Shape sensitivities in a Navier-Stokes flow with convective and gray bodies radiative thermal transfer

Jerome Monnier

OPTIMAL CONTROL APPLICATIONS AND METHODS

Optim. Control Appl. Meth. 2003; **00**:1–6 *Prepared using* ocaauth.cls *[Version: 2002/11/11 v1.00]*

Shape Sensitivities in a Navier-Stokes Flow with Convective and Gray Bodies Radiative Thermal Transfer†

J. Monnier

Laboratoire de Modelisation et Calcul (LMC-IMAG), Tour IRMA, BP 53, F-38041 Grenoble Cedex 9, France

SUMMARY

We study a shape optimal design problem for a forced convection flow: the steady-state Navier-Stokes equations coupled with an integro-differential thermal model. The thermal transfers are convective, diffusive and radiative with multiple reflections (model of gray bodies, radiosity equation). The inverse problem consists in minimizing a smooth cost function which depends on the solution, with respect to the domain of the equations. We prove the differentiability of the solution with respect to the domain. It follows the cost function differentiability. We introduce the adjoint state equation and obtain the exact differential of the cost function. The computational method of shape sensitivities and the optimization process are presented too. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: Shape sensitivities, optimal design, Navier-Stokes, radiative heat transfer, gray bodies, radiosity, coupled flow problem.

*Correspondence to: Laboratoire de Modelisation et Calcul (LMC-IMAG), Tour IRMA, BP 53, F-38041 Grenoble Cedex 9, France. http://www-lmc.imag.fr/lmc-edp/Jerome.Monnier

†Projet IDOPT (CNRS-INPG-INRIA-UJF)

Received 1999

Revised 2002

1. INTRODUCTION

We consider in the following a shape optimal design problem for a forced convection flow. We assume the flow to be viscous, incompressible and steady-state. We describe it by the Navier-Stokes equations weakly coupled with a thermal model taking into account conductive, convective and radiative heat exchanges with multiple reflections (the surfaces are gray, opaque and separated by a radiatively non-participating media -radiosity equation-). The problem is to optimize the domain shape of the equations in order to minimize a general smooth cost function that depends on the solution.

As an industrial application, one can cite the cooling problem under a car bonnet studied in [1, 2], or other shape optimal design problems in a forced convection flow based on the present radiative model.

Many authors studied shape optimal design problems in a Navier-Stokes flow without heat transfer, steady-state or not, and using gradient type algorithm, see e.g. [3, 4, 5, 6, 7]... Shape optimal design problems in a Navier-Stokes flow with convective and diffusive heat transfer have been studied for example in [8, 9]. In [2], the authors study a shape optimal design problem in a potential flow coupled with the present thermal model.

In the present paper, we study a shape optimal design problem in a Navier-Stokes flow with diffusive, convective and radiative thermal transfer of gray bodies.

The results obtained are the differentiability of the solution with respect to the domain, the differentiability of the cost function follows. Using an adjoint state method, we give an expression of the exact differential of the cost function. The shape gradient is obtained when discretizing this expression using a finite element method for example. Let us point out that we do not need to derive the angle factors. We present the optimization process required to solve numerically the shape inverse problem. We refer to [10] for a detailed finite element discretization of the present thermal model and to [11, 2] for the details of implementation of the shape problem.

We notice that all the studies cited previously are based on algorithms of gradient type; nevertheless, one could consider an other approach based on a fictitious domain method, see e.g. [12].

This paper is organized as follows. We present in Section 2 the physical model and the inverse shape problem. The cost function considered is general and depends smoothly on the solution. In Section 3, we recall existence and uniqueness results for the solution of the state equation and existence and uniqueness results for the linearized system. (These last results are used for the proof of the solution differentiability and for the adjoint state equation analysis). In Section 4, we recall the classical notion of shape derivatives based on the transport method. The domains considered are Lipschitz open sets and because of mixed type boundary conditions, the domain transformations are $\mathcal{C}^1$ homeomorphisms. Next, we analyze mathematically the shape optimal design problem: the differentiability of the solution and the cost function are proved and an expression of its exact differential is obtained using an adjoint state (Theorem 4.1). Finally, we present the computational method of shape sensitivities and the optimization process based on a gradient type minimization algorithm such as BFGS.

2. POSITION OF THE PROBLEM

2.0.1. The direct problem We consider a stationary fluid flow with heat transfer in a two-dimensional or three-dimensional bounded domain. The fluid is viscous, incompressible and the heat transfer are convective, diffusive and radiative. The model of radiation takes into account the emission, the reflection and the absorption of the radiant energy. The emitted and reflected radiation are diffusely distributed. The surfaces are assumed to be opaque and to behave like gray bodies i.e. the radiative exchanges do not depend on the wave length. In addition, we assume that the surfaces are separated by a non participating media, hence the radiative heat transfer is apparent in the boundary conditions and is described by the radiosity (the radiosity is the radiant energy which flows away from a surface), see e.g. [13].

The unknowns of the full model are the fluid velocity $\tilde{\vec{u}}$, the fluid pressure p, the fluid temperature $\tilde{T}$ and the radiosity w. Let us define the following dimensionless numbers: the Reynolds number $Re = \frac{L^* U^*}{\nu}$, where ν, U^* and L^* are respectively the fluid viscosity, the characteristic velocity and the characteristic length of the flow; the Peclet number $Pe = \frac{\rho\ C_p\ U^*\ L^*}{\lambda}$ where ρ, λ and C_p are respectively the density, the thermal conductivity and the specific heat (at constant pressure) of the fluid; the Biot number Bi, $Bi = \frac{h\ L^*}{\lambda}$ where h is the thermal transfer coefficient on the boundary (Bi does not depend on $\tilde{\vec{u}}$, it is constant); and the dimensionless reals $\delta_1 = \frac{L^* \sigma (T^*)^3}{\lambda}$ and $\delta_2 = \frac{L^* w^*}{\lambda T^*}$, where T^* is a characteristic temperature, w^* is a characteristic radiative energy and σ is the Stephan-Boltzmann's constant.

Let ω be a Lipschitz bounded open set of $I\!R^d$ (d=2 or 3), the dimensionless model is the following. Find $(\tilde{\vec{u}}, p)$ satisfying:

$$-\frac{1}{Re}\,\Delta\tilde{\vec{u}} \;+\; (\tilde{\vec{u}}\cdot\nabla)\tilde{\vec{u}} \;+\; \vec{\nabla}p = 0 \qquad \text{in } \omega \tag{1}$$

$$\operatorname{div}(\tilde{\vec{u}}) = 0 \qquad \text{in } \omega \tag{2}$$

$$\tilde{\vec{u}} = 0 \qquad \text{on } \gamma_0 \tag{3}$$

$$\tilde{\vec{u}} = \vec{g} \qquad \text{on } \gamma_1 \tag{4}$$

where $(\vec{u}\cdot\nabla) = \sum_{i=1}^{d} u_i\partial_i$ and $\gamma_0 \cup \gamma_1 = \partial\omega$. The function $\vec{g}$ belongs to $(H^1(\omega))^d$, it satisfies $\operatorname{div}(\vec{g}) = 0$ and it vanishes in $\omega \setminus \mathcal{V}_1$, where $\mathcal{V}_1$ is a neighborhood "small enough" of γ_1.

Given the fluid velocity $\tilde{\vec{u}}$, find $(\tilde{T}, w)$ satisfying:

$$-\frac{1}{Pe}\Delta\tilde{T} \;+\; \tilde{\vec{u}}\;\vec{\nabla}\tilde{T} = 0 \qquad \text{in } \omega \tag{5}$$

$$\tilde{T} = T_d \qquad \text{on } \gamma_d \tag{6}$$

$$\frac{\partial\tilde{T}}{\partial n} = 0 \qquad \text{on } \gamma_n \tag{7}$$

$$-\frac{\partial\tilde{T}}{\partial n} = \Phi(\tilde{T}, w) \qquad \text{on } \gamma_f \tag{8}$$

$$(I - A)w \;=\; \varepsilon\frac{\delta_1}{\delta_2}\tilde{T}^4 \qquad \text{on } \partial\omega \tag{9}$$

where

$$\Phi(\tilde{T}, w) \;=\; Bi\;(\tilde{T} - T_0) \;+\; \frac{\varepsilon}{(1-\varepsilon)}(\delta_1\;\tilde{T}^4 - \delta_2\;w) \tag{10}$$

$$A\;w(x) \;=\; (1 - \varepsilon(x))\int_{\partial\omega}\phi(x,y)\;w(y)\;ds(y) \tag{11}$$

and I is the mapping identity. We have $\gamma_d \cup \gamma_n \cup \gamma_f = \partial\omega$, the measure of γ_d is strictly positive. The temperatures T_d and T_0 are positives and belong respectively to $H^1(\omega)\cap L^\infty(\omega)$ and $L^\infty(\gamma_f)$. Moreover, T_d is assumed to vanish in $\omega \setminus \mathcal{V}_d$, where $\mathcal{V}_d$ is a neighborhood "small enough" of γ_d.

The kernel $\phi(x,y) \in L^1(\partial\omega\times\partial\omega)$ is the angle factor, it is positive, symmetric. The function

$\varepsilon(x)$ is the emissivity coefficient, it satisfies: $0 < \varepsilon(x) < 1$. $\varepsilon(x)$ is assumed to be a Lipschitz function. We refer to [10] for a detailed description of the thermal model (5)-(9).

In order to have the thermal problem well posed, we need the following assumption:

Assumption 2.1. The parts of the boundary γ_n and γ_f are such that $\vec{\tilde{u}}.\vec{n} \geq 0$ on $\gamma_n \cup \gamma_f$ i.e. the fluid flow is outgoing where the boundary temperature is not given.

We set $\vec{u} = \vec{\tilde{u}} - \vec{g}$ and $T = \tilde{T} - T_d$. Then, the problem (1)-(9) is equivalent to find $(\vec{u}, p)$ satisfying:

$$-\frac{1}{Re}\,\Delta\vec{u} \;+\; (\vec{u}\cdot\nabla)\vec{u} \;+\; (\vec{u}\cdot\nabla)\vec{g} \;+\; (\vec{g}\cdot\nabla)\vec{u} \;+\; \vec{\nabla}p \;=\; \vec{f} \quad \text{in } \omega \tag{12}$$

$$\operatorname{div}(\vec{u}) = 0 \quad \text{in } \omega \tag{13}$$

$$\vec{u} = 0 \quad \text{on } \partial\omega \tag{14}$$

with $\vec{f} = -(\vec{g}\cdot\nabla)\vec{g} + \frac{1}{Re}\Delta\vec{g},\ \vec{f} \in (H^{-1}(\omega))^d$. Then, to find (T, w) satisfying:

$$-\frac{1}{Pe}\Delta T \;+\; (\vec{u}+\vec{g})\;\vec{\nabla}T \;=\; h \quad \text{in } \omega \tag{15}$$

$$T = 0 \quad \text{on } \gamma_d \tag{16}$$

$$-\frac{\partial T}{\partial n} = -e \quad \text{on } \gamma_n \tag{17}$$

$$-\frac{\partial T}{\partial n} = \Phi(T + T_d, w) - e \quad \text{on } \gamma_f \tag{18}$$

$$(I - A)w \;=\; \varepsilon\frac{\delta_1}{\delta_2}(T + T_d)^4 \quad \text{on } \partial\omega \tag{19}$$

with $h = \frac{1}{Pe}\Delta T_d - (\vec{u}+\vec{g})\nabla T_d,\ h \in H^{-1}(\omega)$ and $e = -\frac{\partial T_d}{\partial n},\ e \in H^{-\frac{1}{2}}(\partial\omega)$.

The mathematical and numerical analysis of the thermal model (5)-(9) are detailed in [10]. We recall the main results of these analysis in next section.

2.0.2. The shape optimal design problem We consider the following abstract shape optimal design problem. Let $J_\omega(y)$ be the observation function, we define the cost function:

$$j(\omega) = J_\omega(y^\omega) \tag{20}$$

where $y^\omega = (u^\omega, p^\omega, T^\omega, w^\omega)$ is the solution of the equations (12)-(19) posed in ω.

The minimization problem is:

$$\left| \begin{array}{l} \text{Find } \omega^\star \text{ such that:} \\ j(\omega^\star) \;=\; \min\limits_{\omega} j(\omega) \end{array} \right. \tag{21}$$

Example. In the cooling problem studied in [11, 1, 2], the authors consider γ_h a part of the boundary (a hose shape), $\gamma_h \subset \gamma_f \subset \partial\omega$, and they seek to minimize $j(\omega) = J_\omega(y^\omega) = \frac{1}{2}\int_{\gamma_h}(T^\omega + T_d)^2\, ds$ with respect to the shape of γ_h, T_d being the given temperature on γ_d.

Remark 2.1. In the present paper, we consider a shape optimal design problem for a forced convection flow with gray bodies radiative thermal transfer. Such flows appear in many industrial applications. The observation function considered here is general. In order to apply the present analysis, it suffices that $J_\omega(y)$ considered depends smoothly on $y = (u, p, T, w)$ or depends on some of these variables only. Of course, it is implicit that this function depends on T or w, otherwise the thermal model would be useless.

In a practical point of view and for a specific problem with equations (12)-(19) as direct model, it is straightforward to apply the equations and the method presented in the sequel, see especially Section 4.3 and 4.4.

3. THE STATE EQUATION

We write the variational formulation of the direct problem in appropriate Sobolev spaces and we obtain the state equation. We recall some results of existence and uniqueness of the solution (Proposition 3.1). Also, we recall a result of existence and uniqueness of the solution to the linearized problem (Proposition 3.2).

From now, we voluntarily omit the arrows above vector entities.

We define $V_0^u(\omega) = (H_0^1(\omega))^d$, $V_0^T(\omega) = \{t \in H^1(\omega);\ t|_{\gamma_d} = 0\}$ and:

$$V_0(\omega) = V_0^u(\omega) \times L_0^2(\omega) \times V_0^T(\omega) \times L^2(\partial\omega)$$

with $L_0^2(\omega) = \{q \in L^2(\omega);\ \int_\omega q\ dx = 0\}$.

The state equation is:

$$\begin{cases} \text{Find } y^\omega = (u^\omega, p^\omega, T^\omega, w^\omega) \in V_0(\omega) \text{ such that } : \\ \forall z = (v, q, t, r) \in V_0(\omega),\ E_\omega(y^\omega, z) \ = \ 0 \end{cases} \tag{22}$$

where $E_\omega(y, z)$ is the sum of the variational formulations of the Navier-Stokes equations, the thermal partial differential equation and the integral equation. This state equation (22) is equivalent to:

$$\begin{cases} \text{Find } (u^\omega, p^\omega) \in V_0^u(\omega) \times L_0^2(\omega) \text{ such that:} \\ \forall v \in V_0^u(\omega), \quad \dfrac{1}{Re} \displaystyle\sum_{i=1}^{d} \int_\omega < \nabla u_i, \nabla v_i > \ dx \\ \quad + \displaystyle\sum_{i,j=1}^{d} \int_\omega [u_i \partial_i u_j v_j + g_i \partial_i u_j v_j + u_i \partial_i g_j v_j]\ dx \\ \quad - \displaystyle\int_\omega p \ \ \text{div} v\ dx \ = \ < f, v >_{H^{-1} \times H^1(\omega)} \\ \forall q \in L_0^2(\omega), \quad \displaystyle\int_\omega \text{div} u\ q\ dx \ = \ 0 \end{cases} \tag{23}$$

where $< .,. >$ denotes the inner product in $I\!R^d$.

$$\begin{cases} \text{Given } u^\omega, \text{ find } (T^\omega, w^\omega) \in V_0^T(\omega) \times L^2(\partial\omega) \text{ such that: } \forall t \in V_0^T(\omega), \\ \displaystyle\int_\omega < \nabla T, \nabla t > \; dx \; + \; Pe \int_\omega < (u^\omega + g), \nabla T > \; t \; dx \\ \displaystyle + \int_{\gamma_f} \Phi(T + T_d, w) t ds \; = \; Pe < h, t >_{H^{-1} \times H^1(\omega)} \; + \; < e, t >_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}(\gamma_f \cup \gamma_n)} \\ \forall r \in L^2(\partial\omega), \\ \displaystyle\int_{\partial\omega} (I - A) w \; r \; ds = \frac{\delta_1}{\delta_2} \int_{\partial\omega} \varepsilon \; (T + T_d)^4 \; r \; ds \end{cases} \tag{24}$$

3.0.3. Existence and uniqueness of the physical solution

We set:

$$T_{inf} = Min(\inf_{\gamma_d} T_d, \; \inf_{\gamma_f} T_0) \text{ and } T_{sup} = Max(\sup_{\gamma_d} T_d, \; \sup_{\gamma_f} T_0)$$

We have

Proposition 3.1. If the viscosity and the thermal conductivity are large enough, then there exists a unique solution $y^\omega = (u^\omega, p^\omega, T^\omega, w^\omega) \in V_0(\omega)$ to (22) such that $\tilde{T}^\omega = (T^\omega + T_d)$ satisfies the weak maximum principle: $T_{inf} \leq \tilde{T} \leq T_{sup}$ a.e.

Proof. One knows that there exists $(u^\omega, p^\omega) \in V_0^u(\omega) \times L_0^2(\omega)$, solution of (23) and if the viscosity ν is large enough (or equivalently if the Reynolds number is small enough) then the solution is unique (see e.g. [14]). In others respects, there exists a solution to (24) -see [1]-. And if the thermal conductivity λ is large enough then there exists a unique $(T^\omega, w^\omega) \in V_0^T(\omega) \times L^2(\partial\omega)$ satisfying the weak maximum principle, see [10]. ■

Let us notice that since $\tilde{T}^\omega$ satisfies the weak maximum principle and T_d belongs to $L^\infty(\omega)$, T^ω belongs to $H^1(\omega) \cap L^\infty(\omega)$ and the terms $\int_{\gamma_f} (T^\omega + T_d)^4 t \; ds$ and $\int_{\partial\omega} (T^\omega + T_d)^4 r \; ds$ make sense with test functions in $t \in H^1(\omega)$ and $r \in L^2(\partial\omega)$ (even in three dimensions of space).

3.0.4. The linearized problem In next Section, we study the differentiability of the solution y^ω with respect to ω using the implicit function theorem (see Lemma 4.1). Then, we need to consider the linearized state equation:

$$\left\{\begin{array}{l}\text{Find } \eta^\omega = (\eta^\omega_u, \eta^\omega_p, \eta^\omega_T, \eta^\omega_w) \in V_0(\omega) \text{ such that :}\\ \forall z = (v, q, t, r) \in V_0(\omega),\ \dfrac{\partial E_\omega}{\partial y}(y^\omega, z).\eta^\omega \ = \ 0\end{array}\right. \tag{25}$$

Proposition 3.2. If the viscosity ν and the thermal conductivity λ are large enough then the linearized problem (25) is well posed.

Proof. We develop the expression of (25) and we obtain the linearized Navier-Stokes equations weakly coupled with the linearized thermal model. Then, one knows that if the viscosity ν is large enough then the linearized Navier-Stokes equations are well posed (see e.g. [15], Lemma IV.3.2); and if the thermal conductivity λ is large enough then the linearized thermal equations are well posed (see [10], Proposition 4). Hence the result. ■

4. THE SHAPE OPTIMAL DESIGN PROBLEM

We seek to solve the shape optimal design problem (12)-(19)(20)(21) using an optimization algorithm of gradient type. Before computing the shape gradient, we define in Section 4.1 the admissible domains space (Lipschitz domains) and we recall the classical definition of shape derivatives based on domain deformations (method of transport with $\mathcal{C}^1$ transformations). Let us point out that we require $\mathcal{C}^1$ homeomorphisms (and not Lipschitz homeomorphisms) since the boundary conditions are mixed type -see Remark 4.3-.

Then, we prove in Section 4.2 the differentiability of the state equation solution with respect to the domain (Lemma 4.2). The differentiability of the cost function follows.

In Section 4.3, we give an expression of the exact differential of the cost function.

In order to compute the exact differential (or the shape gradient), few approaches are possible. The direct differentiation, it requires to derive the state with respect to the shape variables. In practice, it implies to solve as much PDEs systems as discrete shape variables. In order to avoid this extra computational cost, we use in the present article the classical adjoint state method which requires to solve only one linear extra PDE system, [16, 17]. Then, to obtain the shape gradient, again two approaches are possible.

The first one is to discretize the equations, using a finite element method for example, and to derive the discrete equations. We obtain the discrete shape gradient. The second one is to calculate the expression of the exact differential of the cost function and to discretize it. We obtain the discretized continuous shape gradient.

In the present study, we follow this last approach. In Theorem 4.1, we introduce the adjoint state equation and obtain an expression of the exact differential of the cost function.

Let us point out that when using conforming finite elements, the discrete shape gradient and the discretized continuous shape gradient are equal, [18].

In Section 4.4, we present the optimization process required to solve numerically the shape optimal design problem.

4.1. Domain variations and shape derivatives

We consider family of Lipschitz domains. (Domains with corners are admissible and the main properties in Sobolev spaces are valid).

We define the space of admissible domains in a classical manner as the set of domains homeomorphic to a reference domain, see (27). Also, we define the derivative with respect to the domain in a classical manner as the derivative of the transported function with respect to the transformation, see (29). We refer to [19, 20, 21, 17, 22, 23, 11, 5, 2].

The regularity of the transformations depends on the problem considered. In the present study, the direct problem is second order and the boundary conditions are mixed type. It leads us to consider $\mathcal{C}^1$ homeomorphisms in order to well define the transported boundary integrals.

4.1.1. The admissible domains space Let $\hat{\Omega}$ be a bounded open subset of $I\!R^d$ with a Lipschitz boundary, $\partial\hat{\Omega} = \hat{\Gamma}' \cup \hat{\Gamma}$, $\hat{\Gamma}' \cap \hat{\Gamma} = \emptyset$, -Fig. 1-.

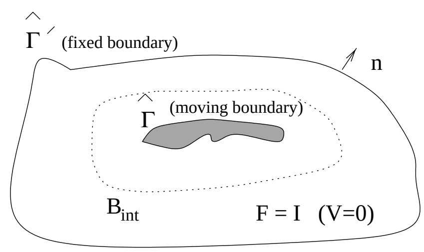

Figure 1. Reference domain $\hat{\Omega}$

We denote by B_{int} a neighborhood of $\hat{\Gamma}$, B_{int} large enough and $B_{int} \supset\supset \hat{\Gamma}$, -Fig. 1-.

We set the function space:

$$\hat{\mathcal{F}} = \{\hat{F},\ \hat{F} \text{ bijection of } \hat{\Omega} \text{ onto } \hat{F}(\hat{\Omega});\ \hat{F} \in \mathcal{C}^1(\bar{\hat{\Omega}}, I\!R^d),\ \hat{F}^{-1} \in \mathcal{C}^1(\bar{\hat{F}}(\hat{\Omega}), I\!R^d)\} \tag{26}$$

and its affine subspace:

$$\hat{\mathcal{F}}_0 = \{\hat{F} \in \hat{\mathcal{F}};\ \hat{F} = I \text{ in } \hat{\Omega} \setminus B_{int}\}$$

where I denotes the identity of $I\!R^d$.

Then, we define the admissible domains space $\mathcal{D}$ as follows

$$\mathcal{D} = \{\omega = \hat{F}_0(\hat{\Omega});\ \ \hat{F}_0 \in \hat{\mathcal{F}}_0\} \tag{27}$$

Remark 4.1. In the present configuration, the boundary part $\hat{\Gamma}'$ is fixed and we seek to optimize the shape of $\hat{\Gamma}$. A such situation arises in many problems like for example in the cooling problem studied in [2] or in the minimization drag problem studied in [5].

Remark 4.2. If $\hat{F}$ is close enough to I in $\hat{\mathcal{F}}_0$ (($\hat{F} - I$) small enough) then $\hat{F}(\hat{\Omega})$ is an open set of $I\!R^d$ with a Lipschitz boundary , see ([5], Lemma 3), and $F(\hat{\Gamma}) \subset B_{int}$.
In others respects, let us recall that an open set with a Lipschitz boundary has an external normal n almost everywhere, and boundary integrals can be defined.

4.1.2. Shape derivative of a real valued function For $\hat{F}_0 \in \hat{\mathcal{F}}_0$, $(\hat{F}_0 - I)$ small enough, we define the domain Ω by $\Omega = \hat{F}_0(\hat{\Omega})$ and $\Gamma = \hat{F}_0(\hat{\Gamma})$.

We set the homeomorphisms space defined in Ω, Fig. 2:

$$\mathcal{F} = \{F,\ F = \hat{F} \circ \hat{F}_0^{-1},\ \hat{F} \in \hat{\mathcal{F}}\}$$

and its affine subspace:

$$\mathcal{F}_0 = \{F,\ F = \hat{F} \circ \hat{F}_0^{-1},\ \hat{F} \in \hat{\mathcal{F}}_0\}$$

Let $F \in \mathcal{F}_0$, we define $\omega = F(\Omega)$ and $V \in \mathcal{C}^1(\bar{\Omega}, I\!R^d)$ by: $V = F - I$. We have $V = 0$ in $\hat{\Omega} \setminus B_{int}$.

For a given cost function j, $j : \omega \in \mathcal{D} \mapsto j(\omega) \in I\!R$, we define the "transported" cost function $\bar{j}$ by:

$$\bar{j} : \mathcal{F}_0 \to I\!R\ :\ F \mapsto \bar{j}(F) = j(F(\Omega)) = j(\omega) \tag{28}$$

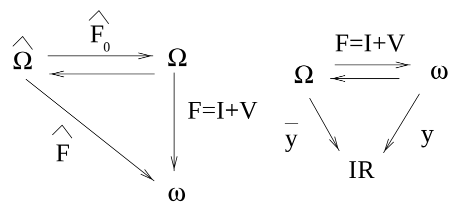

Figure 2. Change of variables

Then, the derivative with respect to the domain is classically defined as the transported function $\bar{\jmath}$ derived with respect to the transformation F. We have (see e.g. [19, 2] for more details):

$$\frac{dj}{d\omega}(\Omega) \cdot V \;=\; \frac{d\bar{\jmath}}{dF}(I) \cdot V \;, \quad \forall V \in \mathcal{C}^1(\bar{\Omega}, I\!R^d) \tag{29}$$

Remark 4.3. In the present study, we consider an open set family of $I\!R^d$ *with Lipschitz boundary. (Therefore, we can consider domains with corners).*

If the direct problem is a Dirichlet type boundary value problem, if the observation function is a volume integral then no boundary integrals appear in the variational formulations. In that case (and if the direct problem is second order), $\hat{\mathcal{F}}$ *defined as the Lipschitz homeomorphisms set is restrictive enough, see e.g. [19, 11, 5, 2]. In others words, we do not require* $\mathcal{C}^1$ *transformations, Lipschitz ones are regular enough.*

On the other hand, if the boundary conditions are Neumann type or mixed type, and/or if the observation function is a boundary integral (like in the present study), then one needs regular transformations in order to transport the boundary integrals on the reference domain Ω*. As a matter of fact, we need the terms* $\partial_i V_j$*,* $1 \leq i, j \leq d$*, well defined on the boundary (see Theorem 4.1 and the equations presented in Section 4.3).*

For a standard second order problem with Neumann type boundary value conditions, $W^{2,\infty}$

homeomorphisms are considered in [19], bounded $\mathcal{C}^2$ homeomorphisms are considered in [24] and in [23], $\mathcal{C}^1$ homeomorphisms slightly different from the present ones are considered.

4.2. Differentiability of the solution

In this paragraph we prove the differentiability of the solution of the state equation with respect to the domain (Lemma 4.2). To this end, we transport the state equation on the reference domain, see (31). The transported equations are detailed.

Let Ω be the "reference" domain, $\Omega = \hat{F}_0(\hat{\Omega})$, $\hat{F}_0 \in \hat{\mathcal{F}}_0$, $(\hat{F}_0 - I)$ small enough. We consider an open set $\omega = F(\Omega)$ where $F \in \mathcal{F}_0$.

4.2.1. Boundary conditions and data regularity The function F is continuous from Ω into $\mathbb{R}^d$, hence $\partial\omega$, the boundary of $\omega = F(\Omega)$, is the image of $\partial\Omega$ by F. We denote by $\gamma_0,\ \gamma_1,\ \gamma_d,\ \gamma_n$ and γ_f the image by F of $\Gamma_0,\ \Gamma_1,\ \Gamma_d,\ \Gamma_n, \Gamma_f$ respectively, $\Gamma_0 \cup \Gamma_1 = \partial\Omega$ and $\Gamma_d \cup \Gamma_n \cup \Gamma_f = \partial\Omega$.

Assumption 4.1. The Dirichlet boundary conditions are posed on the fixed boundary $\hat{\Gamma}'$ (i.e. Γ_1 and Γ_d are included in $\hat{\Gamma}'$). Moreover, the Dirichlet data denoted g and T_d vanish in B_{int}. In others words, the neighborhood $\mathcal{V}_1$ and $\mathcal{V}_d$ introduced in Section 2, are strictly included in $\hat{\Omega} \setminus B_{int}$ (see Fig. 1).

Let us suppose given $\bar{T}_0$ a positive function in $L^\infty(\Gamma_f)$, $\bar{\phi} \in L^1(\partial\Omega \times \partial\Omega)$ and $\bar{\varepsilon} \in W^{1,\infty}(\partial\Omega)$. We denote with $^-$ the variables defined in Ω, Fig. 2.

We denote: $T_0 = \bar{T}_0 \circ F^{-1},\ \phi = \bar{\phi} \circ F^{-1},\ \varepsilon = \bar{\varepsilon} \circ F^{-1}$.

Hence for $F \in \mathcal{F}_0$, we have: $T_0 \in L^\infty(\gamma_f)$, $\phi \in L^1(\partial\omega \times \partial\omega)$, $\varepsilon \in W^{1,\infty}(\partial\omega)$.

We assume that the parts of the boundary Γ_n and Γ_f are such that $\bar{\bar{u}}.\bar{n} \geq 0$ on $\Gamma_n \cup \Gamma_f$, hence for $F \in \mathcal{F}_0$, Assumption 2.1 holds.

4.2.2. Functional spaces We want the Hilbert space $V_0(\omega)$ preserved by the change of variable. To this end and following [26], we introduce the space of pressures:

$$L^2_{\#}(F) = \{q \in L^2(F(\Omega));\ \int_{F(\Omega)} \frac{q}{|detDF|}\, dx \ = \ 0\}$$

This space $L^2_{\#}(F)$ is isomorph to the space $L^2_0(F(\Omega))$ and from now, we consider:

$$V_0(\omega) = V_0(F(\Omega)) \ = V_0^u(F(\Omega)) \times L^2_{\#}(F) \times V_0^T(F(\Omega)) \times L^2(F(\partial\Omega)) \tag{30}$$

Then, it follows

Lemma 4.1. For $F \in \mathcal{F}_0$, the mapping $z \in V_0(F(\Omega)) \mapsto z \circ F \in V_0(\Omega)$ is an isomorphism. ■

Let us notice that we could not consider divergence free velocity fields because $div(u) = 0$ does not imply $div(u \circ F) = 0$.

4.2.3. Transport of the state equation In order to study the solution differentiability (with respect to the domain) and in order to compute the differential of the cost function, we need to transport the equations on the "reference" domain $\Omega = F^{-1}(\omega)$ as follows.

For any $y,\ z \in V_0(\omega)$, we let:

$$\bar{E}(F;\bar{y},\bar{z}) = E_{F(\Omega)}(\bar{y} \circ F^{-1}, \bar{z} \circ F^{-1}) = E_\omega(y,z)$$

$$\bar{J}(F;\bar{y}) = J_{F(\Omega)}(\bar{y} \circ F^{-1}) = J_\omega(y)$$

with $\omega = F(\Omega),\ y = \bar{y} \circ F^{-1},\ z = \bar{z} \circ F^{-1}$ (see Fig. 2).

The mapping $\bar{J}$ is supposed to be of class $\mathcal{C}^1(\mathcal{F} \times V_0(\Omega))$.

The transported state equation is:

$$\left\{ \begin{array}{l} \text{Find } \bar{y}^F = (\bar{u}^F, \bar{p}^F, \bar{T}^F, \bar{w}^F) \in V_0(\Omega) \text{ such that:} \\ \forall \bar{z} = (\bar{v}, \bar{q}, \bar{t}, \bar{r}) \in V_0(\Omega),\ \ \bar{E}(F; \bar{y}^F, \bar{z}) \ = \ 0 \end{array} \right. \tag{31}$$

where $\bar{y}^F = y^\omega \circ F$, y^ω is solution of (22).

Remark 4.4. Since the mapping $z \in V_0(F(\Omega)) \mapsto z \circ F \in V_0(\Omega)$ is an isomorphism for $F \in \mathcal{F}_0$, if (22) has a unique solution (see Proposition 3.1) then the transported state equation (31) has a unique solution $\bar{y}^F \in V_0(\Omega)$.

We have -see e.g [19] for the change of variables in integrals-:

$$\begin{aligned}
\bar{E}(F;\bar{y},\bar{z}) \;&=\; E_{F(\Omega)}(\bar{y}\circ F^{-1},\bar{z}\circ F^{-1})\\
= \frac{1}{Re}\sum_{i=1}^{d}\int_{\Omega} &< {}^{T}(DF^{-1}\circ F)\nabla\bar{u}_i,\ {}^{T}(DF^{-1}\circ F)\nabla\bar{v}_i > \ |detDF| d\bar{x}\\
&+ \sum_{j=1}^{d}\int_{\Omega} < {}^{T}(DF^{-1}\circ F)\nabla\bar{u}_j,\bar{u} > \ \bar{v}_j\ |detDF|\ d\bar{x}\\
&- \int_{\Omega}\bar{p}\ {}^{T}(DF^{-1}\circ F) : D\bar{v}\ |detDF|\ d\bar{x}\\
&+ \int_{\Omega} < {}^{T}(DF^{-1}\circ F) : D\bar{u}\ \bar{q}\ |detDF|\ d\bar{x}\\
&+ \int_{\Omega} < {}^{T}(DF^{-1}\circ F)\nabla\bar{T},\ {}^{T}(DF^{-1}\circ F)\nabla\bar{t} > \ |detDF|\ d\bar{x}\\
&+ Pe\int_{\Omega} < \bar{u},\ {}^{T}(DF^{-1}\circ F)\nabla\bar{T} > \ \bar{t}\ |detDF|\ d\bar{x}\\
&+ \int_{\Gamma_f}\phi(\bar{T},\bar{w})\bar{t}\ Jac(F)d\bar{s}\\
&+ \int_{\partial\Omega}\bar{w}\bar{r}\ Jac(F)d\bar{s}\\
- \int_{\partial\Omega}\int_{\partial\Omega}(1-\bar{\varepsilon})\bar{\phi}(\bar{x},\bar{y})\bar{w}(\bar{y})\bar{r}(\bar{x})&(Jac(F)(\bar{x}))\ (Jac(F)(\bar{y}))d\bar{s}(\bar{x})\ d\bar{s}(\bar{y})\\
&- \frac{\delta_1}{\delta_2}\int_{\partial\Omega}\bar{\varepsilon}\ \bar{T}^4\ \bar{r}\ Jac(F)\ d\bar{s}
\end{aligned}$$

with $Jac(F) \;=\; |detDF|\ \|\ {}^{T}DF^{-1}.n\|_{I\!R^d}$, $A : B = \sum_{i,j=1}^{d} a_{ij}b_{ij} = tr(A\ {}^{T}B)$ and $< .,. >$ denotes the scalar product in $I\!R^d$.

4.2.4. Transport of the minimization problem Let

$$J_\omega : V_0(\omega) \rightarrow I\!R : y \mapsto J_\omega(y)$$

be the observation function. We assume that this observation function is continuously differentiable. We define the cost function

$$j : \mathcal{D} \to I\!R; \ \omega \mapsto j(\omega) = J_\omega(y^\omega)$$

where y^ω is the unique physical solution of the state equation (22).

The problem we seek to solve is

$$\left| \begin{array}{l} \text{Find } \omega^\star \in \mathcal{D} \text{ such that :} \\ j(\omega^\star) \ = \ \min_{\mathcal{D}} j(\omega) \end{array} \right.$$

Then, we write $\bar{j}(F) = \bar{J}(F; \bar{y}^F)(= j(\omega))$.

From now, the minimization problem we solve is:

$$\left| \begin{array}{l} \text{Find } F^\star \in \mathcal{F}_0 \text{ such that :} \\ \bar{j}(F^\star) \ = \ \min_{F \in \mathcal{F}_0} \bar{j}(F) \end{array} \right.$$

This problem is a classical optimal control problem in a Banach space.

4.2.5. Differentiability of the solution We have

Lemma 4.2. Let the viscosity ν and the thermal conductivity λ be large enough and let $\bar{y}^F \in V_0(\Omega)$ be the the unique physical solution of (31). Then, there exists a neighborhood $\mathcal{V}_I$ of the identity mapping in $\mathcal{F}_0$ such that the mapping $F \in \mathcal{V}_I \subset \mathcal{F}_0 \mapsto \bar{y}^F \in V_0(\Omega)$ is of class C^1.

Proof. The proof is based on the implicit function theorem.

One can easily verify that the mapping $\bar{E}(F; \bar{y}, \bar{z})$ is C^1 with respect to $(F; \bar{y})$.

In others respects, the linearized problem is well posed when the viscosity ν and the thermal conductivity λ are large enough (Proposition 3.1). Then, it follows from the implicit function theorem that under these conditions, the equation (31) defines a C^1-mapping $F \mapsto \bar{y}^F : \mathcal{F}_0 \to V_0(\Omega)$ in a neighborhood $\mathcal{V}_I$ of I. ∎

Remark 4.5. Under Assumption 4.1, we have $g \circ F = g$ and $T_d \circ F = T_d$ for $F \in \mathcal{F}_0$, F small enough. Then, we proved that the physical solution of (1)-(9) is continuously differentiable with respect to the domain i.e. the mapping

$$F \in \mathcal{F}_0 \mapsto \bar{\bar{y}}^F = (\bar{y}^F + (g, 0, T_d, 0)) \in V(\Omega)$$

with $V(\Omega) = (H^1(\Omega))^d \times L^2_{\#}(I) \times H^1(\Omega) \times L^2(\partial\Omega)$, is of class C^1 in a neighborhood $\mathcal{V}_I$ of I. On the other hand, if Assumption 4.1 does not hold and if $F \in \mathcal{F}$, then one must take care that in general $g \circ F \neq g$ and $T_d \circ F \neq T_d$. And, for a function $g \in H^{m+1}(\mathbb{R}^d)$, $m = 0$ or 1, the mapping $F \mapsto g \circ F$ is of class C^1 from $\mathcal{F}$ into $H^m(\mathbb{R}^d)$ (see [19], Lemma IV.4.4).

4.3. The exact differential of the cost function

We present in Theorem 4.1 below, the exact differential of the cost function and the adjoint state equation. The adjoint state equation is a system of linear equations solved in the reverse way compared to the state equation.

Let us recall that we introduce the adjoint state equation in order to avoid extra computations as much extra PDEs systems as discrete shape variables.

We derive below integrals with respect to domain. To this end, we need

Lemma 4.3.

i) The application $F \mapsto |det(DF)|$ is differentiable from $\mathcal{F}$ onto $\mathcal{C}^0(\bar{\Omega}; \mathbb{R})$. And for all

$V \in \mathcal{C}^1(\bar{\Omega}, I\!R^d)$,

$$\frac{d(|det(DF)|)}{dF}(I).V \;=\; div(V)$$

ii) The application $F \mapsto (DF^{-1} \circ F)$ *is differentiable from* $\mathcal{F}$ *onto* $\mathcal{C}^0(\bar{\Omega}; I\!R^d)$*. And for all* $V \in \mathcal{C}^1(\bar{\Omega}, I\!R^d)$,

$$\frac{d(DF^{-1} \circ F)}{dF}(I).V \;=\; -DV$$

iii) The application $F \mapsto (\|\, ^T DF^{-1}.n\|_{I\!R^d})$ *is differentiable from* $\mathcal{F}$ *onto* $L^\infty(\partial\Omega)$*. And for all* $V \in \mathcal{C}^1(\bar{\Omega}, I\!R^d)$,

$$\frac{d(\|\, ^T DF^{-1}.n\|_{I\!R^d})}{dF}(I).V \;=\; -< n,\, ^T DVn >$$

■

See e.g. ([19], chap. IV) for the proof.

Now, we can establish the main result of the present article:

Theorem 4.1. If the viscosity ν *and the thermal conductivity* λ *are large enough then there exists a neighborhood* $\mathcal{V}_I$ *of the identity mapping in* $\mathcal{F}_0$ *such that the cost function* $j : \omega \in \mathcal{D} \mapsto j(\omega) \in I\!R$ *is of class* C^1 *for all* $\omega = F(\Omega),\ F \in \mathcal{V}_I \subset \mathcal{F}_0$.

Furthermore, for all $V \in \mathcal{C}^1(\bar{\Omega}, I\!R^d)$*, we have:*

$$\frac{dj}{d\omega}(\Omega).V \;=\; \frac{\partial J_\Omega}{\partial \omega}(y^\Omega).V \;-\; \frac{\partial E_\Omega}{\partial \omega}(y^\Omega, \pi^\Omega).V \tag{32}$$

where y^Ω *is the solution of the state equation (22) posed in* Ω *and* π^Ω *is the unique solution* *of the adjoint state equation:*

$$\left\{\begin{array}{l} Find\ \pi^{\Omega} = (\vartheta^{\Omega}, \rho^{\Omega}, \theta^{\Omega}, \kappa^{\Omega}) \in V_0(\Omega)\ such\ that: \\ \forall z = (v, q, t, r) \in V_0(\Omega), \\ \dfrac{\partial E_{\Omega}}{\partial y}(y^{\Omega}, \pi^{\Omega}).z \ = \ \dfrac{\partial J_{\Omega}}{\partial y}(y^{\Omega}).z \end{array}\right. \tag{33}$$

All the terms of the cost differential (32) and the adjoint state equation (33) being detailed below.

Proof. The observation function $\bar{J}(F, \bar{y})$ is of class C^1 and the solution $\bar{y}^F$ is of class C^1 in $\mathcal{V}_I$ (Lemma 4.2), hence the cost function $j(\omega)$ is also of class C^1 for $F \in \mathcal{V}_I$.

The computation method of the derivatives with respect to ω has been described previously: we transport the mappings to the "reference" domain Ω -see (31)- then, by definition, the partial derivative with respect to ω is the partial derivative of the transported mappings with respect to F. The detailed computations are presented below.

In others respects, the expression (32) of the differential of the cost function follows from the classical adjoint state method, [16, 17]. ■

Let us detail the expressions (32) and (33).

The term $\frac{\partial E_{\Omega}}{\partial \omega}(y^{\Omega}, \pi^{\Omega}).V$ in (32) can be expressed by:

$$\begin{aligned} \frac{\partial E_{\Omega}}{\partial \omega}(y^{\Omega}, \pi^{\Omega}).V = (\frac{\partial E^u_{\Omega}}{\partial \omega} \ + \ \frac{\partial E^p_{\Omega}}{\partial \omega})(y^{\Omega}, \pi^{\Omega}).V \\ +(\frac{\partial E^T_{\Omega}}{\partial \omega} \ + \ \frac{\partial E^w_{\Omega}}{\partial \omega})(y^{\Omega}, \pi^{\Omega}).V \end{aligned}$$

Let us consider the first term of $E^u_{\Omega}(y, \pi)$:

$$I_{\omega}(u, v) \ = \ \frac{1}{Re}\sum_{i=1}^{d}\int_{\omega} < \nabla u_i, \nabla v_i > \ dx$$

We define the transported application (see section 4.1):

$$\begin{aligned}\bar{I}(F;\bar{u},\bar{v}) &= I_{F(\Omega)}(\bar{u}\circ F^{-1},\bar{v}\circ F^{-1}) = I_\omega(u,v)\\ &= \frac{1}{Re}\sum_{i=1}^{d}\int_\Omega < {}^T(DF^{-1}\circ F)\nabla\bar{u}_i,\ {}^T(DF^{-1}\circ F)\nabla\bar{v}_i > \ |detDF|d\bar{x}\end{aligned}$$

Then,

$$\frac{\partial I_\Omega}{\partial\omega}(u,v).V = \frac{\partial\bar{I}}{\partial F}(I;\bar{u},\bar{v}).V$$

Using Lemma 4.3, it gives

$$\frac{\partial I_\Omega}{\partial\omega}(u,v).V = \frac{1}{Re}\sum_{i=1}^{d}[\int_\Omega < \nabla u_i,\nabla v_i > \mathrm{div}V dx - \int_\Omega < (DV + {}^TDV)\nabla u_i,\nabla v_i > dx]$$

The computations of the other terms are similar, see e.g. [19, 11, 23, 2].

With the notations $\tilde{u}=u+g$ and $\tilde{T}=T+T_d$, we obtain:

$$\begin{aligned}\frac{\partial E^u_\Omega}{\partial\omega}(y,\pi).V &= \frac{1}{Re}\sum_{i=1}^{d}[\int_\Omega < \nabla\tilde{u}_i,\nabla\vartheta_i > \mathrm{div}V dx\\ &\quad - \int_\Omega < (DV + {}^TDV)\nabla\tilde{u}_i,\nabla\vartheta_i > dx]\\ &+\int_\Omega < D\tilde{u}\tilde{u},\vartheta > \ \mathrm{div}V\ dx - \int_\Omega < D\tilde{u}.DV\tilde{u},\vartheta > \ dx\\ &\quad -\int_\Omega p\ \mathrm{div}\vartheta\ \mathrm{div}V\ dx + \int_\Omega p\ {}^TDV.D\vartheta\ dx\end{aligned}$$

where $A.B = \sum_{i,j=1}^{d} a_{ij} b_{ij}$ =trace($A\ {}^T B$), A and B being two real matrix $d \times d$.

$$\frac{\partial E^p_\Omega}{\partial \omega}(y,\pi).V \;=\; \int_\Omega \mathrm{div}\tilde{u}\ \rho\ \mathrm{div}V\ dx - \int_\Omega {}^T DV.D\tilde{u}\ \rho\ dx$$

$$\begin{aligned}\frac{\partial E^T_\Omega}{\partial \omega}(y,\pi).V \;=\; & \int_\Omega < \nabla\tilde{T}, \nabla\theta > \mathrm{div}V dx \\ & - \int_\Omega < (DV + {}^T DV)\nabla\tilde{T}, \nabla\theta > \ dx \\ + Pe \int_\Omega < \tilde{u}, \nabla\tilde{T} > \ \theta\ \mathrm{div}V\ dx \ - \ & Pe \int_\Omega < \tilde{u},\ {}^T DV\nabla\tilde{T} > \ \theta\ dx \\ & + \int_{\Gamma_f} Bi\ (\tilde{T} - T_0)\ \theta\ \ < \tau, DV\tau > \ ds \\ & + \int_{\Gamma_f} \frac{\varepsilon}{(1-\varepsilon)} (\delta_1 \tilde{T}^4 - \delta_2 w)\ \theta\ \ < \tau, DV\tau > \ ds\end{aligned}$$

where τ is the unit tangent vector to $\partial\Omega$.

$$\begin{aligned}\frac{\partial E^w_\Omega}{\partial \omega}(y,\pi).V \;=\; & \int_{\partial\Omega} (w - \varepsilon \frac{\delta_1}{\delta_2}\tilde{T}^4)\ \kappa\ \ < \tau, DV\tau > \ ds \\ - & \int_{\partial\Omega} (1-\varepsilon(x))[\int_{\partial\Omega} \phi(x,y) w(y) (< \tau, DV\tau > (y) \\ & + \ < \tau, DV\tau > (x))\ ds(y)]\ \kappa(x)\ ds(x)\end{aligned}$$

Let us point out that we do not need to derive the angle factor $\phi(x, y)$.

4.3.1. The adjoint state equation The adjoint state equation (33) is equivalent to a system of linear partial differential equations and an integral equation. These equations are solved in the reverse way compared to the state equation and each of them is the adjoint equation of the corresponding equation to the direct problem. They are:

$$
\left\{
\begin{array}{l}
\text{Find } (\theta^\Omega, \kappa^\Omega) \in V_0^T(\Omega) \times L^2(\partial\Omega) \text{ such that: } \forall r \in L^2(\partial\Omega), \\
\displaystyle\int_{\partial\Omega} \kappa\ r\ ds\ -\ \int_{\partial\Omega} (1-\varepsilon(x))[\int_{\partial\Omega} \phi(x,y)\ r(y)\ ds(y)]\ \kappa(x)\ ds(x) \\
\qquad\qquad \displaystyle = \delta_2 \int_{\Gamma_f} \frac{\varepsilon}{(1-\varepsilon)}\ \theta\ r\ ds\ +\ \frac{\partial J_\Omega}{\partial w}(y^\Omega).r \\
\forall t \in V_0^T(\Omega), \\
\displaystyle\int_\Omega < \nabla\theta, \nabla t >\ dx\ +\ Pe \int_\Omega < \tilde{u}^\Omega, \nabla t >\ \theta\ dx \\
\displaystyle +\ \int_{\Gamma_f} [Bi +\ 4\delta_1 \frac{\varepsilon}{(1-\varepsilon)} (\tilde{T}^\Omega)^3]\ \theta\ t\ ds\ =\ 4\frac{\delta_1}{\delta_2} \int_{\Gamma_f} \varepsilon(\tilde{T}^\Omega)^3\ \kappa\ t\ ds\ +\ \frac{\partial J_\Omega}{\partial T}(y^\Omega).t
\end{array}
\right.
\tag{34}
$$

$$
\left\{
\begin{array}{l}
\text{Given } \theta^\Omega, \text{ find } (\vartheta^\Omega, \rho^\Omega) \in V_0^u(\Omega) \times L_0^2(\Omega) \text{ such that: } \forall v \in V_0^u(\Omega), \\
\displaystyle \frac{1}{Re} \sum_{i=1}^d \int_\Omega < \nabla\vartheta_i, \nabla v_i > dx\ +\ \sum_{i,j=1}^d [\int_\Omega \tilde{u}_i^\Omega \partial_i v_j\ \vartheta_j\ dx\ +\ \int_\Omega v_i\ \partial_i \tilde{u}_j^\Omega\ \vartheta_j\ dx] \\
\displaystyle +\ Pe \int_\Omega < v, \nabla\tilde{T}^\Omega >\ \theta^\Omega\ dx + \int_\Omega \rho\ \mathrm{div} v\ dx\ =\ \frac{\partial J_\Omega}{\partial u}(y^\Omega).v \\
\displaystyle \forall q \in L_0^2(\Omega), \quad \int_\Omega \mathrm{div}\vartheta\ q\ dx\ =\ -\frac{\partial J_\Omega}{\partial p}(y^\Omega).q
\end{array}
\right.
\tag{35}
$$

4.4. Computation of shape sensitivities and optimization process

We present briefly below the optimization process and the method to compute numerically the shape gradient.

In order to compute a (local) optimum to the minimization problem (21), it remains to:

- Discretize and implement the state equation (22) and the adjoint state equation (33) using a finite element method for example.
- Discretize the cost function and its differential (32) (numerical integration). We obtain the shape gradient.
- Call an optimization algorithm of gradient type, BFGS for example.

The optimization process is schemed in Fig. 3.

The discretization of the state equation (22) (Navier-Stokes flow with convective and radiative gray bodies thermal transfer) using few first or second order finite element methods is detailed in [25]. A conforming finite element discretization and implementation of the present shape inverse problem with a potential flow replacing the Navier-Stokes flow, are detailed in [2]. (The displacement field V is computed in the domain by solving Laplace problems).

In the present approach, we compute the discretized continuous shape gradient and not the discrete shape gradient. Using conforming finite elements, they are equals, [18].
The continuous approach needs more sophisticated mathematical tools but presents the advantage to be probably more intelligible to read. Also, if we change the discretization scheme we do not have to re-derive the code (using automatic differentiation for example). This is especially true when using C++ finite elements libraries.

ACKNOWLEDGEMENT

The author would like to thank the referees for their helpful and fruitful comments.

REFERENCES

1. Chenais D, Monnier J, Vila JP., Shape optimal design for a fluid-heat coupled system, *Appl. Math. Comp. Sc.*, **6**, 2, 245–261 , 1996.
2. Chenais D, Monnier J, Vila JP., A shape optimal design problem with convective radiative thermal transfer. Analysis and implementation, *J. Optim Th. Appl.*, **110**, 75–117 , 2001.

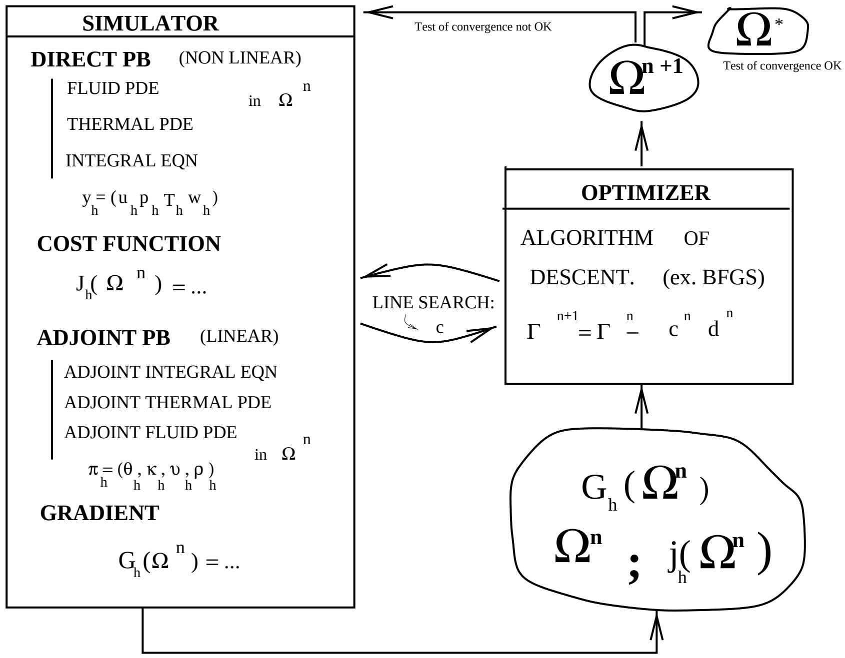

Figure 3. Optimization process

3. Pironneau O., On optimum design in fluid mechanics, *J. Fluid Mech.*, **64**, part. I, 97–110 , 1974.

4. Mohammadi B, Pironneau O., *Applied Shape Optimization for Fluids*, Numerical Mathematics Scientific Computation. Oxford , 2001.

5. Bello JA, Fernandez-Cara E, Lemoine J, Simon J., The differentiability of the drag with respect to the variations of a lipschitz domain in a Navier-Stokes flow, *SIAM J. Control Optim.*, **35**, 2, 626–640, 1997.

6. McDavid RM, Dantzig JA., Design sensitivity finite element analysis of free surface flows with application to optimal design of casting rigging systems, *Int. J. Num. Meth. Fl.*, **28**, 419–442 , 1998.

7. Laporte E., Optimisation de formes pour écoulements instationnaires, Thesis, Ecole Polytechnique , 1998.

8. Wang Z, Tortorelli DA, Dantzig JA., Sensitivity analysis and optimization of coupled thermal and flow problems with applications to contraction design, *Int. J. Num. Meth. Fl.*, **23**, 991–1020 , 1996.

9. Dziri R, Zolésio JP., Shape existence in Navier-Stokes flow with heat convection. *Ann. Sc. Norm. Super. Pisa*, **24**, 1, 165–192, 1997.

10. Monnier J, Vila JP., Convective and radiative thermal transfer with multiple reflections. Analysis and

approximation by a finite element method., *Math. Models Meth. Appl. Sc.*, **11**, 2 , 2001.

11. Monnier J., Conception optimale de forme pour un système couplé fluide-thermique. Application à l'aérothermique d'un véhicule. Thesis, University of Nice - Sophia Antipolis , 1995.

12. Glowinski R, Kearsley AJ, Pan TW, Périaux J., Numerical simulations and optimal shape for viscous flow by a fictitious domain method, *Int. J. Num. Meth. Fl.*, **20**, 695–711 , 1995.

13. Sparrow EM, Cess RD., *Radiation Heat Transfer*, Mc Graw-Hill Book Co. New-York , 1978.

14. Lions J.L., *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Gauthier-Villars, 1969.

15. Girault V, Raviart PA., *Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms*, Springer-Verlag, Berlin , 1986.

16. Lions JL., *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag , 1971.

17. Céa J., Conception optimale ou identification de formes - Calcul rapide de la dérivée directionnelle de la fonction coût, *M2AN*, **20**, 3, 371–402 , 1986.

18. Chenais D, Lods V., Comparison between the discrete gradient the discretized continuous gradient for nonconforming finite element methods, *C. R. Acad. Sci., Paris*, **I 319**, 5, 493–498, , 1994.

19. Murat F, Simon J., *Sur le contrôle par un domaine géométrique*, Publication of L.A. 189, University P. et M. Curie Paris VI, 1976.

20. Céa J., Problems of shape optimal design, Optimization of distributed parameter structures, Sijthoff Noordhoff, Alphen aan den Rijn, The Nertherlands, Eds. E.J. Haug J. Céa, **II**, 1981.

21. Rousselet B., Quelques résultats en optimisation de domaine, Thèse détat, University of Nice - Sophia Antipolis , 1982.

22. Masmoudi M., Outils pour la conception optimale de formes, Thèse d'état, University of Nice , 1987.

23. Guillaume P., Dérivées d'ordre supérieur en conception optimale de forme, Thesis, Université P. Sabatier of Toulouse , 1994.

24. Simon J., Variations with respect to domain for Neumann conditions, In: *Control of distributed parameter systems*, Proceedings of the 4th IFAC Symposium, Los-Angeles, 1986.

25. Monnier J., Free convection with radiative thermal transfer of grey bodies. Analysis and approximation by finite element methods, *Math. Models Meth. Appl. Sc.*, 1383–1424, **10**, 9 , 2000.

26. Bello J.A, Fernandez-Cara E, Simon J., The variation of the drag with respect to the domain in Navier-Stokes flow, First Franco-Romanian conference on optimization, optimal control, partial differential equations, International series of numerical mathematics, **107**, 287–296, Birkhauser, 1992.

