Transport equations with singular coefficients. Application to the gyro-kinetic models in plasma physics

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Abstract: The subject matter of this paper concerns the asymptotic regimes for transport equations with singular coefficients. Such models arise for example in plasma physics, when dealing with charged particles moving under the action of strong magnetic fields. These regimes are motivated by the magnetic confinement fusion. The stiffness of the coefficients comes from the multi-scale character of the problem. According to the different possible orderings between the typical physical scales (Larmor radius, Debye length, cyclotronic frequency, plasma frequency) we distinguish several regimes. From the mathematical point of view the analysis of such regimes reduces to stability properties for transport equations whose coefficients have different magnitude orders, depending on some small parameter. The main purpose is to derive limit models by letting the small parameter vanish. In the magnetic confinement context these asymptotics can be assimilated to homogenization procedures with respect to the fast cyclotronic movement of particles around the magnetic lines. We justify rigorously the convergence towards these limit models and we investigate the well-posedness of them.

Key-words: Transport equations, Vlasov equation, Gyro-kinetic models

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Résumé : Dans cet article nous étudions le comportement asymptotique des équations de transport dont les coefficients sont des différents ordres de grandeur. Parmi les applications visées on rappelle l'approximation centre-guide et le régime du rayon de Larmor fini dans la physique des plasmas magnétisés. L'objectif est d'identifier les modèles asymptotiques et de justifier rigoureusement ces limites.

Mots-clés : Equations de Vlasov-Maxwell, Approximation centre-guide, Energie modulée

1 Introduction

Motivated by the magnetic confinement fusion, the study of strong magnetic field effect is now of crucial importance. We are concerned with the dynamics of a population of charged particles interacting through electro-magnetic fields. We consider a population of non relativistic electrons whose density is denoted by f. This particle density satisfies the Vlasov equation

$$\partial_t f + \frac{p}{m_{\mathrm{e}}} \cdot \nabla_x f - e\left(E(t,x) + \frac{p}{m_{\mathrm{e}}} \wedge B(t,x)\right) \cdot \nabla_p f = 0$$

where $-e < 0$ is the electron charge and $m_{\mathrm{e}} > 0$ is the electron mass. The self-consistent electro-magnetic field (E, B) verifies the Maxwell equations

$$\partial_t E - c_0^2 \,\mathrm{curl}_x B = \frac{e}{\varepsilon_0} \int_{\mathbb{R}^3} \frac{p}{m_{\mathrm{e}}} f \,\mathrm{d}p, \quad \partial_t B + \mathrm{curl}_x E = 0$$

$$\mathrm{div}_x E = \frac{e}{\varepsilon_0}\left(n - \int_{\mathbb{R}^3} f \,\mathrm{d}p\right), \quad \mathrm{div}_x B = 0.$$

Here ε_0 is the vacuum permittivity, c_0 is the light speed in the vacuum and n is the concentration of the background ion distribution. One of the asymptotic regimes we wish to address here is the gyro-kinetic model with finite Larmor radius. Let us denote by ω_{p} the plasma frequency

$$\omega_{\mathrm{p}}^2 = \frac{e^2 n}{m_{\mathrm{e}} \varepsilon_0}$$

and by ω_{c} the cyclotronic frequency

$$\omega_{\mathrm{c}} = \frac{eB}{m_{\mathrm{e}}}.$$

Assuming that the cyclotronic frequency is much larger than the plasma frequency we deduce that the typical magnetic field magnitude satisfies

$$B = \frac{m_{\mathrm{e}} \omega_{\mathrm{p}}}{e} \cdot \frac{\omega_{\mathrm{c}}}{\omega_{\mathrm{p}}} = \frac{m_{\mathrm{e}} \omega_{\mathrm{p}}}{e} \cdot \frac{1}{\varepsilon}$$

where $\omega_{\mathrm{c}}/\omega_{\mathrm{p}} = 1/\varepsilon$, $0 < \varepsilon \ll 1$. We assume also that the electron momentum in the plane orthogonal to the magnetic field is much larger than the thermal momentum p_{th} given by

$$\frac{p_{\mathrm{th}}^2}{m_{\mathrm{e}}} = K_B T_{\mathrm{th}}$$

where K_{B} is the Boltzmann constant and T_{th} is the temperature. Note that in this case the Larmor radius corresponding to the cyclotronic frequency ω_{c} and the typical momentum $\frac{p_{\mathrm{th}}}{\varepsilon}$ remains of order of the Debye length

$$\rho_{\mathrm{L}} = \frac{p_{\mathrm{th}}}{\varepsilon m_{\mathrm{e}} \omega_{\mathrm{c}}} = \frac{p_{\mathrm{th}}}{m_{\mathrm{e}} \omega_{\mathrm{p}}} = \left(\frac{\varepsilon_0 K_{\mathrm{B}} T_{\mathrm{th}}}{e^2 n}\right)^{1/2} = \lambda_{\mathrm{D}}.$$

This model is called the finite Larmor radius regime. For example in the two dimensional setting and assuming that the magnetic field has a constant direction

$$f = f(t,x,p), \ \ (E,B) = (E_1, E_2, 0, 0, 0, B_3)(t,x), \ \ (t,x,p) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2$$

we are led, up to a multiplicative constant of order 1, to the following Vlasov equation (see [8], [2])

$$\partial_t f^\varepsilon + \frac{p}{\varepsilon} \cdot \nabla_x f^\varepsilon - \left(E^\varepsilon(t,x) + B_3^\varepsilon(t,x) \frac{{}^\perp p}{\varepsilon} \right) \cdot \nabla_p f^\varepsilon = 0 \tag{1}$$

where the notation ${}^\perp p$ stands for $(p_2, -p_1)$ for any $p = (p_1, p_2) \in \mathbb{R}^2$. When the typical momentum is supposed of order of the thermal momentum, the Larmor radius vanishes as the magnetic field becomes very large; we are dealing with the guiding-center approximation. The guiding-center approximation for the Vlasov-Maxwell system was studied in [4] by the modulated energy method, see also [3], [6] for other results obtained by this method. The analysis of the Vlasov or Vlasov-Poisson equations with large external magnetic field have been carried out in [9], [11], [10], [5].

For simplifying we assume that the self-consistent electric field in the Vlasov equation derives from a potential which is determined by solving the Poisson equation

$$E^\varepsilon = \nabla_x \phi^\varepsilon, \ \ \Delta_x \phi^\varepsilon = 1 - \int_{\mathbb{R}^2} f^\varepsilon \, dp.$$

We suppose also that $B_3 = B_3(x)$ is a given stationary external magnetic field. The Vlasov equation leads naturally to problems like

$$\partial_t u^\varepsilon + a(t,y) \cdot \nabla_y u^\varepsilon + \frac{b(y)}{\varepsilon} \cdot \nabla_y u^\varepsilon = 0, \ \ (t,y) \in \mathbb{R}_+ \times \mathbb{R}^m \tag{2}$$

with the initial condition

$$u^\varepsilon(0,y) = u_0^\varepsilon(y), \ \ y \in \mathbb{R}^m.$$

For example (1) can be recast in the form (2) by taking $m = 4, y = (x,p) \in \mathbb{R}^2 \times \mathbb{R}^2$, $u^\varepsilon(t,y) = f^\varepsilon(t,x,p)$, $a(t,y) = -(0,0,E(t,x))$, $b(y) = (p, -B_3(x)^\perp p)$.

In this work we focuss on the linear transport equation (2) when a and b are given smooth fields. Formally, multiplying (2) by ε one gets $b(y) \cdot \nabla_y u^\varepsilon = \mathcal{O}(\varepsilon)$, saying that the variation of u^ε along the trajectories of b vanishes as ε goes to zero. Following this observation it may seem reasonable to interpret the asymptotic $\varepsilon \searrow 0$ in (2) as homogenization procedure with respect to the flow of b. More precisely we appeal here to the ergodic theory.

By Hilbert's method we have the formal expansion

$$u^\varepsilon = u + \varepsilon u_1 + \varepsilon^2 u_2 + ... \tag{3}$$

and thus, plugging the ansatz (3) in (2) yields the equations

$$\varepsilon^{-1} \ : \ b(y) \cdot \nabla_y u = 0 \tag{4}$$

$$\varepsilon^0 \;:\; \partial_t u + a(t,y)\cdot\nabla_y u + b(y)\cdot\nabla_y u_1 = 0 \tag{5}$$

$$\varepsilon^1 \;:\; \partial_t u_1 + a(t,y)\cdot\nabla_y u_1 + b(y)\cdot\nabla_y u_2 = 0 \tag{6}$$

$$\vdots$$

The operator $\mathcal{T} = b(y)\cdot\nabla_y$ will play a crucial role in our analysis: the equation (4) says that at any time $t \in \mathbb{R}_+$ the leading order term in the expansion (3) belongs to the kernel of $\mathcal{T}$. Unfortunately this information (which will be interpreted later on as a constraint) is not sufficient for uniquely determining u. The use of (5) is mandatory, despite the coupling with the next term u_1 in the asymptotic expansion (3). Actually, at least in a first step, we do not need all the information in (5), but only some consequence of it, such that, supplemented by the constraint (4), it will allow us to determine u. Since we need to eliminate u_1 in (5), the idea is to projet (5) at any time $t \in \mathbb{R}_+$ to the orthogonal complement of the image of $\mathcal{T}$, for example in $L^2(\mathbb{R}^m)$. Indeed, we will see that this consequence of (5) together with the constraint (4) provide a well-posed limit model for $u = \lim_{\varepsilon\searrow 0} u^\varepsilon$. And the same procedure applies for computing $u_1, u_2, ...$ For example, once we have determined u, by (5) we know the image by $\mathcal{T}$ of u_1

$$\mathcal{T} u_1 = -\partial_t u - a(t,y)\cdot\nabla_y u. \tag{7}$$

Projecting now (6) on the orthogonal complement of the image of $\mathcal{T}$ we eliminate u_2 and one gets another equation for u_1, which combined to (7) provides a well-posed problem for u_1.

Our paper is organized as follows. In Section 2 we recall some notions of ergodic theory. We introduce the average over a flow associated to a smooth field and we discuss the main properties of this operator. Section 3 is devoted to the study of the limit model. We prove existence, uniqueness and regularity results. The convergence towards the limit model is justified rigorously in Section 4. Based on the concept of prime integrals, an equivalent limit model is derived in Section 5. We end this paper with some examples.

2 Ergodic theory and average over a flow

We assume that $b : \mathbb{R}^m \to \mathbb{R}^m$ is a given field satisfying

$$b \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^m) \tag{8}$$

$$\mathrm{div}_y b = 0 \tag{9}$$

and the growth condition

$$\exists\, C > 0 \;:\; |b(y)| \le C(1 + |y|), \;\; y \in \mathbb{R}^m. \tag{10}$$

Under the above hypotheses the characteristic flow $Y = Y(s;y)$ is well defined

$$\frac{dY}{ds} = b(Y(s;y)), \quad (s,y) \in \mathbb{R} \times \mathbb{R}^m \tag{11}$$

$$Y(0;y) = y, \quad y \in \mathbb{R}^m, \tag{12}$$

and has the regularity $Y \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^m)$. By (9) we deduce that for any $s \in \mathbb{R}$, the map $y \to Y(s;y)$ is measure preserving

$$\int_{\mathbb{R}^m} \theta(Y(s;y)) \, \mathrm{d}y = \int_{\mathbb{R}^m} \theta(y) \, \mathrm{d}y, \ \ \forall \, \theta \in L^1(\mathbb{R}^m).$$

We have the following standard result concerning the kernel of $u \to \mathcal{T} u = \mathrm{div}_y(b(y)u(y))$.

Proposition 2.1 *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$. Then $\mathrm{div}_y(b(y)u(y)) = 0$ in $\mathcal{D}'(\mathbb{R}^m)$ iff for any $s \in \mathbb{R}$ we have $u(Y(s;y)) = u(y)$ for a.a. $y \in \mathbb{R}^m$.*

Remark 2.1 *Sometimes we will write $u \in \ker \mathcal{T}$ meaning that u is constant along the characteristics, i.e., $u(Y(s;y)) = u(y)$ for all $s \in \mathbb{R}$ and a.a. $y \in \mathbb{R}^m$.*

For any $q \in [1,+\infty]$ we denote by $\mathcal{T}_q$ the linear operator defined by $\mathcal{T}_q u = \mathrm{div}_y(b(y)u(y))$ for any u in the domain

$$\mathrm{D}_q = \{u \in L^q(\mathbb{R}^m) \ : \ \mathrm{div}_y(b(y)u(y)) \in L^q(\mathbb{R}^m)\}.$$

Thanks to Proposition 2.1 we have for any $q \in [1,+\infty]$

$$\ker \mathcal{T}_q = \{u \in L^q(\mathbb{R}^m) \ : \ u(Y(s;y)) = u(y), \ s \in \mathbb{R}, \ \text{a.e. } y \in \mathbb{R}^m\}.$$

For any continuous function $h \in C([a,b];L^q(\mathbb{R}^m))$, with $q \in [1,+\infty]$, we denote by $\int_a^b h(t) \, \mathrm{d}t \in L^q(\mathbb{R}^m)$ the Riemann integral of the function $t \to h(t) \in L^q(\mathbb{R}^m)$ on the interval $[a,b]$. It is easily seen by the construction of the Riemann integral that for any function $\varphi \in L^{q'}(\mathbb{R}^m)$ (where $1/q + 1/q' = 1$) we have

$$\int_{\mathbb{R}^m} \left(\int_a^b h(t) \, \mathrm{d}t \right)(y) \varphi(y) \, \mathrm{d}y = \int_a^b \left(\int_{\mathbb{R}^m} h(t,y)\varphi(y) \, \mathrm{d}y \right) \, \mathrm{d}t \tag{13}$$

implying that

$$\left\| \int_a^b h(t) \, \mathrm{d}t \right\|_{L^q(\mathbb{R}^m)} \le \int_a^b \|h(t)\|_{L^q(\mathbb{R}^m)} \, \mathrm{d}t.$$

Moreover, by Fubini theorem we have

$$\int_a^b \left(\int_{\mathbb{R}^m} h(t,y)\varphi(y) \, \mathrm{d}y \right) \, \mathrm{d}t = \int_{\mathbb{R}^m} \left(\int_a^b h(t,y) \, \mathrm{d}t \right) \varphi(y) \, \mathrm{d}y$$

which together with (13) yields

$$\left(\int_a^b h(t)\ \mathrm{d}t\right)(y) = \int_a^b h(t,y)\ \mathrm{d}t, \ \text{ a.e. } y \in \mathbb{R}^m.$$

Consider now a function $u \in L^q(\mathbb{R}^m)$. Observing that for any $q \in [1,+\infty)$ the application $s \to u(Y(s;\cdot))$ belongs to $C(\mathbb{R}; L^q(\mathbb{R}^m))$, we deduce that for any $T > 0$ the function $\langle u\rangle_T := \frac{1}{T}\int_0^T u(Y(s;\cdot))\ \mathrm{d}s$ is well defined as a element of $L^q(\mathbb{R}^m)$ and $\|\langle u\rangle_T\|_{L^q(\mathbb{R}^m)} \leq \|u\|_{L^q(\mathbb{R}^m)}$. Observe that for any function $h \in L^\infty([a,b]; L^\infty(\mathbb{R}^m))$, the map $\varphi \in L^1(\mathbb{R}^m) \to \int_a^b \int_{\mathbb{R}^m} h(t,y)\varphi(y)\ \mathrm{d}y\ \mathrm{d}t$ belongs to $(L^1(\mathbb{R}^m))\,' = L^\infty(\mathbb{R}^m)$. Therefore there is a unique function in $L^\infty(\mathbb{R}^m)$, denoted $\int_a^b h(t)\ \mathrm{d}t$, such that for any $\varphi \in L^1(\mathbb{R}^m)$ we have

$$\int_{\mathbb{R}^m}\left(\int_a^b h(t)\ \mathrm{d}t\right)(y)\varphi(y)\ \mathrm{d}y = \int_a^b \left(\int_{\mathbb{R}^m} h(t,y)\varphi(y)\ \mathrm{d}y\right)\ \mathrm{d}t.$$

In particular we have

$$\left\|\int_a^b h(t)\ \mathrm{d}t\right\|_{L^\infty(\mathbb{R}^m)} \leq \int_a^b \|h(t)\|_{L^\infty(\mathbb{R}^m)}\ \mathrm{d}t$$

and as before

$$\left(\int_a^b h(t)\ \mathrm{d}t\right)(y) = \int_a^b h(t,y)\ \mathrm{d}t, \ \text{ a.e. } y \in \mathbb{R}^m.$$

Notice that for any function $u \in L^\infty(\mathbb{R}^m)$, the map $s \to u(Y(s;\cdot))$ belongs to $L^\infty(\mathbb{R}; L^\infty(\mathbb{R}^m))$ and thus we deduce that for any $T > 0$ the function $\langle u\rangle_T := \frac{1}{T}\int_0^T u(Y(s;\cdot))\ \mathrm{d}s$ is well defined as a element of $L^\infty(\mathbb{R}^m)$ and $\|\langle u\rangle_T\|_{L^\infty(\mathbb{R}^m)} \leq \|u\|_{L^\infty(\mathbb{R}^m)}$.

Obviously, when u belongs to $\ker \mathcal{T}_q$ we have $\langle u\rangle_T = u$ for any $q \in [1,+\infty]$ and $T > 0$. Generally, when $q \in (1,+\infty)$ we prove the weak convergence of $\langle u\rangle_T$ as T goes to $+\infty$ towards some element in $\ker \mathcal{T}_q$.

Proposition 2.2 *Assume that $q \in (1,+\infty)$ and $u \in L^q(\mathbb{R}^m)$. Then there is a unique function $\langle u\rangle \in \ker \mathcal{T}_q$ such that for any $\varphi \in \ker \mathcal{T}_{q\,'}$ we have*

$$\int_{\mathbb{R}^m}(u(y) - \langle u\rangle(y))\varphi(y)\ \mathrm{d}y = 0. \tag{14}$$

Moreover we have the weak convergences in $L^q(\mathbb{R}^m)$

$$\langle u\rangle = \lim_{T\to+\infty}\frac{1}{T}\int_0^T u(Y(s;\cdot))\ \mathrm{d}s = \lim_{T\to+\infty}\frac{1}{T}\int_{-T}^0 u(Y(s;\cdot))\ \mathrm{d}s = \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^T u(Y(s;\cdot))\ \mathrm{d}s$$

and the inequality $\|\langle u\rangle\|_{L^q(\mathbb{R}^m)} \leq \|u\|_{L^q(\mathbb{R}^m)}$. In particular the application $u \in L^q(\mathbb{R}^m) \to \langle u\rangle \in L^q(\mathbb{R}^m)$ is linear, continuous and $\|\langle\cdot\rangle\|_{\mathcal{L}(L^q(\mathbb{R}^m),L^q(\mathbb{R}^m))} \leq 1$.

Proof. We start by checking the uniqueness. Consider two functions $u_1, u_2 \in \ker \mathcal{T}_q$ satisfying

$$\int_{\mathbb{R}^m} (u(y) - u_1(y))\varphi(y)\ \mathrm{d}y = \int_{\mathbb{R}^m} (u(y) - u_2(y))\varphi(y)\ \mathrm{d}y = 0$$

for any $\varphi \in \ker \mathcal{T}_{q\,'}$. We deduce that

$$\int_{\mathbb{R}^m} (u_1(y) - u_2(y))\varphi(y)\ \mathrm{d}y = 0, \ \ \forall\, \varphi \in \ker \mathcal{T}_{q\,'}.$$

Taking $\varphi = |u_1 - u_2|^{q-2}(u_1 - u_2) \in \ker \mathcal{T}_{q\,'}$ we deduce that $\int_{\mathbb{R}^m} |u_1 - u_2|^q\ \mathrm{d}y = 0$ saying that $u_1 = u_2$. In order to justify the existence of $\langle u \rangle$ consider a sequence $(T_n)_n$ such that $\lim_{n\to+\infty} T_n = +\infty$ and $(\langle u \rangle_{T_n})_n$ converges weakly in $L^q(\mathbb{R}^m)$ towards some function $\tilde{u} \in L^q(\mathbb{R}^m)$. Observe that $\tilde{u} \in \ker \mathcal{T}_q$. For this it is sufficient to prove that for any $t \in \mathbb{R}$ and $\psi \in L^{q\,'}(\mathbb{R}^m)$ we have

$$\int_{\mathbb{R}^m} \tilde{u}(y)\psi(Y(-t;y))\ \mathrm{d}y = \int_{\mathbb{R}^m} \tilde{u}(y)\psi(y)\ \mathrm{d}y. \tag{15}$$

Indeed, by using the weak convergence $\lim_{n\to+\infty} \langle u \rangle_{T_n} = \tilde{u}$ we deduce

$$\begin{aligned}
\int_{\mathbb{R}^m} \tilde{u}(y)\psi(Y(-t;y))\ \mathrm{d}y \;&=\; \lim_{n\to+\infty} \int_{\mathbb{R}^m} \langle u \rangle_{T_n}(y)\psi(Y(-t;y))\ \mathrm{d}y\\
&=\; \lim_{n\to+\infty} \frac{1}{T_n}\int_0^{T_n}\int_{\mathbb{R}^m} u(Y(s;y))\psi(Y(-t;y))\ \mathrm{d}y\ \mathrm{d}s\\
&=\; \lim_{n\to+\infty} \frac{1}{T_n}\int_0^{T_n}\int_{\mathbb{R}^m} u(Y(s+t;y))\psi(y)\ \mathrm{d}y\ \mathrm{d}s\\
&=\; \lim_{n\to+\infty} \frac{1}{T_n}\int_t^{t+T_n}\int_{\mathbb{R}^m} u(Y(s;y))\psi(y)\ \mathrm{d}y\ \mathrm{d}s\\
&=\; \lim_{n\to+\infty} \frac{1}{T_n}\int_{T_n}^{t+T_n}\int_{\mathbb{R}^m} u(Y(s;y))\psi(y)\ \mathrm{d}y\ \mathrm{d}s\\
&-\; \lim_{n\to+\infty} \frac{1}{T_n}\int_0^{t}\int_{\mathbb{R}^m} u(Y(s;y))\psi(y)\ \mathrm{d}y\ \mathrm{d}s\\
&+\; \lim_{n\to+\infty} \int_{\mathbb{R}^m} \langle u \rangle_{T_n}(y)\psi(y)\ \mathrm{d}y.
\end{aligned} \tag{16}$$

It is easily seen that

$$\frac{1}{T_n}\left|\int_{T_n}^{t+T_n}\int_{\mathbb{R}^m} u(Y(s;y))\psi(y)\ \mathrm{d}y\ \mathrm{d}s\right| \le \frac{|t|}{T_n}\|u\|_{L^q(\mathbb{R}^m)}\|\psi\|_{L^{q\,'}(\mathbb{R}^m)} \tag{17}$$

and

$$\frac{1}{T_n}\left|\int_0^{t}\int_{\mathbb{R}^m} u(Y(s;y))\psi(y)\ \mathrm{d}y\ \mathrm{d}s\right| \le \frac{|t|}{T_n}\|u\|_{L^q(\mathbb{R}^m)}\|\psi\|_{L^{q\,'}(\mathbb{R}^m)}. \tag{18}$$

Combining (16), (17), (18) yields (15), implying that

$$\tilde{u}(Y(s;y)) = \tilde{u}(y), \quad s \in \mathbb{R}, \quad \text{a.e. } y \in \mathbb{R}^m.$$

We claim that $\tilde{u}$ satisfies (14). For any $\varphi \in \ker \mathcal{T}_{q\,'}$ and $s \in \mathbb{R}$ we have $u\varphi \in L^1(\mathbb{R}^m)$ and thus by change of variable along the characteristics we obtain

$$\int_{\mathbb{R}^m} u(y)\varphi(y) \, \mathrm{d}y = \int_{\mathbb{R}^m} u(Y(s;y))\varphi(Y(s;y)) \, \mathrm{d}y = \int_{\mathbb{R}^m} u(Y(s;y))\varphi(y) \, \mathrm{d}y.$$

Taking the average on $[0, T_n]$ one gets

$$\int_{\mathbb{R}^m} u(y)\varphi(y) \, \mathrm{d}y = \int_{\mathbb{R}^m} \left(\frac{1}{T_n} \int_0^{T_n} u(Y(s;\cdot)) \, \mathrm{d}s \right)(y)\varphi(y) \, \mathrm{d}y = \int_{\mathbb{R}^m} \langle u \rangle_{T_n}(y)\varphi(y) \, \mathrm{d}y.$$

Since $\varphi \in L^{q\,'}(\mathbb{R}^m)$ we obtain thanks to the weak convergence $\lim_{n\to+\infty} \langle u \rangle_{T_n} = \tilde{u}$ in $L^q(\mathbb{R}^m)$ that

$$\int_{\mathbb{R}^m} (u(y) - \tilde{u}(y))\varphi(y) \, \mathrm{d}y = 0, \quad \forall \, \varphi \in \ker \mathcal{T}_{q\,'}.$$

Therefore the existence of the element $\langle u \rangle$ in (14) is guaranteed, and by the uniqueness of such element we deduce also the convergence $\lim_{T\to+\infty} \frac{1}{T} \int_0^T u(Y(s;\cdot)) \, \mathrm{d}s = \langle u \rangle$ weakly in $L^q(\mathbb{R}^m)$. Similarly one gets

$$\lim_{T\to+\infty} \frac{1}{T} \int_{-T}^0 u(Y(s;\cdot)) \, \mathrm{d}s = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^T u(Y(s;\cdot)) \, \mathrm{d}s = \langle u \rangle \;\text{ weakly in } L^q(\mathbb{R}^m).$$

Since for any $T > 0$ we have $\|\langle u \rangle_T\|_{L^q(\mathbb{R}^m)} \leq \|u\|_{L^q(\mathbb{R}^m)}$ we deduce that $\|\langle u \rangle\|_{L^q(\mathbb{R}^m)} \leq \|u\|_{L^q(\mathbb{R}^m)}$. The linearity of $\langle \cdot \rangle$ follows immediately and we have $\|\langle \cdot \rangle\|_{\mathcal{L}(L^q(\mathbb{R}^m), L^q(\mathbb{R}^m))} \leq 1$.
□

Corollary 2.1 *Assume that $q \in (1, +\infty)$ and $u \in L^q(\mathbb{R}^m)$. Let us denote by $\langle u \rangle \in L^q(\mathbb{R}^m)$ the function constructed in Proposition 2.2.*
a) If $u \geq -M$ for some constant $M \geq 0$ then $\langle u \rangle \geq -M$.
b) If $u \leq M$ for some constant $M \geq 0$ then $\langle u \rangle \leq M$.

Proof. a) For any $T > 0$ and a.a. $y \in \mathbb{R}^m$ we have $\langle u \rangle_T(y) = \frac{1}{T} \int_0^T u(Y(s;y)) \, \mathrm{d}s \geq -M$. By using the weak convergence $\lim_{T\to+\infty} \langle u \rangle_T = \langle u \rangle$ in $L^q(\mathbb{R}^m)$ we have for any non negative function $\varphi \in C_c(\mathbb{R}^m)$

$$\int_{\mathbb{R}^m} (\langle u \rangle + M)\varphi(y) \, \mathrm{d}y = \lim_{T\to+\infty} \int_{\mathbb{R}^m} (\langle u \rangle_T + M)\varphi(y) \, \mathrm{d}y \geq 0$$

implying that $\langle u \rangle \geq -M$. The second item follows in a similar manner. □

We can prove that the operator $\langle\cdot\rangle$ is local with respect to the trajectories.

Corollary 2.2 *Let $A \subset \mathbb{R}^m$ be a invariant set under the flow Y (i.e., $Y(s;A) \subset A$ for any $s \in \mathbb{R}$). Then for any $u \in L^q(\mathbb{R}^m)$ with $q \in (1,+\infty)$ we have*

$$\langle \mathbf{1}_A u\rangle = \mathbf{1}_A \langle u\rangle.$$

In particular if $u_1, u_2 \in L^q(\mathbb{R}^m)$ satisfy $u_1 = u_2$ on A, then $\langle u_1\rangle = \langle u_2\rangle$ on A.

Proof. For any $\varphi \in \ker \mathcal{T}_{q'}$ we have $\int_{\mathbb{R}^m}(u - \langle u\rangle)\varphi \,\mathrm{d}y = 0$. Since A is invariant under the flow, the function $\mathbf{1}_A\varphi$ belongs to $\ker \mathcal{T}_{q'}$ and thus $\int_{\mathbb{R}^m}(u-\langle u\rangle)\mathbf{1}_A\varphi \,\mathrm{d}y = 0$ which says that $\langle \mathbf{1}_A u\rangle = \mathbf{1}_A\langle u\rangle$. If $u_1, u_2 \in L^q(\mathbb{R}^m)$ coincide on A then $\mathbf{1}_A(u_1 - u_2) = 0$. Consequently we have

$$\mathbf{1}_A\langle u_1 - u_2\rangle = \langle \mathbf{1}_A(u_1 - u_2)\rangle = 0$$

saying that $\langle u_1\rangle = \langle u_2\rangle$ on A. $\square$

Corollary 2.3 *Assume that $1 < q_1 < q_2 < +\infty$ and $u \in L^{q_1}(\mathbb{R}^m) \cap L^{q_2}(\mathbb{R}^m)$. We denote by $\langle u\rangle^{(q)}$ the function of $L^q(\mathbb{R}^m)$ constructed in Proposition 2.2 for $q \in \{q_1, q_2\}$. Then we have $\langle u\rangle^{(q_1)} = \langle u\rangle^{(q_2)} \in \ker \mathcal{T}_{q_1} \cap \ker \mathcal{T}_{q_2}$.*

Proof. For any $T > 0$ and $\varphi \in C_c(\mathbb{R}^m)$ we have

$$\int_{\mathbb{R}^m}\left(\frac{1}{T}\int_0^T u(Y(s;\cdot))\,\mathrm{d}s\right)(y)\varphi(y)\,\mathrm{d}y = \frac{1}{T}\int_0^T\left(\int_{\mathbb{R}^m} u(Y(s;y))\varphi(y)\,\mathrm{d}y\right)\mathrm{d}s. \tag{19}$$

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T u(Y(s;\cdot))\,\mathrm{d}s = \langle u\rangle^{(q_1)} \text{ weakly in } L^{q_1}(\mathbb{R}^m)$$

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T u(Y(s;\cdot))\,\mathrm{d}s = \langle u\rangle^{(q_2)} \text{ weakly in } L^{q_2}(\mathbb{R}^m).$$

Therefore, passing to the limit for $T \to +\infty$ in (19) yields

$$\int_{\mathbb{R}^m}\langle u\rangle^{(q_1)}\varphi(y)\,\mathrm{d}y = \lim_{T\to+\infty}\frac{1}{T}\int_0^T\int_{\mathbb{R}^m} u(Y(s;y))\varphi(y)\,\mathrm{d}y\,\mathrm{d}s = \int_{\mathbb{R}^m}\langle u\rangle^{(q_2)}\varphi(y)\,\mathrm{d}y$$

implying that $\langle u\rangle^{(q_1)} = \langle u\rangle^{(q_2)} \in \ker \mathcal{T}_{q_1} \cap \ker \mathcal{T}_{q_2}$. $\square$

It is possible to prove that the convergences in Proposition 2.2 are strong. This is the object of the next proposition.

Proposition 2.3 *Assume that $q \in (1,+\infty)$ and $u \in L^q(\mathbb{R}^m)$. Then*

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T u(Y(s;\cdot))\,\mathrm{d}s = \lim_{T\to+\infty}\frac{1}{T}\int_{-T}^0 u(Y(s;\cdot))\,\mathrm{d}s = \langle u\rangle \textit{ strongly in } L^q(\mathbb{R}^m).$$

Proof. We analyze first the case $q = 2$, which corresponds to the mean ergodic theorem, or von Neumann's ergodic theorem (see [12], pp. 57). For the sake of the completeness we detail here its proof. Recall that the adjoint of $\mathcal{T}_2$ satisfies

$$\mathrm{D}(\mathcal{T}_2^\star) = \mathrm{D}(\mathcal{T}_2), \;\; \mathcal{T}_2^\star u = -\mathcal{T}_2 u, \;\; \forall\, u \in \mathrm{D}(\mathcal{T}_2).$$

Therefore we have $\ker \mathcal{T}_2 = \ker \mathcal{T}_2^\star = (\mathrm{range}\mathcal{T}_2)^\perp$, implying the orthogonal decomposition of $L^2(\mathbb{R}^m)$

$$\ker \mathcal{T}_2 \oplus \overline{\mathrm{range}\mathcal{T}_2} = (\mathrm{range}\mathcal{T}_2)^\perp \oplus \left((\mathrm{range}\mathcal{T}_2)^\perp\right)^\perp = L^2(\mathbb{R}^m).$$

By Proposition 2.2 we know that for any $u \in L^2(\mathbb{R}^m)$, the function $\langle u \rangle^{(2)}$ is the orthogonal projection of u on the closed subspace $\ker \mathcal{T}_2$ and thus we have the decomposition $u = \langle u \rangle^{(2)} + (u - \langle u \rangle^{(2)})$ with $\langle u \rangle^{(2)} \in \ker \mathcal{T}_2$ and $u - \langle u \rangle^{(2)} \in \overline{\mathrm{range}\mathcal{T}_2}$. As seen before, for any $T > 0$ we have

$$\langle \langle u \rangle^{(2)} \rangle_T = \frac{1}{T} \int_0^T \langle u \rangle^{(2)}(Y(s;\cdot))\; \mathrm{d}s = \langle u \rangle^{(2)}$$

and thus

$$\lim_{T\to+\infty} \langle u \rangle_T = \langle u \rangle^{(2)} + \lim_{T\to+\infty} \langle u - \langle u \rangle^{(2)} \rangle_T, \;\; \text{strongly in } L^2(\mathbb{R}^m).$$

In order to prove that $\lim_{T\to+\infty} \langle u \rangle_T = \langle u \rangle^{(2)}$ strongly in $L^2(\mathbb{R}^m)$ it remains to check that $\lim_{T\to+\infty} \langle v \rangle_T = 0$, strongly in $L^2(\mathbb{R}^m)$ for any $v \in \overline{\mathrm{range}\mathcal{T}_2}$. Consider first $v = \mathcal{T}_2 w$ for some $w \in \mathrm{D}_2$. Let us consider a sequence $(w_n)_n \subset C^1_c(\mathbb{R}^m)$ such that

$$\lim_{n\to+\infty} (w_n, \mathcal{T}_2 w_n) = (w, \mathcal{T}_2 w), \;\; \text{strongly in } L^2(\mathbb{R}^m).$$

We have for any $y \in \mathbb{R}^m$

$$\begin{aligned}
\langle \mathcal{T}_2 w_n \rangle_T(y) &= \frac{1}{T} \int_0^T (\mathcal{T}_2 w_n)(Y(s;y))\; \mathrm{d}s \\
&= \frac{1}{T} \int_0^T \frac{d}{ds} \{w_n(Y(s;y))\}\; \mathrm{d}s \\
&= \frac{1}{T} (w_n(Y(T;y)) - w_n(y))
\end{aligned}$$

and therefore

$$\|\langle \mathcal{T}_2 w_n \rangle_T\|_{L^2(\mathbb{R}^m)} \le \frac{2}{T} \|w_n\|_{L^2(\mathbb{R}^m)}.$$

Passing to the limit for $n \to +\infty$ one gets $\|\langle v \rangle_T\|_{L^2(\mathbb{R}^m)} \le \frac{2}{T} \|w\|_{L^2(\mathbb{R}^m)}$, implying that $\lim_{T\to+\infty} \langle v \rangle_T = 0$ strongly in $L^2(\mathbb{R}^m)$. Consider now a function $v \in \overline{\mathrm{range}\mathcal{T}_2}$. For any $\delta > 0$ there exists $v_\delta \in \mathrm{range}\mathcal{T}_2$ such that $\|v - v_\delta\|_{L^2(\mathbb{R}^m)} < \delta$. We have

$$\begin{aligned}
\|\langle v \rangle_T\|_{L^2(\mathbb{R}^m)} &\le \|\langle v - v_\delta \rangle_T\|_{L^2(\mathbb{R}^m)} + \|\langle v_\delta \rangle_T\|_{L^2(\mathbb{R}^m)} \\
&\le \|v - v_\delta\|_{L^2(\mathbb{R}^m)} + \|\langle v_\delta \rangle_T\|_{L^2(\mathbb{R}^m)} \\
&\le \delta + \|\langle v_\delta \rangle_T\|_{L^2(\mathbb{R}^m)}.
\end{aligned}$$

Passing to the limit for $T \to +\infty$ we obtain

$$\limsup_{T\to+\infty} \|\langle v\rangle_T\|_{L^2(\mathbb{R}^m)} \le \delta, \;\; \forall\, \delta > 0$$

and consequently $\lim_{T\to+\infty} \|\langle v\rangle_T\|_{L^2(\mathbb{R}^m)} = 0$ for any $v \in \overline{\mathrm{range}\mathcal{T}_2}$.

Consider now the general case $q \in (1, +\infty)$. By density arguments it is sufficient to treat the case of functions $u \in C_c(\mathbb{R}^m)$. Since $C_c(\mathbb{R}^m) \subset L^r(\mathbb{R}^m)$ for any $r \in (1, +\infty)$ we deduce thanks to Corollary 2.3 that $\langle u\rangle \in L^r(\mathbb{R}^m)$ and $\|\langle u\rangle\|_{L^r(\mathbb{R}^m)} \le \|u\|_{L^r(\mathbb{R}^m)}$ for any $r \in (1, +\infty)$. By the previous step we know that $\lim_{T\to+\infty}\langle u\rangle_T = \langle u\rangle$ strongly in $L^2(\mathbb{R}^m)$ and it is easily seen that $\langle u\rangle \in L^1(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$ and satisfies $\|\langle u\rangle\|_{L^1(\mathbb{R}^m)} \le \|u\|_{L^1(\mathbb{R}^m)}, \|\langle u\rangle\|_{L^\infty(\mathbb{R}^m)} \le \|u\|_{L^\infty(\mathbb{R}^m)}$ (use for example the convergence $\lim_{T\to+\infty}\langle u\rangle_T = \langle u\rangle$ in $\mathcal{D}'(\mathbb{R}^m)$ and the bounds $\|\langle u\rangle_T\|_{L^1(\mathbb{R}^m)} \le \|u\|_{L^1(\mathbb{R}^m)}$, $\|\langle u\rangle_T\|_{L^\infty(\mathbb{R}^m)} \le \|u\|_{L^\infty(\mathbb{R}^m)}$ for any $T > 0$). If $q \in (1, 2)$ we have by interpolation inequalities

$$\begin{aligned}
\|\langle u\rangle_T - \langle u\rangle\|_{L^q(\mathbb{R}^m)} &\le \|\langle u\rangle_T - \langle u\rangle\|_{L^1(\mathbb{R}^m)}^{\frac{2}{q}-1} \|\langle u\rangle_T - \langle u\rangle\|_{L^2(\mathbb{R}^m)}^{2-\frac{2}{q}} \\
&\le \left(2\|u\|_{L^1(\mathbb{R}^m)}\right)^{\frac{2}{q}-1} \|\langle u\rangle_T - \langle u\rangle\|_{L^2(\mathbb{R}^m)}^{2-\frac{2}{q}} \to 0 \;\text{ as } T \to +\infty.
\end{aligned}$$

If $q \in (2, +\infty)$ we have

$$\begin{aligned}
\|\langle u\rangle_T - \langle u\rangle\|_{L^q(\mathbb{R}^m)} &\le \|\langle u\rangle_T - \langle u\rangle\|_{L^2(\mathbb{R}^m)}^{\frac{2}{q}} \|\langle u\rangle_T - \langle u\rangle\|_{L^\infty(\mathbb{R}^m)}^{1-\frac{2}{q}} \\
&\le \left(2\|u\|_{L^\infty(\mathbb{R}^m)}\right)^{1-\frac{2}{q}} \|\langle u\rangle_T - \langle u\rangle\|_{L^2(\mathbb{R}^m)}^{\frac{2}{q}} \to 0 \;\text{ as } T \to +\infty.
\end{aligned}$$

□

It is also possible to define the operator $\langle\cdot\rangle$ for functions in $L^1(\mathbb{R}^m)$ and $L^\infty(\mathbb{R}^m)$. These constructions are a little bit more delicate and require some additional hypotheses on the flow. As usual we introduce the relation on $\mathbb{R}^m \times \mathbb{R}^m$ given by

$$y_1 \sim y_2 \;\text{ iff }\; \exists\, s \in \mathbb{R} \;\text{ such that } y_2 = Y(s; y_1).$$

Using the properties of the flow it is immediate that the above relation is an equivalence relation. The classes of $\mathbb{R}^m$ with respect to $\sim$ are the orbits. For any measurable set $A \subset \mathbb{R}^m$ observe that $\mathbf{1}_A$ is constant along the flow iff A is the union of a certain subset of orbits. We will write also $\mathbf{1}_A \in \ker\mathcal{T}$ for such sets $A \subset \mathbb{R}^m$. Let us denote by $\mathcal{A}$ the family

$$\mathcal{A} = \{A \text{ measurable set of } \mathbb{R}^m \;:\; \mathbf{1}_A \in \ker\mathcal{T}\}.$$

We consider the family $\mathcal{A}_0$ of sets $A \in \mathcal{A}$ such that the only integrable function on A, constant along the flow, is the trivial one. We make the following hypothesis: there are a set $\mathcal{O} \in \mathcal{A}_0$ and a function $\xi : \mathbb{R}^m \setminus \mathcal{O} \to (0, +\infty)$ such that

$$\xi(y) = \xi(Y(s; y)), \;\; s \in \mathbb{R}, \;\; y \in \mathbb{R}^m \setminus \mathcal{O}, \;\; \int_{\mathbb{R}^m\setminus\mathcal{O}} \xi(y)\,\mathrm{d}y < +\infty. \tag{20}$$

We check easily that if such a couple $(\mathcal{O}, \xi)$ exists, then the set $\mathcal{O}$ is unique up to a negligible set. Let us analyze some examples.
Example 1 We consider $m = 2$, $b(y) = (1, 0)$. In this case we have $(Y_1, Y_2)(s; y) = (y_1 + s, y_2)$, $s \in \mathbb{R}$, $y \in \mathbb{R}^2$ and thus the constant functions along the flow are the functions depending only on y_2. We claim that $\mathcal{O} = \mathbb{R}^2$. Indeed, let $f = f(y_2) \in L^1(\mathbb{R}^2)$. Therefore we have

$$\int_{\mathbb{R}^2} |f(y_2)| \, \mathrm{d}y = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y_2)| \, \mathrm{d}y_2 \right) \, \mathrm{d}y_1 < +\infty$$

implying that $\int_{\mathbb{R}} |f(y_2)| \mathrm{d}y_2 = 0$ which says that $f = 0$. In this case (20) is trivially satisfied.
Example 2 We consider $m = 2$, $b(y) = {}^\perp y = (y_2, -y_1)$. The flow is given by

$$Y(s; y) = \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix} y, \quad s \in \mathbb{R}, \quad y \in \mathbb{R}^2$$

and the functions constant along the trajectories are $f = f(|y|)$. In particular $y \to \mathrm{e}^{-|y|}$ belongs to $L^1(\mathbb{R}^2)$ implying that $\mathcal{O} = \emptyset$ and that (20) holds true (with $\xi(y) = \mathrm{e}^{-|y|} > 0$ on $\mathbb{R}^2$).
Example 3 We consider $m = 2$ and $b(y) = (y_2, -\sin y_1)$. It is easily seen that $\psi(y) = \frac{1}{2}(y_2)^2 - \cos y_1$ is constant along the flow. Actually the constant functions along the trajectories are the functions depending only on $\frac{1}{2}(y_2)^2 - \cos y_1 = \psi$. We claim that $\mathcal{O} = \{y \in \mathbb{R}^2 \ : \ \psi(y) > 1\} = \mathcal{O}_1 \cup \mathcal{O}_2$ where

$$\mathcal{O}_1 = \{y \in \mathbb{R}^2 \ : \ y_2 > 2\,|\cos(y_1/2)|\}, \quad \mathcal{O}_2 = \{y \in \mathbb{R}^2 \ : \ y_2 < -2\,|\cos(y_1/2)|\}.$$

Indeed, let $f((y_2)^2/2 - \cos y_1)$ be a function in $L^1(\mathcal{O})$. In particular we have

$$\int_{\mathcal{O}_1} |f((y_2)^2/2 - \cos y_1)| \, \mathrm{d}y < +\infty.$$

Performing the change of variable $x_1 = y_1 \in \mathbb{R}$, $x_2 = (y_2)^2/2 - \cos y_1 > 1$ we obtain

$$\begin{aligned} \int_{\mathcal{O}_1} |f((y_2)^2/2 - \cos y_1)| \, \mathrm{d}y &= \int_{\mathbb{R}} \left(\int_1^{+\infty} \frac{|f(x_2)|}{\sqrt{2(x_2 + \cos x_1)}} \, \mathrm{d}x_2 \right) \, \mathrm{d}x_1 \\ &\geq \int_{\mathbb{R}} \left(\int_1^{+\infty} \frac{|f(x_2)|}{\sqrt{2(x_2 + 1)}} \, \mathrm{d}x_2 \right) \, \mathrm{d}x_1. \end{aligned}$$

Therefore $\int_1^{+\infty} \frac{|f(x_2)|}{\sqrt{2(x_2+1)}} \, \mathrm{d}x_2 = 0$ saying that $f((y_2)^2/2 - \cos y_1) = 0$ on $\mathcal{O}_1$. Similarly we obtain $f((y_2)^2/2 - \cos y_1) = 0$ on $\mathcal{O}_2$. Observe also that (20) holds true. Indeed we have

$$\mathbb{R}^2 \setminus \mathcal{O} = \{y \in \mathbb{R}^2 \ : \ -1 \leq \psi(y) \leq 1\} = \{y \in \mathbb{R}^2 \ : \ |y_2| \leq 2|\cos(y_1/2)|\} = \cup_{k \in \mathbb{Z}} A_k$$

where

$$A_k = A + (2\pi k, 0), \quad A = \{y \in [-\pi, \pi) \times \mathbb{R} \ : \ |y_2| \leq 2|\cos(y_1/2)|\}$$

and $\int_A \; \mathrm{d}y = 16$. Therefore we can consider the function

$$\xi(y) = \sum_{k \in \mathbb{Z}} \frac{1}{2^{|k|}} \mathbf{1}_{A_k}(y)$$

which is strictly positif on $\mathbb{R}^2 \setminus \mathcal{O}$, is constant along the flow and

$$\int_{\mathbb{R}^2 \setminus \mathcal{O}} \xi(y) \; \mathrm{d}y = \sum_{k \in \mathbb{Z}} \frac{1}{2^{|k|}} \cdot 16 = 48 < +\infty.$$

Under the hypothesis (20) we have, for $q = 1$, a similar result as those in Proposition 2.2.

Proposition 2.4 *Assume that (20) holds and $u \in L^1(\mathbb{R}^m)$. Then there is a unique function $\langle u \rangle \in \ker \mathcal{T}_1$ such that $\langle u \rangle|_{\mathcal{O}} = 0$ and for any $\varphi \in \ker \mathcal{T}_\infty$ we have*

$$\int_{\mathbb{R}^m \setminus \mathcal{O}} (u(y) - \langle u \rangle(y))\varphi(y) \; \mathrm{d}y = 0. \tag{21}$$

Moreover we have the inequality $\|\langle u \rangle\|_{L^1(\mathbb{R}^m)} \leq \|u\|_{L^1(\mathbb{R}^m)}$. In particular the application $u \in L^1(\mathbb{R}^m) \to \langle u \rangle \in L^1(\mathbb{R}^m)$ is linear, continuous and $\|\langle \cdot \rangle\|_{\mathcal{L}(L^1(\mathbb{R}^m), L^1(\mathbb{R}^m))} \leq 1$.

Proof. Consider a sequence $(u_n)_n \subset C_c(\mathbb{R}^m)$ satisfying $\lim_{n \to +\infty} u_n = u$ in $L^1(\mathbb{R}^m)$. For any $n \in \mathbb{N}$, $q \in (1, +\infty)$ the function u_n belongs to $L^q(\mathbb{R}^m)$ and by Proposition 2.2 and Corollary 2.3 we know that there is $\langle u_n \rangle \in \ker \mathcal{T}_q$, $\forall\, q \in (1, +\infty)$ satisfying

$$\int_{\mathbb{R}^m} (u_n(y) - \langle u_n \rangle(y))\varphi(y) \; \mathrm{d}y = 0, \;\; \forall\, \varphi \in \ker \mathcal{T}_{q\,'}, \;\; q \in (1, +\infty). \tag{22}$$

In particular we have

$$\int_{\mathbb{R}^m} (u_n(y) - u_l(y))\varphi(y) \; \mathrm{d}y = \int_{\mathbb{R}^m} (\langle u_n \rangle(y) - \langle u_l \rangle(y))\varphi(y) \; \mathrm{d}y.$$

Taking $\varphi = |\langle u_n \rangle - \langle u_l \rangle|^{q-2}(\langle u_n \rangle - \langle u_l \rangle) \in \ker \mathcal{T}_{q\,'}$ implies

$$\int_{\mathbb{R}^m} |\langle u_n \rangle - \langle u_l \rangle|^q \; \mathrm{d}y \leq \left(\int_{\mathbb{R}^m} |u_n - u_l|^q \; \mathrm{d}y \right)^{1/q} \left(\int_{\mathbb{R}^m} |\langle u_n \rangle - \langle u_l \rangle|^q \; \mathrm{d}y \right)^{1/q\,'}$$

and finally one gets

$$\int_{\mathbb{R}^m} |\langle u_n \rangle - \langle u_l \rangle|^q \; \mathrm{d}y \leq \int_{\mathbb{R}^m} |u_n - u_l|^q \; \mathrm{d}y. \tag{23}$$

By Fatou's lemma we deduce that

$$\int_{\mathbb{R}^m} |\langle u_n \rangle - \langle u_l \rangle| \; \mathrm{d}y \leq \liminf_{q \searrow 1} \int_{\mathbb{R}^m} |\langle u_n \rangle - \langle u_l \rangle|^q \; \mathrm{d}y$$

and by dominated convergence theorem we have

$$\lim_{q\searrow 1}\int_{\mathbb{R}^m}|u_n-u_l|^q\,\mathrm{d}y=\int_{\mathbb{R}^m}|u_n-u_l|\,\mathrm{d}y.$$

Therefore, passing to the limit for $q\searrow 1$ in (23) yields

$$\int_{\mathbb{R}^m}|\langle u_n\rangle-\langle u_l\rangle|\,\mathrm{d}y\le\int_{\mathbb{R}^m}|u_n-u_l|\,\mathrm{d}y$$

saying that $(\langle u_n\rangle)_n$ is a Cauchy sequence in $L^1(\mathbb{R}^m)$. Let us denote by $\langle u\rangle$ the limit of $(\langle u_n\rangle)_n$ in $L^1(\mathbb{R}^m)$. Since $(\langle u_n\rangle)_n$ are constant along the flow we check easily that $\langle u\rangle$ is also constant along the flow. Moreover, $\langle u\rangle$ belongs to $L^1(\mathbb{R}^m)$ and by the construction of $\mathcal{O}$ we deduce that $\langle u\rangle=0$ on $\mathcal{O}$. Consider a function $\varphi\in\ker\mathcal{T}_\infty$. Applying (22) with

$$\left(\xi^{1/q}+|\langle u_n\rangle|\right)^{q-1}\varphi\,\mathbf{1}_{\mathbb{R}^m\setminus\mathcal{O}}\in\ker\mathcal{T}_{q'}$$

(where $\xi(\cdot)$ is the function appearing in (20)) we deduce that

$$\int_{\mathbb{R}^m\setminus\mathcal{O}}u_n\left(\xi^{1/q}+|\langle u_n\rangle|\right)^{q-1}\varphi\,\mathrm{d}y=\int_{\mathbb{R}^m\setminus\mathcal{O}}\langle u_n\rangle\left(\xi^{1/q}+|\langle u_n\rangle|\right)^{q-1}\varphi\,\mathrm{d}y.\tag{24}$$

We keep n fixed for the moment and we intend to pass to the limit for $q\searrow 1$ in the above equality. We use the trivial inequality $x^z\le 1+x$, for any $x>0$, $z\in(0,1)$. One gets for any $q\in(1,2)$

$$\left((\xi(y))^{1/q}+|\langle u_n\rangle(y)|\right)^{q-1}\le 1+(\xi(y))^{1/q}+|\langle u_n\rangle(y)|\le 2+\xi(y)+|\langle u_n\rangle(y)|$$

and thus

$$\begin{aligned}\left|u_n(y)\left((\xi(y))^{1/q}+|\langle u_n\rangle(y)|\right)^{q-1}\varphi(y)\right| &\le \|\varphi\|_{L^\infty(\mathbb{R}^m)}\|u_n\|_{L^\infty(\mathbb{R}^m)}(\xi(y)+|\langle u_n\rangle(y)|)\\ &+ 2\|\varphi\|_{L^\infty(\mathbb{R}^m)}|u_n(y)|\in L^1(\mathbb{R}^m\setminus\mathcal{O}).\end{aligned}$$

Since $\xi>0$ on $\mathbb{R}^m\setminus\mathcal{O}$ we have the pointwise convergence

$$\lim_{q\searrow 1}u_n(y)\left((\xi(y))^{1/q}+|\langle u_n\rangle(y)|\right)^{q-1}\varphi(y)=u_n(y)\varphi(y),\ \ y\in\mathbb{R}^m\setminus\mathcal{O}$$

and thus we deduce by Lebesgue's theorem

$$\lim_{q\searrow 1}\int_{\mathbb{R}^m\setminus\mathcal{O}}u_n(y)\left((\xi(y))^{1/q}+|\langle u_n\rangle(y)|\right)^{q-1}\varphi(y)\,\mathrm{d}y=\int_{\mathbb{R}^m\setminus\mathcal{O}}u_n(y)\varphi(y)\,\mathrm{d}y.\tag{25}$$

By similar arguments we can pass to the limit for $q \searrow 1$ in the right hand side of (24) (for this observe also that, by Corollary 2.1, we have $\|\langle u_n\rangle\|_{L^\infty(\mathbb{R}^m)} \leq \|u_n\|_{L^\infty(\mathbb{R}^m)}$)

$$\lim_{q\searrow 1}\int_{\mathbb{R}^m\setminus\mathcal{O}} \langle u_n\rangle(y)\left((\xi(y))^{1/q}+|\langle u_n\rangle(y)|\right)^{q-1}\varphi(y)\;\mathrm{d}y = \int_{\mathbb{R}^m\setminus\mathcal{O}}\langle u_n\rangle(y)\varphi(y)\;\mathrm{d}y. \tag{26}$$

Combining (24), (25), (26) yields

$$\int_{\mathbb{R}^m\setminus\mathcal{O}}(u_n(y)-\langle u_n\rangle(y))\varphi(y)\;\mathrm{d}y = 0,\;\;\forall\,\varphi\in\ker\mathcal{T}_\infty.$$

Passing now to the limit for $n\to+\infty$ implies

$$\int_{\mathbb{R}^m\setminus\mathcal{O}}(u(y)-\langle u\rangle(y))\varphi(y)\;\mathrm{d}y = 0,\;\;\forall\,\varphi\in\ker\mathcal{T}_\infty. \tag{27}$$

We consider the function $\varphi = \mathrm{sgn}\langle u\rangle$. Since $\langle u\rangle$ is constant along the flow, we have $\varphi\in\ker\mathcal{T}_\infty$ and therefore we deduce thanks to (27)

$$\int_{\mathbb{R}^m}|\langle u\rangle|\;\mathrm{d}y = \int_{\mathbb{R}^m\setminus\mathcal{O}}|\langle u\rangle|\;\mathrm{d}y = \int_{\mathbb{R}^m\setminus\mathcal{O}}u\;\mathrm{sgn}\langle u\rangle\;\mathrm{d}y \leq \int_{\mathbb{R}^m\setminus\mathcal{O}}|u|\;\mathrm{d}y \leq \int_{\mathbb{R}^m}|u|\;\mathrm{d}y.$$

The uniqueness of the function $\langle u\rangle$ constructed above is immediate. Indeed, let us consider two functions $u_1, u_2\in\ker\mathcal{T}_1$ satisfying

$$\int_{\mathbb{R}^m\setminus\mathcal{O}}(u-u_1)\varphi\;\mathrm{d}y = \int_{\mathbb{R}^m\setminus\mathcal{O}}(u-u_2)\varphi\;\mathrm{d}y = 0,\;\;\forall\,\varphi\in\ker\mathcal{T}_\infty.$$

By the definition of $\mathcal{O}$ we have $u_1 = u_2 = 0$ on $\mathcal{O}$ and taking $\varphi = \mathrm{sgn}(u_1-u_2)\in\ker\mathcal{T}_\infty$ we deduce

$$\int_{\mathbb{R}^m\setminus\mathcal{O}}|u_1-u_2|\;\mathrm{d}y = \int_{\mathbb{R}^m\setminus\mathcal{O}}(u_1-u_2)\varphi\;\mathrm{d}y = 0.$$

Finally $u_1 = u_2$ on $\mathbb{R}^m$. The linearity of the application $u\in L^1(\mathbb{R}^m)\to\langle u\rangle\in L^1(\mathbb{R}^m)$ follows easily by using the characterization (21). $\square$

Employing similar arguments as those in the proof of Proposition 2.2 we analyze the operator $\langle\cdot\rangle$ in the $L^\infty(\mathbb{R}^m)$ setting.

Proposition 2.5 *Assume that (20) holds and $u\in L^\infty(\mathbb{R}^m)$. Then there is a unique function $\langle u\rangle\in\ker\mathcal{T}_\infty$ such that $\langle u\rangle = 0$ on $\mathcal{O}$ and for any $\varphi\in\ker\mathcal{T}_1$ we have*

$$\int_{\mathbb{R}^m\setminus\mathcal{O}}(u(y)-\langle u\rangle(y))\varphi(y)\;\mathrm{d}y = 0.$$

Moreover we have the weak $\star$ convergence in $L^\infty(\mathbb{R}^m\setminus\mathcal{O})$

$$\langle u\rangle = \lim_{T\to+\infty}\frac{1}{T}\int_0^T u(Y(s;\cdot))\;\mathrm{d}s$$

and the inequality $\|\langle u\rangle\|_{L^\infty(\mathbb{R}^m)} \leq \|u\|_{L^\infty(\mathbb{R}^m)}$*. In particular the application* $u \in L^\infty(\mathbb{R}^m) \to \langle u\rangle \in L^\infty(\mathbb{R}^m)$ *is linear, continuous and* $\|\langle\cdot\rangle\|_{\mathcal{L}(L^\infty(\mathbb{R}^m),L^\infty(\mathbb{R}^m))} \leq 1$.

Proof. In order to prove the uniqueness, consider $u_1, u_2 \in \ker \mathcal{T}_\infty$ satisfying $u_1 = u_2 = 0$ on $\mathcal{O}$ and $\int_{\mathbb{R}^m\setminus\mathcal{O}}(u_1 - u_2)\varphi \,\mathrm{d}y = 0$ for any $\varphi \in \ker \mathcal{T}_1$. By Proposition 2.4 we know that for any $\psi \in L^1(\mathbb{R}^m)$ there is $\langle\psi\rangle \in \ker \mathcal{T}_1$ such that

$$\int_{\mathbb{R}^m\setminus\mathcal{O}}(\psi - \langle\psi\rangle)v \,\mathrm{d}y = 0, \ \ \forall\, v \in \ker \mathcal{T}_\infty.$$

In particular we have for $v = u_1 - u_2 \in \ker \mathcal{T}_\infty$

$$\int_{\mathbb{R}^m}(u_1 - u_2)\psi \,\mathrm{d}y = \int_{\mathbb{R}^m\setminus\mathcal{O}}(u_1 - u_2)\psi \,\mathrm{d}y = \int_{\mathbb{R}^m\setminus\mathcal{O}}(u_1 - u_2)\langle\psi\rangle \,\mathrm{d}y = 0, \ \ \forall\, \psi \in L^1(\mathbb{R}^m)$$

implying that $u_1 = u_2$. The existence follows by considering $(T_n)_n$ such that $\lim_{n\to+\infty} T_n = +\infty$ and

$$\langle u\rangle_{T_n} \rightharpoonup \tilde{u} \ \text{ weakly } \star \text{ in } \ L^\infty(\mathbb{R}^m \setminus \mathcal{O})$$

for some function $\tilde{u} \in L^\infty(\mathbb{R}^m \setminus \mathcal{O})$. As in the proof of Proposition 2.2 we check that

$$\tilde{u} \in \ker \mathcal{T}_\infty, \ \ \int_{\mathbb{R}^m\setminus\mathcal{O}}(u - \tilde{u})\varphi \,\mathrm{d}y = 0, \ \ \forall\, \varphi \in \ker \mathcal{T}_1, \ \ \|\tilde{u}\|_{L^\infty(\mathbb{R}^m\setminus\mathcal{O})} \leq \|u\|_{L^\infty(\mathbb{R}^m\setminus\mathcal{O})}.$$

We take $\langle u\rangle = \tilde{u}\, \mathbf{1}_{\mathbb{R}^m\setminus\mathcal{O}}$. □

We inquire now about the symmetry between the operators $\langle\cdot\rangle^{(q)}$, $\langle\cdot\rangle^{(q\,')}$ when $q, q\,'$ are conjugate exponents. We have the natural duality result.

Proposition 2.6 *a) Assume that* $q, q\,' \in (1, +\infty)$, $1/q + 1/q\,' = 1$, $u \in L^q(\mathbb{R}^m)$, $\varphi \in L^{q\,'}(\mathbb{R}^m)$*. Then*

$$\int_{\mathbb{R}^m} u\, \langle\varphi\rangle^{(q\,')} \,\mathrm{d}y = \int_{\mathbb{R}^m}\langle u\rangle^{(q)}\varphi \,\mathrm{d}y.$$

b) In particular $\langle\cdot\rangle^{(2)}$ *is symmetric on* $L^2(\mathbb{R}^m)$ *and coincides with the orthogonal projection on* $\ker \mathcal{T}_2$*. Moreover we have the orthogonal decomposition* $L^2(\mathbb{R}^m) = \ker \mathcal{T}_2 \oplus \ker\langle\cdot\rangle^{(2)}$.
c) Assume that (20) holds and that $u \in L^1(\mathbb{R}^m), \varphi \in L^\infty(\mathbb{R}^m)$*. We denote by* $\langle u\rangle^{(1)}$, $\langle\varphi\rangle^{(\infty)}$ *the functions constructed in Propositions 2.4, 2.5 respectively. Then*

$$\int_{\mathbb{R}^m} u\, \langle\varphi\rangle^{(\infty)} \,\mathrm{d}y = \int_{\mathbb{R}^m}\langle u\rangle^{(1)}\varphi \,\mathrm{d}y.$$

Proof. a) The function $\langle\varphi\rangle^{(q\,')}$ belongs to $\ker \mathcal{T}_{q\,'}$ and therefore

$$\int_{\mathbb{R}^m}(u - \langle u\rangle^{(q)})\langle\varphi\rangle^{(q\,')} \,\mathrm{d}y = 0. \tag{28}$$

Similarly $\langle u\rangle^{(q)}$ belongs to $\ker \mathcal{T}_q$ and thus

$$\int_{\mathbb{R}^m} (\varphi - \langle \varphi \rangle^{(q\,')}) \langle u\rangle^{(q)} \;\mathrm{d}y = 0. \tag{29}$$

Combining (28), (29) yields

$$\int_{\mathbb{R}^m} u \;\langle \varphi \rangle^{(q\,')} \;\mathrm{d}y = \int_{\mathbb{R}^m} \langle u\rangle^{(q)} \langle \varphi \rangle^{(q\,')} \;\mathrm{d}y = \int_{\mathbb{R}^m} \langle u\rangle^{(q)} \varphi \;\mathrm{d}y.$$

b) When $q = 2$ we obtain

$$\int_{\mathbb{R}^m} u \;\langle \varphi \rangle^{(2)} \;\mathrm{d}y = \int_{\mathbb{R}^m} \langle u\rangle^{(2)} \varphi \;\mathrm{d}y, \;\; \forall \, u, \varphi \in L^2(\mathbb{R}^m).$$

By the characterization in Proposition 2.2 we deduce that $\langle \cdot \rangle^{(2)} = \mathrm{Proj}_{\ker \mathcal{T}_2}$. Since $\ker \mathcal{T}_2$ is closed we have the orthogonal decomposition

$$L^2(\mathbb{R}^m) = \ker \mathcal{T}_2 \oplus (\ker \mathcal{T}_2)^{\perp} = \ker \mathcal{T}_2 \oplus \ker \langle \cdot \rangle^{(2)}.$$

c) By Proposition 2.4 we know that

$$\int_{\mathbb{R}^m \setminus \mathcal{O}} (u - \langle u\rangle^{(1)}) \langle \varphi \rangle^{(\infty)} \;\mathrm{d}y = 0.$$

By construction we have $\langle \varphi \rangle^{(\infty)} = 0$ on $\mathcal{O}$ and thus we have also

$$\int_{\mathbb{R}^m} (u - \langle u\rangle^{(1)}) \langle \varphi \rangle^{(\infty)} \;\mathrm{d}y = 0.$$

By Proposition 2.5 we deduce that

$$\int_{\mathbb{R}^m \setminus \mathcal{O}} (\varphi - \langle \varphi \rangle^{(\infty)}) \langle u\rangle^{(1)} \;\mathrm{d}y = 0.$$

Since $\langle u\rangle^{(1)} = 0$ on $\mathcal{O}$, the above equality can be written

$$\int_{\mathbb{R}^m} (\varphi - \langle \varphi \rangle^{(\infty)}) \langle u\rangle^{(1)} \;\mathrm{d}y = 0.$$

Finally we obtain

$$\int_{\mathbb{R}^m} u \;\langle \varphi \rangle^{(\infty)} \;\mathrm{d}y = \int_{\mathbb{R}^m} \langle u\rangle^{(1)} \langle \varphi \rangle^{(\infty)} \;\mathrm{d}y = \int_{\mathbb{R}^m} \langle u\rangle^{(1)} \varphi \;\mathrm{d}y.$$

$\square$

The following result is a straightforward consequence of the characterizations for $\langle\cdot\rangle^{(r)}$ with $r \in [1, +\infty]$.

Corollary 2.4 *Let $u \in L^p(\mathbb{R}^m)$, $v \in L^q(\mathbb{R}^m)$ and $1/r = 1/p + 1/q$ with $p, q, r \in [1, +\infty]$. Assume that u is constant along the flow. Then*

$$\langle uv\rangle^{(r)} = u\ \langle v\rangle^{(q)}.$$

Proof. We distinguish several cases.
a) $p, q, r \in (1, +\infty)$. Take any function $\varphi \in \ker \mathcal{T}_{r\,'}$ (with $1/r + 1/r\,' = 1$) and observe that $\varphi u \in \ker \mathcal{T}_{q\,'}$ (with $1/q + 1/q\,' = 1$). Therefore we know that

$$\int_{\mathbb{R}^m} (v - \langle v\rangle^{(q)})\varphi u \ \mathrm{d}y = 0$$

saying that $\langle uv\rangle^{(r)} = u\ \langle v\rangle^{(q)}$.
b) $r \in (1, +\infty), p = r, q = +\infty$ (we assume that (20) holds). For any function $\varphi \in \ker \mathcal{T}_{r\,'}$ we have $\varphi u \in \ker \mathcal{T}_1$ and thus

$$\int_{\mathbb{R}^m \setminus \mathcal{O}} (v - \langle v\rangle^{(\infty)})\varphi u \ \mathrm{d}y = 0.$$

Since $\varphi u = 0$ on $\mathcal{O}$ (as function in $\ker \mathcal{T}_1$) we deduce that

$$\int_{\mathbb{R}^m} (v - \langle v\rangle^{(\infty)})\varphi u \ \mathrm{d}y = 0$$

implying that $\langle uv\rangle^{(r)} = u\ \langle v\rangle^{(\infty)}$.
The other cases are: c) $r \in (1, +\infty),\ p = +\infty,\ q = r$, d) $r = 1,\ p, q \in (1, +\infty)$, e) $r = p = 1,\ q = +\infty$, f) $r = q = 1,\ p = +\infty$, g) $r = p = q = +\infty$. They follow in similar way and are left to the reader. $\square$

By the orthogonal decompositions in Propositions 2.3 and 2.6 we deduce that $\ker\langle\cdot\rangle^{(2)} = \overline{\mathrm{range}\mathcal{T}_2}$. We have the general result.

Proposition 2.7 *Assume that $q \in (1, +\infty)$. Then $\ker\langle\cdot\rangle^{(q)} = \overline{\mathrm{range}\mathcal{T}_q}$.*

Proof. For any $v = \mathcal{T}_q u \in \mathrm{range}\mathcal{T}_q$ and $\varphi \in \ker \mathcal{T}_{q\,'}$ we have

$$\int_{\mathbb{R}^m} (v - 0)\varphi \ \mathrm{d}y = \int_{\mathbb{R}^m} \mathcal{T}_q u\ \varphi \ \mathrm{d}y = -\int_{\mathbb{R}^m} u \mathcal{T}_{q\,'}\varphi \ \mathrm{d}y = 0$$

saying that $\langle v\rangle^{(q)} = 0$. Therefore $\mathrm{range}\mathcal{T}_q \subset \ker\langle\cdot\rangle^{(q)}$ and also $\overline{\mathrm{range}\mathcal{T}_q} \subset \ker\langle\cdot\rangle^{(q)}$. Consider now a linear form h on $L^q(\mathbb{R}^m)$ vanishing on $\mathrm{range}\mathcal{T}_q$. There is $v \in L^{q\,'}(\mathbb{R}^m)$ such that $h(w) = \int_{\mathbb{R}^m} wv \ \mathrm{d}y$ for any $w \in L^q(\mathbb{R}^m)$. In particular

$$\int_{\mathbb{R}^m} \mathcal{T}_q u\ v \ \mathrm{d}y = 0,\ \ \forall\, u \in \mathrm{D}_q$$

saying that $v \in \ker \mathcal{T}_{q\,'}$. For any $\varphi \in \ker \langle \cdot \rangle^{(q)}$ we can write by Proposition 2.6

$$h(\varphi) = \int_{\mathbb{R}^m} v\varphi \,\mathrm{d}y = \int_{\mathbb{R}^m} \langle v \rangle^{(q\,')} \varphi \,\mathrm{d}y = \int_{\mathbb{R}^m} v \,\langle \varphi \rangle^{(q)} \,\mathrm{d}y = 0$$

and thus h vanishes on $\ker \langle \cdot \rangle^{(q)}$. Consequently we have $\overline{\mathrm{range}\mathcal{T}_q} = \ker \langle \cdot \rangle^{(q)}$. □

At this stage let us point out that if $\mathrm{range}\mathcal{T}_q$ is closed, then $\ker\langle \cdot \rangle^{(q)} = \mathrm{range}\mathcal{T}_q$ saying that the equation $\mathcal{T}_q u = f \in L^q(\mathbb{R}^m)$ is solvable iff $\langle f \rangle^{(q)} = 0$. Let us indicate a simple situation where the above characterization for the range of $\mathcal{T}_q$ occurs.

Proposition 2.8 *Assume that all the trajectories are closed, uniformly in time i.e.,*

$$\exists\, T > 0 \;:\; \forall\, y \in \mathbb{R}^m,\ \exists\, T_y \in [0, T] \text{ such that } Y(T_y; y) = y.$$

Then for any $q \in (1, +\infty)$ the range of $\mathcal{T}_q$ is closed and we have $\mathrm{range}\mathcal{T}_q = \ker\langle \cdot \rangle^{(q)}$.

Proof. By Proposition 2.7 we have $\mathrm{range}\mathcal{T}_q \subset \overline{\mathrm{range}\mathcal{T}_q} = \ker\langle \cdot \rangle^{(q)}$. Conversely, assume that $f \in \ker\langle \cdot \rangle^{(q)}$ and let us check that $f \in \mathrm{range}\mathcal{T}_q$. For any $\mu > 0$ let $u_\mu \in L^q(\mathbb{R}^m)$ solving

$$\mu u_\mu + \mathcal{T}_q u_\mu = f. \tag{30}$$

It is easily seen that the unique solution of the above equation is

$$u_\mu = \int_{-\infty}^0 \mathrm{e}^{\mu s} f(Y(s; \cdot)) \,\mathrm{d}s. \tag{31}$$

Observe that we are done if we prove that $(\|u_\mu\|_{L^q(\mathbb{R}^m)})_{\mu>0}$ is bounded. Indeed, in this case we can extract a sequence $(\mu_n)_n$ converging towards 0 such that $\lim_{n\to+\infty} u_{\mu_n} = u$ weakly in $L^q(\mathbb{R}^m)$. Passing to the limit in the weak formulation of (30) we deduce that $u \in \mathrm{D}_q$ and $f = \mathcal{T}_q u \in \mathrm{range}\mathcal{T}_q$. In order to estimate $(\|u_\mu\|_{L^q(\mathbb{R}^m)})_{\mu>0}$ we use the immediate lemma, whose proof is left to the reader.

Lemma 2.1 *Let $g : \mathbb{R} \to \mathbb{R}$ be a locally integrable T periodic function. Then for any $t \in \mathbb{R}^\star$ we have*

$$\left| \frac{1}{t}\int_0^t g(s) \,\mathrm{d}s - \frac{1}{T}\int_0^T g(s) \,\mathrm{d}s \right| \le \frac{2}{|t|}\int_0^T |g(s)| \,\mathrm{d}s.$$

By Proposition 2.3 we know that

$$\lim_{s\to-\infty} \left(-\frac{1}{s}\int_s^0 f(Y(\tau; \cdot)) \,\mathrm{d}\tau \right) = \langle f \rangle^{(q)} = 0, \text{ strongly in } L^q(\mathbb{R}^m).$$

In particular we have the pointwise convergence

$$\lim_{k\to+\infty} \left(-\frac{1}{s_k}\int_{s_k}^0 f(Y(\tau; y)) \,\mathrm{d}\tau \right) = 0, \text{ a.e. } y \in \mathbb{R}^m$$

for some sequence $(s_k)_k$ verifying $\lim_{k\to+\infty} s_k = -\infty$. Observe that

$$\left\| \int_0^T |f(Y(\tau;\cdot))| \,\mathrm{d}\tau \right\|_{L^q(\mathbb{R}^m)} \leq T\|f\|_{L^q(\mathbb{R}^m)} < +\infty$$

and thus, for a.a. $y \in \mathbb{R}^m$ the function $\tau \to f(Y(\tau;y))$ is locally integrable. Since the function $\tau \to f(Y(\tau;y))$ is T_y periodic, we have by Lemma 2.1

$$\frac{1}{T_y}\int_0^{T_y} f(Y(\tau;y))\,\mathrm{d}\tau = \lim_{k\to+\infty}\left(-\frac{1}{s_k}\int_{s_k}^0 f(Y(\tau;y))\,\mathrm{d}\tau\right) = 0, \text{ a.e. } y \in \mathbb{R}^m$$

and

$$\begin{aligned}
\left\| -\frac{1}{s}\int_s^0 f(Y(\tau;\cdot))\,\mathrm{d}\tau \right\|_{L^q(\mathbb{R}^m)} &\leq \left\| \frac{2}{|s|}\int_0^{T_y} |f(Y(\tau;y))|\,\mathrm{d}\tau \right\|_{L^q(\mathbb{R}^m)} \\
&\leq \left\| \frac{2}{|s|}\int_0^{T} |f(Y(\tau;y))|\,\mathrm{d}\tau \right\|_{L^q(\mathbb{R}^m)} \\
&\leq \frac{2T}{|s|}\|f\|_{L^q(\mathbb{R}^m)}
\end{aligned}$$

implying that

$$\left\| \int_s^0 f(Y(\tau;\cdot))\mathrm{d}\tau \right\|_{L^q(\mathbb{R}^m)} \leq 2T\|f\|_{L^q(\mathbb{R}^m)}. \tag{32}$$

Coming back to (31) one gets after integration by parts

$$u_\mu = -\int_{-\infty}^0 \mathrm{e}^{\mu s}\frac{d}{ds}\left\{\int_s^0 f(Y(\tau;\cdot))\mathrm{d}\tau\right\}\,\mathrm{d}s = \int_{-\infty}^0 \mu\mathrm{e}^{\mu s}\int_s^0 f(Y(\tau;\cdot))\mathrm{d}\tau\,\mathrm{d}s$$

and therefore, combining with (32) yields

$$\|u_\mu\|_{L^q(\mathbb{R}^m)} \leq 2T\|f\|_{L^q(\mathbb{R}^m)}, \ \ \forall\, \mu > 0.$$

$\square$

Remark 2.2 *The hypotheses in Proposition 2.8 are verified in the case of a constant magnetic field $B = (0,0,B_3), B_3 \neq 0$. By direct computation we observe that all the trajectories of*

$$\frac{d}{ds}(X_1,X_2) = (P_1,P_2), \ \ \frac{d}{ds}(P_1,P_2) = -B_3(P_2,-P_1)$$

are $\frac{2\pi}{B_3}$ periodic.

Generally we have the following characterization for $\ker\langle\cdot\rangle^{(q)} = \overline{\mathrm{range}\mathcal{T}_q}$.

Proposition 2.9 *Let f be a function in $L^q(\mathbb{R}^m)$ for some $q \in (1, +\infty)$. For any $\mu > 0$ we denote by u_μ the unique solution of (30). Then the following statements are equivalet*
a) $\langle f\rangle^{(q)} = 0$.
b) $\lim_{\mu\searrow 0}(\mu u_\mu) = 0$ *in* $L^q(\mathbb{R}^m)$.

Proof. Assume that b) holds true. Applying the operator $\langle\cdot\rangle^{(q)}$ in (30) one gets

$$\langle f\rangle^{(q)} = \langle \mu u_\mu\rangle^{(q)} + \langle \mathcal{T}_q u_\mu\rangle^{(q)} = \langle \mu u_\mu\rangle^{(q)},\ \ \forall\ \mu > 0$$

and therefore

$$\langle f\rangle^{(q)} = \lim_{\mu\searrow 0}\langle \mu u_\mu\rangle^{(q)} = \langle \lim_{\mu\searrow 0}(\mu u_\mu)\rangle^{(q)} = 0.$$

Conversely, suppose that a) holds true. Considering the function $G(s;y) = \int_s^0 f(Y(\tau;y))\,\mathrm{d}\tau$ we obtain by the formula (31) (use the inequality $\|G(s;\cdot)\|_{L^q(\mathbb{R}^m)} \leq |s|\|f\|_{L^q(\mathbb{R}^m)}$ in order to justify the integration by parts)

$$u_\mu = -\int_{-\infty}^0 \mathrm{e}^{\mu s}\frac{\partial G}{\partial s}(s;\cdot)\,\mathrm{d}s = \int_{-\infty}^0 \mu s\,\mathrm{e}^{\mu s}\frac{G(s;\cdot)}{s}\,\mathrm{d}s = \frac{1}{\mu}\int_{-\infty}^0 t\,\mathrm{e}^t\frac{G(t\mu^{-1};\cdot)}{t\mu^{-1}}\,\mathrm{d}t.$$

We know that $\|G(t\mu^{-1})/(t\mu^{-1})\|_{L^q(\mathbb{R}^m)} \leq \|f\|_{L^q(\mathbb{R}^m)}$ and by Proposition 2.3 we have for any $t < 0$

$$\lim_{\mu\searrow 0}\frac{G(t\mu^{-1};\cdot)}{t\mu^{-1}} = \lim_{\mu\searrow 0}\frac{\int_{t/\mu}^0 f(Y(s;\cdot))\,\mathrm{d}s}{t/\mu} = -\langle f\rangle^{(q)} = 0,\ \ \text{strongly in}\ \ L^q(\mathbb{R}^m).$$

Consequently, by the dominated convergence theorem, one gets

$$\|\mu u_\mu\|_{L^q(\mathbb{R}^m)} \leq \int_{-\infty}^0 |t|\mathrm{e}^t\left\|\frac{G(t\mu^{-1};\cdot)}{t\mu^{-1}}\right\|_{L^q(\mathbb{R}^m)}\mathrm{d}t \to 0\ \text{ as } \mu \searrow 0.$$

$\square$

Remark 2.3 *With the above notations we have* $\|\mu u_\mu\|_{L^q(\mathbb{R}^m)} \leq \|f\|_{L^q(\mathbb{R}^m)},\ \forall\ \mu > 0$.

Up to this point we have investigated the properties of $\langle\cdot\rangle^{(q)}$ operating from $L^q(\mathbb{R}^m)$ to $L^q(\mathbb{R}^m)$ with $q \in [1, +\infty]$. In view of further regularity results for transport equations with singular coefficients we investigate now how $\langle\cdot\rangle^{(q)}$ acts on some particular subspaces of smooth functions. For this purpose we recall here the following basic results concerning the derivation operators along fields in $\mathbb{R}^m$. For any $\xi = (\xi_1(y), ..., \xi_m(y))$, where $y \in \mathbb{R}^m$, we denote by L_ξ the operator $\xi\cdot\nabla_y$. A direct computation shows that for any smooth fields ξ, η, the commutator between L_ξ, L_η is still a first order operator, given by

$$[L_\xi, L_\eta] := L_\xi L_\eta - L_\eta L_\xi = L_\chi$$

where χ is the Poisson bracket of ξ and η

$$\chi = [\xi, \eta], \ \ [\xi, \eta]_i = (\xi \cdot \nabla_y)\eta_i - (\eta \cdot \nabla_y)\xi_i = L_\xi(\eta_i) - L_\eta(\xi_i), \ \ i \in \{1, ..., m\}.$$

It is well known (see [1], pp. 93) that L_ξ, L_η commute (or equivalently the Poisson bracket $[\xi, \eta]$ vanishes) iff the flows corresponding to ξ, η, let say Z_1, Z_2, commute

$$Z_1(s_1; Z_2(s_2; y)) = Z_2(s_2; Z_1(s_1; y)), \ \ s_1, s_2 \in \mathbb{R}, \ \ y \in \mathbb{R}^m.$$

Consider a smooth field c in involution with b and having bounded divergence

$$c \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^m), \ \ \mathrm{div}_y c \in L^\infty(\mathbb{R}^m), \ \ [c, b] = 0$$

and let us denote by Z the flow associated to c (we assume that Z is well defined for any $(s, y) \in \mathbb{R} \times \mathbb{R}^m$). For any $h \in \mathbb{R}$ we denote by τ_h the map associating to a function u its translation on a time h along the flow Z

$$(\tau_h u)(y) = u(Z(h; y)), \ \ y \in \mathbb{R}^m \ \ h \in \mathbb{R}.$$

We claim that for any $h \in \mathbb{R}$ the operators $\langle \cdot \rangle^{(q)}$ and τ_h commute. We use the following easy lemma.

Lemma 2.2 *Let c be a smooth field in involution with b. Then the divergence of c is invariant along the flow of b.*

Proof. For any $i \in \{1, ..., m\}$ we have

$$\sum_{j=1}^m b_j \frac{\partial c_i}{\partial y_j} = \sum_{j=1}^m c_j \frac{\partial b_i}{\partial y_j}.$$

Multiplying by $\frac{\partial \varphi}{\partial y_i}$, where $\varphi \in C^2_c(\mathbb{R}^m)$, and integrating with respect to $y \in \mathbb{R}^m$ yield

$$\sum_{1 \le i,j \le m} \int_{\mathbb{R}^m} b_j \left\{ \frac{\partial}{\partial y_j}\left(c_i \frac{\partial \varphi}{\partial y_i}\right) - c_i \frac{\partial^2 \varphi}{\partial y_j \partial y_i} \right\} \mathrm{d}y = \sum_{1 \le i,j \le m} \int_{\mathbb{R}^m} c_j \left\{ \frac{\partial}{\partial y_j}\left(b_i \frac{\partial \varphi}{\partial y_i}\right) - b_i \frac{\partial^2 \varphi}{\partial y_j \partial y_i} \right\} \mathrm{d}y.$$

After integration by parts and by taking into account that $\mathrm{div}_y b = 0$ one gets

$$\int_{\mathbb{R}^m} (\mathrm{div}_y c)\ (b \cdot \nabla_y \varphi)\ \mathrm{d}y = \int_{\mathbb{R}^m} (\mathrm{div}_y b)\ (c \cdot \nabla_y \varphi)\ \mathrm{d}y = 0$$

saying that $\mathrm{div}_y(b\ \mathrm{div}_y c) = 0$ in $\mathcal{D}'(\mathbb{R}^m)$. Our conclusion follows by Proposition 2.1. $\square$

Proposition 2.10 *Assume that c is a smooth field in involution with b, with bounded divergence and well defined flow. Then for any $q \in (1, +\infty)$ the operator $\langle \cdot \rangle^{(q)}$ commutes with the translations along the flow of c*

$$\langle u \circ Z(h; \cdot)\rangle^{(q)} = \langle u \rangle^{(q)} \circ Z(h; \cdot), \ \ u \in L^q(\mathbb{R}^m), \ \ h \in \mathbb{R}.$$

Moreover, under the hypothesis (20) the above conclusion holds true when $q \in \{1, +\infty\}$.

Proof. First of all observe that τ_h maps $L^q(\mathbb{R}^m)$ to $L^q(\mathbb{R}^m)$ (use Liouville's theorem and the hypothesis $\mathrm{div}_y c \in L^\infty(\mathbb{R}^m)$). Assume that $q \in (1, +\infty)$. By Proposition 2.2 we know that for any $\varphi \in \mathcal{T}_{q\,'}$ we have

$$\int_{\mathbb{R}^m} (u - \langle u \rangle^{(q)})\varphi \,\mathrm{d}y = 0. \tag{33}$$

We denote by φ_{-h} the function

$$\varphi_{-h}(z) = \varphi(Z(-h;z))\,\mathrm{e}^{-\int_0^h (\mathrm{div}_y c)(Z(-t;z))\,\mathrm{d}t}.$$

Notice that $\varphi_{-h} \in \ker \mathcal{T}_{q\,'}$. Indeed, replacing z by $Y(s;y)$ and by taking into account that the flows Y and Z commute we obtain

$$\varphi(Z(-h;Y(s;y))) = \varphi(Y(s;Z(-h;y))) = \varphi(Z(-h;y)).$$

Thanks to Lemma 2.2 we have

$$(\mathrm{div}_y c)(Z(-t;Y(s;y))) = (\mathrm{div}_y c)(Y(s;Z(-t;y))) = (\mathrm{div}_y c)(Z(-t;y)).$$

Consequently one gets $\varphi_{-h}(Y(s;y)) = \varphi_{-h}(y)$ and it is easily seen that φ_{-h} belongs to $L^{q\,'}(\mathbb{R}^m)$. Applying (33) with the trial function φ_{-h} and using the variable change $z = Z(h;y)$ we deduce that

$$\int_{\mathbb{R}^m} (u(Z(h;y)) - \langle u \rangle^{(q)}(Z(h;y))\varphi(y)\,\mathrm{d}y = 0.$$

Observe also that $\langle u \rangle^{(q)}(Z(h;\cdot))$ belongs to $L^q(\mathbb{R}^m)$ and that it is invariant along the flow of b

$$\langle u \rangle^{(q)}(Z(h;Y(s;y))) = \langle u \rangle^{(q)}(Y(s;Z(h;y))) = \langle u \rangle^{(q)}(Z(h;y)).$$

Consequently, by Proposition 2.2 we deduce that $\langle u \circ Z(h;\cdot)\rangle^{(q)} = \langle u \rangle^{(q)} \circ Z(h;\cdot)$. Assume now that $q \in \{1, +\infty\}$ and that (20) holds. Observe that the set $\mathcal{O}$ in (20) is left invariant under the flow of c. Indeed, for any $h \in \mathbb{R}$ let us denote by ξ_h the function $y \in \mathbb{R}^m \to \xi(Z(h;y))$, where $\xi \in \ker \mathcal{T}_1$, $\xi > 0$ on $\mathbb{R}^m \setminus \mathcal{O}$, $\xi = 0$ on $\mathcal{O}$. As before we check that $\xi_h \in \ker \mathcal{T}_1$. Consider the set $A = Z(-h;\mathbb{R}^m \setminus \mathcal{O}) \cap \mathcal{O}$, which is left invariant under the flow of b, and the function $\eta = \mathbf{1}_A \xi_h$. It is easily seen that $\eta \in \ker \mathcal{T}_1$ and that $\eta > 0$ on A. By the definition of $\mathcal{O}$ we have $\int_{\mathcal{O}} \eta \,\mathrm{d}y = 0$ and thus A is a negligible set. Therefore, up to a negligible set we have the inclusion $Z(-h;\mathbb{R}^m \setminus \mathcal{O}) \subset \mathbb{R}^m \setminus \mathcal{O}$ implying that, up to a negligible set, we have $Z(h;\mathbb{R}^m \setminus \mathcal{O}) = \mathbb{R}^m \setminus \mathcal{O}$ for any $h \in \mathbb{R}$. From now on we can use the same arguments as in the case $q \in (1, +\infty)$ replacing the integrations over $\mathbb{R}^m$ by integrations over $\mathbb{R}^m \setminus \mathcal{O}$. Our conclusion follows thanks to the Propositions 2.4, 2.5. The details are left to the reader. $\square$

Remark 2.4 *In particular we have* $[b, b] = 0$ *and therefore* $\langle\cdot\rangle^{(q)}$ *commutes with the translations along the flow of* b. *We have for any* $h \in \mathbb{R}$, $u \in L^q(\mathbb{R}^m)$, $q \in [1, +\infty]$

$$\langle u(Y(h; \cdot))\rangle^{(q)} = \langle u\rangle^{(q)}(Y(h; \cdot)) = \langle u\rangle^{(q)}.$$

We shall show that for any smooth field c in involution with b, the operator $\langle\cdot\rangle^{(q)}$ commutes with $c \cdot \nabla_y$. We denote by $\mathcal{T}_q^c$ the operator given by

$$\mathrm{D}(\mathcal{T}_q^c) = \{u \in L^q(\mathbb{R}^m) \ : \ \mathrm{div}_y(cu) \in L^q(\mathbb{R}^m)\}, \ \ \mathcal{T}_q^c u = \mathrm{div}_y(cu) - (\mathrm{div}_y c)u, \ \ u \in \mathrm{D}(\mathcal{T}_q^c).$$

We have the standard result (see [7], Proposition IX.3, pp. 153 for similar results).

Lemma 2.3 *Assume that* $q \in (1, +\infty)$ *and let* u *be a function in* $L^q(\mathbb{R}^m)$. *Then the following statements are equivalent*
a) $u \in \mathrm{D}(\mathcal{T}_q^c)$.
b) $(h^{-1}(u(Z(h; \cdot)) - u))_h$ *is bounded in* $L^q(\mathbb{R}^m)$.
Moreover, for any $u \in \mathrm{D}(\mathcal{T}_q^c)$ *we have the convergence*

$$\lim_{h \to 0} \frac{u(Z(h; \cdot)) - u}{h} = \mathcal{T}_q^c u, \ \ \textit{strongly in} \ \ L^q(\mathbb{R}^m).$$

Proposition 2.11 *Under the hypotheses of Proposition 2.10, assume that* $u \in \mathrm{D}(\mathcal{T}_q^c)$ *for some* $q \in (1, +\infty)$. *Then* $\langle u\rangle^{(q)} \in \mathrm{D}(\mathcal{T}_q^c)$ *and* $\mathcal{T}_q^c \langle u\rangle^{(q)} = \langle \mathcal{T}_q^c u\rangle^{(q)}$.

Proof. For any $h \in \mathbb{R}^\star$ we have thanks to Proposition 2.10

$$\left\langle \frac{u(Z(h; \cdot)) - u}{h} \right\rangle^{(q)} = \frac{\langle u\rangle^{(q)}(Z(h; \cdot)) - \langle u\rangle^{(q)}}{h}. \tag{34}$$

Since $u \in \mathrm{D}(\mathcal{T}_q^c)$ we know by Lemma 2.3 that

$$\lim_{h \to 0} \frac{u(Z(h; \cdot)) - u}{h} = \mathcal{T}_q^c u, \ \ \text{strongly in} \ \ L^q(\mathbb{R}^m).$$

By the continuity of $\langle\cdot\rangle^{(q)}$ we deduce that $(h^{-1}(\langle u\rangle^{(q)}(Z(h; \cdot)) - \langle u\rangle^{(q)}))_h$ is bounded in $L^q(\mathbb{R}^m)$ and consequently, using one more time Lemma 2.3 and (34) one gets

$$\langle u\rangle^{(q)} \in \mathrm{D}(\mathcal{T}_q^c), \ \ \mathcal{T}_q^c \langle u\rangle^{(q)} = \lim_{h \to 0} \frac{\langle u\rangle^{(q)}(Z(h; \cdot)) - \langle u\rangle^{(q)}}{h} = \langle \mathcal{T}_q^c u\rangle^{(q)}.$$

□

Remark 2.5 *In particular Proposition 2.11 applies for* $c = b$. *Actually, for any* $u \in \mathrm{D}(\mathcal{T}_q)$, $q \in [1, +\infty]$ *we have* $\mathcal{T}_q \langle u\rangle^{(q)} = \langle \mathcal{T}_q u\rangle^{(q)} = 0$.

Remark 2.6 *Under the hypotheses of Proposition 2.10 we check immediately thanks to Lemma 2.3 that if* $u \in \mathrm{D}(\mathcal{T}_q^c)$*, then for any* $s \in \mathbb{R}$, $u \circ Y(s;\cdot) \in \mathrm{D}(\mathcal{T}_q^c)$ *and*

$$\mathcal{T}_q^c(u \circ Y(s;\cdot)) = (\mathcal{T}_q^c u) \circ Y(s;\cdot).$$

In particular if $u \in \ker \mathcal{T}_q \cap \mathrm{D}(\mathcal{T}_q^c)$ *then* $\mathcal{T}_q^c u \in \ker \mathcal{T}_q$.

The last result in this section states that $\langle\cdot\rangle^{(q)}$ commutes with the time derivation. The proof is standard and comes easily by observing that

$$\frac{\langle u(t+h)\rangle^{(q)} - \langle u(t)\rangle^{(q)}}{h} = \left\langle \frac{u(t+h)-u(t)}{h} \right\rangle^{(q)}$$

and by adapting the arguments in Lemma 2.3.

Proposition 2.12 *Assume that* $u \in W^{1,p}([0,T];L^q(\mathbb{R}^m))$ *for some* $p,q \in (1,+\infty)$*. Then the application* $(t,y) \to \langle u(t,\cdot)\rangle^{(q)}(y)$ *belongs to* $W^{1,p}([0,T];L^q(\mathbb{R}^m))$ *and we have* $\partial_t \langle u\rangle^{(q)} = \langle \partial_t u\rangle^{(q)}$.

3 Well-posedness of the limit model

This section is devoted to the study of the limit model, when ε goes to 0, for the transport equation

$$\begin{cases} \partial_t u^\varepsilon + a(t,y)\cdot\nabla_y u^\varepsilon + \dfrac{b(y)}{\varepsilon}\cdot\nabla_y u^\varepsilon = 0, & (t,y) \in (0,T)\times\mathbb{R}^m \\ u^\varepsilon(0,y) = u_0^\varepsilon(y), & y \in \mathbb{R}^m. \end{cases} \tag{35}$$

Recall that b is a given smooth field satisfying (8), (9), (10). We assume that a satisfies the conditions

$$a \in L^1([0,T];W^{1,\infty}(\mathbb{R}^m)),\ \ \mathrm{div}_y a = 0. \tag{36}$$

Based on Hilbert's expansion method we have obtained (see (4), (5)) the formula $u^\varepsilon = u + \varepsilon u_1 + \varepsilon^2 \mathcal{O}(\varepsilon)$ where

$$b(y)\cdot\nabla_y u = 0,\ \ \partial_t u + a(t,y)\cdot\nabla_y u + b(y)\cdot\nabla_y u_1 = 0.$$

Projecting the second equation on the kernel of $\mathcal{T}$ leads to the model

$$\partial_t\langle u\rangle + \langle a(t)\cdot\nabla_y u(t)\rangle = 0,\ \ (t,y) \in (0,T)\times\mathbb{R}^m.$$

Notice that $\mathcal{T}u = 0$ and thus $\langle u\rangle = u$. Finally we obtain

$$\begin{cases} \partial_t u + \langle a(t)\cdot\nabla_y u(t)\rangle = 0,\ \ b(y)\cdot\nabla_y u = 0, & (t,y)\in(0,T)\times\mathbb{R}^m \\ u(0,y) = u_0(y), & y \in \mathbb{R}^m. \end{cases} \tag{37}$$

We work in the $L^q(\mathbb{R}^m)$ setting, with $q \in (1, +\infty)$. For any $\varphi \in \ker \mathcal{T}_{q'}$ we have

$$\int_{\mathbb{R}^m} (a(t,y) \cdot \nabla_y u - \langle a(t) \cdot \nabla_y u(t) \rangle^{(q)}) \varphi(y) \, \mathrm{d}y = 0$$

and we introduce the notion of weak solution for (37) as follows.

Definition 3.1 *Assume that $u_0 \in \ker \mathcal{T}_q$, $f \in L^1([0,T]; \ker \mathcal{T}_q)$ (i.e., $f \in L^1([0,T]; L^q(\mathbb{R}^m))$ and $f(t) \in \ker \mathcal{T}_q$, $t \in [0,T]$). We say that $u \in L^\infty([0,T]; \ker \mathcal{T}_q)$ is a weak solution for*

$$\begin{cases} \partial_t u + \langle a(t) \cdot \nabla_y u(t) \rangle^{(q)} = f(t,y), \;\; \mathcal{T}_q u = 0, & (t,y) \in (0,T) \times \mathbb{R}^m \\ u(0,y) = u_0(y), & y \in \mathbb{R}^m \end{cases} \tag{38}$$

iff for any $\varphi \in C^1_c([0,T) \times \mathbb{R}^m)$ satisfying $\mathcal{T}\varphi = 0$ we have

$$\int_0^T \!\! \int_{\mathbb{R}^m} u(t,y)(\partial_t \varphi + \mathrm{div}_y(\varphi a)) \, \mathrm{d}y\mathrm{d}t + \int_{\mathbb{R}^m} u_0(y) \varphi(0,y) \, \mathrm{d}y + \int_0^T \!\! \int_{\mathbb{R}^m} f(t,y) \varphi(t,y) \, \mathrm{d}y\mathrm{d}t = 0. \tag{39}$$

We start by establishing existence and regularity results for the solution of (38).

Proposition 3.1 *Assume that $u_0 \in \ker \mathcal{T}_q$, $f \in L^1([0,T]; \ker \mathcal{T}_q)$ for some $q \in (1, +\infty)$. Then there is at least a weak solution $u \in L^\infty([0,T]; \ker \mathcal{T}_q)$ of (38) satisfying*

$$\|u(t)\|_{L^q(\mathbb{R}^m)} \le \|u_0\|_{L^q(\mathbb{R}^m)} + \int_0^t \|f(s)\|_{L^q(\mathbb{R}^m)}, \;\; t \in [0,T].$$

Moreover, if $u_0 \ge 0$ and $f \ge 0$ then $u \ge 0$.

Proof. For any $\varepsilon > 0$ there is a unique weak solution u^ε of

$$\begin{cases} \partial_t u^\varepsilon + a(t,y) \cdot \nabla_y u^\varepsilon + \frac{b(y)}{\varepsilon} \cdot \nabla_y u^\varepsilon = f(t,y), & (t,y) \in (0,T) \times \mathbb{R}^m \\ u^\varepsilon(0,y) = u_0(y), & y \in \mathbb{R}^m. \end{cases} \tag{40}$$

The solution is given by

$$u^\varepsilon(t,y) = u_0(Z^\varepsilon(0;t,y)) + \int_0^t f(s, Z^\varepsilon(s;t,y)) \, \mathrm{d}s, \;\; (t,y) \in [0,T] \times \mathbb{R}^m$$

where Z^ε are the characteristics corresponding to the field $a + \varepsilon^{-1} b$. Multiplying by $u^\varepsilon(t,y)|u^\varepsilon(t,y)|^{q-2}$ and integrating with respect to $y \in \mathbb{R}^m$, we obtain thanks to Hölder's inequality

$$\|u^\varepsilon\|_{L^q(\mathbb{R}^m)} \le \|u_0\|_{L^q(\mathbb{R}^m)} + \int_0^t \|f(s)\|_{L^q(\mathbb{R}^m)}, \;\; t \in [0,T].$$

We extract a sequence $(\varepsilon_k)_k$ converging towards 0 such that $u^{\varepsilon_k} \rightharpoonup u$ weakly $\star$ in $L^\infty([0,T]; L^q(\mathbb{R}^m))$ for some function $u \in L^\infty([0,T]; L^q(\mathbb{R}^m))$ satisfying

$$\|u\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} \le \|u_0\|_{L^q(\mathbb{R}^m)} + \|f\|_{L^1([0,T];L^q(\mathbb{R}^m))}.$$

By the weak formulation of (40) with a function $\varphi \in C^1_c([0,T)\times\mathbb{R}^m)$ we deduce that

$$\int_0^T\!\!\int_{\mathbb{R}^m} u^{\varepsilon_k}\left(\partial_t\varphi + \left(a + \frac{b}{\varepsilon_k}\right)\cdot\nabla_y\varphi\right)\,\mathrm{d}y\mathrm{d}t + \int_{\mathbb{R}^m} u_0\varphi(0,y)\,\mathrm{d}y + \int_0^T\!\!\int_{\mathbb{R}^m} f\varphi\,\mathrm{d}y\mathrm{d}t = 0. \quad (41)$$

Multiplying by ε_k and passing to the limit as $k\to+\infty$ one gets easily by Proposition 2.1 that $u(t)\in\ker\mathcal{T}_q$, $t\in[0,T)$. If the test function in (41) verifies $\mathcal{T}\varphi = 0$ we obtain

$$\int_0^T\!\!\int_{\mathbb{R}^m} u^{\varepsilon_k}\left(\partial_t\varphi + a\cdot\nabla_y\varphi\right)\,\mathrm{d}y\mathrm{d}t + \int_{\mathbb{R}^m} u_0\varphi(0,y)\,\mathrm{d}y + \int_0^T\!\!\int_{\mathbb{R}^m} f\varphi\,\mathrm{d}y\mathrm{d}t = 0.$$

Passing to the limit for $k\to+\infty$ we deduce that the weak $\star$ limit u satisfies the weak formulation of (38). If $u_0\geq 0, f\geq 0$ then $u^\varepsilon\geq 0$ for any $\varepsilon>0$ and thus the solution constructed above is non negative. $\square$

Whereas Proposition 3.1 yields a satisfactory theoretical result for solving the limit model (38), its numerical approximation remains a difficult problem. The main drawback of the weak formulation (39) is the particular form of the trial functions $\varphi\in\ker\mathcal{T}\cap C^1_c([0,T)\times\mathbb{R}^m)$. Generally, the choice of such test functions could be a difficult task. Accordingly, we are looking for a strong formulation of (38). Therefore we inquire about the smoothness of the solution. A complete regularity analysis can be carried out under the following hypothesis: we will assume that the field a is a linear combination of fields in involution with $b^0 := b$

$$a(t,y) = \sum_{i=0}^r \alpha_i(t,y)b^i(y),\;\; b^i\in W^{1,\infty}(\mathbb{R}^m),\;\; [b^i,b]=0,\; i\in\{1,...,r\} \quad (42)$$

where $(\alpha_i)_i$ are smooth coefficients verifying

$$\alpha_i\in L^1([0,T];L^\infty(\mathbb{R}^m)),\;\; b^j\cdot\nabla_y\alpha_i\in L^1([0,T];L^\infty(\mathbb{R}^m)),\;\; i,j\in\{0,1,...,r\}. \quad (43)$$

For any $i\in\{1,...,r\}$ we denote by $\mathcal{T}^i_q : \mathrm{D}(\mathcal{T}^i_q)\subset L^q(\mathbb{R}^m)\to L^q(\mathbb{R}^m)$ the operator given by

$$\mathrm{D}(\mathcal{T}^i_q) = \{u\in L^q(\mathbb{R}^m)\;:\;\mathrm{div}_y(b^iu)\in L^q(\mathbb{R}^m)\},\;\; \mathcal{T}^i_q u = \mathrm{div}_y(b^iu) - (\mathrm{div}_y b^i)u,\;\; u\in\mathrm{D}(\mathcal{T}^i_q)$$

and by Y^i the flow associated to b^i. Since $[b^i,b]=0$ then Y^i commutes with Y for any $i\in\{1,...,r\}$.

Proposition 3.2 *Assume that (42), (43) hold,* $u_0\in\ker\mathcal{T}_q\cap(\cap_{i=1}^r\mathrm{D}(\mathcal{T}^i_q))$, $f\in L^1([0,T];\ker\mathcal{T}_q\cap(\cap_{i=1}^r\mathrm{D}(\mathcal{T}^i_q)))$ *(i.e.,* $f\in L^1([0,T];L^q(\mathbb{R}^m))$, $\mathcal{T}_q f = 0$ *and* $\mathcal{T}^i_q f\in L^1([0,T];L^q(\mathbb{R}^m))$, $i\in\{1,...,r\}$*) and let us denote by u the weak solution of (38) constructed in Proposition 3.1. Then we have* $u(t)\in\ker\mathcal{T}_q\cap(\cap_{i=1}^r\mathrm{D}(\mathcal{T}^i_q))$, $t\in[0,T]$ *and*

$$\begin{aligned}\|\partial_t u\|_{L^1([0,T];L^q(\mathbb{R}^m))} &+ \sum_{i=1}^r\|\mathcal{T}^i_q u\|_{L^\infty([0,T];L^q(\mathbb{R}^m))}\leq C(\|f\|_{L^1([0,T];L^q(\mathbb{R}^m))}\\ &+ \sum_{i=1}^r\|\mathcal{T}^i_q f\|_{L^1([0,T];L^q(\mathbb{R}^m))} + \sum_{i=1}^r\|\mathcal{T}^i_q u_0\|_{L^q(\mathbb{R}^m)})\end{aligned}$$

for some constant depending on $\sum_{0\leq i,j\leq r} \|b^i\cdot\nabla_y\alpha_j\|_{L^1([0,T];L^\infty(\mathbb{R}^m))}$, $\sum_{i=0}^r \|\alpha_i\|_{L^1([0,T];L^\infty(\mathbb{R}^m))}$. *Moreover, if* $f \in L^\infty([0,T];L^q(\mathbb{R}^m))$, $\alpha_i \in L^\infty([0,T];L^\infty(\mathbb{R}^m))$ *for any* $i \in \{1,...,r\}$ *then* $\partial_t u \in L^\infty([0,T];L^q(\mathbb{R}^m))$.

Proof. For any $\varepsilon > 0$ let u^ε be the solution of (40). We intend to estimate $\|\mathcal{T}_q u^\varepsilon\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} + \sum_{i=1}^r \|\mathcal{T}_q^i u^\varepsilon\|_{L^\infty([0,T];L^q(\mathbb{R}^m))}$ and $\|\partial_t u^\varepsilon\|_{L^1([0,T];L^q(\mathbb{R}^m))}$ uniformly with respect to $\varepsilon > 0$. Consider the sequences of smooth functions $(u_{0n})_n$, $(f_n)_n$ such that

$$\lim_{n\to+\infty} u_{0n} = u_0,\ \ \lim_{n\to+\infty} \mathcal{T}_q^i u_{0n} = \mathcal{T}_q^i u_0,\ i\in\{0,1,...,r\}\ \text{ in } L^q(\mathbb{R}^m)$$

$$\lim_{n\to+\infty} f_n = f,\ \ \lim_{n\to+\infty} \mathcal{T}_q^i f_n = \mathcal{T}_q^i f,\ i\in\{0,1,...,r\}\ \text{ in } L^1([0,T];L^q(\mathbb{R}^m))$$

and let us denote by $(u_n^\varepsilon)_n$ the solutions of (40) corresponding to the initial conditions $(u_{0n})_n$ and the source terms $(f_n)_n$. Actually $(u_n^\varepsilon)_n$ are strong solutions. It is easily seen that

$$\|u_n^\varepsilon(t) - u^\varepsilon(t)\|_{L^q(\mathbb{R}^m)} \leq \|u_{0n}-u_0\|_{L^q(\mathbb{R}^m)} + \int_0^t \|f_n(s)-f(s)\|_{L^q(\mathbb{R}^m)}\,\mathrm{d}s,\ \ t\in[0,T]$$

and therefore $\lim_{n\to+\infty} u_n^\varepsilon = u^\varepsilon$ in $L^\infty([0,T];L^q(\mathbb{R}^m))$. Assume for the moment that ε, n are fixed and let us estimate $\sum_{i=0}^r \|\mathcal{T}_q^i u_n^\varepsilon\|_{L^\infty([0,T];L^q(\mathbb{R}^m))}$ and $\|\partial_t u_n^\varepsilon\|_{L^1([0,T];L^q(\mathbb{R}^m))}$. Take $h\in\mathbb{R}$, $i\in\{1,...,r\}$ and consider the functions

$$u_{nh}^\varepsilon(t,y) = u_n^\varepsilon(t,Y^i(h;y)),\ \ a_h(t,y) = \frac{\partial Y^i}{\partial y}(-h;Y^i(h;y))a(t,Y^i(h;y))$$

$$b_h(y) = \frac{\partial Y^i}{\partial y}(-h;Y^i(h;y))b(Y^i(h;y)),\ \ u_{0nh}(y) = u_{0n}(Y^i(h;y)),\ \ f_{nh}(t,y) = f_n(t,Y^i(h;y)).$$

A direct computation shows that

$$\begin{cases} \partial_t u_{nh}^\varepsilon + a_h(t,y)\cdot\nabla_y u_{nh}^\varepsilon + \frac{b_h(y)}{\varepsilon}\cdot\nabla_y u_{nh}^\varepsilon = f_{nh}(t,y), & (t,y)\in(0,T)\times\mathbb{R}^m \\ u_{nh}^\varepsilon(0,y) = u_{0nh}(y), & y\in\mathbb{R}^m. \end{cases} \tag{44}$$

Combining with the formulation (40) of u_n^ε one gets

$$\begin{cases} \partial_t\left(\frac{u_{nh}^\varepsilon - u_n^\varepsilon}{h}\right) + \frac{a_h - a}{h}\cdot\nabla_y u_{nh}^\varepsilon + a(t,y)\cdot\nabla_y\left(\frac{u_{nh}^\varepsilon-u_n^\varepsilon}{h}\right) & \\ +\frac{b_h - b}{\varepsilon h}\cdot\nabla_y u_{nh}^\varepsilon + \frac{b(y)}{\varepsilon}\cdot\nabla_y\left(\frac{u_{nh}^\varepsilon-u_n^\varepsilon}{h}\right) = \frac{f_{nh}-f_n}{h}, & (t,y)\in(0,T)\times\mathbb{R}^m \\ \frac{u_{nh}^\varepsilon(0,y)-u_n^\varepsilon(0,y)}{h} = \frac{u_{0nh}(y)-u_{0n}(y)}{h}, & y\in\mathbb{R}^m. \end{cases} \tag{45}$$

Obviously we have

$$\lim_{h\to0}\frac{u_{nh}^\varepsilon - u_n^\varepsilon}{h} = \lim_{h\to0}\frac{u_n^\varepsilon(t,Y^i(h;y)) - u_n^\varepsilon(t,y)}{h} = b^i(y)\cdot\nabla_y u_n^\varepsilon(t,y) = \mathcal{T}_q^i u_n^\varepsilon$$

$$\lim_{h\to0}\frac{f_{nh}-f_n}{h} = \lim_{h\to0}\frac{f_n(t,Y^i(h;y)) - f_n(t,y)}{h} = b^i(y)\cdot\nabla_y f_n(t,y) = \mathcal{T}_q^i f_n$$

$$\lim_{h\to 0}\frac{u_{0nh}-u_{0n}}{h}=\lim_{h\to 0}\frac{u_{0n}(Y^i(h;y))-u_{0n}(y)}{h}=b^i(y)\cdot\nabla_y u_{0n}(y)=\mathcal{T}_q^i u_{0n}.$$

Taking the derivatives with respect to y and then with respect to h in the equality $Y^i(-h;Y^i(h;y))=y$, we deduce after some easy manipulations that

$$\lim_{h\to 0}\frac{1}{h}\left\{\frac{\partial Y^i}{\partial y}(-h;Y^i(h;y))-I_m\right\}=-\frac{\partial b^i}{\partial y}(y).$$

By direct computations we obtain immediately

$$\lim_{h\to 0}\frac{a_h-a}{h}=(b^i\cdot\nabla_y)a-(a\cdot\nabla_y)b^i=[b^i,a]$$

$$\lim_{h\to 0}\frac{b_h-b}{h}=(b^i\cdot\nabla_y)b-(b\cdot\nabla_y)b^i=[b^i,b]=0.$$

By passing to the limit for $h\to 0$ in (45) we deduce that $\mathcal{T}_q^i u_n^\varepsilon$ solves weakly the problem

$$\begin{cases}\partial_t(\mathcal{T}_q^i u_n^\varepsilon)+a\cdot\nabla_y(\mathcal{T}_q^i u_n^\varepsilon)+\dfrac{b}{\varepsilon}\cdot\nabla_y(\mathcal{T}_q^i u_n^\varepsilon)=\mathcal{T}_q^i f_n-[b^i,a]\cdot\nabla_y u_n^\varepsilon\\ \mathcal{T}_q^i u_n^\varepsilon(0,\cdot)=\mathcal{T}_q^i u_{0n}.\end{cases}\tag{46}$$

As in the proof of Proposition 3.1 we obtain for any $t\in[0,T]$ and $i\in\{1,...,r\}$

$$\|\mathcal{T}_q^i u_n^\varepsilon(t)\|_{L^q(\mathbb{R}^m)}\le\|\mathcal{T}_q^i u_{0n}\|_{L^q(\mathbb{R}^m)}+\int_0^t\|\mathcal{T}_q^i f_n(s)-[b^i,a(s)]\cdot\nabla_y u_n^\varepsilon(s)\|_{L^q(\mathbb{R}^m)}\,\mathrm{d}s.\tag{47}$$

Since $a=\sum_{k=0}^r\alpha_k b^k$ we obtain by direct computation, with the notation $\mathcal{T}_q^0:=\mathcal{T}_q$

$$[b^i,a]=\sum_{k=0}^r(\mathcal{T}_q^i\alpha_k)b^k$$

and therefore

$$[b^i,a]\cdot\nabla_y u_n^\varepsilon=\sum_{k=0}^r(\mathcal{T}_q^i\alpha_k)(\mathcal{T}_q^k u_n^\varepsilon).$$

Consequently (47) implies

$$\begin{aligned}\|\mathcal{T}_q^i u_n^\varepsilon(t)\|_{L^q(\mathbb{R}^m)}&\le\|\mathcal{T}_q^i u_{0n}\|_{L^q(\mathbb{R}^m)}+\int_0^t\|\mathcal{T}_q^i f_n(s)\|_{L^q(\mathbb{R}^m)}\,\mathrm{d}s\\&+\int_0^t\sum_{k=0}^r\|b^i\cdot\nabla_y\alpha_k(s)\|_{L^\infty(\mathbb{R}^m)}\|\mathcal{T}_q^k u_n^\varepsilon(s)\|_{L^q(\mathbb{R}^m)}\,\mathrm{d}s.\end{aligned}\tag{48}$$

Actually (48) holds also for b^i replaced by $b^0 = b$ since $[b, b] = 0$

$$\begin{aligned}\|\mathcal{T}_q^0 u_n^\varepsilon(t)\|_{L^q(\mathbb{R}^m)} &\leq \|\mathcal{T}_q^0 u_{0n}\|_{L^q(\mathbb{R}^m)} + \int_0^t \|\mathcal{T}_q^0 f_n(s)\|_{L^q(\mathbb{R}^m)}\,\mathrm{d}s \\ &+ \int_0^t \sum_{k=0}^r \|b^0 \cdot \nabla_y \alpha_k(s)\|_{L^\infty(\mathbb{R}^m)} \|\mathcal{T}_q^k u_n^\varepsilon(s)\|_{L^q(\mathbb{R}^m)}\,\mathrm{d}s. \qquad (49)\end{aligned}$$

Summing up the above inequalities one gets

$$\begin{aligned}\sum_{i=0}^r \|\mathcal{T}_q^i u_n^\varepsilon(t)\|_{L^q(\mathbb{R}^m)} &\leq \sum_{i=0}^r \|\mathcal{T}_q^i u_{0n}\|_{L^q(\mathbb{R}^m)} + \int_0^t \sum_{i=0}^r \|\mathcal{T}_q^i f_n(s)\|_{L^q(\mathbb{R}^m)}\,\mathrm{d}s \\ &+ \sum_{i=0}^r \sum_{k=0}^r \int_0^t \|b^i \cdot \nabla_y \alpha_k(s)\|_{L^\infty(\mathbb{R}^m)} \|\mathcal{T}_q^k u_n^\varepsilon(s)\|_{L^q(\mathbb{R}^m)}. \qquad (50)\end{aligned}$$

By Gronwall's lemma we deduce that for any $t \in [0, T]$

$$\sum_{i=0}^r \|\mathcal{T}_q^i u_n^\varepsilon\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} \leq C \sum_{i=0}^r \left\{ \|\mathcal{T}_q^i u_{0n}\|_{L^q(\mathbb{R}^m)} + \|\mathcal{T}_q^i f_n\|_{L^1([0,T];L^q(\mathbb{R}^m))} \right\} \qquad (51)$$

for some constant depending on $\sum_{0 \leq i,j \leq r} \|b^i \cdot \nabla_y \alpha_j\|_{L^1([0,T];L^\infty(\mathbb{R}^m))}$. After extraction eventually we can assume that $(\mathcal{T}_q^i u_n^\varepsilon)_n$ converges weakly $\star$ in $L^\infty([0,T]; L^q(\mathbb{R}^m))$ towards some function $w^i \in L^\infty([0,T]; L^q(\mathbb{R}^m))$ for any $i \in \{0, 1, ..., r\}$. Since we know that $\lim_{n \to +\infty} u_n^\varepsilon = u^\varepsilon$ in $L^\infty([0,T]; L^q(\mathbb{R}^m))$ it is easily seen that

$$u^\varepsilon(t) \in \cap_{i=0}^r \mathrm{D}(\mathcal{T}_q^i), \;\; \mathcal{T}_q^i u^\varepsilon(t) = w^i(t), \;\; t \in [0, T].$$

Moreover, passing to the limit with respect to n in (51) and taking into account that $\lim_{n \to +\infty} \mathcal{T}_q u_{0n} = \mathcal{T}_q u_0 = 0$ in $L^q(\mathbb{R}^m)$ and $\lim_{n \to +\infty} \mathcal{T}_q f_n = \mathcal{T}_q f = 0$ in $L^1([0,T]; L^q(\mathbb{R}^m))$ we obtain

$$\sum_{i=1}^r \|\mathcal{T}_q^i u^\varepsilon\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} \leq C \sum_{i=1}^r \{ \|\mathcal{T}_q^i u_0\|_{L^q(\mathbb{R}^m)} + \|\mathcal{T}_q^i f\|_{L^1([0,T];L^q(\mathbb{R}^m))} \}. \qquad (52)$$

Recall that the weak solution u constructed in Proposition 3.1 has been obtained by taking a weak $\star$ limit point of the family $(u^\varepsilon)_{\varepsilon > 0}$ in $L^\infty([0,T]; L^q(\mathbb{R}^m))$. Therefore we deduce by passing to the limit for $\varepsilon \searrow 0$ in (52) that $u(t) \in \cap_{i=1}^r \mathrm{D}(\mathcal{T}_q^i)$, $t \in [0, T]$ and

$$\sum_{i=1}^r \|\mathcal{T}_q^i u\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} \leq C \sum_{i=1}^r \{ \|\mathcal{T}_q^i u_0\|_{L^q(\mathbb{R}^m)} + \|\mathcal{T}_q^i f\|_{L^1([0,T];L^q(\mathbb{R}^m))} \}. \qquad (53)$$

Since $\mathcal{T}_q u = 0$, observe also that

$$\|a(t) \cdot \nabla_y u(t)\|_{L^q(\mathbb{R}^m)} = \| \sum_{i=1}^r \alpha_i(t) b^i \cdot \nabla_y u(t) \|_{L^q(\mathbb{R}^m)} \leq \sum_{i=1}^r \|\alpha_i(t)\|_{L^\infty(\mathbb{R}^m)} \|\mathcal{T}_q^i u(t)\|_{L^q(\mathbb{R}^m)}$$

and thus

$$\begin{aligned}
\|\partial_t u\|_{L^1([0,T];L^q(\mathbb{R}^m))} &= \|f - \langle a \cdot \nabla_y u\rangle^{(q)}\|_{L^1([0,T];L^q(\mathbb{R}^m))} \\
&\leq \|f\|_{L^1([0,T];L^q(\mathbb{R}^m))} + \sum_{i=1}^{r} \|\mathcal{T}_q^i u\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} \|\alpha_i\|_{L^1([0,T];L^\infty(\mathbb{R}^m))} \\
&\leq \|f\|_{L^1([0,T];L^q(\mathbb{R}^m))} + C\sum_{i=1}^{r} \{\|\mathcal{T}_q^i f\|_{L^1([0,T];L^q(\mathbb{R}^m))} + \|\mathcal{T}_q^i u_0\|_{L^q(\mathbb{R}^m)}\}.
\end{aligned}$$

When f belongs to $L^\infty([0,T];L^q(\mathbb{R}^m))$ and $\alpha_i \in L^\infty([0,T];L^\infty(\mathbb{R}^m))$ for any $i \in \{1,...,r\}$ we obtain

$$\begin{aligned}
\|\partial_t u\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} &\leq \|f\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} + \sum_{i=1}^{r} \|\alpha_i\|_{L^\infty([0,T];L^\infty(\mathbb{R}^m))} \|\mathcal{T}_q^i u\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} \\
&\leq \|f\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} + C\sum_{i=1}^{r} \{\|\mathcal{T}_q^i f\|_{L^1([0,T];L^q(\mathbb{R}^m))} + \|\mathcal{T}_q^i u_0\|_{L^q(\mathbb{R}^m)}\}.
\end{aligned}$$

□

Thanks to the previous regularity result we are able to establish the existence of strong solution for (38).

Definition 3.2 *Under the hypotheses (42), (43), (20) we say that u is a strong solution of (38) iff $u \in L^\infty([0,T];L^q(\mathbb{R}^m))$, $\partial_t u \in L^1([0,T];L^q(\mathbb{R}^m))$, $\mathcal{T}_q^i u \in L^\infty([0,T];L^q(\mathbb{R}^m))$ for any $i \in \{1,...,r\}$ and*

$$\begin{cases} \partial_t u + \sum_{i=1}^r \langle \alpha_i(t)\rangle^{(\infty)} \mathcal{T}_q^i u(t) = f(t), & t \in (0,T) \\ u(0) = u_0. \end{cases} \tag{54}$$

Corollary 3.1 *Assume that (42), (43), (20) hold. Then for any $u_0 \in (\cap_{i=1}^r \mathrm{D}(\mathcal{T}_q^i)) \cap \ker \mathcal{T}_q$ and $f \in L^1([0,T];(\cap_{i=1}^r \mathrm{D}(\mathcal{T}_q^i)) \cap \ker \mathcal{T}_q)$, there is a strong solution u for (38) verifying*

$$\begin{aligned}
\|\partial_t u\|_{L^1([0,T];L^q(\mathbb{R}^m))} &+ \sum_{i=1}^{r} \|\mathcal{T}_q^i u\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} \leq C\|f\|_{L^1([0,T];L^q(\mathbb{R}^m))} \\
&+ C\sum_{i=1}^{r} \{\|\mathcal{T}_q^i f\|_{L^1([0,T];L^q(\mathbb{R}^m))} + \|\mathcal{T}_q^i u_0\|_{L^q(\mathbb{R}^m)}\}.
\end{aligned} \tag{55}$$

Proof. Let u be the solution constructed in Proposition 3.2. This function has the regularity in (55), satisfies $\mathcal{T}_q u = 0$ and

$$\int_0^T\!\!\int_{\mathbb{R}^m} u(\partial_t \varphi + \mathrm{div}_y(\varphi a))\,\mathrm{d}y\mathrm{d}t + \int_{\mathbb{R}^m} u_0 \varphi(0,y)\,\mathrm{d}y + \int_0^T\!\!\int_{\mathbb{R}^m} f\varphi\,\mathrm{d}y\mathrm{d}t = 0 \tag{56}$$

for any function $\varphi \in C^1_c([0,T)\times\mathbb{R}^m)$ verifying $\mathcal{T}\varphi = 0$. Since $a = \sum_{i=0}^r \alpha_i b^i$ and $\mathcal{T}_q u = 0$ one gets

$$\begin{aligned}\int_0^T\!\!\int_{\mathbb{R}^m} u \,\mathrm{div}_y(a\varphi)\,\mathrm{d}y\mathrm{d}t &= \int_0^T\!\!\int_{\mathbb{R}^m} u\,\mathrm{div}_y\{(\sum_{i=0}^r \alpha_i b^i)\varphi\}\,\mathrm{d}y\mathrm{d}t \\ &= -\sum_{i=1}^r \int_0^T\!\!\int_{\mathbb{R}^m} \alpha_i\varphi\mathcal{T}_q^i u\,\mathrm{d}y\mathrm{d}t\end{aligned}$$

implying that

$$\int_0^T\!\!\int_{\mathbb{R}^m}(\partial_t u + \sum_{i=1}^r \alpha_i \mathcal{T}_q^i u)\varphi\,\mathrm{d}y\mathrm{d}t = \int_0^T\!\!\int_{\mathbb{R}^m} f\varphi\,\mathrm{d}y\mathrm{d}t. \tag{57}$$

Using now the properties of the operators $\langle\cdot\rangle^{(q)}, \langle\cdot\rangle^{(q\,')}$ we obtain for any $t\in[0,T]$

$$\begin{aligned}\int_{\mathbb{R}^m}\varphi(t)\ \sum_{i=1}^r \alpha_i(t)\mathcal{T}_q^i u(t)\,\mathrm{d}y &= \int_{\mathbb{R}^m}\langle\varphi(t)\rangle^{(q\,')}\sum_{i=1}^r \alpha_i(t)\mathcal{T}_q^i u(t)\,\mathrm{d}y \\ &= \sum_{i=1}^r\int_{\mathbb{R}^m}\varphi(t)\ \langle\alpha_i(t)\mathcal{T}_q^i u(t)\rangle^{(q)}\,\mathrm{d}y\end{aligned} \tag{58}$$

(we have used the equality $\langle\varphi(t)\rangle^{(q\,')} = \varphi(t)$ which is valid since $\mathcal{T}_{q\,'}\varphi = 0$). Combining (57), (58) yields

$$\int_0^T\!\!\int_{\mathbb{R}^m}\left(\partial_t u + \sum_{i=1}^r\left\langle \alpha_i\mathcal{T}_q^i u\right\rangle^{(q)} - f\right)\varphi\,\mathrm{d}y\mathrm{d}t = 0.$$

Observe that the function $\partial_t u + \sum_{i=1}^r \left\langle\alpha_i\mathcal{T}_q^i u\right\rangle^{(q)} - f$ belongs to $\ker\mathcal{T}_q$ and thus we obtain

$$\partial_t u + \sum_{i=1}^r\left\langle\alpha_i\mathcal{T}_q^i u(t)\right\rangle^{(q)} = f(t),\ \ t\in(0,T).$$

Since for any $i\in\{1,...,r\}$ we have $u(t)\in\ker\mathcal{T}_q\cap \mathrm{D}(\mathcal{T}_q^i)$, we deduce by Remark 2.6 that $\mathcal{T}_q^i u(t)\in\ker\mathcal{T}_q$. Therefore, by Corollary 2.4 we obtain

$$\langle\alpha_i(t)\mathcal{T}_q^i u(t)\rangle^{(q)} = \langle\alpha_i(t)\rangle^{(\infty)}\mathcal{T}_q^i u(t).$$

Finally u solves

$$\left\{\begin{array}{ll}\partial_t u + \sum_{i=1}^r\langle\alpha_i(t)\rangle^{(\infty)}\mathcal{T}_q^i u(t) = f(t), & t\in(0,T)\\ u(0) = u_0. & \end{array}\right. \tag{59}$$

$\square$

Remark 3.1 *Notice that if* u *is a strong solution of (59) whose initial condition belongs to* $\ker \mathcal{T}_q$ *then the constraint* $\mathcal{T}_q u = 0$ *is automatically satisfied. Indeed, we have*

$$\sum_{i=1}^{r} \langle \alpha_i(t) \rangle^{(\infty)} \mathcal{T}_q^i u(t) \in \ker \mathcal{T}_q \ \ t \in [0,T]$$

and therefore $\partial_t u \in \ker \mathcal{T}_q$. *We deduce that* $\partial_t \mathcal{T}_q u = 0$ *implying that* $\mathcal{T}_q u(t) = \mathcal{T}_q u_0 = 0$ *for* $t \in [0,T]$.

As usual, the existence of strong solution for the adjoint problem implies the uniqueness of weak solution.

Proposition 3.3 *Assume that (42), (43) hold. Then for any* $u_0 \in \ker \mathcal{T}_q$ *and* $f \in L^1([0,T]; \ker \mathcal{T}_q)$, *with* $q \in (1, +\infty)$, *there is at most one weak solution of (38).*

Proof. Let $u \in L^\infty([0,T]; \ker \mathcal{T}_q)$ be any weak solution of (38) with vanishing initial condition and source term. We will show that $u = 0$. We know that

$$\int_0^T \int_{\mathbb{R}^m} u(\partial_t \theta + a \cdot \nabla_y \theta) \, \mathrm{d}y\mathrm{d}t = 0 \tag{60}$$

for any function $\theta \in C_c^1([0,T) \times \mathbb{R}^m)$ satisfying $\mathcal{T}\theta = 0$. Consider a function $\eta = \eta(t) \in C([0,T])$ and $\psi = \psi(y) \in (\cap_{i=1}^r \mathrm{D}(\mathcal{T}_{q'}^i)) \cap \ker \mathcal{T}_{q'}$. By Corollary 3.1 there is a strong solution $\tilde{\varphi}$ of

$$\begin{cases} \partial_t \tilde{\varphi} - \langle a(T-t) \cdot \nabla_y \tilde{\varphi} \rangle^{(q')} = \eta(T-t)\psi(y), & (t,y) \in (0,T) \times \mathbb{R}^m \\ \tilde{\varphi}(0,y) = 0. & y \in \mathbb{R}^m \end{cases}$$

satisfying $\tilde{\varphi}, \mathcal{T}_{q'}^i \tilde{\varphi} \in L^\infty([0,T]; L^{q'}(\mathbb{R}^m))$, $\partial_t \tilde{\varphi} \in L^1([0,T]; L^{q'}(\mathbb{R}^m))$. It is easily seen that $\varphi(t,y) = \tilde{\varphi}(T-t,y)$ has the same regularity as $\tilde{\varphi}$, $\varphi(t) \in \ker \mathcal{T}_{q'}$ and

$$\begin{cases} -\partial_t \varphi - \langle a(t) \cdot \nabla_y \varphi \rangle^{(q')} = \eta(t)\psi(y), & (t,y) \in (0,T) \times \mathbb{R}^m \\ \varphi(T,y) = 0, & y \in \mathbb{R}^m. \end{cases}$$

We use now (60) with the function φ (observe that the formulation (60) still holds true for trial functions having the regularity of φ)

$$\begin{aligned} 0 &= \int_0^T \int_{\mathbb{R}^m} u(\partial_t \varphi + a \cdot \nabla_y \varphi) \, \mathrm{d}y\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{R}^m} u \partial_t \varphi \, \mathrm{d}y\mathrm{d}t + \int_0^T \int_{\mathbb{R}^m} \langle u(t) \rangle^{(q)} a \cdot \nabla_y \varphi \, \mathrm{d}y\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{R}^m} u(\partial_t \varphi + \langle a(t) \cdot \nabla_y \varphi \rangle^{(q')}) \, \mathrm{d}y\mathrm{d}t \\ &= -\int_0^T \eta(t) \int_{\mathbb{R}^m} u(t,y)\psi(y) \, \mathrm{d}y \, \mathrm{d}t. \end{aligned}$$

We deduce that $\int_{\mathbb{R}^m} u(t,y)\psi(y) \, \mathrm{d}y = 0$ for any $t \in [0,T]$ and any $\psi \in (\cap_{i=1}^r \mathrm{D}(\mathcal{T}_{q'}^i)) \cap \ker \mathcal{T}_{q'}$. Since $u(t) \in \ker \mathcal{T}_q$ it follows that $u(t) = 0$, $t \in [0,T]$. □

Remark 3.2 *The uniqueness of the weak solution for (38) guarantees the uniqueness of the strong solution in Corollary 3.1.*

In order to establish the conservation of the L^q norm for weak solutions without source term we use the easy lemma.

Lemma 3.1 *Let $\beta \in W^{1,\infty}(\mathbb{R}^m)$ be a smooth function and c a smooth field with bounded divergence. Assume that $v \in \mathrm{D}(c\cdot\nabla_y) \subset L^q(\mathbb{R}^m)$ for some $q \in (1,+\infty)$. Then we have*

$$\int_{\mathbb{R}^m} \beta(y)(c\cdot\nabla_y)v\;|v|^{q-2}v\;\mathrm{d}y = -\frac{1}{q}\int_{\mathbb{R}^m}|v|^q\;\mathrm{div}_y(\beta c)\;\mathrm{d}y.$$

Corollary 3.2 *Assume that (42), (43) hold and that $u_0 \in \ker\mathcal{T}_q$, $f \in L^1([0,T];\ker\mathcal{T}_q)$ for some $q\in(1,+\infty)$. Then the weak solution of (38) satisfies for any $t\in[0,T]$*

$$\frac{1}{q}\int_{\mathbb{R}^m}|u(t,y)|^q\;\mathrm{d}y = \frac{1}{q}\int_{\mathbb{R}^m}|u_0(y)|^q\;\mathrm{d}y + \int_0^t\int_{\mathbb{R}^m} f(s,y)|u(s,y)|^{q-2}u(s,y)\;\mathrm{d}y\;\mathrm{d}s.$$

In particular, when $f=0$ the L^q norm is preserved.

Proof. Consider the sequences $(u_{0n})_n$ and $(f_n)_n$ such that $\lim_{n\to+\infty} u_{0n} = u_0$ in $L^q(\mathbb{R}^m)$, $\lim_{n\to+\infty} f_n = f$ in $L^1([0,T];L^q(\mathbb{R}^m))$. Let us denote by u, u_n the unique solutions associated to $(u_0,f),(u_{0n},f_n)$ respectively. Thanks to the uniqueness result of Proposition 3.3 we deduce by Proposition 3.1 that

$$\|u_n - u\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} \le \|u_{0n}-u_0\|_{L^q(\mathbb{R}^m)} + \|f_n - f\|_{L^1([0,T];L^q(\mathbb{R}^m))}$$

and therefore it is sufficient to analyze the case of strong solutions $(u_n)_n$. Taking into account that $|u_n|^{q-2}u_n \in \ker\mathcal{T}_{q'}$ we have by Lemma 3.1

$$\begin{aligned}
\int_{\mathbb{R}^m}\langle a(t)\cdot\nabla_y u_n(t)\rangle^{(q)}|u_n|^{q-2}u_n\;\mathrm{d}y &= \int_{\mathbb{R}^m} a(t)\cdot\nabla_y u_n(t)\langle|u_n(t)|^{q-2}u_n(t)\rangle^{(q')}\;\mathrm{d}y\\
&= \int_{\mathbb{R}^m} a(t)\cdot\nabla_y u_n(t)|u_n(t)|^{q-2}u_n(t)\;\mathrm{d}y\\
&= \int_{\mathbb{R}^m}\sum_{i=0}^r \alpha_i(t,y)\mathcal{T}_q^i u_n(t)|u_n|^{q-2}u_n\;\mathrm{d}y\\
&= -\frac{1}{q}\int_{\mathbb{R}^m}\mathrm{div}_y\Big(\sum_{i=0}^r\alpha_i b^i\Big)|u_n|^q\;\mathrm{d}y\\
&= -\frac{1}{q}\int_{\mathbb{R}^m}|u_n|^q\mathrm{div}_y a\;\mathrm{d}y = 0.
\end{aligned}$$

Our conclusion follows immediately by multiplying the equation $\partial_t u_n + \langle a(t)\cdot\nabla_y u_n(t)\rangle^{(q)} = f_n(t)$ by $|u_n(t)|^{q-2}u_n(t)$ and integrating with respect to $y\in\mathbb{R}^m$. $\square$

Naturally we can obtain more smoothness for the solution provided that the data are more regular. We present here a simplified version for the homogeneous problem. The proof is a direct consequence of Propositions 3.2, 2.11.

Proposition 3.4 *Assume that (42), (43) hold and let us denote by u the solution of (38) with $f = 0$ and the initial condition u_0 satisfying for some $q \in (1, +\infty)$*

$$u_0 \in (\cap_{i=1}^r \mathrm{D}(\mathcal{T}_q^i)) \cap \ker \mathcal{T}_q, \;\; \mathcal{T}_q^j u_0 \in \cap_{i=1}^r \mathrm{D}(\mathcal{T}_q^i), \; \forall\, j \in \{1, ..., r\}.$$

Then we have

$$\sum_{i=1}^r \sum_{j=1}^r \|\mathcal{T}_q^i \mathcal{T}_q^j u\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} \leq C \left(\sum_{i=1}^r \sum_{j=1}^r \|\mathcal{T}_q^i \mathcal{T}_q^j u_0\|_{L^q(\mathbb{R}^m)} + \sum_{i=1}^r \|\mathcal{T}_q^i u_0\|_{L^q(\mathbb{R}^m)} \right)$$

with C depending on $\sum_{1\leq i,j,k\leq r} \|\mathcal{T}_q^i \mathcal{T}_q^j \alpha_k\|_{L^1([0,T];L^\infty(\mathbb{R}^m))}$, $\sum_{1\leq i,j\leq r} \|\mathcal{T}_q^i \alpha_j\|_{L^1([0,T];L^\infty(\mathbb{R}^m))}$ and

$$\|\partial_t^2 u\|_{L^1([0,T];L^q)} + \sum_{i=1}^r \|\partial_t \mathcal{T}_q^i u\|_{L^1([0,T];L^q)} \leq C \left(\sum_{i=1}^r \sum_{j=1}^r \|\mathcal{T}_q^i \mathcal{T}_q^j u_0\|_{L^q} + \sum_{i=1}^r \|\mathcal{T}_q^i u_0\|_{L^q} \right)$$

with C depending on $\sum_{1\leq i,j,k\leq r} \|\mathcal{T}_q^i \mathcal{T}_q^j \alpha_k\|_{L^1([0,T];L^\infty(\mathbb{R}^m))}$, $\sum_{1\leq i,j\leq r} \|\mathcal{T}_q^i \alpha_j\|_{L^1([0,T];L^\infty(\mathbb{R}^m))}$, $\sum_{i=1}^r \|\alpha_i\|_{L^1([0,T];L^\infty(\mathbb{R}^m))}$ and $\sum_{i=1}^r \|\partial_t \alpha_i\|_{L^1([0,T];L^\infty(\mathbb{R}^m))}$.

Proof. We only sketch the arguments. For any $j \in \{1, ..., r\}$ we have formally

$$\begin{aligned}
\mathcal{T}_q^j (a(t) \cdot \nabla_y u) &= \mathcal{T}_q^j \left(\sum_{i=1}^r \alpha_i \mathcal{T}_q^i u \right) \\
&= \sum_{i=1}^r \left(\mathcal{T}_q^j \alpha_i\right) \left(\mathcal{T}_q^i u\right) + \sum_{i=1}^r \left(\alpha_i \mathcal{T}_q^j \left(\mathcal{T}_q^i u\right)\right) \\
&= (a \cdot \nabla_y) \left(\mathcal{T}_q^j u\right) + \sum_{i=1}^r \left(\mathcal{T}_q^j \alpha_i\right) \left(\mathcal{T}_q^i u\right).
\end{aligned}$$

Applying now Proposition 2.11 yields

$$\begin{aligned}
\mathcal{T}_q^j \langle a(t) \cdot \nabla_y u(t) \rangle^{(q)} &= \langle \mathcal{T}_q^j (a(t) \cdot \nabla_y u(t)) \rangle^{(q)} \\
&= \langle (a(t) \cdot \nabla_y)(\mathcal{T}_q^j u) \rangle^{(q)} + \sum_{i=1}^r \langle (\mathcal{T}_q^j \alpha_i)(\mathcal{T}_q^i u) \rangle^{(q)}
\end{aligned}$$

and therefore $v^j := \mathcal{T}_q^j u$ satisfies the problem

$$\begin{cases} \partial_t v^j + \langle a(t) \cdot \nabla_y v^j \rangle^{(q)} = -\sum_{i=1}^r \langle (\mathcal{T}_q^j \alpha_i)(\mathcal{T}_q^i u) \rangle^{(q)} =: -f^j, & (t, y) \in (0, T) \times \mathbb{R}^m \\ v^j(0, y) = \mathcal{T}_q^j u_0(y), & y \in \mathbb{R}^m. \end{cases} \quad (61)$$

The above computations can be justified rigorously by observing that, thanks to Proposition 3.2, we already know that $u(t) \in \mathrm{D}(\mathcal{T}_q^j) \cap \ker \mathcal{T}_q$, $t \in [0,T]$, and that for any smooth function φ we have

$$\begin{aligned}\int_{\mathbb{R}^m} u \,\mathrm{div}_y(\mathrm{div}_y(\varphi\, b^j)a)\;\mathrm{d}y &= -\int_{\mathbb{R}^m} \mathrm{div}_y(\varphi\, b^j)(\sum_{i=1}^r \alpha_i \mathcal{T}_q^i u)\;\mathrm{d}y \\ &= \int_{\mathbb{R}^m} \varphi \sum_{i=1}^r (b^j \cdot \nabla_y \alpha_i)\mathcal{T}_q^i u\;\mathrm{d}y - \int_{\mathbb{R}^m} \mathrm{div}_y(\varphi a)\mathcal{T}_q^j u\;\mathrm{d}y.\end{aligned} \tag{62}$$

Notice also by Lemma 2.2 and Remark 2.6 that, if φ is a smooth function such that $\mathcal{T}\varphi = 0$, then $\mathrm{div}_y(\varphi\, b^j) = \varphi\,\mathrm{div}_y b^j + b^j \cdot \nabla_y \varphi$ remains constant along the flow of b. Therefore, for any smooth function φ, compactly supported in $[0,T) \times \mathbb{R}^m$, such that $\mathcal{T}\varphi = 0$, we can apply the weak formulation (39) with the trial function $\mathrm{div}_y(\varphi\, b^j)$

$$\int_0^T\!\!\int_{\mathbb{R}^m} u(\partial_t \mathrm{div}_y(\varphi\, b^j) + \mathrm{div}_y(\mathrm{div}_y(\varphi\, b^j)a))\;\mathrm{d}y\mathrm{d}t + \int_{\mathbb{R}^m} u_0(y)\mathrm{div}_y(\varphi(0,y)\; b^j)\;\mathrm{d}y = 0. \tag{63}$$

But we can write

$$\int_0^T\!\!\int_{\mathbb{R}^m} u\partial_t \mathrm{div}_y(\varphi\, b^j)\;\mathrm{d}y\mathrm{d}t = -\int_0^T\!\!\int_{\mathbb{R}^m} (\mathcal{T}_q^j u)\partial_t\varphi\;\mathrm{d}y\mathrm{d}t \tag{64}$$

$$\int_{\mathbb{R}^m} u_0(y)\mathrm{div}_y(\varphi(0,y)\; b^j)\;\mathrm{d}y = -\int_{\mathbb{R}^m} (\mathcal{T}_q^j u_0)\varphi(0,y)\;\mathrm{d}y. \tag{65}$$

Combining (62), (63), (64), (65) shows that $v^j = \mathcal{T}_q^j u$ solves weakly the problem (61). As in the proof of Proposition 3.2 (see formula (53)) we obtain

$$\sum_{i=1}^r \|\mathcal{T}_q^i v^j(t)\|_{L^q(\mathbb{R}^m)} \;\leq\; C\sum_{i=1}^r \left\{ \|\mathcal{T}_q^i\mathcal{T}_q^j u_0\|_{L^q(\mathbb{R}^m)} + \int_0^t \|\mathcal{T}_q^i f^j(s)\|_{L^q(\mathbb{R}^m)}\;\mathrm{d}s \right\} \tag{66}$$

for some constant C depending on $\sum_{1\leq i,j\leq r} \|\mathcal{T}_q^i \alpha_j\|_{L^1([0,T];L^\infty(\mathbb{R}^m))}$. By Proposition 2.11 we obtain

$$\begin{aligned}\|\mathcal{T}_q^i f^j(s)\|_{L^q(\mathbb{R}^m)} &= \left\| \sum_{k=1}^r \mathcal{T}_q^i \langle (\mathcal{T}_q^j \alpha_k(s))(\mathcal{T}_q^k u(s))\rangle^{(q)} \right\|_{L^q(\mathbb{R}^m)} \\ &= \left\| \sum_{k=1}^r \left\{ \langle (\mathcal{T}_q^i\mathcal{T}_q^j \alpha_k(s)) v^k(s)\rangle^{(q)} + \langle (\mathcal{T}_q^j \alpha_k(s))(\mathcal{T}_q^i v^k(s))\rangle^{(q)} \right\} \right\|_{L^q(\mathbb{R}^m)} \\ &\leq \sum_{k=1}^r \left\{ \|\mathcal{T}_q^i\mathcal{T}_q^j \alpha_k(s)\|_{L^\infty} \|v^k(s)\|_{L^q} + \|\mathcal{T}_q^j \alpha_k(s)\|_{L^\infty} \|\mathcal{T}_q^i v^k(s)\|_{L^q} \right\}.\end{aligned}$$

Since by Proposition 3.2 we already know that

$$\sum_{k=1}^{r} \|v^k\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} \le C \sum_{k=1}^{r} \|\mathcal{T}_q^k u_0\|_{L^q(\mathbb{R}^m)}$$

therefore, summing up the inequalities (66) for $j \in \{1, ..., r\}$ implies the first assertion in our conclusion, by Gronwall's lemma. The estimate for $\partial_t \mathcal{T}_q^j u$ comes immediately by (61) and the estimate for $\partial_t^2 u$ follows easily by using the equality

$$\partial_t^2 u + \sum_{i=1}^{r} \left\{ \langle \partial_t \alpha_i \mathcal{T}_q^i u \rangle^{(q)} + \langle \alpha_i \partial_t \mathcal{T}_q^i u \rangle^{(q)} \right\} = 0.$$

$\square$

4 Convergence towards the limit model

This section is devoted to the asymptotic behavior of the solutions $(u^\varepsilon)_{\varepsilon>0}$ of

$$\left\{ \begin{array}{ll} \partial_t u^\varepsilon + a(t,y) \cdot \nabla_y u^\varepsilon + \dfrac{b(y)}{\varepsilon} \cdot \nabla_y u^\varepsilon = 0, & (t,y) \in (0,T) \times \mathbb{R}^m \\ u^\varepsilon(0,y) = u_0^\varepsilon(y), & y \in \mathbb{R}^m. \end{array} \right. \tag{67}$$

We assume that b, a satisfy the hypotheses (8), (9), (10), (42) and we work in the $L^2(\mathbb{R}^m)$ setting ($q = 2$). Motivated by Hilbert's expansion method, we intend to show the convergence of $(u^\varepsilon)_{\varepsilon>0}$ as ε goes to 0 towards the solution u of

$$\left\{ \begin{array}{ll} \partial_t u + \langle a(t) \cdot \nabla_y u(t) \rangle^{(2)} = 0, & (t,y) \in (0,T) \times \mathbb{R}^m \\ u(0,y) = u_0(y), & y \in \mathbb{R}^m. \end{array} \right. \tag{68}$$

Our main result is the following.

Theorem 4.1 *Assume that $(\alpha_i)_{i\in\{1,...,r\}}$ are smooth and satisfy*

$$\sum_{i=1}^{r} \|\alpha_i\|_{L^1([0,T];L^\infty(\mathbb{R}^m))} + \sum_{i=1}^{r} \|\partial_t \alpha_i\|_{L^1([0,T];L^\infty(\mathbb{R}^m))} < +\infty$$

$$\sum_{i=1}^{r} \sum_{j=1}^{r} \|\mathcal{T}_2^i \alpha_j\|_{L^1([0,T];L^\infty(\mathbb{R}^m))} + \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{k=1}^{r} \|\mathcal{T}_2^i \mathcal{T}_2^j \alpha_k\|_{L^1([0,T];L^\infty(\mathbb{R}^m))} < +\infty.$$

Suppose that

$$u_0 \in (\cap_{i=1}^r \mathrm{D}(\mathcal{T}_2^i)) \cap \ker \mathcal{T}_2, \ \ \mathcal{T}_2^j u_0 \in \cap_{i=1}^r \mathrm{D}(\mathcal{T}_2^i), \ \forall\, j \in \{1, ..., r\}$$

and that $(u_0^\varepsilon)_{\varepsilon>0}$ are smooth initial conditions such that $\lim_{\varepsilon\searrow 0} u_0^\varepsilon = u_0$ in $L^2(\mathbb{R}^m)$. We denote by u^ε, u the solutions of (67), (68) respectively. Then we have $\lim_{\varepsilon\searrow 0} u^\varepsilon = u$, in $L^\infty([0,T]; L^2(\mathbb{R}^m))$.

Proof. By the Propositions 3.2, 3.3 and Corollary 3.2 there is a unique strong solution u for (68), satisfying $\|u(t)\|_{L^2(\mathbb{R}^m)} = \|u_0\|_{L^2(\mathbb{R}^m)}$ for any $t \in [0,T]$ and

$$\|\partial_t u\|_{L^\infty([0,T];L^2(\mathbb{R}^m))} + \sum_{i=1}^r \|\mathcal{T}_2^i u\|_{L^\infty([0,T];L^2(\mathbb{R}^m))} \le C \sum_{i=1}^r \|\mathcal{T}_2^i u_0\|_{L^2(\mathbb{R}^m)}.$$

Since $u(t) \in \ker \mathcal{T}_2$, $t \in [0,T]$, we have

$$\langle \partial_t u + a(t)\cdot\nabla_y u(t)\rangle^{(2)} = \partial_t \langle u\rangle^{(2)} + \langle a(t)\cdot\nabla_y u(t)\rangle^{(2)} = \partial_t u + \langle a(t)\cdot\nabla_y u(t)\rangle^{(2)} = 0$$

and thus by Proposition 2.9 there are $(v_\mu)_{\mu>0}$ such that

$$\partial_t u + a(t,y)\cdot\nabla_y u + \mu v_\mu(t,y) + \mathcal{T}_2 v_\mu = 0, \;\; \lim_{\mu\searrow 0}(\mu v_\mu(t)) = 0 \text{ in } \;L^2(\mathbb{R}^m),\; t\in[0,T]. \quad (69)$$

Moreover, by Remark 2.3 we know that

$$\begin{aligned}
\|\mu v_\mu\|_{L^\infty([0,T];L^2(\mathbb{R}^m))} &\le \|\partial_t u + a(t)\cdot\nabla_y u\|_{L^\infty([0,T];L^2(\mathbb{R}^m))}\\
&\le \|\partial_t u\|_{L^\infty([0,T];L^2(\mathbb{R}^m))}\\
&+ C\sum_{i=1}^r \|\alpha_i\|_{W^{1,1}([0,T];L^\infty(\mathbb{R}^m))}\|\mathcal{T}_2^i u\|_{L^\infty([0,T];L^2(\mathbb{R}^m))}\\
&\le C\sum_{i=1}^r \|\mathcal{T}_2^i u_0\|_{L^2(\mathbb{R}^m)}. \qquad (70)
\end{aligned}$$

Combining (67), (68) and the equation $\mathcal{T}_2 u = 0$ yields

$$\left(\partial_t + a(t,y)\cdot\nabla_y + \frac{b(y)}{\varepsilon}\cdot\nabla_y\right)(u^\varepsilon - u - \varepsilon v_\mu) = \mu v_\mu - \varepsilon(\partial_t v_\mu + a(t,y)\cdot\nabla_y v_\mu). \quad (71)$$

We investigate now the regularity of v_μ. By Remark 2.3 we have

$$\mu\|\partial_t v_\mu(t)\|_{L^2(\mathbb{R}^m)} \le \left\|\partial_t^2 u + \sum_{i=1}^r \partial_t\alpha_i \mathcal{T}_2^i u + \sum_{i=1}^r \alpha_i(t)\partial_t\mathcal{T}_2^i u\right\|_{L^2(\mathbb{R}^m)}$$

and thus Proposition 3.4 implies

$$\mu\|\partial_t v_\mu\|_{L^1([0,T];L^2(\mathbb{R}^m))} \le C\left(\sum_{i=1}^r\sum_{j=1}^r \|\mathcal{T}_2^i\mathcal{T}_2^j u_0\|_{L^2(\mathbb{R}^m)} + \sum_{i=1}^r \|\mathcal{T}_2^i u_0\|_{L^2(\mathbb{R}^m)}\right). \quad (72)$$

Applying now the operator $\mathcal{T}_2^i$, $i\in\{0,1,...,r\}$, in (69), yields

$$\partial_t\mathcal{T}_2^i u + \sum_{j=1}^r\{(\mathcal{T}_2^i\alpha_j)(\mathcal{T}_2^j u) + \alpha_j(\mathcal{T}_2^i\mathcal{T}_2^j u)\} + \mu\mathcal{T}_2^i v_\mu + \mathcal{T}_2\mathcal{T}_2^i v_\mu = 0.$$

By Remark 2.3 and Proposition 3.4 we obtain as before

$$\mu\|\mathcal{T}_2^i v_\mu(t)\|_{L^2(\mathbb{R}^m)} \leq \|\partial_t \mathcal{T}_2^i u(t) + \sum_{j=1}^r \{(\mathcal{T}_2^i \alpha_j(t))(\mathcal{T}_2^j u(t)) + \alpha_j(t)(\mathcal{T}_2^i\mathcal{T}_2^j u(t))\}\|_{L^2(\mathbb{R}^m)}$$

implying that

$$\mu \sum_{i=0}^r \|\mathcal{T}_2^i v_\mu\|_{L^1([0,T];L^2(\mathbb{R}^m))} \leq C\left(\sum_{i=1}^r\sum_{j=1}^r \|\mathcal{T}_2^i\mathcal{T}_2^j u_0\|_{L^2(\mathbb{R}^m)} + \sum_{i=1}^r \|\mathcal{T}_2^i u_0\|_{L^2(\mathbb{R}^m)}\right). \quad (73)$$

Multiplying (71) by $u^\varepsilon - u - \varepsilon v_\mu$ and integrating over $\mathbb{R}^m$ yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(u^\varepsilon - u - \varepsilon v_\mu)(t)\|^2_{L^2(\mathbb{R}^m)} &\leq \|\mu v_\mu(t)\|_{L^2(\mathbb{R}^m)}\|(u^\varepsilon - u - \varepsilon v_\mu)(t)\|_{L^2(\mathbb{R}^m)} \\ &+ \varepsilon\left\|\partial_t v_\mu(t) + \sum_{i=0}^r \alpha_i(t)\mathcal{T}_2^i v_\mu(t)\right\|_{L^2(\mathbb{R}^m)} \\ &\times \|(u^\varepsilon - u - \varepsilon v_\mu)(t)\|_{L^2(\mathbb{R}^m)}\end{aligned}$$

and we deduce that

$$\frac{d}{dt}\|(u^\varepsilon - u - \varepsilon v_\mu)(t)\|_{L^2(\mathbb{R}^m)} \leq \|\mu v_\mu(t)\|_{L^2(\mathbb{R}^m)} + C\varepsilon(\|\partial_t v_\mu(t)\|_{L^2(\mathbb{R}^m)} + \sum_{i=0}^r \|\mathcal{T}_2^i v_\mu(t)\|_{L^2(\mathbb{R}^m)}).$$

Combining with (72), (73), we obtain for any $t \in [0,T]$

$$\begin{aligned}\|(u^\varepsilon - u - \varepsilon v_\mu)(t)\|_{L^2(\mathbb{R}^m)} &\leq \|u_0^\varepsilon - u_0 - \varepsilon v_\mu(0)\|_{L^2(\mathbb{R}^m)} + \int_0^T \|\mu v_\mu(s)\|_{L^2(\mathbb{R}^m)}\,\mathrm{d}s \\ &+ C\frac{\varepsilon}{\mu}(\|\mu\partial_t v_\mu\|_{L^1([0,T];L^2(\mathbb{R}^m))} + \sum_{i=0}^r \|\mu\mathcal{T}_2^i v_\mu\|_{L^1([0,T];L^2(\mathbb{R}^m))}) \\ &\leq \|u_0^\varepsilon - u_0 - \varepsilon v_\mu(0)\|_{L^2(\mathbb{R}^m)} + \int_0^T \|\mu v_\mu(s)\|_{L^2(\mathbb{R}^m)}\,\mathrm{d}s + C\frac{\varepsilon}{\mu}.\end{aligned}$$

Consequently one gets by (70) for any $t \in [0,T]$

$$\begin{aligned}\|(u^\varepsilon - u)(t)\|_{L^2(\mathbb{R}^m)} &\leq \|u_0^\varepsilon - u_0\|_{L^2(\mathbb{R}^m)} + \frac{\varepsilon}{\mu}(\|\mu v_\mu(t)\|_{L^2(\mathbb{R}^m)} + \|\mu v_\mu(0)\|_{L^2(\mathbb{R}^m)}) \\ &+ C\frac{\varepsilon}{\mu} + \|\mu v_\mu\|_{L^1([0,T];L^2(\mathbb{R}^m))} \\ &\leq \|u_0^\varepsilon - u_0\|_{L^2(\mathbb{R}^m)} + C\frac{\varepsilon}{\mu} + \|\mu v_\mu\|_{L^1([0,T];L^2(\mathbb{R}^m))}.\end{aligned}$$

Since the functions $t \to \|\mu v_\mu(t)\|_{L^2(\mathbb{R}^m)}$ converge pointwise to 0 as $\mu \searrow 0$ (cf. (69)) and they are uniformly bounded on $[0,T]$ (cf. (70)) we deduce by dominated convergence theorem that

$$\lim_{\mu \searrow 0} \|\mu v_\mu\|_{L^1([0,T];L^2(\mathbb{R}^m))} = 0.$$

In particular, for $\mu = \varepsilon^\delta$, with $\delta \in (0,1)$ we have

$$\|u^\varepsilon - u\|_{L^\infty([0,T];L^2(\mathbb{R}^m))} \le \|u_0^\varepsilon - u_0\|_{L^2(\mathbb{R}^m)} + C\varepsilon^{1-\delta} + \|\varepsilon^\delta v_{\varepsilon^\delta}\|_{L^1([0,T];L^2(\mathbb{R}^m))} \to 0, \text{ as } \varepsilon \searrow 0.$$

$\square$

5 The limit model in terms of prime integrals

In the previous section we have derived a limit model for the transport equation (67) based on the computation of the fields $(b^i)_i$ in involution with b. We investigate now the same limit model from the view point of prime integral concept. Surely, this approach will provide an equivalent analysis. Nevertheless, in practical situations (see the examples in the next section) the computations simplify when prime integrals are employed. We assume that there are $m-1$ prime integrals, independent on $\mathbb{R}^m$, associated to the field b

$$b \cdot \nabla_y \psi^i = 0, \;\; i \in \{1, ..., m-1\} \tag{74}$$

$$\text{rank}\left(\frac{\partial \psi^i}{\partial y_j}(y)\right)_{(m-1)\times m} = m-1, \;\; y \in \mathbb{R}^m. \tag{75}$$

Let us recall, that generally, around any non singular point y_0 of b (*i.e.,* $b(y_0) \neq 0$) there are $(m-1)$ independent prime integrals, defined only locally, in a small enough neighborhood of y_0 (see [1], pp. 95). For any $y \in \mathbb{R}^m$ we denote by $M(y)$ the matrix whose lines are $\nabla_y \psi^1, ..., \nabla_y \psi^{m-1}$ and b. The hypotheses (74), (75) imply that $\det M(y) \neq 0$ for any $y \in \mathbb{R}^m$. The idea is to search for fields $c = c(y)$ such that $c(y) \cdot \nabla_y u$ remains constant along the flow of b for any function u which is constant along the same flow. If u is constant on the characteristics of b, there is a function $v = v(z) : \mathbb{R}^{m-1} \to \mathbb{R}$ such that

$$u(y) = v(\psi^1(y), ..., \psi^{m-1}(y)), \;\; y \in \mathbb{R}^m.$$

Therefore one gets

$$\frac{\partial u}{\partial y_j} = \sum_{k=1}^{m-1} \frac{\partial v}{\partial z_k}(\psi^1(y), ..., \psi^{m-1}(y)) \frac{\partial \psi^k}{\partial y_j}$$

implying that

$$c \cdot \nabla_y u = \sum_{k=1}^{m-1} \frac{\partial v}{\partial z_k}(\psi^1(y), ..., \psi^{m-1}(y)) \sum_{j=1}^{m} \frac{\partial \psi^k}{\partial y_j} c_j = (\nabla_z v)(\psi(y)) \cdot \frac{\partial \psi}{\partial y} c(y).$$

In particular, if $\frac{\partial \psi}{\partial y} c(y)$ do not depend on y, the directional derivative $c \cdot \nabla_y$ remains constant along the trajectories of b. Actually, the following more general result holds.

Lemma 5.1 *Assume that (74), (75) hold and let c be a smooth field such that $y \to \frac{\partial \psi}{\partial y}(y) c(y)$ and $y \to b(y) \cdot c(y)$ are constant along the flow of b. Then we have*

$$[c, b](y) = \frac{c(y) \cdot \left(\nabla_y \frac{|b|^2}{2} + \mathcal{T} b \right)}{|b(y)|^2} \; b(y), \quad y \in \mathbb{R}^m.$$

Proof. For any $i \in \{1, ..., m-1\}$ we have

$$[c, b]_i = \sum_{j=1}^{m} c_j \frac{\partial b_i}{\partial y_j} - \sum_{j=1}^{m} b_j \frac{\partial c_i}{\partial y_j} = \left(\frac{\partial b}{\partial y} c - \mathcal{T} c \right)_i . \tag{76}$$

By the hypotheses we know that $y \to M(y) c(y)$ is constant along the flow of b and thus

$$0 = \mathcal{T}(M(y) c(y)) = (\mathcal{T} M) \; c(y) + M(y) \; \mathcal{T} c$$

implying that

$$\mathcal{T} c = -M^{-1}(y) (\mathcal{T} M) c(y), \quad y \in \mathbb{R}^m. \tag{77}$$

For any $i \in \{1, ..., m-1\}$ we have $\sum_{k=1}^{m} b_k(y) \frac{\partial \psi^i}{\partial y_k} = 0$ and by taking the derivative with respect to y_j, $j \in \{1, ..., m\}$ one gets

$$\sum_{k=1}^{m} \frac{\partial \psi^i}{\partial y_k} \frac{\partial b_k}{\partial y_j} + \sum_{k=1}^{m} b_k(y) \frac{\partial^2 \psi^i}{\partial y_k \partial y_j} = 0$$

saying that

$$\left(M(y) \frac{\partial b}{\partial y} \right)_{ij} + (\mathcal{T} M)_{ij} = 0, \quad i \in \{1, ..., m-1\}, \quad j \in \{1, ..., m\}. \tag{78}$$

Notice also that

$$\left(M(y) \frac{\partial b}{\partial y} \right)_{mj} + (\mathcal{T} M)_{mj} = \sum_{k=1}^{m} b_k(y) \frac{\partial b_k}{\partial y_j} + \mathcal{T} b_j = \frac{\partial}{\partial y_j} \left(\frac{1}{2} |b|^2 \right) + \mathcal{T} b_j . \tag{79}$$

Combining (78), (79) yields

$$M(y) \frac{\partial b}{\partial y} + \mathcal{T} M = \begin{pmatrix} \mathcal{O}_{(m-1) \times m} \\ \nabla_y \left(\frac{|b|^2}{2} \right) + \mathcal{T} b \end{pmatrix} \tag{80}$$

and thus by (76), (77), (80) one gets

$$[c, b] = \left(\frac{\partial b}{\partial y} + M^{-1}(y) \mathcal{T} M \right) c(y) = M^{-1}(y) \begin{pmatrix} \mathcal{O}_{(m-1) \times m} \\ L^m(y) \end{pmatrix} c(y) \tag{81}$$

where $L^m(y)$ is the line $\nabla_y\left(\frac{|b|^2}{2}\right)+\mathcal{T}b$. Observe that the last column $C_m(y)$ of the matrix $M^{-1}(y)$ solves the linear system $M(y)C_m(y)=(0,...,0,1)^t$. Since $b(y)/|b(y)|^2$ solves also the above system, we deduce that $C_m=b/|b|^2$. It is easily seen that for any $i,j\in\{1,...,m\}$

$$\left(M^{-1}(y)\begin{pmatrix}\mathcal{O}_{(m-1)\times m}\\ L^m\end{pmatrix}\right)_{ij}=C^i_m L^m_j=(C_m\otimes L^m)_{ij}$$

and consequently

$$[c,b]=C_m(y)(L^m(y)\cdot c(y))=\frac{c(y)\cdot\left(\nabla_y\frac{|b|^2}{2}+\mathcal{T}b\right)}{|b(y)|^2}\,b(y),\ \ y\in\mathbb{R}^m.$$

$\square$

For any $i\in\{1,...,m-1\}$ let us denote by $c^i(y)$ the unique solution of the linear system

$$M(y)c^i(y)=e^i:=(\delta_{ij})_{1\le j\le m}$$

where δ_{ij} are the Kronecker's symbols. Notice that $M(y)\frac{b(y)}{|b(y)|^2}=e^m$ and thus $c^1(y),...,c^{m-1}(y),b(y)$ are linearly independent at any $y\in\mathbb{R}^m$. According to Lemma 5.1 we have for any $i\in\{1,...,m-1\}$

$$(c^i\cdot\nabla_y)(b\cdot\nabla_y)-(b\cdot\nabla_y)(c^i\cdot\nabla_y)=\frac{c^i(y)\cdot\left(\nabla_y\frac{|b|^2}{2}+(b(y)\cdot\nabla_y)b\right)}{|b(y)|^2}\,(b\cdot\nabla_y).$$

In particular, for any function u constant along the flow of b, the directional derivative $c^i\cdot\nabla_y u$ remains constant along the same flow for any $i\in\{1,...,m-1\}$. We denote by $\beta_0,\beta_1,...,\beta_{m-1}$ the coordinates of a with respect to $b,c^1,...,c^{m-1}$ and we assume that $(\beta_i)_i$ are smooth and bounded

$$a(t,y)=\beta_0(t,y)b(y)+\sum_{i=1}^{m-1}\beta_i(t,y)c^i(y),\ \ (t,y)\in[0,T]\times\mathbb{R}^m. \tag{82}$$

Thanks to Corollary 2.4, one gets for any function $u\in(\cap_{i=1}^{m-1}\mathrm{D}(\mathcal{T}_q^{c^i}))\cap\ker\mathcal{T}_q$

$$\langle a(t)\cdot\nabla_y u(t)\rangle^{(q)}=\left\langle\sum_{i=1}^{m-1}\beta_i(t)c^i(y)\cdot\nabla_y u(t)\right\rangle^{(q)}=\sum_{i=1}^{m-1}\langle\beta_i(t)\rangle^{(\infty)}c^i(y)\cdot\nabla_y u(t).$$

It remains to compute $(\beta_i)_i$. Multiplying (82) by $M(y)$ yields

$$M(y)a(t,y)=\beta_0(t,y)|b(y)|^2e^m+\sum_{i=1}^{m-1}\beta_i(t,y)e^i$$

implying that

$$\beta_i(t,y) = M(y)a(t,y)\cdot e^i, \;\; i\in\{1,...,m-1\}, \;\; \beta_0(t,y)|b(y)|^2 = M(y)a(t,y)\cdot e^m$$

or equivalently to

$$\beta_i(t,y) = a(t,y)\cdot\nabla_y\psi^i, \;\; i\in\{1,...,m-1\}, \;\; \beta_0(t,y) = \frac{a(t,y)\cdot b(y)}{|b(y)|^2}.$$

Finally one gets the following form of the limit model

$$\partial_t u + \sum_{i=1}^{m-1}\langle a(t)\cdot\nabla_y\psi^i\rangle^{(\infty)} M^{-1}(y)e^i\cdot\nabla_y u = 0 \tag{83}$$

supplemented by the constraint $\mathcal{T}_q u = 0$. Actually we check that this constraint is a consequence of the equation (83), provided that the initial condition satisfies $\mathcal{T}_q u_0 = 0$. Indeed, by Lemma 5.1 it is easily seen that for any $i\in\{1,...,m-1\}$ we have

$$\begin{aligned}
\mathcal{T}_q\left(\langle a(t)\cdot\nabla_y\psi^i\rangle^{(\infty)} c^i\cdot\nabla_y u\right) &= \langle a(t)\cdot\nabla_y\psi^i\rangle^{(\infty)}\mathcal{T}_q(c^i\cdot\nabla_y u)\\
&= \langle a(t)\cdot\nabla_y\psi^i\rangle^{(\infty)}(c^i\cdot\nabla_y)\mathcal{T}_q u\\
&- \langle a(t)\cdot\nabla_y\psi^i\rangle^{(\infty)}\frac{c^i(y)\cdot\left(\nabla_y\frac{|b|^2}{2} + (b(y)\cdot\nabla_y)b\right)}{|b(y)|^2}\,\mathcal{T}_q u.
\end{aligned}$$

Therefore, by applying $\mathcal{T}_q$ to (83) we obtain

$$\begin{aligned}
\partial_t\mathcal{T}_q u &+ \sum_{i=1}^{m-1}\langle a(t)\cdot\nabla_y\psi^i\rangle^{(\infty)}(c^i\cdot\nabla_y)\mathcal{T}_q u\\
&- \left(\sum_{i=1}^{m-1}\langle a(t)\cdot\nabla_y\psi^i\rangle^{(\infty)}\frac{c^i(y)\cdot\left(\nabla_y\frac{|b|^2}{2} + (b(y)\cdot\nabla_y)b\right)}{|b(y)|^2}\right)\mathcal{T}_q u = 0
\end{aligned}$$

and thus it is clear that if $\mathcal{T}_q u_0 = 0$, then $\mathcal{T}_q u(t) = 0$, $t\in[0,T]$.

5.1 Examples

We apply the previous theoretical results in two particular cases. The first example treats the general case in two dimensions. The second one concerns the finite Larmor radius regime, in the particular case of a constant magnetic field.

Two dimensional case

We consider $m = 2$ and let $a = a(t, y), b = b(y)$ be two smooth fields on $\mathbb{R}^2$ such that $b(y) \neq 0, \forall\, y \in \mathbb{R}^2$. We assume that $\mathrm{div}_y b = 0$ and thus there is a function $\psi = \psi(y)$ such that

$$b(y) = \left(\frac{\partial \psi}{\partial y_2}, -\frac{\partial \psi}{\partial y_1} \right) = {}^\perp \nabla_y \psi, \quad y \in \mathbb{R}^2.$$

Actually ψ is a prime integral of the flow associated to b, since $b(y) \cdot \nabla_y \psi = 0$ for any $y \in \mathbb{R}^2$. The matrix $M(y)$ is given by

$$M(y) = \begin{pmatrix} \frac{\partial \psi}{\partial y_1} & \frac{\partial \psi}{\partial y_2} \\ \frac{\partial \psi}{\partial y_2} & -\frac{\partial \psi}{\partial y_1} \end{pmatrix}$$

and the vector c^1 is equal to $-{}^\perp b/|b|^2$. Writing a as a linear combination of the vectors b, c^1 one gets

$$a(t, y) = (a \cdot b) \frac{b}{|b|^2} + (a \cdot {}^\perp b) \frac{{}^\perp b}{|b|^2} = \frac{(a \cdot b)}{|b|^2} b - (a \cdot {}^\perp b) c^1.$$

In this case we obtain the limit model

$$\begin{cases} \partial_t u + \dfrac{\langle a(t) \cdot^\perp b \rangle^{(\infty)}}{|b(y)|^2} \ {}^\perp b \cdot \nabla_y u = 0, & (t, y) \in (0, T) \times \mathbb{R}^2 \\ u(0, y) = u_0(y), & y \in \mathbb{R}^2. \end{cases}$$

Finite Larmor radius regime

We investigate now the finite Larmor radius regime (1) with a constant magnetic field $B_3 \neq 0$. We have $m = 4$, $y = (x, p) \in \mathbb{R}^2 \times \mathbb{R}^2$, $\tilde{a}(t, y) = (0, 0, -E_1(t, x), -E_2(t, x))$, $\tilde{b}(y) = (p_1, p_2, -B_3 p_2, B_3 p_1) = (p, -B_3\, {}^\perp p)$. The characteristic flow $Y = (X, P)$ associated to $\tilde{b}$ satisfies

$$\frac{dX}{ds} = P(s; x, p), \quad \frac{dP}{ds} = -B_3\, {}^\perp P(s; x, p).$$

It is easily seen that a set of independent prime integrals are given by

$$\tilde{\psi}^1(x, p) = B_3 x_2 + p_1, \quad \tilde{\psi}^2(x, p) = -B_3 x_1 + p_2, \quad \tilde{\psi}^3(p) = \frac{1}{2} |p|^2$$

and thus we need to invert the matrix

$$\tilde{M}(p) = \begin{pmatrix} 0 & B_3 & 1 & 0 \\ -B_3 & 0 & 0 & 1 \\ 0 & 0 & p_1 & p_2 \\ p_1 & p_2 & -B_3 p_2 & B_3 p_1 \end{pmatrix}.$$

In order to simplify our computations it is very convenient to introduce the new variable $z = x - \frac{{}^\perp p}{B_3} = (-\tilde{\psi}^2, \tilde{\psi}^1)/B_3$ and the new unknown $g^\varepsilon(t, z, p) = f^\varepsilon(t, x, p)$. The equation for g^ε becomes

$$\partial_t g^\varepsilon + \frac{1}{B_3} {}^\perp E \left(t, z + \frac{{}^\perp p}{B_3} \right) \cdot \nabla_z g^\varepsilon - E \left(t, z + \frac{{}^\perp p}{B_3} \right) \cdot \nabla_p g^\varepsilon - \frac{1}{\varepsilon} B_3\, {}^\perp p \cdot \nabla_p g^\varepsilon = 0$$

and thus the fields to analyze in this case are

$$a(t,z,p) = \left(\frac{1}{B_3}{}^{\perp}E\left(t, z + \frac{{}^{\perp}p}{B_3}\right), -E\left(t, z + \frac{{}^{\perp}p}{B_3}\right)\right), \quad b(p) = (0,0,-B_3{}^{\perp}p).$$

A set of independent prime integrals is given by

$$\psi^1 = z_1, \quad \psi^2 = z_2, \quad \psi^3 = \frac{1}{2}|p|^2.$$

The matrix to be inverted is

$$M(p) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p_1 & p_2 \\ 0 & 0 & -B_3 p_2 & B_3 p_1 \end{pmatrix}.$$

It is easily seen that M^{-1} is given by

$$M^{-1}(p) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{p_1}{|p|^2} & -\frac{p_2}{B_3|p|^2} \\ 0 & 0 & \frac{p_2}{|p|^2} & \frac{p_1}{B_3|p|^2} \end{pmatrix}.$$

In view of (83) we need to compute $\langle a(t) \cdot \nabla_{(z,p)}\psi^i\rangle^{(\infty)}$, $i \in \{1,2,3\}$. A direct computation shows that the flow $(Z,P)(s;z,p)$ associated to b is given by

$$Z(s;z,p) = z, \quad P(s;z,p) = R(sB_3)p, \quad R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Consequently the constant functions along the flow are the functions with radial symmetry with respect to p. Observe also that the hypothesis (20) holds true with $\mathcal{O} = \emptyset$ and $\xi(z,p) = \mathrm{e}^{-|z|^2-|p|^2}$, $(z,p) \in \mathbb{R}^4$. Since all the trajectories are $2\pi/B_3$ periodic, we have

$$\langle u\rangle^{(\infty)}(z,p) = \frac{B_3}{2\pi}\int_0^{\frac{2\pi}{B_3}} u(z, R(sB_3)p)\,\mathrm{d}s = \frac{1}{2\pi}\int_0^{2\pi} u(z, R(\theta)p)\,\mathrm{d}\theta$$

for any bounded function $u \in L^\infty(\mathbb{R}^4)$. We have

$$\langle a(t) \cdot \nabla_{(z,p)}\psi^1\rangle^{(\infty)} = \left\langle \frac{1}{B_3}E_2\left(t, z + \frac{{}^{\perp}p}{B_3}\right)\right\rangle^{(\infty)} = \frac{1}{2\pi B_3}\int_0^{2\pi} E_2\left(t, z + \frac{{}^{\perp}(R(\theta)p)}{B_3}\right)\,\mathrm{d}\theta$$

$$\langle a(t) \cdot \nabla_{(z,p)}\psi^2\rangle^{(\infty)} = -\left\langle \frac{1}{B_3}E_1\left(t, z + \frac{{}^{\perp}p}{B_3}\right)\right\rangle^{(\infty)} = -\frac{1}{2\pi B_3}\int_0^{2\pi} E_1\left(t, z + \frac{{}^{\perp}(R(\theta)p)}{B_3}\right)\,\mathrm{d}\theta.$$

We claim that the coefficient $\langle a(t) \cdot \nabla_{(z,p)} \psi^3 \rangle^{(\infty)}$ vanishes. Indeed

$$\langle a(t) \cdot \nabla_{(z,p)} \psi^3 \rangle^{(\infty)} = -\frac{B_3}{2\pi} \int_0^{\frac{2\pi}{B_3}} E\left(t, z + \frac{{}^\perp P(s; z, p)}{B_3}\right) \cdot P(s; z, p) \, \mathrm{d}s.$$

Taking into account that $E(t)$ derives from a potential $\phi(t)$ and that

$$\frac{d}{ds}\phi\left(t, z + \frac{{}^\perp P(s; z, p)}{B_3}\right) = E\left(t, z + \frac{{}^\perp P(s; z, p)}{B_3}\right) \cdot P(s; z, p)$$

we deduce that

$$\langle a(t) \cdot \nabla_{(z,p)} \psi^3 \rangle^{(\infty)} = -\frac{B_3}{2\pi} \int_0^{\frac{2\pi}{B_3}} \frac{d}{ds}\phi\left(t, z + \frac{{}^\perp P(s; z, p)}{B_3}\right) \, \mathrm{d}s = 0.$$

Plugging into (83) all these computations yields the limit model

$$\partial_t g + \frac{1}{2\pi B_3} \int_0^{2\pi} {}^\perp E\left(t, z + \frac{{}^\perp(R(\theta)p)}{B_3}\right) \, \mathrm{d}\theta \cdot \nabla_z g = 0$$

which is equivalent to

$$\partial_t f + \frac{1}{2\pi B_3} \int_0^{2\pi} {}^\perp E\left(t, x - \frac{{}^\perp p}{B_3} + \frac{{}^\perp(R(\theta)p)}{B_3}\right) \, \mathrm{d}\theta \cdot \nabla_x f = 0.$$

Therefore the finite Larmor radius regime leads to a transport equation for the particle density, whose advection field is given by a gyro-average type operator. For more details, the reader can refer to [2] where a complete analysis of the coupled Vlasov-Poisson equations (with finite Larmor radius) was performed.

From the application point of view (plasma confinement) a much interesting case would be that of a variable magnetic field. And surely, the numerical approximation of these models is of crucial importance for simulating tokamak regimes. These topics will be the object of future works.

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