) \leq c(\eta)\ \|f\|\|g\|$$

In the above displayed formula, $c(\eta) < \infty$ stands for a finite constant whose value only depends on the measure η. Under our assumptions, the following almost sure convergence result readily follows

$$\lim_{p\to\infty} \|C'_p(f,g) - C_\mu(f,g)\| = 0 \tag{4.11}$$

We let π_n be the distribution flow of the random variables X_n. We proved in [3] that the quantities

$$\frac{1}{n+1} \sum_{0 \le p \le n} (h(X_p) - \Phi(\eta_p)(h))$$

converge almost surely to 0 for any function $h \in \mathcal{B}(S)$ as $n \to \infty$. On the other hand, we also have the almost sure convergence result

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{0\le p\le n} \Phi(\eta_p)(h) = \Phi(\eta)(h)$$

This also implies that

$$\lim_{n\to\infty} \mathbb{E}\left(\frac{1}{n+1} \sum_{0\le p\le n} h(X_p) \right) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{0\le p\le n} \pi_p(h) = \Phi(\eta)(h)$$

from which we conclude that $\lim_{n\to\infty} \pi_n(h) = \Phi(\eta)(h)$. On the other hand, using the Burkholder-Davis-Gundy type inequalities for martingales, we find that for any weight array function $(v_n(p))_{0\le p\le n}$ and any $m \ge 1$, we have

$$\mathbb{E}\left(\left| \sum_{0\le p\le n} v_n(p)(h(X_p) - \pi_p(h)) \right|^m \right)^{\frac{1}{m}} \le a(m) \left(\sum_{0\le p\le n} v_n(p)^2 \right)^{\frac{1}{2}} \operatorname{osc}(h)$$

for some constants $a(m)$ whose values only depend on the parameter m. Thus, if we set $v_n(p) = w_n(p)^2$, we find that

$$\begin{aligned} & \mathbb{E}\left(\left| \textstyle\sum_{0\le p\le n} w_n^2(p)(h(X_p) - \pi_p(h)) \right|^m \right)^{\frac{1}{m}} \\ & \le a(m) \left(\textstyle\sum_{0\le p\le n} w_n(p)^4 \right)^{\frac{1}{2}} \operatorname{osc}(h) \\ & \le a(m)\ w_n(0) \left(\textstyle\sum_{0\le p\le n} w_n(p)^2 \right)^{\frac{1}{2}} \operatorname{osc}(h) \end{aligned}$$

Under our assumptions on the weight functions w, if we take $h = C_\eta(f,g)$ then by (4.11) we obtain the following almost sure convergence result

$$\begin{aligned} \lim_{n\to\infty} \sum_{p=0}^{n} w_n(p)^2\ C'_p(f,g)(X_p) &= \lim_{n\to\infty} \sum_{p=0}^{n} w_n(p)^2\ C_\eta(f,g)(X_p) \\ &= \lim_{n\to\infty} \sum_{p=0}^{n} w_n(p)^2\ \pi_p(C_\eta(f,g)) = \Phi(\eta)(C_\eta(f,g)) \end{aligned}$$

Therefore, we also have the almost sure convergence result

$$\lim_{n\to\infty} \sum_{p=0}^{n} w_n(p)^2\ \mathbb{E}(\Delta\mathbb{M}_{p+1}(f)\Delta\mathbb{M}_{p+1}(g) \mid \mathcal{F}_p) = \Phi(\eta)(C_\eta(f,g))$$

Since we have $\vee_{0\le p\le n} w_n(p) = w_n(0) \to 0$, as $n \to \infty$, the Lindeberg condition is satisfied and we conclude that the sequence of random fields $W'_n(\zeta)$ defined in (4.10) converges in law to the Gaussian random field V as $n \to \infty$. This ends the proof of the theorem. ■

We end this section with an alternative and simpler representation of the covariance function of the random field V presented in proposition 4.6. We have

$$C_\eta(f,f)(x) = \int M_\eta(x,dy)\ \left[P_\eta(f)(y) - M_\eta(P_\eta(f))(x)\right]^2$$

Using the decomposition

$$\begin{aligned} P_\eta(f)(y) - M_\eta(P_\eta(f))(x) &= [P_\eta(f)(y) - P_\eta(f)(x)] + [P_\eta(f)(x) - M_\eta(P_\eta(f))(x)] \\ &= [P_\eta(f)(y) - P_\eta(f)(x)] + [f(x) - \Phi(\eta)(f)] \end{aligned}$$

and the fact that

$$\begin{aligned} \int M_\eta(x,dy)\ [P_\eta(f)(y) - P_\eta(f)(x)] &= [M_\eta(P_\eta(f))(x) - P_\eta(f)(x)] \\ &= -[f(x) - \Phi(\eta)(f)] \end{aligned}$$

we readily prove the formula

$$C_\eta(f,f)(x) = \int M_\eta(x,dy)\ \left[P_\eta(f)(y) - P_\eta(f)(x)\right]^2 - \left[f(x) - \Phi(\eta)(f)\right]^2$$

On the other hand, recalling that $\Phi(\eta) = \Phi(\eta)M_\eta$ and using again the Poisson equation we also have

$$\int \Phi(\eta)(dx)M_\eta(x,dy)\ \left[P_\eta(f)(y) - P_\eta(f)(x)\right]^2 = 2\ \Phi(\eta)\left[P_\eta(f)\ (f - \Phi(\eta)(f))\right]$$

and

$$\begin{aligned} &2\ \Phi(\eta)\left[P_\eta(f)\ (f - \Phi(\eta)(f))\right] \\ &= 2\ \Phi(\eta)\left[(f - \Phi(\eta)(f))^2\right] + 2\textstyle\sum_{n\geq 1}\ \Phi(\eta)\left[M_\eta^n(f - \Phi(\eta)(f))\ (f - \Phi(\eta)(f))\right] \end{aligned}$$

In summary, we have proved the following proposition.

Proposition 4.7 *The limiting covariance function presented in theorem 4.6 is alternatively defined for any function $f \in \mathcal{B}(S)$ by the following formula*

$$\begin{aligned} &\Phi(\eta)\left[C_\eta(f,f)\right] \\ &= \Phi(\eta)\left[(f - \Phi(\eta)(f))^2\right] + 2\textstyle\sum_{n\geq 1}\ \Phi(\eta)\left[(f - \Phi(\eta)(f))\ M_\eta^n(f - \Phi(\eta)(f))\right] \end{aligned}$$

5 A fluctuation theorem for local interaction fields

5.1 Introduction

This section presents the fluctuation analysis of a class of weighted random fields associated with the i-MCMC model. Following the local fluctuation analysis for time inhomogenous Markov chain models presented in section 4.3, we introduce the following weighted random fields models.

Definition 5.1 *We consider the flow of random measures*

$$\forall l \geq 0 \quad \forall n \geq 0 \qquad \delta_n^{(l)} := \left[\delta_{X_n^{(l)}} - \Phi_l\left(\eta_{n-1}^{(l-1)}\right)\right]$$

For $n = 0$, we use the convention $\Phi_l(\eta_{-1}) = \nu_l$ so that $\delta_0^{(l)} = \left[\delta_{X_0^{(l)}} - \nu_l\right]$. We associate with a sequence of weight array functions $(w^{(l)})_{l\geq 0} \in \mathcal{W}^{\mathbb{N}}$, the flow of random fields $(W_n^{(l)}(\delta^{(l)}))_{l\geq 0}$ on the sets of functions $(\mathcal{B}(S^{(l)}))_{l\geq 0}$ given by

$$\forall l \geq 0 \quad \forall n \geq 0 \qquad W_n^{(l)}(\delta^{(l)}) := \sum_{0\leq p\leq n} w_n^{(l)}(p)\ \delta_p^{(l)}$$

We observe that the regularity conditions (2.3) and (2.4) ensure that the collection of Markov operators $M_\eta^{(l)}$ and their fixed point measures mappings $\Phi_l(\eta)$ satisfy the regularity conditions (4.1) and (4.2) introduced in section 4.1. Also observe that the i-MCMC chain $X^{(l+1)}$ is a time inhomogeneous model of the form (4.3) with a collection of elementary transitions $M^{(l+1)}_{\eta_n^{(l)}}$ that depends on the flow of occupation measures $\eta_n^{(l)}$ associated with the i-MCMC chain at level l. Note that in this situation, condition (4.4) is satisfied with

$$\forall n \geq 0 \qquad \|\eta_n^{(l)} - \eta_{n-1}^{(l)}\| \leq \tau^{(l)}(n) := \frac{2}{n+1}$$

and we have

$$\overline{\tau}^{(l)}(n) := \sum_{0\leq p\leq n} \tau^{(l)}(p) = 2\sum_{0\leq p\leq n} \frac{1}{p+1} \leq 2(1 + \log(n+1))$$

Finally, we recall that for any $m \geq 1$ we have

$$\forall l \geq 0 \quad \forall f \in \mathcal{B}_1(S^{(l)}) \qquad \mathbb{E}(|\eta_n^{(l)}(f) - \pi^{(l)}(f)|^m)^{\frac{1}{m}} \leq b(m)\ c(l)\ \frac{1}{\sqrt{n+1}} \tag{5.1}$$

for a collection of finite constants $b(m)$ whose values only depend on the parameter m (see for instance [3]). Using theorem 4.6, we can prove that the random fields

$$V_n^{(l)} := W_n^{(l)}(\delta^{(l)}) \tag{5.2}$$

associated with a given weight array function $w^{(l)} \in \mathcal{W}$ where

$$\lim_{n\to\infty} (w_n^{(l)}(0)\log(n)) = 0$$

converges in law to a Gaussian random field $V^{(l)}$ as $n \to \infty$ such that

$$\forall (f,g) \in \mathcal{B}(S^{(l)})^2 \qquad \mathbb{E}(V^{(l)}(f)V^{(l)}(g)) = \pi^{(l)}\left[C^{(l)}(f,g)\right] \tag{5.3}$$

In the above display, the covariance functions $C^{(l)}(f,g)$ are defined in terms of the resolvent operator $P^{(l)}_{\pi^{(l-1)}}$ associated with the Markov transition $M^{(l)}_{\pi^{(l-1)}}$ and the fixed point measure $\Phi_l(\pi^{(l-1)}) = \pi^{(l)}$ with the following formula

$$C^{(l)}(f,g)$$

$$:= M^{(l)}_{\pi^{(l-1)}}\left[\left(P^{(l)}_{\pi^{(l-1)}}(f) - M^{(l)}_{\pi^{(l-1)}}P^{(l)}_{\pi^{(l-1)}}(g)\right)\left(P^{(l)}_{\pi^{(l-1)}}(g) - M^{(l)}_{\pi^{(l-1)}}P^{(l)}_{\pi^{(l-1)}}(g)\right)\right]$$

The main objective of this section is to prove the following theorem.

Theorem 5.2 *We consider a collection of weight array functions* $(w^{(l)})_{l\geq 0} \in \mathcal{W}^{\mathbb{N}}$ *where* $\lim_{n\to\infty}(w^{(l)}_n(0)\log(n)) = 0$ *for any* $l \geq 0$*. In this situation, the corresponding flow of weighted random fields* $(V^{(l)}_n)_{l\geq 0}$ *defined in (5.2), converges in law, as* n *tends to infinity and in the sense of finite dimensional distributions, to a sequence of independent and centered Gaussian fields* $\left(V^{(l)}\right)_{l\geq 0}$ *with covariance functions defined in (5.3).*

Using this result, the proof of the functional central limit theorem 2.1 follows exactly the same arguments as the one we used in the proof of theorem 2.1 presented in section 6 in the recent article [4].

Proof of theorem 2.1 :
We let $\mathbb{S}^k := \mathbb{S}\mathbb{S}^{k-1}$ be the k-th iterate of the mapping $\mathbb{S} : \eta \in \mathcal{M}(S^{(l)})^{\mathbb{N}} \mapsto \mathbb{S}(\eta) = (\mathbb{S}_n(\eta))_{n\geq 0} \in \mathcal{M}(S^{(l)})^{\mathbb{N}}$ defined for any $\eta = (\eta_n)_{n\geq 0} \in \mathcal{M}(S^{(l)})^{\mathbb{N}}$ by

$$\forall n \geq 0 \qquad \mathbb{S}_n(\eta) = \frac{1}{n+1}\sum_{0\leq p\leq n} \eta_p$$

We observe that the time averaged semigroup S^k can be rewritten in terms of the following weighted summations

$$\mathbb{S}^k_n(\eta) = \frac{1}{n+1}\sum_{0\leq p\leq n} s^{(k)}_n(p)\ \eta_p$$

with the weight array functions $s^{(k)}_n := (s^{(k)}_n(p))_{0\leq p\leq n}$ defined by the induction

$$\forall k \geq 1 \quad \forall 0 \leq p \leq n \qquad s^{(k+1)}_n(p) = \sum_{p\leq q\leq n} \frac{1}{(q+1)}\ s^{(k)}_n(q) \quad \text{and} \quad s^{(1)}_n(p) := 1$$

We also known from proposition 6.1 in [4] that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{0\leq q\leq n} s^{(k+1)}_n(q)^2 = (2k)!/k!^2$$

and for any $k \geq 1$, the weight array functions $w^{(k)}$ defined by

$$\forall n \geq 0 \quad \forall 0 \leq p \leq n \qquad w^{(k)}_n(p) := s^{(k)}_n(p)/\sqrt{\sum_{0\leq q\leq n} s^{(k)}_n(q)^2}$$

belong the set $\mathcal{W}$ introduced in definition 4.3, with $\lim_{n\to\infty}(w^{(k)}_n(0)\log(n)) = 0$.

On the other hand, using the proposition 5.2 in [4], we have the following multilevel expansion

$$\eta^{(k)}_n - \pi^{(k)} = \sum_{0\leq l\leq k} \mathbb{S}^{(l+1)}_n(\delta^{(k-l)})\ D_{(k-l)+1,k} + \Xi^{(k)}_n \tag{5.4}$$

where $\Xi^{(k)} = (\Xi^{(k)}_n)_{n\geq 0}$ is a flow of signed and random measures such that

$$\forall m \geq 1 \qquad \sup_{f\in\mathcal{B}_1(S^{(k)})} \mathbb{E}(|\Xi^{(k)}_n(f)|^m)^{\frac{1}{m}} \leq b(m)\ c(k)\ (\log(n+1))^k/(n+1)$$

In the above display, $b(m)$ stand for some constant whose value only depend on the parameter m. This multilevel expansion implies that

$$\sqrt{(n+1)}\left[\eta_n^{(k)}-\pi^{(k)}\right]$$

$$= \sum_{0\leq l\leq k}\sqrt{\tfrac{1}{n+1}\sum_{0\leq q\leq n} s_n^{(l+1)}(q)^2}\; W_n^{(k-l)}(\delta^{(k-l)})\; D_{(k-l)+1,k}+\Xi_n^{(k)}$$

with the weighted distribution flow mappings $W^{(k-l)}$ associated with the weight functions $w^{(l+1)}$ and a remainder signed measure $\Xi_n^{(k)}$ such that

$$\sup_{f\in\mathcal{B}_1(S^{(k)})}\mathbb{E}(|\Xi_n^{(k)}(f)|)\leq c(k)\;(\log{(n+1)})^k/\sqrt{(n+1)}$$

The proof of theorem 2.1 is now a direct consequence of theorem 5.2. This ends the proof of the theorem. ■

5.2 A martingale limit theorem

This section is mainly concerned with the proof of theorem 5.2. We following the same lines of arguments as the ones we used in section 4.3 devoted to the fluctuations of weighted occupation measures associated with time inhomogeous Markov chain models.

Firstly, we fix some notation. For any $k\geq 0$ and any $\mu\in\mathcal{P}(S^{(k-1)})$, we let $P_\mu^{(k)}$ be the resolvent operator associated with the Markov transition $M_\mu^{(k)}$ and the fixed point measure $\Phi_k(\mu)\in\mathcal{P}(S^{(k)})$. We also set

$$p^{(k)}(n_k):=2n_k/(1-m_k(n_k))$$

with the pair of parameters (n_k,m_k) defined in (2.3).

Using proposition 4.5, we find that the weighted measures $W_n^{(k)}(\delta^{(k)})$ satisfy the following decomposition

$$W_n^{(k)}(\delta^{(k)})(f)=\sum_{0\leq p\leq n} w_n^{(k)}(p)\;\Delta\mathbb{M}_{p+1}^{(k)}(f)+\mathbb{L}_n^{(k)}(f)$$

for any $f\in\mathcal{B}(S^{(k)})$ with the martingale increments

$$\Delta\mathbb{M}_{p+1}^{(k)}(f)=\mathbb{M}_{p+1}^{(k)}(f)-\mathbb{M}_p^{(k)}(f)=\left(P^{(k)}_{\eta_{p-1}^{(k-1)}}(f)(X_{p+1}^{(k)})-M^{(k)}_{\eta_p^{(k-1)}}P^{(k)}_{\eta_{p-1}^{(k-1)}}(f)(X_p^{(k)})\right)$$

and the remainder signed measure $\mathbb{L}_n^{(k)}$ which are such that

$$\|\mathbb{L}_n^{(k)}\|\leq\left\{w_n^{(k)}(0)\;\left(1+2p^{(k)}(n_k)(1+\log{(n+1)})\right)\;(2+p^{(k)}(n_k))\;\Lambda\right\}\longrightarrow_{n\to\infty}0$$

We consider a sequence of functions $f=(f^i)_{1\leq i\leq d}$, with $d\geq 1$, and $f^i=(f_k^i)_{k\geq 0}\in\prod_{k\geq 0}\mathcal{B}(S^{(k)})$, and we let $\mathcal{W}^{(n)}(f)=(\mathcal{W}^{(n)}(f^i))_{1\leq i\leq d}$ be the $\mathbb{R}^d$-valued and $\mathcal{F}^{(n)}$-adapted process defined for any $l\geq 0$ and any $1\leq i\leq d$ by

$$\mathcal{W}_l^{(n)}(f^i):=\sum_{0\leq k\leq l} W_n^{(k)}(\delta^{(k)})(f_k^i)$$

From the previous discussion, we find that

$$\mathcal{W}_l^{(n)}(f^i) = \mathcal{M}_l^{(n)}(f^i) + \mathcal{L}_l^{(n)}(f^i)$$

with the $\mathcal{F}^{(n)}$-martingale $\mathcal{M}_l^{(n)}(f^i)$ given below

$$\mathcal{M}_l^{(n)}(f^i) := \sum_{0\leq k\leq l} \Delta\mathcal{M}_k^{(n)}(f^i) \quad \text{with} \quad \Delta\mathcal{M}_k^{(n)}(f^i) := \sum_{0\leq p\leq n} w_n^{(k)}(p)\ \Delta\mathbb{M}_{p+1}^{(k)}(f) \tag{5.5}$$

and the remainder bias type measure $\mathcal{L}_l^{(n)} = \sum_{0\leq k\leq l} \mathbb{L}_n^{(k)}$, such that

$$\lim_{n\to\infty} \|\mathcal{L}_l^{(n)}\| = 0$$

The theorem 5.2 is now a direct consequence of the following proposition (see for instance the arguments used in section 4.2 in [4]).

Proposition 5.3 *The sequence of martingales $\mathcal{M}_l^{(n)}(f)$ defined in (5.5) converges in law, as $n\to\infty$, to an $\mathbb{R}^d$-valued and Gaussian martingale $\mathcal{M}_l(f) = (\mathcal{M}_l(f^i))_{1\leq i\leq d}$ such that for any $l\geq 0$ and any pair of indexes $1\leq i,j\leq d$*

$$\langle \mathcal{M}(f^i), \mathcal{M}(f^j)\rangle_l = \sum_{0\leq k\leq l} \pi^{(k)}\left[C^{(k)}(f^i,f^j)\right]$$

with the local covariance functions $\pi^{(k)}\left[C^{(k)}(f^i,f^j)\right]$ defined in (5.3).

Proof:
The proof of the proposition follows the same lines of arguments as those we used in the proof of theorem 4.6. Firstly, we consider the decomposition

$$\mathcal{M}_l^{(n)}(f^i) = \sum_{i=0}^{l(n+1)+n} \mathcal{V}_i^{(n)}(f)$$

where for every $0\leq i\leq l(n+1)+n$, with $i=k(n+1)+p$ for some $0\leq k\leq l$, and $0\leq p\leq n$

$$\mathcal{V}_i^{(n)}(f) := w_n^{(k)}(p)\ \Delta\mathbb{M}_{p+1}^{(k)}(f_k)$$

We further denote by $\mathcal{G}_i^{(n)}$ the σ-field generated by the pair of random variables $(X_p^{(k)}, X_{p+1}^{(k)})$ for any pair of parameters (k,p) such that $k(n+1)+p\leq i$. By construction, for any flow of functions $f=(f_l)_{l\geq 0}$ and $g=(g_l)_{l\geq 0}\in\prod_{l\geq 0}\mathcal{B}(S^{(l)})$ and for every $0\leq i\leq l(n+1)+n$, with $i=k(n+1)+p$ for some $0\leq k\leq l$, and $0\leq p\leq n$, we find that

$$\begin{aligned}\mathbb{E}(\mathcal{V}_i^{(n)}(f)\mid\mathcal{G}_{i-1}^{(n)}) &= 0\\ \mathbb{E}(\mathcal{V}_i^{(n)}(f)\mathcal{V}_i^{(n)}(g)\mid\mathcal{G}_{i-1}^{(n)}) &= w_n^{(k)}(p)^2\ C_p^{(k)}(f,g)(X_p^{(k)})\end{aligned}$$

with the local covariance function

$$\begin{aligned}C_p^{(k)}(f,g) := M^{(k)}_{\eta_p^{(k-1)}}\Bigg[&\left(P^{(k)}_{\eta_{p-1}^{(k-1)}}(f_k) - M^{(k)}_{\eta_p^{(k-1)}}P^{(k)}_{\eta_{p-1}^{(k-1)}}(f_k)\right)\\ &\times\left(P^{(k)}_{\eta_{p-1}^{(k-1)}}(g_k) - M^{(k)}_{\eta_p^{(k-1)}}P^{(k)}_{\eta_{p-1}^{(k-1)}}(g_k)\right)\Bigg]\end{aligned}$$

Under our Lipschitz regularity conditions (2.3) and (2.4), the proposition 4.2 applies to the fixed point mappings Φ_k and the resolvent operators $P_\mu^{(k)}$. As in the proof of theorem 4.6, after some tedious but elementary calculations we obtain

$$\|C_p^{(k)}(f,g) - C^{(k)}(f,g)\|$$

$$\leq c(k)\ \left\{\int \left|\left[\eta_{p-1}^{(k-1)} - \pi^{(k-1)}\right](h)\right| \Upsilon^{(k),1}_{\pi^k,\pi^{(k-1)}}((f_k,g_k),dh)\right.$$

$$\left. + \int \left|\left[\eta_{p}^{(k-1)} - \pi^{(k-1)}\right](h)\right| \Upsilon^{(k),2}_{\pi^k,\pi^{(k-1)}}((f_k,g_k),dh)\right\}$$

where $\Upsilon^{(k),i}_{\pi^k,\pi^{(k-1)}}$, $i=1,2$, is a pair of bounded integral operator from $\mathcal{B}(S^{(k)})^2$ into $\mathcal{B}(S^{(k-1)})$ such that

$$\int\ \|h\|\ \Upsilon^{(k),i}_{\pi^k,\pi^{(k-1)}}((f_k,g_k),dh) \leq c(k)\ \|f_k\|\|g_k\|$$

Combining the generalized Minkowski integral inequality with (5.1) we prove the following almost sure converge result

$$\lim_{p\to\infty} \|C_p^{(k)}(f,g) - C^{(k)}(f,g)\| = 0$$

We let $\pi_n^{(k)}$ be the distribution flow of the random variables $X_n^{(k)}$. One the one hand, since the quantities $\eta_n^{(k)}(h)$ converge almost surely to $\pi^{(k)}(h)$, as $n\to\infty$, for every $k\geq 0$ and any function $h\in\mathcal{B}(S^{(k)})$, we also have that

$$\lim_{n\to\infty} \mathbb{E}(\eta_n^{(k)}(h)) = \lim_{n\to\infty} \frac{1}{n+1}\sum_{0\leq p\leq n} \pi_p^{(k)}(h) = \pi^{(k)}(h) \Longrightarrow \lim_{n\to\infty} \pi_n^{(k)}(h) = \pi^{(k)}(h)$$

On the other hand, using the Burkholder-Davis-Gundy type inequalities for martingales, we find that for any weight array function $(v_n(p))_{0\leq p\leq n}$ and any $m\geq 1$, we have

$$\mathbb{E}\left(\left|\sum_{0\leq p\leq n} v_n(p)(h(X_p^{(k)}) - \pi_p^{(k)}(h))\right|^m\right)^{\frac{1}{m}} \leq a(m)\ \left(\sum_{0\leq p\leq n} v_n(p)^2\right)^{\frac{1}{2}}\ \mathrm{osc}(h)$$

for some constants $a(m)$ whose values only depend on the parameter m. Thus, if we set $v_n(p) = w_n(p)^2$, we find that

$$\mathbb{E}\left(\left|\sum_{0\leq p\leq n} w_n^2(p)(h(X_p^{(k)}) - \pi_p^{(k)}(h))\right|^m\right)^{\frac{1}{m}}$$

$$\leq a(m)\ \left(\sum_{0\leq p\leq n}\ w_n(p)^4\right)^{\frac{1}{2}}\ \mathrm{osc}(h)$$

$$\leq a(m)\ w_n(0)\ \left(\sum_{0\leq p\leq n}\ w_n(p)^2\right)^{\frac{1}{2}}\ \mathrm{osc}(h)$$

Under our assumptions on the weight functions $w^{(k)}$, if we take $h = C^{(k)}(f,g)$ then we obtain the following almost sure convergence results

$$\begin{aligned}\lim_{n\to\infty}\sum_{p=0}^{n} w_n^{(k)}(p)^2\ C_p^{(k)}(f,g)(X_p^{(k)}) &= \lim_{n\to\infty}\sum_{p=0}^{n} w_n^{(k)}(p)^2\ C^{(k)}(f,g)(X_p^{(k)})\\ &= \lim_{n\to\infty}\sum_{p=0}^{n} w_n^{(k)}(p)^2\ \pi_p^{(k)}(C^{(k)}(f,g))\\ &= \pi^{(k)}(C^{(k)}(f,g))\end{aligned}$$

This yields the almost sure convergence

$$\lim_{n\to\infty}\langle \mathcal{M}^{(n)}(f), \mathcal{M}^{(n)}(g)\rangle_l = \mathcal{C}_l^{(k)}(f,g) := \sum_{0\leq k\leq l}\pi^{(k)}(C^{(k)}(f,g))$$

Using the same arguments as the ones we used in the proof of theorem 4.4 in [4], we conclude that the $\mathbb{R}^d$-valued martingale $\mathcal{M}_l^{(n)}(f)$ converges in law, as n tends to infinity, to a martingale $\mathcal{M}_l(f)$ with a predictable bracket given for any air of indexes $1 \leq j, j' \leq d$ by

$$\langle \mathcal{M}(f^j), \mathcal{M}(f^{j'})\rangle_l = \mathcal{C}_l^{(k)}(f^j, f^{j'})$$

This ends the proof of the proposition. ∎

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