An existence and uniqueness result in bone remodeling theory

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Existence and uniqueness of solution to an adaptive elasticity model

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Abstract. In this work we study the existence and unicity of solutions to an *Adaptive Elasticity Model* applied to bone remodeling. Specifically, we consider the model derived by Cowin and Hegedus, directly from continuum mechanics theory. We use a fixed point argument in order to prove the existence of solutions and a straightforward adaptation of the Cowin and Nachlinger analysis in order to prove uniqueness.

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1 Introduction

As mentionned in the first paragraph of the introduction of the work by Cowin and Hegedus, [1], living bone is continuously adapting itself to external stimuli. This process termed collectively "remodeling" is responsible for the continuous adaptation of the bone structure. Since this remodeling process takes place either when living bone is subjected to prolonged straining (which tends to make bone stiffer and more dense), or when the bone is not subjected to its usual strain level (which has the effect of making it weaker and, consequently, less stiff and more porous), and since this has an enormous effect in the overall behavior and health of the entire body, to be able to predict bone remodeling is of great importance.

Several remodeling theories have been proposed. Some are more of an empirical nature, others of a theoretical kind but all of them try to fit some experimental data. However, in spite of the great progress in experiments done in the last decades it is always extremely difficult to get reliable data for the human case. This is one of the reasons why a good theoretical model which can be easily discretized leading to robust numerical algorithms is extremely important.

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Among all the proposed models the one studied by Cowin and Hegedus, [1], [4], possesses several advantages, namely:

i) it is derived from Continuum Mechanics in the standard way,

ii) in all its generality it includes most of the models usually used and can be adapted in order to include some of the most important experimental data available.

iii) it is a nonlinear model generalizing nonlinear elasticity theory,

iv) its linearization leads to a generalization of classical elasticity.

On the other hand, and since it constitutes a generalization of classical elasticity theory it can be used in order to study the functional adaptation of the bone structure in the framework of optimization theory of structural mechanics leading to new insights in the understanding of Wolff's law, for example.

In order to possess reliable remodeling theories that lead to robust algorithms and optimization techniques, one needs to have a sound mathematical study of these different models. This is one of the motivations of the present work. We take the model proposed by Cowin and Hegedus and not only do we point out the major mathematical difficulties that arise in its study but also give an insight on what are the possible modifications that one may have to perform in order to address the issues of existence and uniqueness for more complex models. We use a fixed point argument in order to prove the existence of solutions and a straightfortward adaptation of the Cowin and Nachlinger analysis, [2], in order to prove uniqueness.

We shall now briefly outline the contents of the paper. In the next section we describe the bone remodeling model under study in a rather mathematical framework. We clearly outline the hypothesis on the data and the regularity on the material coefficients. We point out two major difficulties on the analysis which are related to the imposition of the condition that the volume fraction belongs to the interval $[0, 1]$. We indicate how to overcome this difficulty by using a truncation and a mollification, interpreting the results physically afterwards.

In section three we study the so called remodeling equation for the variable e, which stands for *a measure of the change in volume fraction from a reference configuration*. We prove some estimates both for e and $\dot{e}$.

In the fourth section we study the quasi-static elasticity equations associated with this model. With the help of the previous estimates we are able to prove some estimates for the three-dimensional displacement field u. Finally, in section five we use a fixed point argument in order to prove existence of solution to the model and a modification of an argument of Cowin and Nachlinger in order to obtain uniqueness. As a byproduct, and due to the formulation used, we also deduce a regularity result from [5].

2 The physical model

Notations Let Ω be a bounded open set of $I\!R^n$ ($n = 2$ or 3) of class C^2 and independent of time t. Let $T > 0$ be a real parameter and denote: $Q =]0, T[\times\Omega$, $\bar{Q} = [0, T] \times \bar{\Omega}$ and $\Sigma =]0, T[\times\partial\Omega$. Let $\mathcal{R}$ be the set of infinitesimal rigid displacements, $\mathcal{R} = \{v/v = a + b \wedge x;\ a, b \in I\!R^3\}$. Let q be a real number, $1 \leq q \leq \infty$, and m be a positive integer, and define the following spaces: $V^{m,q} = (W^{m,q}(\Omega)/\mathcal{R})^n$, $V^m = (H^m(\Omega)/\mathcal{R})^n$, $\mathcal{L}^q = (L^q(\Omega))^{n\times n}$, $\mathcal{W}^{m,q} = (W^{m,q}(\Omega))^{n\times n}$, $\mathcal{H}^m = (H^m(\Omega))^{n\times n}$, $\mathcal{C}^m = (C^m(\bar{\Omega}))^{n\times n}$ where $W^{m,q}(\Omega)$ and $H^m(\Omega)$ are the classical Sobolev spaces and $C^m(\bar{\Omega})$ is the space of functions m continuously differentiable in $\bar{\Omega}$. We denote by $C^m([0, T]; V)$ the space of functions g such that $g(t) \in V$ for all $[0, T]$ and the function $t \in [0, T] \mapsto g(t) \in V$ is m times continuously differentiable with respect to t. If V is a Banach space then $C^m([0, T]; V)$ equipped with its usual norm defined by $||g||_{C^m(V)} = \sum_{k=0}^{m}(\max_{(t\in[0,T])} ||\frac{\partial^k g(t)}{\partial^k t})||_V)$ is also a Banach space.

We denote the displacements vector field by $u = (u_i)$, $1 \leq i \leq n$, the Cauchy stress tensor by $\sigma = (\sigma_{ij})$, $1 \leq i, j \leq n$, the linearized strain tensor by $\varepsilon = (\varepsilon_{ij})$, $1 \leq i, j \leq n$ and the measure of the change in volume fraction from a reference configuration by e. They are all functions of time t and of the space variable x.

Data of the problem We suppose given:
.the open set Q of $I\!R \times I\!R^n$,
.the density γ of the full elastic material which is supposed to be constant.
.the reference volume fraction ξ_0. It belongs to $C^1(\bar{\Omega})$ and there exist constants ξ_0^{min} and ξ_0^{max} such that: $0 < \xi_0^{min} \leq \xi_0(x) \leq \xi_0^{max} < 1$ in $\bar{\Omega}$.
.the coefficients of elasticity denoted by $a_{ijkm}(e)$, $1 \leq i, j, k, m \leq n$. They are continuously differentiable with respect to e, they satisfy the conditions of symmetry : $\forall e$, $a_{ijkm}(e) = a_{jikm}(e) = a_{kmij}(e)$, $1 \leq i, j, k, m \leq n$, and they also satisfy the following ellipticity condition:

$$(\xi_0 + e)\ a_{ijkm}(e)\varepsilon_{ij}\varepsilon_{km} \geq N\varepsilon_{ij}\varepsilon_{ij} \quad \forall\varepsilon_{ij}\ \in I\!R^{n\times n} \text{ with } \varepsilon_{ij} = \varepsilon_{ji} \qquad (1)$$

where N is a strictly positive constant (independent of e, t and x). Let us notice that this inequality (1) implies the assumption: $(\xi_0(x)+e(t, x)) > 0 \quad \forall(t, x) \in Q$.
.the body load $f_i \in C^1([0, T])$, $1 \leq i \leq n$. f_i is independent of x and depends on t.
.the normal traction on the boundary $\partial\Omega$, $F_i \in C^1([0, T]; W^{1-\frac{1}{p},p}(\partial\Omega))$, $1 \leq i \leq n$ with $p > n$.
.the constitutive function $a(e)$ and the remodeling rate coefficients $A_{km}(e)$, $1 \leq k, m \leq n$, which are continuously differentiable with respect to e.
.the initial value of the change in volume fraction $e_0(x)$, which belongs to $C^1(\bar{\Omega})$.

We shall employ the usual summation and differentiation conventions. Moreover, given a function $g(t,x)$ we denote by $\dot{g}$ its partial derivative with respect to t and by $\partial_j g$ its partial derivative with respect to x_j.

The model of Cowin and Hegedus The model derived from the continuum mechanic laws in [1] and [4] is the following:

$$(\mathcal{P}^I) \quad \left\{ \begin{array}{ll} \text{Find } (u,e) \text{ which satisfies:} & \\ \quad -\partial_j \sigma_{ij} = \gamma(\xi_0 + e) f_i & \text{in } Q \\ \sigma_{ij} = (\xi_0 + e)\ a_{ijkm}(e)\ \varepsilon_{km}(u) & \\ \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}) & \\ \quad \sigma_{ij} n_j = F_i & \text{on } \Sigma_t \\ \quad u = u_d & \text{on } \Sigma_d \\ \quad \dot{e} = a(e) + A_{km}(e)\ \varepsilon_{km}(u) & \text{in } Q \\ \quad e(x,0) = e_0(x) & \text{in } \bar{\Omega} \end{array} \right.$$

where n_j are the components of the outward unit normal n to $\partial\Omega$, $\Sigma_d =]0,T[\times\Gamma_d$, $\text{mes}(\Gamma_d) > 0$, $\Sigma_t =]0,T[\times(\partial\Omega\backslash\Gamma_d)$ and the displacement u_d on Γ_d is given.

Remark 1 *The functions $a_{ijkm}(e)$, $a(e)$ and $A_{ij}(e)$ characterize the material properties and there is very few experimental data on these functions. We can make a polynomial approximation of these functions as in [4] and as a first approximation, we could write:*

$$a_{ijkm}(e) = \frac{1}{\xi_0 + e}(\xi_0\ a^0_{ijkm} + e\ a^1_{ijkm}) \tag{2}$$

$$a(e) = a_0 + a_1\ e + a_2\ e^2 \tag{3}$$

$$A_{ij}(e) = A^0_{ij} + e\ A^1_{ij} \tag{4}$$

where a^0_{ijkm}, a^1_{ijkm}, a_0, a_1, a_2, A^0_{ij} and A^1_{ij} are constants representing the material properties. Let us notice that polynomial approximations of these functions satisfy the regularity assumptions made previously.

Cowin and Nachlinger proved in [2] that if the solution (u,e) of $(\mathcal{P}^I)$ exists and is regular enough, then it is unique.

In this work we mainly address the question of existence. As a by-product we also obtain regularity and from a straightforward adaptation of the method of Cowin and Nachlinger [2] uniqueness follows.

Let us try to establish a way of proving existence and see what are the problems involved. We try to write the model of Cowin and Hegedus $(\mathcal{P}^I)$ as a fixed point problem. If we seek the displacement u in the space $C^0([0,T];V^{m,p})$ with $m \geq 1$ and p such that $n < p < \infty$, then $\varepsilon(u)$ belongs to $C^0([0,T];\mathcal{W}^{m-1,p})$.

First, let us consider the case $m = 1$ i.e. $\varepsilon(u) \in C^0([0,T];\mathcal{L}^p)$. Then, the solution e of (11)(12) does not necessarily belong to $C^0([0,T];L^1(\Omega))$. To illustrate this, let us consider the following ordinary differential equation: $\dot{e} = g(x)e$; $e(0,x) = 1$ with $g(x) = x^{\frac{-1}{2p}}$. The function g belongs to $L^p(]0,1[)$ and the solution of this equation, $e(t,x) = \exp(g(x)t)$, does not belong to $C^0([0,T];L^1(]0,1[))$. This counter-example, due to L. Sanchez, shows that the regularity L^p of a parameter in an ordinary differential equation is "not preserved", as it is with the spaces C^p.

Let us now consider the case $m \geq 2$. It follows from the Sobolev imbedding theorem that the strain tensor $\varepsilon(u)$ belongs to $C^0([0,T];\mathcal{C}^{m-2})$, then under sufficient assumptions of regularity on the elastic coefficients $a_{ijkm}(e)$ and on ξ_0, we have $(\xi_0 + e)a_{ijkm}(e)$ in $C^0([0,T];C^{m-2}(\bar{\Omega}))$. Finally, we deduce from regularity results in elastostatic (see [5], Theorem 7.6 p 83) that u belongs to $C^0([0,T];V^{m-1,p})$ and not to $C^0([0,T];V^{m,p})$. From this analysis it is clear that if one wants to prove existence of solutions with the help of a classical fixed point method, one needs some additional regularity. This can be achieved through truncation and mollification.

Truncation and mollification Let $\eta > 0$, η a small parameter, and denote by $\mathcal{P}_\eta(e)$ a truncation operator of class C^1, such that:

$$\mathcal{P}_\eta(e)(x) = \begin{cases} -\xi_0(x) + \frac{\eta}{2} & \text{if} \quad e(x) \leq -\xi_0 + \frac{\eta}{2} \\ e(x) & \text{if} \quad \eta - \xi_0 \leq e(x) \leq 1 - \xi_0 - \eta \\ 1 - \xi_0(x) & \text{if} \quad e(x) \geq 1 - \xi_0 \end{cases} \tag{5}$$

Consequently, if $e(x) \in C^1(\bar{\Omega})$ then $(\xi_0 + \mathcal{P}_\eta(e))(x) \in C^1(\bar{\Omega})$ and

$$0 < \frac{\eta}{2} \leq (\xi_0 + \mathcal{P}_\eta(e))(x) \leq 1 \quad \forall x \in \bar{\Omega}$$

Let the function:

$$w(x) = \begin{cases} e^{\left(\frac{1}{|x|^2-1}\right)} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

and let $\rho > 0$ be a positive real number. We classicaly define the mollifier $w_\rho(x) = c\frac{1}{\rho^n}w(\frac{x}{\rho})$ with $c = (\int w)^{-1}$. It satisfies $w_\rho \in C_c^\infty(\mathbb{R}^n)$, $w_\rho \geq 0$, Supp $w_\rho \subset B(0,\rho)$ and $\int w_\rho = 1$.

Let a function $g \in C^0([0,T];C^0(\bar{\Omega}))$, denote by $\bar{g}(t)$ an extension of $g(t)$ to $\mathbb{R}^n$ such that $\bar{g}(t) \in C^0(\mathbb{R}^n)$ and define the operator M_ρ from $C^0(\bar{\Omega})$ into $C^\infty(\mathbb{R}^n)$ such that:

$$M_\rho(g(t)) = w_\rho * \bar{g}(t)$$

where $w_\rho * \bar{g}(t) = \int_{\mathbb{R}^n} w_\rho(x-y)\ \bar{g}(t,y)\ dy$.

We define the coefficients of elasticity $c_{ijkm}(e)$, $1 \leq k, m \leq n$, of non local type as follows:

$$c_{ijkm}(e) = (\xi_0 + M_\rho \circ \mathcal{P}_\eta(e)) \; a_{ijkm}(M_\rho \circ \mathcal{P}_\eta(e)) \tag{6}$$

It follows from the properties of symmetry and ellipticity of the coefficients a_{ijkm} and the properties of $\mathcal{P}_\eta(e)$ that these coefficients c_{ijkm} satisfy:

$$c_{ijkm}(e)\varepsilon_{ij}\varepsilon_{km} \geq N\varepsilon_{ij}\varepsilon_{ij} \quad \forall \varepsilon_{ij} \in I\!\!R^{n\times n} \text{ with } \varepsilon_{ij} = \varepsilon_{ji}$$

The model of non local type The problem for which we prove, in the sequel, the existence (regularity) and uniqueness of the solution is the following: find (u, e) which satisfy (in the sense of distributions)

$$-\partial_j \sigma_{ij} = \gamma(\xi_0 + \mathcal{P}_\eta(e))f_i \quad \text{in } Q \tag{7}$$

$$\sigma_{ij} = c_{ijkm}(e)\varepsilon_{km}(u) \tag{8}$$

$$\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{9}$$

$$\sigma_{ij} n_j = F_i \quad \text{on } \Sigma \tag{10}$$

$$\dot{e} = a(e) + A_{km}(e) \; \varepsilon_{km}(u) \quad \text{in } Q \tag{11}$$

$$e(x, 0) = e_0(x) \quad \text{in } \bar{\Omega} \tag{12}$$

where n_j are the components of the unit outward normal n to $\partial\Omega$. We assume that the resultant of external forces is null:

$$\forall w \in \mathcal{R}, \quad \int_\Omega \gamma(\xi_0 + \mathcal{P}_\eta(e))f_i(t)w_i \; dx + \int_{\partial\Omega} F_i(t)w_i \; ds \; = \; 0 \quad \text{in } [0, T] \tag{13}$$

Let us notice that for all $t \in [0, T]$ such that $f(t) = 0$, (13) becomes:

$$\forall w \in \mathcal{R}, \quad \int_{\partial\Omega} F_i(t)w_i \; ds \; = \; 0 \tag{14}$$

and for all $t \in [0, T]$ such that $f(t) \neq 0$ ($f(t)$ is assumed to be constant on x), (13) becomes:

$$\forall w \in \mathcal{R} - \{0\}, \quad f_i(t) \; = \; \frac{-\int_{\partial\Omega} F_j(t)w_j \; ds}{\int_\Omega \gamma(\xi_0 + \mathcal{P}_\eta(e))w_i \; dx} \tag{15}$$

The relation (14) does not depend on e while (15) depends on it.

Comments and remarks

1. Truncation is a way of imposing the physical condition that the volume fraction belongs to the interval $]0, 1]$. This can also be done using other methods

but one needs some regularity in order to study the coupling between equations (7) and (11) and this is the reason for the C^1 truncation.
2. The mollification can be regarded as a nonlocal constitutive law. This has also a physical meaning since it asserts that the state of stress at a point depends on the stress levels in a neighborhood of that point. In this case its necessity arises from the fact that one needs some regularity estimates in the displacement field u in order to once again be able to study the coupling between equations (7) and (11).
3. The fact that we have a pure Neumann problem is required in order to have some regularity results for the displacement field u in the framework of elasticity theory. This is a realistic situation for it corresponds to the case in which a bone is subjected to external and to body forces only (no imposed displacements on the boundary).
4. The given forces f and F have to satisfy the relation (13). As already mentionned and due to the fact that $f(t)$ is constant on x, this relation becomes (14) if $f(t) = 0$ and (15) if $f(t) \neq 0$. Moreover, relation (14) does not depend on e.
In the sequel we prove the existence of solutions to (7)-(12) with the help of the Schauder fixed point theorem. The proof can be done if the application $e \mapsto f$ where f satisfies (15), is continuous from $C^1([0,T];C^0(\bar{\Omega}))$ into $(C^1([0,T])^n$, which, under the previous assumptions, follows from (15).
Now, let us assume, for a moment, that f depended on t and x. In that case and in order to prove the existence, we should have to study the correspondence $e \mapsto (f, F)$, where (f, F) satisfies (15), but which is not an application. This is the reason why we consider only body forces constant on x. Physically, it is not a very restrictive assumption because not only are gravity forces constant on x, but also the most important deformations and remodelling phenomena are due to the forces applied on the surface of the bone.

3 The Differential equation of remodeling

We have the following result

Proposition 1 *Assume that $\varepsilon(u)$ is given in $C^0([0,T];\mathcal{C}^0)$. Then, there exists a unique e in $C^1([0,T];C^0(\bar{\Omega}))$ solution to (11)(12). Furthermore, there exists a positive constant c such that:*

$$\begin{aligned} \|e\|_{C^1(C^0)} \;\leq\;& \{c \;+\; c\|\varepsilon(u)\|_{C^0(C^0)}\} \;\times\; \{\; \|e_0\|_{C^0(\bar{\Omega})} \\ &+\; T\;[\;\|a(e_0)\|_{C^0(\bar{\Omega})} \;+\; \|A(e_0)\|_{C^0(\bar{\Omega})}\|\varepsilon(u)\|_{C^0(C^0)}\;] \\ &\times\; \exp[\;T\;(k_1 \;+\; k_2\|\varepsilon(u)\|_{C^0(C^0)})\;]\;\} \end{aligned} \tag{16}$$

Proof. The result of existence and uniqueness of e follows from the Cauchy-Lipschitz-Picard theorem. We define: $f(e,v) = a(e) + A_{km}(e)\varepsilon_{km}(v)$. Then, for all $t \in [0,T]$,

$$||e(t) - e_0||_{C^0(\bar{\Omega})} \leq \int_0^T ||f(e,v) - f(e_0,v)||_{C^0(\bar{\Omega})} dt + \int_0^T ||f(e_0,v)||_{C^0(\bar{\Omega})} dt$$

Using the mean value theorem, we can write:

$$\begin{aligned} ||e(t) - e_0||_{C^0(\bar{\Omega})} \;\; \leq \;\; & \int_0^T (k_1 \; + \; k_2||\varepsilon(u)||_{C^0(\bar{\Omega})}) \; ||e - e_0||_{C^0(\bar{\Omega})} dt \\ & + \;\; T \; ||a(e_0)||_{C^0(\bar{\Omega})} \; + \; ||A(e_0)||_{C^0(\bar{\Omega})} \int_0^T ||\varepsilon(u)||_{C^0(\bar{\Omega})} dt \end{aligned}$$

where k_1 and k_2 depend on the first derivative of $a(e)$ and $A_{km}(e)$. Then, we have:

$$\begin{aligned} ||e(t) - e_0||_{C^0(\bar{\Omega})} \;\; \leq \;\; & T \; [||a(e_0)||_{C^0(\bar{\Omega})} \; + \; ||A(e_0)||_{C^0(\bar{\Omega})} \; ||\varepsilon(u)||_{C^0(C^0)}] \\ & + \; \int_0^T (k_1 \; + \; k_2||\varepsilon(u)||_{C^0(\bar{\Omega})}) \; ||e - e_0||_{C^0(\bar{\Omega})} dt \end{aligned}$$

We deduce from Gronwall's lemma that for all $t \in [0,T]$,

$$\begin{aligned} ||e(t)||_{C^0(\bar{\Omega})} \;\; \leq \;\; & ||e_0||_{C^0(\bar{\Omega})} \; + \; T \; [\; ||a(e_0)||_{C^0(\bar{\Omega})} \; + \; ||A(e_0)||_{C^0(\bar{\Omega})} ||\varepsilon(u)||_{C^0(C^0)} \;] \\ & \times \; \exp[\; T \; (k_1 \; + \; k_2||\varepsilon(u)||_{C^0(C^0)}) \;] \end{aligned} \tag{17}$$

In addition, we deduce from (11) that

$$||\dot{e}||_{C^0(C^0)} \;\leq\; c \; ||e||_{C^0(C^0)} \; [1 \; + \; ||\varepsilon(u)||_{C^0(C^0)}] \tag{18}$$

where c depends on the constants of continuity of functions $a(e)$ and $A_{km}(e)$. By combining (17) and (18), we obtain (16). ■

4 The quasi-static elasticity system

First, let us recall the following result which can be seen in Valent[5]:
Let the time t be fixed in $[0,T]$, let $c_{ijkm}(e)$, $1 \leq i,j,k,m \leq n$, be given in $C^1(\bar{\Omega})$ and let (13) be satisfied. Then, the mapping $u \mapsto (-div(\sigma), \sigma|_{\partial\Omega}.n)$ is a (linear) homeomorphism from $V^{2,p}$ onto $(L^p(\Omega))^n \times (W^{1-\frac{1}{p},p}(\partial\Omega))^n$, $n < p < \infty$.
The existence and uniqueness of u in V^1 follows from the Korn inequality, see e.g. [3]. The idea to prove the above regularity result is to reduce the elasto-static system to the Laplace operator and use a method of continuity, (see [5], Theorem 7.6 p 83). Let us notice that the displacement u is unique in $V^{2,p}$ i.e. unique up to the addition of a rigid displacement.

We have the following lemma which will be important in the sequel:

Lemma 1 *Let $e \in C^1([0,T]; C^0(\bar{\Omega}))$, then $c_{ijkm}(e) \in C^1([0,T]; C^1(I\!R^n))$, $1 \leq i,j,k,m \leq n$, and*

$$||c_{ijkm}(e(t))||_{C^1(\bar{\Omega})} \leq c \quad \text{in } [0,T] \tag{19}$$

where c is a positive constant, which depends notably on $||w_\rho||_{W^{1,1}(I\!R^n)}$, $||\xi_0||_{C^1(\bar{\Omega})}$, $||a_{ijkm}(f)||_{C^1([-\xi_0^{max}, 1-\xi_0^{min}]}$ but is independent of e.

Proof. First, let us notice that by construction, if $e(t) \in C^0(\bar{\Omega})$ then $(M_\rho \circ \mathcal{P}_\eta(e(t))) \in C^\infty(I\!R^n)$ and

$$-\xi_0^{max} \leq M_\rho \circ \mathcal{P}_\eta(e(t)) \leq 1 - \xi_0^{min} \quad \text{ in } Q \tag{20}$$

In addition, $\frac{\partial(M_\rho \circ \mathcal{P}_\eta(e(t)))}{\partial x} = \frac{\partial w_\rho}{\partial x} * \mathcal{P}_\eta(e(t))$ hence

$$-\xi_0^{max} \int_{B(0,\rho)} \frac{\partial w_\rho}{\partial x}\, dx \leq \frac{\partial(M_\rho \circ \mathcal{P}_\eta(e(t)))}{\partial x} \leq (1-\xi_0^{min}) \int_{B(0,\rho)} \frac{\partial w_\rho}{\partial x}\, dx \quad \text{ in } Q \tag{21}$$

One knows that $c_{ijkm}(e(t))$ belongs to $C^1(I\!R^n)$ and if $e(.)$ is C^1 with respect to t then $c_{ijkm}(e(.))$ is also C^1 with respect to t.
Let us prove (19). By definition:

$$||c_{ijkm}(e(t))||_{C^1(\bar{\Omega})} \;=\; \max_{x\in\bar{\Omega}} |c_{ijkm}(e(t))| + \max_{x\in\bar{\Omega}} |\frac{\partial c_{ijkm}(e(t))}{\partial x}|$$

It follows from (20) that: $|c_{ijkm}(e(t))| \leq 2\,\xi_0^{max}\; \max_{(g\in[-\xi_0^{max}, 1-\xi_0^{min}])} |a_{ijkm}(g)|$ and,

$$\begin{aligned} |\, \frac{\partial c_{ijkm}(e(t))}{\partial x} \,| \;&\leq\; |\, [\frac{\partial \xi_0}{\partial x} \;+\; \frac{\partial(M_\rho \circ \mathcal{P}_\eta(e(t)))}{\partial x}]\; a_{ijkm}(M_\rho \circ \mathcal{P}_\eta(e(t)))\,| \\ &+\; |\, [\xi_0 + M_\rho \circ \mathcal{P}_\eta(e(t))]\; \frac{\partial a_{ijkm}}{\partial e}(M_\rho \circ \mathcal{P}_\eta(e(t)))\; \frac{\partial(M_\rho \circ \mathcal{P}_\eta(e(t)))}{\partial x}\,| \end{aligned}$$

Hence from (20) and (21):

$$|\, \frac{\partial c_{ijkm}(e(t))}{\partial x} \,| \leq c_{(||w_\rho||_{W^{1,1}(I\!R^n)}, ||\xi_0||_{C^1(\bar{\Omega})}, ||a_{ijkm}(g)||_{C^1([-\xi_0^{max}, 1-\xi_0^{min}]})}$$

From which the result follows. ■

Proposition 2 *Let $e(t,x)$ be given in $C^1([0,T]; C^0(\bar{\Omega}))$. Then, there exists a unique solution $u \in C^1([0,T]; V^{2,p})$ to (7)-(10). Furthermore, there exists a positive constant c independent of e such that:*

$$||u||_{C^1(W^{2,p})} \;\leq\; c\; (\gamma ||f||_{C^1(L^p)} \;+\; ||F||_{C^1(W^{1-\frac{1}{p},p})}) \tag{22}$$

Proof. Let $e(t,x) \in C^1([0,T]; C^0(\bar{\Omega}))$ be given. Thanks to the results of existence, uniqueness and regularity for the solution of the static elasticity problem, recalled above, and thanks to Lemma 1, there exists a unique solution $u(t) \in V^{2,p}$, for all $t \in [0,T]$ fixed, to (7)-(10). Furthermore:

$$||u(t)||_{W^{2,p}(\Omega)} \ \leq \ c \ (\gamma ||\xi_0 + \mathcal{P}_\eta(e(t))||_{L^\infty(\Omega)} ||f(t)||_{L^p(\Omega)} \ + \ ||F(t)||_{W^{1-\frac{1}{p},p}(\partial\Omega)}) \tag{23}$$

where the constant c depends notably on $||c_{ijkm}(e(t))||_{C^1(\bar{\Omega})}$, N, $\partial\Omega$, n and p. It follows from Lemma 1 and the definition of $\mathcal{P}_\eta$ that there exists a positive constant c, independent of e, such that:

$$||u(t)||_{W^{2,p}(\Omega)} \ \leq \ c \ (\gamma ||f(t)||_{L^p(\Omega)} \ + \ ||F(t)||_{W^{1-\frac{1}{p},p}(\partial\Omega)}) \tag{24}$$

Let us prove that u is continuous with respect to t. We denote $h_i(e) = \gamma(\xi_0 + \mathcal{P}_\eta(e))f_i$. The right hand side $h_i(e)$ of (7) is C^1 with respect to t. Let $u(t)$ and $u(t+\tau)$ be the two solutions to (7)-(10) at times t and $(t+\tau)$, $(t,(t+\tau)) \in [0,T]^2$. We have:

$$-\partial_j[\ \sigma_{ij}(t+\tau) - \sigma_{ij}(t)\ ] \ = \ [h_i(e(t+\tau)) \ - \ h_i(e(t))] \quad \text{in } Q \tag{25}$$

$$[\ \sigma_{ij}(t+\tau) - \sigma_{ij}(t)\ ]\ n_j = F_i(t+\tau) - F_i(t) \qquad \text{on } \Sigma \tag{26}$$

Let us write:

$$\begin{aligned} \sigma_{ij}(t+\tau) - \sigma_{ij}(t) \ &= \ [c_{ijkm}(e(t+\tau)) \ - \ c_{ijkm}(e(t))]\ \varepsilon_{km}(u(t)) \\ &\quad + c_{ijkm}(e(t+\tau))\ \varepsilon_{km}(u(t+\tau) - u(t)) \end{aligned} \tag{27}$$

We pass to the limit when τ tends to 0 into (25) (26) and, thanks to (27), we obtain that $\lim_{(\tau\to 0)}(u(t+\tau) - u(t))$ exists and equals 0 for all $t \in [0,T]$. Therefore, $u \in C^0([0,T]; V^{2,p})$. In order to prove that u is continuously differentiable with respect to time, we differenciate equations (7)-(12) as follows:

$$-\partial_j[\ c_{ijkm}(e)\ \varepsilon_{km}(\dot{u})\ ] \ = \ h_i'(e)\ \dot{e} \ + \ \partial_j[c_{ijkm}'(e)\ \dot{e}\ \varepsilon_{km}(u)] \quad \text{in } Q \tag{28}$$

$$\dot{\sigma}_{ij} n_j = \dot{F}_i \qquad \text{on } \Sigma \tag{29}$$

The right hand side of (28) is continuous with respect to t. Define now $g_i(e,\dot{e},u) = h_i'(e)\ \dot{e} + \partial_j[c_{ijkm}'(e)\ \dot{e}\ \varepsilon_{km}(u)]$, we then have:

$$\begin{aligned} &-\partial_j[(c_{ijkm}(e(t+\tau)) \ - \ c_{ijkm}(e(t)))\ \varepsilon_{km}(\dot{u}(t))] \\ &-\partial_j[c_{ijkm}(e(t+\tau))\varepsilon_{km}(\dot{u}(t+\tau) - \dot{u}(t))] = g_i(e,\dot{e},u)(t+\tau) - g_i(e,\dot{e},u)(t) \end{aligned} \tag{30}$$

Passing to the limit when τ tends to 0 into (30) we obtain that $\lim_{(\tau\to 0)}(\dot{u}(t+\tau) - \dot{u}(t))$ exists and equals 0 for all $t \in [0,T]$, and so $u \in C^1([0,T]; V^{2,p})$. Finally, we deduce (22) from (24). ■

Let us notice that in our model, the body load f is constant in x, hence $||f(t)||_{L^p(\Omega)} = |f(t)|^p mes(\Omega)$.

5 Existence, uniqueness and regularity of the solution

Theorem 1 *Under conditions of Section 2, there exists a unique solution* (u, e) *in* $C^1([0,T];V^{2,p}) \times C^1([0,T];C^0(\bar{\Omega}))$*, which satisfies (7)-(12).*

Proof. It is done in three steps.

First step: Existence of solutions. We prove the existence of a solution (u, e) in $C^0([0,T];(C^1(\bar{\Omega})/\mathcal{R})^n) \times C^1([0,T];C^0(\bar{\Omega}))$ to (7)-(12). In order to do so we use Schauder's fixed point theorem. We consider the operator

$$O : C^0([0,T];\mathcal{C}^0) \to C^1([0,T];C^0(\bar{\Omega}));\ \varepsilon(v) \mapsto e$$

where e is the unique solution to (11)(12), together with the operator

$$E : C^1([0,T];C^0(\bar{\Omega})) \to C^1([0,T];V^1);\ e \mapsto u$$

where u is the unique solution to (7)-(10). We now define the operator T as follows:

$$\begin{array}{lcr}
v \in C^0([0,T];(C^1(\bar{\Omega})/\mathcal{R})^n) & \xrightarrow{T} & u \in C^0([0,T];(C^1(\bar{\Omega})/\mathcal{R})^n) \\
\downarrow\varepsilon : \text{ strain tensor} & & \uparrow i : \text{ injection} \\
\varepsilon(v) \in C^0([0,T];\mathcal{C}^0) & & u \in C^1([0,T];V^{2,p}),\ p > n \\
\downarrow O : \text{ O.D.E.} & & \uparrow R : \text{ Regularity} \\
e \in C^1([0,T];C^0(\bar{\Omega})) & \xrightarrow{E:Elasticity} & u \in C^1([0,T];V^1)
\end{array}$$

We have $T = i \circ R \circ E \circ O \circ \varepsilon$. Let us point out that u does not refer to the solution of (7)-(12) with the given strengths f and F which satisfy (13). Indeed, for all $t \in [0,T]$ such that $f(t) \neq 0$, the relation of compatibility (13) depends on e which depends on u -see (14) and (15)-. Hence, in order to write correctly (7)-(12) as a fixed point problem, it remains to specify that we consider a strenght f which satisfies (15) then, f depends continuously on e. Indeed, we have -see (15)-:

$$\forall w \in \mathcal{R}-\{0\},\ f_i^v(t) = \frac{-\int_{\partial\Omega} F_j(t) w_j \ ds}{\int_\Omega \gamma(\xi_0 + \mathcal{P}_\eta(e^v)) w_i \ dx} \quad \forall t \in [0,T] \text{ such that } f_i(t) \neq 0$$

where $e^v = (O \circ \varepsilon)(v)$. It follows that the operator $e^v \mapsto f^v$ is continuous from $C^1([0,T];C^0(\bar{\Omega}))$ into $(C^1([0,T])^n$.
Let us study the operator T. The operator O is continuous from $C^0([0,T];\mathcal{C}^0)$ into $C^1([0,T];C^0(\bar{\Omega}))$ (Proposition 1) and the operator E is continuous from $C^1([0,T];C^0(\bar{\Omega}))$ into $C^1([0,T];V^1)$, hence the operator T is continuous in $C^0([0,T];(C^1(\bar{\Omega})/\mathcal{R})^n)$. Since the injection i from $C^1([0,T];V^{2,p})$, $p > n$, into $C^0([0,T];(C^1(\bar{\Omega})/\mathcal{R})^n)$ is compact, we conclude that T is compact.

From (22) one deduces that $u = (R \circ E \circ O \circ \varepsilon)(v)$ belongs to a closed ball of $C^1([0,T]; V^{2,p})$, which we denote by B. (Let us point out that the ball B is independent of e, hence independent of v, thanks to Lemma 1).
Finally, the range of the space $C^0([0,T]; (C^1(\bar{\Omega})/\mathcal{R})^n)$ by T is the ball $i(B) \subset C^0([0,T]; (C^1(\bar{\Omega})/\mathcal{R})^n)$. In vertu of the Schauder fixed point theorem, T has a fixed point in $i(B)$ denoted u. And the strenght considered f^v equals f when $v = u$, then the problem (7)-(12) has a solution (u, e) in $C^0([0,T]; (C^1(\bar{\Omega})/\mathcal{R})^n) \times C^1([0,T]; C^0(\bar{\Omega}))$.

Second step: Uniqueness of the solution. It is easy to verify that the proof of uniqueness of [2] is available with the non local model (7)-(12). This proof requires regularity of the data which is satisfied and a regularity of the solution which is also satisfied when (u, e) belongs to $C^0([0,T]; (C^1(\bar{\Omega})/\mathcal{R})^n) \times C^1([0,T]; C^0(\bar{\Omega}))$. Hence the problem (7)-(12) has a unique solution in $C^0([0,T]; (C^1(\bar{\Omega})/\mathcal{R})^n) \times C^1([0,T]; C^0(\bar{\Omega}))$.

Third step: Regularity of the solution. Thanks to the uniqueness of the solution, we can consider u as the (unique) solution of (7)-(10) with e as a data given in $C^1([0,T]; C^0(\bar{\Omega}))$. Then, we apply Proposition 2 and we obtain that u belongs to $C^1([0,T]; V^{2,p})$. ■

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