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Pascal Morin, Claude Samson

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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N° 6464

Mars 2008

Thème NUM

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--6464--FR+ENG

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Thème NUM — Systèmes numériques
Projet Arobas

Rapport de recherche n° 6464 — Mars 2008 — 49 pages

Abstract: The problem of stabilizing reference trajectories for nonholonomic systems, often referred to as the trajectory tracking problem in the literature on mobile robots, is addressed. The first sections of this report set the theoretical background of the problem, with a focus on controllable driftless systems which are invariant on a Lie group. The interest of the differential geometry framework here adopted comes from the possibility of taking advantage of ubiquitous symmetry properties involved in the motion of mechanical bodies. Theoretical difficulties and impossibilities which set inevitable limits to what is achievable with feedback control are surveyed, and basic control design tools and techniques are recast within the approach here considered. A general method based on the so-called Transverse Function approach –developed by the authors–, yielding feedback controls which unconditionnally achieve the *practical* stabilization of arbitrary reference trajectories, including fixed points and non-admissible trajectories, is recalled. This property singles the proposed solution out of the abundant literature devoted to the subject. It is here complemented with novel results showing how the more common property of asymptotic stabilization of persistently exciting admissible trajectories can also be granted with this type of control. The last section of the report concerns the application of the approach to unicycle-type and car-like vehicles. The versatility and potentialities of the Transverse Function (TF) control approach are illustrated via simulations involving various reference trajectory properties, and a few complementary control issues are addressed. One of them concerns the possiblity of using control degrees of freedom to limit the vehicle's velocity inputs and the number of transient maneuvers associated with the reduction of initially large tracking errors. Another issue, illustrated by the car example, is related to possible extensions of the approach to systems which are not invariant on a Lie group.

Key-words: wheeled robot, nonholonomic system, unicycle, car, stabilization, trajectory tracking, Lie group, transverse function.

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex (France)
Téléphone : +33 4 92 38 77 77 — Télécopie : +33 4 92 38 77 65

Stabilisation de trajectoires pour des systèmes nonholonomes. Aspects théoriques et applications

Résumé : Ce rapport concerne la stabilisation de trajectoires de référence* pour des systèmes nonholonomes. Le cadre théorique ici adopté est celui des systèmes de commande sans dérive contrôlables et invariants par rapport à une opération de groupe de Lie. Son intérêt provient de ce qu'il permet d'exploiter des propriétés de symétrie omniprésentes dans le mouvement des systèmes mécaniques. Après avoir rappelé un certain nombre d'obstructions et limitations intrinsèques à la commande par retour d'état de ces systèmes, certaines techniques existantes de synthèse de commande sont brièvement passées en revue. La méthode de commande par "fonction transverses", développée par les auteurs, est exposée de façon plus détaillée. Elle conduit à la synthèse de commandes par retour d'état permettant de stabiliser de façon pratique des trajectoires de référence *arbitraires*, dont en particulier les points fixes et les trajectoires non-admissibles. Les autres méthodes de commande proposées dans la littérature ne possèdent pas cette propriété. Nous montrons dans ce rapport comment la stabilisation *asymptotique* de certaines trajectoires de référence admissibles vérifiant une propriété "d'excitation persistante" peut également être obtenue avec l'approche fonctions transverses. La dernière partie de ce rapport porte sur l'application de l'approche à des véhicules de type unicycle ou voiture. Des résultats de simulation pour différents types de trajectoires de référence permettent d'illustrer les potentialités de cette méthode de commande, et quelques problèmes complémentaires sont traités. L'un d'eux concerne l'utilisation des degrés de liberté du contrôle, dans le but de limiter les valeurs de commande ainsi que le nombre de manœuvres pendant les phases transitoires de réduction de l'erreur de suivi. Un autre point, illustré par l'exemple de la voiture, concerne des extensions possibles de l'approche à des systèmes qui ne sont pas invariants par rapport à une opération de groupe de Lie.

Mots-clés : Robot à roues, système nonholonome, unicycle, voiture, stabilisation, suivi de trajectoire, groupe de Lie, fonction transverse.

* problème également connu sous le nom de "suivi de trajectoires"

Contents

1 Introduction

Nonholonomic systems, ranging from unicycle and car-like vehicles, possibly equipped with trailers, to more original systems like rolling spheres [4, 13, 17], snake-like robots [14, 15], snakeboards and roller-racers [18, 20], etc., abound in Robotics. All these mechanical systems share strong controllability properties, but the nonholonomic kinematic constraints which characterize their motion render the associated control design problem quite challenging, as illustrated by Brockett's theorem [5] proving the non-existence of pure-state feedbacks for the asymptotic stabilization of fixed points. This difficulty has had the effect of focusing the research on the feedback control of nonholonomic systems on two distinct sub-problems, namely i) fixed point asymptotic stabilization relying on highly nonlinear techniques, and ii) asymptotic stabilization of admissible (feasible) and *persistently exciting* trajectories based on more classical linear and nonlinear techniques –see, e.g., [33] for more details and references on the proposed control methods. Within the stream of papers devoted to these problems, [12] addressed the control of a unicycle-type vehicle in a different way which attracted our attention and inspired the development of the Transverse Function (TF) approach at the core of the present paper. The focus on the aforementionned sub-problems has produced solutions which apply to many practical situations. However, it also matters to realize that this research activity, undertaken during more than a decade, has not –by far– exhausted the subject. In particular, the cases of non-persistently exciting and of non-admissible trajectories have seldom been addressed, nor the transitions between one type of trajectory and another. Moreover, the incompleteness of the results is not merely theoretical. To our knowledge and understanding, none of the control methods and various adaptations which have been proposed performs well in all circumstances. In particular, it has never been proved that a high-level supervisor which implements on-line a switching strategy between two complementary controllers can unconditionally ensure the convergence of tracking errors to zero, even when the class of reference trajectories is restricted to the "small" subclass of admissible ones. As a matter of fact, a conceptually important result by Lizàrraga [22] basically proves that the search for a causal feedback control scheme capable of stabilizing "any" admissible reference trajectory for this type of system is vain. In other words, whatever the chosen control strategy, there always exists an admissible trajectory that this control is unable to stabilize asymptotically, eventhough any admissible trajectory taken separately can be asymptotically stabilized. This limitation, which has no equivalence in Linear Control Theory, is an ever lasting source of frustration that control designers and roboticists have to live with. The TF approach does not (cannot) overcome it, but it goes further than other control methods because it more fully exploits the local controllability property of the systems by providing feedback controllers theoretically capable of stabilizing –in a practical manner defined further in the paper– any trajectory, even non-admissible, with arbitrary tracking precision. Moreover, a proper tuning allows for the asymptotic stabilization of persistently exciting admissible trajectories, thus making these controllers also competitive with classical control laws within their own domain of operation.

This latter feature is one of the original results of the present study, whose other objective is to provide the robotic community with a synthesis of the TF control approach that the authors have been developing for several years. The theoretical foundations of this approach have been published in [29, 30]. Complementary results, some theoretical, others more application-oriented, have also been published in various control journals or conferences. Although an exhaustive presentation of these results is not possible here, the idea is to provide the reader with enough background material and explanations to allow him to successfully implement the approach for robotic applications involving classical systems like unicycle and car-like robots, and also develop new control strategies for other systems.

The report is organized as follows. In Section 2 some properties of kinematic control models of nonholonomic systems are recalled, with a focus on systems which are invariant under a certain Lie group operation. This class of systems contains several examples of interest (unicycles, chained systems, rolling spheres, etc.) and possesses a structure sufficiently rich and general to let its study unveil results which are applicable to many other systems. In particular, the TF approach is best exposed in this framework although it also applies to systems which are not invariant on a Lie group (see Section 5). In fact, this geometric framework has long been exploited in Robotics [6, 19, 36] and the properties recalled in Section 2 are not really new. However, being scattered in the literature on automatic control and differential geometry, they are not always well known to roboticists. Recalling them may not be necessary for a certain number of readers, but is hopefully beneficial to others. Section 3 discusses control issues and contains a brief review of the difficulties associated with the feedback control of nonholonomic systems. Section 4 is devoted to the TF approach. After recalling the basics of the approach –as developed in [30]–, new results about the asymptotic stabilization of persistently exciting admissible reference trajectories, in relation with the choice of transverse functions and their use in the control laws, are presented. Section 5 is dedicated to the application of the TF control approach to unicycle and car-like vehicles. In both cases, several simulation results illustrate various aspects of the controller's performance in relation to the reference trajectory properties: unconditional ultimate uniform boundedness of the tracking errors, maneuvers management during initial transient phases, convergence of the tracking errors to zero when the reference trajectory is admissble and persistently exciting.

2 The geometry of kinematic control models

2.1 Recalls on kinematic models

For completeness, and also to introduce some notation, we recall hereafter well known properties concerning kinematic models of nonholonomic systems (details can be found in most books on mobile robotics, like e.g. [7, 16]). Kinematic equations of nonholonomic mechanical systems are given by driftless control systems of the form

$$\dot{g} = \sum_{i=1}^{m} X_i(g)u_i \tag{1}$$

with g belonging to a n-dimensional manifold G, $X_1, \ldots, X_m$ the system's control vector fields (v.f.) representing admissible directions compatible with the nonholonomic constraints, and $u = (u_1, \ldots, u_m)'$ the control vector, with z' denoting the transpose of a vector z. The system's nonholonomy is characterized by the fact that $m < n = \dim(g)$. The kinematic model of a mechanical system is not unique. It depends on the choice of the state g used to represent the system's configuration and the way $\dot{g}$ is decomposed along m independent directions. For example, a standard model for unicycle-type vehicles is

$$\begin{cases} \dot{x} &= u_1 \cos\theta \\ \dot{y} &= u_1 \sin\theta \\ \dot{\theta} &= u_2 \end{cases} \tag{2}$$

but it is well known that the 3-D chained system can also be used as a local model. Recall that the equations of the n-D chained system with two control inputs are

$$\begin{cases} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= u_1 x_2 \\ &\vdots \\ \dot{x}_n &= u_1 x_{n-1} \end{cases} \tag{3}$$

Similarly, car-like vehicles can be modeled either by the equations

$$\begin{cases} \dot{x} &= u_1 \cos\theta \\ \dot{y} &= u_1 \sin\theta \\ \dot{\theta} &= u_1 (\tan\varphi)/L \\ \dot{\varphi} &= u_2 \end{cases} \tag{4}$$

with $\varphi \in (-\pi/2, \pi/2)$ denoting the steering angle and L the distance between the rear and front wheels' axles, or by the 4-D chained system.

Systems (2), (3), and (4) are particular cases of the general system (1), with $m = 2$. In addition, they are controllable at any point, i.e. the set of points reachable from any point during an arbitrary (non-zero) amount of time by using bounded controls contains a neighborhood of this point. For a driftless system (1) with smooth v.f., local controllability at g is granted by[1] the satisfaction at g of the so-called Lie Algebra Rank Condition (LARC) involving iterated Lie brackets of the system v.f. [8, 37]. This condition requires that one can find n independent vectors in the set

$$\{X_i(g), [X_i, X_j](g), [X_i, [X_j, X_k]](g), \ldots\}$$

with $i, j, k, \ldots \in \{1, \ldots, m\}$, and the Lie bracket $[X, Y]$ of two v.f. X and Y defined (in coordinates x) by $[X, Y](x) = \frac{\partial Y}{\partial x}(x) X(x) - \frac{\partial X}{\partial x}(x) Y(x)$. For instance, for the 3-D chained

[1] and equivalent to, when the control v.f. are real-analytic,

system, the vectors $X_1(x) = (1, 0, x_2)'$, $X_2(x) = (0, 1, 0)'$, and $X_3(x) = [X_1, X_2](x) = (0, 0, -1)'$ form a basis of $\mathbb{R}^3$ for any x. To avoid non-essential technicalities, the following assumptions are made throughout the paper. They are satisfied by Systems (2), (3), and (4).

Assumption 1 *For System* (1)*,*

1. *The state space G is a connected manifold,*
2. *The control v.f. $X_1, \ldots, X_m$ are independent over $\mathbb{R}$, i.e. $(\sum_{i=1}^m \lambda_i X_i(g) = 0\ \forall g) \Longrightarrow \lambda_1 = \cdots = \lambda_m = 0$, with the λ_i's denoting constant scalars.*
3. *The LARC is satisfied at any g.*

2.2 Systems on Lie groups

2.2.1 Definition and characterization

An important structural property of Systems (2) and (3) is that their v.f. are *left-invariant* with respect to a Lie group operation. Recall (see e.g. [46]) that a Lie group G is a smooth manifold endowed with a "smooth" group law $(g_1, g_2) \mapsto g_1 g_2$, i.e. *i)* the mapping is associative, *ii)* there exists an element e (the unit element) such that $ge = eg = g$ for all g, *iii)* for any g, there exists an element g^{-1} (the inverse of g) such that $gg^{-1} = g^{-1}g = e$, *iv)* the mapping $(g_1, g_2) \mapsto g_1 g_2^{-1}$ is smooth. A v.f. X defined on a Lie group G is "left-invariant" if

$$\forall g_1, g_2 \in G, \quad dL_{g_1}(g_2).X(g_2) = X(g_1 g_2)$$

with L_{g_1} the "left translation" by g_1, defined by $L_{g_1}(g_2) = g_1 g_2$, and $df(p)$ denoting the differential of a mapping f, evaluated at p. The set of left-invariant v.f., often denoted as $\mathfrak{g}$, is called the Lie algebra of the group. It is a vector space of the same dimension (over $\mathbb{R}$) as the group. Then, we say that (1) is a system on a Lie group if the associated state space G is a Lie group, and each control v.f. X_i is left-invariant. An equivalent definition in term of trajectories, probably more intuitive, is that given any control input $u(t)$ ($t \in [0, T]$), any solution to the system can be deduced from another solution via a left translation by a constant element. More precisely, if $g_1(t)$ and $g_2(t)$ denote two solutions to (1), then $\forall t \in [0, T]$, $g_2(t) = g_2(0) g_1(0)^{-1} g_1(t)$. This geometric property is shared (at the kinematics level) by all rigid bodies and the associated Lie groups are $SE(2)$, $SO(3)$, $SE(3)$, etc. (see e.g. [36] for a detailed exposition). For example, (2) is a system on the Lie group $SE(2)$, the group operation of which is

$$g_1 g_2 = \begin{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + R(\theta_1) \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\ \theta_1 + \theta_2 \end{pmatrix} \tag{5}$$

with $g_i = (x_i, y_i, \theta_i)$ and $R(\theta)$ the matrix of rotation in the plane of angle θ. The unit element is $e = (0,0,0)$ and the inverse of $g = (x, y, \theta)$ is

$$g^{-1} = \begin{pmatrix} -R(-\theta)\begin{pmatrix} x \\ y \end{pmatrix} \\ -\theta \end{pmatrix} \tag{6}$$

While it is not difficult to guess from the physics that (2) is a system on $SE(2)$ (i.e. the same input applied from two different initial conditions produces the same motions), the fact that (3) is also a system on a Lie group is not so obvious. This raises the following questions. How to determine whether a given system (1) is, or is not, on Lie group? When the test is positively conclusive, how to determine the associated group operation?

First, it matters to notice that the satisfaction of the group property may be local only, i.e. in a neighborhood $\mathcal{U} \subset G$ of a given point g_0. In the special (but nonetheless important) case when the Lie algebra of the control v.f. is nilpotent, e.g. for chained systems, there is an equivalence between the local and global satisfaction of this property. When Assumption 1 is satisfied, there exists around any point g_0 a Lie group operation (defined in a neighborhood of this point) w.r.t. which the control v.f. are left-invariant *if and only if* there exist $n-m$ v.f. $X_{m+1}, \dots, X_n$, consisting of iterated Lie brackets of $X_1, \dots, X_m$, such that *any* other iterated Lie bracket of $X_1, \dots, X_m$ is also a linear combination with *constant coefficients* of $X_1, \dots, X_n$. In other words, $X = \{X_1, \dots, X_n\}$ must be a basis over $\mathbb{R}$ of the Lie algebra generated by the control v.f. In this case, X is also a basis of the Lie group's algebra $\mathfrak{g}$ (i.e. the vector space of left-invariant v.f. on G). For instance, this property is satisfied for System (2) with, e.g., $X_3 = [X_1, X_2]$. It is also satisfied for the chained system (3) by taking, e.g., $X_k = (\mathrm{ad}^{k-2}X_1)(X_2)$ $(k = 3, \dots, n)$ with $(\mathrm{ad}^p X)(Y)$ defined recursively by the relations $(\mathrm{ad}^1 X)(Y) = (\mathrm{ad}X)(Y) = [X, Y]$ and $(\mathrm{ad}^p X)(Y) = [X, (\mathrm{ad}^{p-1}X)(Y)]$ for $p \geq 2$. However, this property is not satisfied for the car model (4). Indeed, although $X_1(g), X_2(g), [X_1, X_2](g), [X_1, [X_1, X_2]](g)$ are independent vectors at any $g = (x, y, \theta, \varphi)$, there does not exist constants $\lambda_1, \dots, \lambda_4$ such that $[X_2, [X_1, X_2]](g) = \lambda_1 X_1(g) + \lambda_2 X_2(g) + \lambda_3 [X_1, X_2](g) + \lambda_4 [X_1, [X_1, X_2]](g)$, $\forall g$. On the other hand, the 4-D chained system is a system on a Lie group, and it is also used as a kinematic model for car-like vehicles. This contradiction is only apparent because the transformation of System (4) into the 4-D chained system involves a change of control variables on top of a change of state coordinates. Whereas the property of left-invariance is conserved by changes of coordinates, a complementary change of control variables is always needed to transform a non-invariant system into an invariant one.

The problem of determining the group operation (e.g. once the Lie group property for the system has been established) is more difficult. A general method, yielding a local expression of the group operation, is based on the use of the exponential mapping and the Baker-Campbell-Hausdorff formula [44]. However, this approach often involves cumbersome calculations. When the state space is $\mathbb{R}^n$, as in the case of the chained system (3), one may proceed as follows. First, the unit element can be chosen as $e = 0$. Then, from the definition of the left-translation operation, for any $i = 1, \dots, n$ and any $x, y \in \mathbb{R}^n$,

$$\begin{aligned}(xy)_i &= L_{x,i}(y) \\ &= L_{x,i}(0) + \int_0^1 dL_{x,i}(ty)y\,dt \\ &= x_i + \int_0^1 dL_{x,i}(ty)y\,dt\end{aligned} \tag{7}$$

with $L_{x,i}$ the i-th component of L_x. Let $X(x)$ denote the matrix composed of the column vectors $X_1(x), \dots, X_n(x)$, with $\{X_1, \dots, X_n\}$ a basis of the Lie algebra generated by the control v.f. It follows from the left-invariance of the X_i's that for any $x, z \in \mathbb{R}^n$, $dL_x(z)X(z) = X(xz)$, so that

$$dL_x(z) = X(xz)X(z)^{-1} \tag{8}$$

If the analytic expression of the term xz in the right-hand side of the above equation was known, then one would obtain an explicit expression for $dL_x(z)$ which could be used in (7). The problem is that this is exactly what we are looking for. Now, in the particular case where, for any $i = 1, \dots, n$, the row i of $X(xz)$ only depends on the components $(xz)_1, \dots, (xz)_{i-1}$ —a type of triangularity property satisfied, e.g., by some homogeneous systems— , then the combination of (7) and (8) allows one to compute the components $(xy)_i$ recursively, for $i = 1, \dots, n$. The application of this method to the chained system (3) yields:

$$(xy)_i = \begin{cases} x_i + y_i & \text{if } i = 1, 2 \\ x_i + y_i + \sum_{j=2}^{i-1} \frac{y_1^{i-j}}{(i-j)!} x_j & \text{otherwise} \end{cases} \tag{9}$$

2.2.2 Tracking error system

If one is interested in the stabilization of a reference trajectory $g_r(.)$ for System (1), one has to define an *error* between the desired (reference) state and the actual state of the system. When the system under consideration is invariant on a Lie group, a "natural" tracking error is $\tilde{g}(t) := g_r(t)^{-1}g(t)$. The problem of stabilizing g_r can then be expressed as the problem of stabilizing the unit element e for the *error system* whose state is $\tilde{g}$, since $\tilde{g}(t) = e$ is equivalent to $g(t) = g_r(t)$. Let us first assume that g_r is constant over time. Then, by the invariance property one has

$$\dot{\tilde{g}} = dL_{g_r^{-1}}(g)\dot{g} = \sum_{i=1}^{m} X_i(\tilde{g})u_i$$

This is the error system equation. The above relation indicates that this equation is the same as the equation of the initial system, thus justifying the adjective "natural" associated with the error $\tilde{g} = g_r^{-1}g$.

Let R_g denote the right-translation operator defined by $R_{g_2}(g_1) := g_1 g_2 \; (= L_{g_1}(g_2))$. When $g_r(t)$ varies with time, the above error equation becomes (see Relation (74) in Appendix A):

$$\begin{aligned}\dot{\tilde{g}} &= dL_{g_r^{-1}}(g)\dot{g} + dR_g(g_r^{-1})\tfrac{d}{dt}g_r^{-1} \\ &= \sum_{i=1}^{m} X_i(\tilde{g})u_i + P(\tilde{g}, g_r, \dot{g}_r)\end{aligned} \tag{10}$$

with

$$P(\tilde{g}, g_r, \dot{g}_r) = -dR_{\tilde{g}}(e)dL_{g_r^{-1}}(g_r)\dot{g}_r$$

Now, if $X = \{X_1, X_2, \ldots, X_n\}$ denotes a basis of the group's Lie algebra, there exists a vector-valued time function $v_r = (v_{r,1}, \ldots, v_{r,n})'$ such that (omitting the time index) $\dot{g}_r = \sum_{i=1}^{n} X_i(g_r)v_{r,i}$. To further simplify the notation, we will write $\dot{g}_r = X(g_r)v_r$. Note that this *notation* coincides, when $G = \mathbb{R}^n$, with the product of the matrix $X(g) = (X_1(g) \; X_2(g) \; \ldots \; X_n(g))$ by the vector v_r. Using this decomposition of $\dot{g}_r$ in the expression of P, one obtains (see Relation (75) in Appendix A):

$$P(\tilde{g}, g_r, \dot{g}_r) = -dL_{\tilde{g}}(e)\mathrm{Ad}(\tilde{g}^{-1})X(e)v_r \tag{11}$$

with Ad the so-called *adjoint representation* defined by

$$\begin{aligned}\mathrm{Ad}(\sigma) &:= dJ_\sigma(e) \\ &= dL_\sigma(\sigma^{-1})dR_{\sigma^{-1}}(e) = dR_{\sigma^{-1}}(\sigma)dL_\sigma(e)\end{aligned} \tag{12}$$

with $J_\sigma(\tau) = \sigma\tau\sigma^{-1}$. From what precedes, a concise way of writing the error-system equation (10) is

$$\dot{\tilde{g}} = X(\tilde{g})(Cu - \mathrm{Ad}^X(\tilde{g}^{-1})v_r) \tag{13}$$

with $C = (I_m \,|\, 0_{m\times(n-m)})'$, I_m the $(m \times m)$ identity matrix, and Ad^X the expression of the Ad operator in the basis X, i.e. the (invertible) matrix-valued function defined by $\mathrm{Ad}(\sigma)X(e)v := X(e)\mathrm{Ad}^X(\sigma)v$. This expression is a generalization of the original system's equation (1) which, with the notation introduced above, writes as

$$\dot{g} = X(g)Cu \tag{14}$$

2.2.3 Linearized equations

Given a control system on $\mathbb{R}^n$

$$\dot{\xi} = f(\xi, u)$$

with $(\xi = 0, u = 0)$ an equilibrium, i.e. such that $f(0,0) = 0$, the linear approximation of this system is

$$\dot{\xi} = A\xi + Bu$$

with $A = \frac{\partial f}{\partial \xi}(0,0)$ and $B = \frac{\partial f}{\partial u}(0,0)$. When this linear approximation is controllable, classical linear control design techniques provide linear feedback control laws $u = K\xi$ which exponentially stabilize $\xi = 0$ for the closed-loop system –the problem reduces to calculating a suitable gain matrix K such that $A + BK$ is Hurwitz stable. Moreover, any of these feedbacks also (locally) exponentially stabilize $\xi = 0$ for the original nonlinear system. This very well-known and rightfully celebrated result illustrates the importance of linear control theory associated with linear approximations which are controllable.

As pointed out above, two issues systematically arise when attempting to apply linear control techniques to nonlinear systems: *i)* the existence of an equilibrium of interest, and *ii)* the controllablity (or at least, the stabilizability) of the linear approximation at this point. Concerning the first one, using the fact that $\mathrm{Ad}(e)$ is the identity operator, Equation (13) tells us that $\tilde{g} = e$ is an equilibrium of the closed-loop system only if v_r belongs to the image of C, i.e. $v_r = Cu_r$ with $u_r \in \mathbb{R}^m$. In view of (14), this just means that $(g_r(t), u_r(t))$ must be one of the system's solutions. It is common to say in this case that the reference trajectory is *feasible*, or *admissible*. We will assume at this point that the reference trajectory is feasible so that the error-system equation can be written as

$$\dot{\tilde{g}} = X(\tilde{g})(C\tilde{u} - (\mathrm{Ad}^X(\tilde{g}^{-1}) - I_n)Cu_r) \tag{15}$$

with $\tilde{u} := u - u_r$. The pair $(\tilde{g}, \tilde{u}) = (e, 0)$ is an equilibrium of this system, and the control objective is to stabilize this point.

Let us now examine the question of controllability of the associated linearized system at this point. First, when a control system evolves on an n-dimensional manifold G, its linearization at an equilibrium point makes sense only after defining coordinates to represent the system's state as a vector in $\mathbb{R}^n$. Local coordinates in the neighborhood of e can be defined in several ways, but the most general methods rely on the *exponential mapping*, $\exp : \mathfrak{g} \longrightarrow G$, which defines a local diffeomorphism from a neighborhood of the origin of $\mathfrak{g}$ to a neighborhood of e. Let us recall that given a v.f. $Y \in \mathfrak{g}$, $\exp(Y)$ denotes the value, at time $t = 1$, of the solution of $\dot{g} = Y(g)$ with initial condition $g(0) = e$. For example, so-called *coordinates of the first kind*, ξ, are defined by the relation $g := \exp(X\xi)$, with X a basis of $\mathfrak{g}$. Let us illustrate this possibility in the case of 3-D chained system which is invariant on the Lie group $\mathbb{R}^3$ endowed with the group operation (9). Note that, since the state manifold is $\mathbb{R}^3$, $g = x$ already defines a system of coordinates. Define the Lie algebra basis as $X = \{X_1, X_2, X_3\}$, with X_1 and X_2 the v.f. of the 3-D chained system and $X_3 = [X_1, X_2] = (0, 0, -1)'$. Then the vector of coordinates ξ of a group's element x is related to the canonical coordinates x_i of x by computing the solution of

$$\begin{cases} \dot{y}_1 = \xi_1 \\ \dot{y}_2 = \xi_2 \\ \dot{y}_3 = \xi_1 y_2 - \xi_3 \end{cases} \quad , \quad y(0) = 0$$

at time $t = 1$ and by setting the result equal to x. This yields

$$x = \exp(X\xi) = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \frac{\xi_1 \xi_2}{2} - \xi_3 \end{pmatrix} \tag{16}$$

One easily verifies that the transformation between the coordinates x and ξ is a global diffeomorphism, so that one or the other can be used indifferently. In the set of coordinates ξ, the group product is defined by

$$\xi\bar{\xi} := \begin{pmatrix} \xi_1 + \bar{\xi}_1 \\ \xi_2 + \bar{\xi}_2 \\ \xi_3 + \bar{\xi}_3 + \frac{1}{2}(\xi_1\bar{\xi}_2 - \bar{\xi}_1\xi_2) \end{pmatrix}$$

and the 3-D chained system writes as

$$\begin{cases} \dot{\xi}_1 = u_1 \\ \dot{\xi}_2 = u_2 \\ \dot{\xi}_3 = \frac{1}{2}(u_2\xi_1 - u_1\xi_2) \end{cases}$$

For the Lie group $\mathbb{R}^4$ endowed with the group operation (9) (i.e. the one associated with the 4-D chained system) and the basis $X = \{X_1, X_2, X_3, X_4\}$, with $X_3 = [X_1, X_2] = (0, 0, -1, 0)'$ and $X_4 = [X_1, X_3] = (0, 0, 0, 1)'$, the following expression of the exp function is obtained:

$$x = \exp(X\xi) = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \frac{\xi_1\xi_2}{2} - \xi_3 \\ \frac{\xi_1^2\xi_2}{6} - \frac{\xi_1\xi_3}{2} + \xi_4 \end{pmatrix} \tag{17}$$

For any system on a Lie group with $\mathbb{R}^n$ as the state manifold one can use either canonical coordinates x or coordinates of the first kind ξ. In what follows, the latter set of coordinates is used due to the general applicability of the relations derived with this representation.

Forthcoming relations involve the adjoint representation ad (recall that $(\text{ad } Y)(Z) = [Y, Z]$). A useful relation between ad and the group's adjoint representation Ad is

$$\frac{d}{dt}_{|t=0} \text{Ad}(\exp(tY))Z(e) = (\text{ad } Y)(Z)(e)$$

In a way similar to the definition of Ad^X, we denote by ad^X the expression of the ad operator in the basis X, i.e. $\forall v_1, v_2 \in \mathbb{R}^n$,

$$X(e)\text{ad}^X(v_1)v_2 = (\text{ad } Xv_1)(Xv_2)(e) = [Xv_1, Xv_2](e)$$

The linearization of (15) at the equilibrium $(\tilde{g}, \tilde{u}) = (e, 0)$, in the coordinates ξ, is (see e.g. [34])

$$\dot{\tilde{\xi}} = -\text{ad}^X(Cu_r)\tilde{\xi} + C\tilde{u} \tag{18}$$

To calculate the state matrix $-\text{ad}^X(Cu_r)$, a useful relation is

$$\text{ad}^X(v) = \big((c_{k1}^j)v | \ldots | (c_{kn}^j)v\big) \tag{19}$$

with (c_{kp}^j) $(p = 1, \ldots, n)$ denoting the matrix whose element at row j and column k is c_{kp}^j, one of the *structure constants* of the original nonlinear system relative to the chosen Lie algebra basis $X = \{X_1, \ldots, X_n\}$. These constants are defined by the relation $[X_k, X_p] = \sum_{j=1}^n X_j c_{kp}^j$. In the case of the n-dimensional chained system, using the fact that $X_{i+1} = [X_1, X_i]$ and that $[X_j, X_k] = 0$ when neither j nor k is equal to 1, one has

$$c_{pq}^r = \begin{cases} 1 & \text{if } p = 1,\ q \neq 1,\ r = q+1 \\ -1 & \text{if } q = 1,\ p \neq 1,\ r = p+1 \\ 0 & \text{otherwise} \end{cases}$$

and, from (19)

$$\mathrm{ad}^X(Cu_r) = \begin{pmatrix} 0 & 0 & \ldots & \ldots & \ldots & 0 \\ 0 & 0 & 0 & \ldots & \ldots & 0 \\ -u_{r,2} & u_{r,1} & 0 & 0 & \ldots & 0 \\ 0 & 0 & u_{r,1} & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & u_{r,1} & 0 \end{pmatrix} \tag{20}$$

Let us mention two important properties of Eq. (18). First, it is completely general for systems on a Lie group. Then, the associated state and control matrix can be computed without determining the coordinates ξ explicitly. This is exploited in [34] to provide a necessary condition for the controllability of System (18) in the case of a constant reference input u_r, by inspection of the control Lie algebra structure only.

3 Control issues

Unless specified otherwise, we assume from now on that the system to be controlled is of the form (1) and is on a Lie group so that all relations derived for these systems apply.

3.1 Exponential stabilization of persistently exciting feasible trajectories

From (18), when u_r is constant, the linearized error system is stabilizable iff the pair $(\mathrm{ad}^X(Cu_r), C)$ is stabilizable. In the case of a chained system, and in view of (20), this condition is equivalent to $(u_{r,1}, u_{r,2}) \neq (0, 0)$ when $n = 3$, and $u_{r,1} \neq 0$ when $n > 3$. These conditions upon u_r may be interpreted as conditions of *persistent excitation* which, if they are satisfied, ensure the stabilizability of the linearized error system and, subsequently, the existence of exponential stabilizers which can be obtained either by applying classical linear control design techniques or via slightly more advanced nonlinear control techniques yielding a larger domain of stability under slightly weaker persistently exciting conditions (for instance, $\forall t: \int_t^{t+T} u_{r,1}(s)^2 ds > \varepsilon$ for some $T, \varepsilon > 0$). However, a systematic shortcoming of these "classical" linear and nonlinear feedback laws is that they fail to asymptotically stabilize fixed points (for which $u_r = 0$). Nor do they usually give satisfactory results when the reference trajectory is not feasible. For instance, the boundedness of the tracking errors may not be ensured.

3.2 Asymptotic stabilization of fixed points

The fact that the linear approximation of the system at a fixed point is not stabilizable does not, by itself, rule out the existence of a pure-state feedback, i.e. a function of $\tilde{x}$, capable of stabilizing this point asymptotically. However, a celebrated topological result by Brockett [5], in the case of differentiable feedbacks, and subsequent extensions dealing with the continuous and discontinuous cases [40, 10], prove that no such feedback can exist for a large class of systems. Basically, this result stipulates that a necessary condition for the origin of a control system $\dot{x} = f(x,u)$ (with $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$) to be asymptotically stabilizable by a feedback $u(x)$ –assuming that $(x,u) = (0,0)$ is an equilibrium of this system– is the local surjectivity of the function f. In other words, the image by f of a neighborhood of $(0,0)$ must contain a neighborhood of zero. In the case of the n-D chained system this condition is clearly not satisfied because the point $x = (0,0,\varepsilon,0,\dots,0)'$ with $\varepsilon \neq 0$ does not belong to the image of the function defined by $f(x,u) = (u_1, u_2, u_1x_2, \dots, u_1x_{n-1})'$. As a matter of fact, this result readily extends to any driftless system whose number of inputs m is smaller than the system's dimension n and whose control vectors $X_1(x_d), \dots, X_m(x_d)$ at the desired point x_d are independent. Nevertheless, it has also been shown, starting with [41], that continuous (and even smooth) *time-varying* feedbacks, i.e. feedbacks which also depend on the exogenous time variable explicitly, can achieve this stabilization objective [9]. For a survey on this type of control, the reader is referred to [33] for instance. For illustration purposes let us just mention the following time-periodic feedback which renders the origin of the 3-D chained system globally asymptotically stable with a uniform exponential rate of convergence [28]:

$$u(x,t) = \begin{pmatrix} -k_1x_1 + \omega\sin(\omega t)\rho(x_2,x_3)^{0.5} \\ -k_2|u_1|\frac{x_2}{\rho(x_2,x_3)^{0.5}} - k_3u_1\frac{x_3}{\rho(x_2,x_3)} \end{pmatrix}$$

with $k_i > 0$ $(i = 1,2,3)$, $\omega \neq 0$, and $\rho(x_2,x_3)$ the real positive root of the polynomial of degree three $P(\alpha,x_2,x_3) = \alpha^3 - \alpha\frac{x_2^2}{k_3} - x_3^2$. Note that this feedback is (by continuity) well defined everywhere and, in particular, at the desired equilibrium $x = 0$. However, it is not Lipschitz continuous, and thus not differentiable, at this point. This "lack" of smoothness is necessary to ensure a fast (exponential) rate of convergence, whatever the initial conditions [26]. However, in the present case, non-smoothness also prevents the control from being robust w.r.t. modeling errors, in the sense that the slightest error in the modeling of the system's v.f. may produce (Lyapunov) instability, and renders its performance extremely sensitive to additive perturbations, as in the case of a very high-gain linear feedback applied to a linear system [23]. To our knowledge, all attempts to achieve fast convergence and robust stability at the same time have failed. For instance, *hybrid* discrete/continuous time feedbacks which are robust to modeling errors upon the system's v.f. have been proposed in [3, 27]. However, they are not robust against small sampling-time period fluctuations.

3.3 Non-existence of "universal" stabilizers

So far, we have seen that asymptotic stabilizers of feasible trajectories can be designed provided that the trajectory under consideration either exhibits "excitation" properties, such as non vanishing generating control inputs, or is a fixed point. The above presentation also points out that different control techniques yielding different control laws have been used, depending on the properties of the reference trajectory. A natural question arising at this point is the existence of "causal" (not depending on the knowledge about future reference inputs) control strategies capable of stabilizing "any" feasible trajectory, whatever the input $u_r(t)$ which generates the trajectory. A conceptually important point is the (probable) non existence of such a control strategy. This result has been proven in [22] for a large class of feedbacks including time-varying ones. It is somewhat unexpected because it goes against common knowledge and expectations associated with the linear case. More precisely, if $x_r(t)$ is a feasible trajectory of the controllable linear system $\dot{x} = Ax + Bu$ (this implies the existence of $u_r(t)$ such that $\dot{x}_r = Ax_r + Bu_r$), then $u(x, x_r, u_r) = K\tilde{x} + u_r$, with $\tilde{x} = x - x_r$ and K such that the matrix $(A + BK)$ is Hurwitz stable, is an asymptotic stabilizer of $\tilde{x} = 0$ for the error system. Such a control law does not exist for chained systems: given a causal feedback control strategy (satisfying a weak set of conditions as specified in [22]) there always exist feasible trajectories which are not asymptotically stabilized by this control.

3.4 The forgotten case: non feasible trajectories

The case of non feasible trajectories is interesting in more than one respect. First, these trajectories cannot, by definition, be asymptotically stabilized (the tracking error cannot converge to zero). However, the property of local controllability of the nonlinear system also tells us that such trajectories can be approximated with arbitrary good precision by feasible ones. Several studies have been devoted to the development of algorithms generating *open-loop* control inputs which solve this problem [21, 43]. On the other hand, the same problem has little been addressed with a feedback control point of view. Several reasons for this lack of interest can be conjectured. One of them is that the general problem of stabilizing non feasible trajectories has never been formulated in a systematic way, even in the case of linear systems. A logical consequence is that feedback control is still largely perceived as a technique to perform asymptotic stabilization with complementary robustness properties. Another one is that the interest of roboticists in a control problem is often correlated to the attention paid by the Automatic Control community to this problem in the first place. The fact that roboticists have not identified the significance of the problem on their own either may also be related to the fact that many mobile robot applications can be addressed by solving simpler output control problems. This conjunction of elements has contributed to maintain the problem of stabilizing non feasible trajectories in the shade.

4 The Transverse Function control approach

The aforementioned difficulties (i.e. unsatisfactory performance of fixed point asymptotic stabilizers, non existence of universal asymptotic stabilizers for feasible trajectories, impossibility of achieving asymptotic stabilization in the case of non feasible trajectories) are the main reasons why we have been advocating for some time that the control design problem for this class of systems should primarily focus on an objective less demanding, and thus more open, than the *asymptotic* stabilization of the origin (i.e. a single point) of some error-system. Such an objective may consist, for instance, in the asymptotic stabilization of an arbitrarily small set containing the origin of the error system, thus leaving the asymptotic stabilization of the origin itself as a particular case and a complementary possibility rather than a systematic requirement. This type of objective (small bounded error) is also more in accordance with what can be achieved in practice with a physical system. For this reason it is common to use the generic denomination of *practical* stabilization when referring to it.

One of the outcomes of the Transverse Function Control approach [30] is to provide feedback controls which ensure uniform practical stabilization of any reference trajectory, whether this trajectory is feasible or not, whether it is persistently exciting or reduced to a fixed point. Moreover, we will see that this type of feedback can also yield asymptotic stabilization in cases when classical control techniques allow for this type of stabilization, i.e. basically when the reference trajectory is persistently exciting. To our knowledge, no other general control approach yielding similar results has been proposed in the literature to date, for the considered class of nonlinear systems. The remainder of the present study is devoted to the application of this approach to chained systems and the adaptation/particularization of the obtained results to unicycle-type and car-like vehicles.

4.1 Basics of the Transverse Function approach

Let:

- G denote the Lie group on which the system's state evolves,
- $X = \{X^1, X^2\}$, with $X^1 = \{X_1, \dots, X_m\}$ and $X^2 = \{X_{m+1}, \dots, X_n\}$, denote a basis of the associated Lie algebra $\mathfrak{g}$,
- $\text{dist}(.,.)$ denote a left-invariant distance on G, i.e. $\forall g_{1,2,3} \in G$, $\text{dist}(g_1 g_2, g_1 g_3) = \text{dist}(g_2, g_3)$,
- f denote a differentiable function from $\mathbb{T}^{n-m}$, the torus of dimension $(n-m)$, to a neighborhood $\mathcal{U} \in G$ of the group's unit element e,
- $\alpha(t) = (\alpha_{m+1}(t), \dots, \alpha_n(t))'$ denote a smooth curve on $\mathbb{T}^{n-m}$.

The decomposition of $\dot{f}$ on the basis X yields the existence of a matrix-valued function A such that, $\forall(\alpha, \dot{\alpha})$

$$\begin{aligned} \dot{f}(\alpha) &= X(f(\alpha))A(\alpha)\dot{\alpha} \\ &= X^1(f(\alpha))A^1(\alpha)\dot{\alpha} + X^2(f(\alpha))A^2(\alpha)\dot{\alpha} \end{aligned} \tag{21}$$

with $A^1(\alpha), A^2(\alpha)$ matrices corresponding to a row decomposition of $A(\alpha)$, i.e.,

$$A(\alpha) = \begin{pmatrix} A^1(\alpha) \\ A^2(\alpha) \end{pmatrix}$$

Define the "modified" tracking error

$$z := \tilde{g} f(\alpha)^{-1} \tag{22}$$

and note that if $f(\alpha)$ is close to e, then z is close to $\tilde{g}$, since $\mathrm{dist}(\tilde{g}, z) = \mathrm{dist}(z^{-1}\tilde{g}, z^{-1}z) = \mathrm{dist}(f(\alpha), e)$. Note also that $z = e$ implies that $\tilde{g} = f(\alpha)$. Therefore, it suffices to have z converge to e in order to have $\tilde{g}$ come close to e. Monitoring the tracking error $\tilde{g}$ via the control of z is the central idea of the Transverse Function approach whose name comes from the specific properties of the function f which make the asymptotic stabilization of $z = e$ a simple control problem. More precisely, by using (13) and Relation (77) in Appendix A, one obtains

$$\dot{z} = X(z)\mathrm{Ad}^X(f(\alpha))(\bar{C}(\alpha)\bar{u} - \mathrm{Ad}^X(\tilde{g}^{-1})v_r) \tag{23}$$

with

$$\begin{aligned} \bar{C}(\alpha) &:= (C \mid -A(\alpha)) \\ &= \begin{pmatrix} I_m & -A^1(\alpha) \\ 0 & -A^2(\alpha) \end{pmatrix} \end{aligned} \tag{24}$$

and $\bar{u}' := (u', \dot{\alpha}') = (u_1, \ldots, u_m, \dot{\alpha}_{m+1}, \ldots, \dot{\alpha}_n)$, which may be seen as an *augmented* n-dimensional control vector composed of the original m control inputs and the $n - m$ time-derivative components of α. Then, if $\bar{C}(\alpha)$ is invertible for any α, the feedback

$$\bar{u} = \bar{C}(\alpha)^{-1} \left(\mathrm{Ad}^X(\tilde{g}^{-1})v_r + \mathrm{Ad}^X(f(\alpha)^{-1})\bar{v} \right) \tag{25}$$

transforms the equation of evolution of z into the system

$$\dot{z} = X(z)\bar{v} \tag{26}$$

Therefore, any asymptotic stabilizer $\bar{v}(z)$ of $z = e$ for this system yields a feedback law $\bar{u}(g, g_r, u_r, \alpha)$ which makes the tracking error $\tilde{g}$ converge to the image set of the function f. The design of such a stabilizer is not difficult because, in view of (26), the variations of z along each of the n possible directions –given by X_i, $i \in \{1, \ldots, n\}$– are directly monitored via an independent control input. For example, in the case of the n-D chained system with the basis X defined earlier, $\bar{v}(z) = (-k_1 z_1, -k_2 z_2, k_3 z_3, \ldots, (-1)^{i-1} k_i z_i, \ldots)'$, with $k_{1,\ldots,n} > 0$, is a global exponential stabilizer of $z = e = 0$.

When $\bar{v}(z)$ is an exponential stabilizer of $z = e$ then, along any solution to the controlled system, $\mathrm{dist}(z(t), e)$ and $|\bar{v}(z(t))|$ converge to zero exponentially. Therefore, in view of (25), when the reference trajectory reduces to a fixed point, i.e. when $u_r = 0$, all components of the extended control $\bar{u}$ also converge to zero exponentially. This in turn implies that the extended state $(\tilde{g}, \alpha)$ converges exponentially to some fixed point $(\tilde{g}^{lim}, \alpha^{lim}) \in G \times \mathbb{T}^{n-m}$, with $\tilde{g}^{lim} = f(\alpha^{lim})$.

4.2 Existence and calculation of transverse functions

In order to apply the control law (25), the matrix $\bar{C}(\alpha)$ must be invertible for every $\alpha \in \mathbb{T}^{n-m}$. From the expression (24) of $\bar{C}$, this property is itself equivalent to the invertibility of $A^2(\alpha)$ for every α. The transverse function theorem given in [30] asserts that the existence of functions f which satisfy this property (of transversality w.r.t. the v.f. $X_1, \dots, X_m$) is equivalent to the satisfaction of the LARC by $X_1, \dots, X_m$, i.e. the controllability of the corresponding driftless system. This theorem also provides a general expression for a family of such functions, the usage of which for the 3-D and 4-D chained systems is detailed next.

In the case of the 3-D chained system, a possible choice is

$$\begin{aligned} f(\alpha) &= \exp(\varepsilon_1 \sin(\alpha) X_1 + \varepsilon_2 \cos(\alpha) X_2) \\ &= \begin{pmatrix} \varepsilon_1 \sin(\alpha) \\ \varepsilon_2 \cos(\alpha) \\ \frac{\varepsilon_1 \varepsilon_2}{4} \sin(2\alpha) \end{pmatrix} \end{aligned} \tag{27}$$

with ε_1 and ε_2 any non-zero real numbers. Note that the second equality in (27) can be deduced from (16) by setting $\xi_1 = \varepsilon_1 \sin(\alpha), \xi_2 = \varepsilon_2 \cos(\alpha)$, and $\xi_3 = 0$. It is simple to check that this function is transversal to the v.f. X_1 and X_2 of the 3-dimensional chained system. Indeed, one has

$$\dot{f}(\alpha) = \begin{pmatrix} \varepsilon_1 \cos(\alpha) \\ -\varepsilon_2 \sin(\alpha) \\ \frac{\varepsilon_1 \varepsilon_2}{2} \cos(2\alpha) \end{pmatrix} \dot{\alpha} = X(f(\alpha)) \begin{pmatrix} \varepsilon_1 \cos(\alpha) \\ -\varepsilon_2 \sin(\alpha) \\ \frac{\varepsilon_1 \varepsilon_2}{2} \end{pmatrix} \dot{\alpha}$$

so that, in this case,

$$A^1(\alpha) = \begin{pmatrix} \varepsilon_1 \cos(\alpha) \\ -\varepsilon_2 \sin(\alpha) \end{pmatrix} \ , \quad A^2(\alpha) = \frac{\varepsilon_1 \varepsilon_2}{2} \tag{28}$$

Note that the Euclidean distance (which is equivalent to a left-invariant distance near the group's unit element) between $f(\alpha)$ and $e = 0$ can be kept as small as desired by choosing $|\varepsilon_1|$ and $|\varepsilon_2|$ small enough.

In the case of the 4-D chained system, a t.f. is defined as the (group) product of two functions: with $\alpha = (\alpha_3, \alpha_4) \in \mathbb{T}^2$,

$$f(\alpha) = f_4(\alpha_4) f_3(\alpha_3)$$

with

$$\begin{aligned} f_3(\alpha_3) &= \exp(\varepsilon_{31} s\alpha_3 X_1 + \varepsilon_{32} c\alpha_3 X_2) \\ f_4(\alpha_4) &= \exp(\varepsilon_{41} s\alpha_4 X_1 + \varepsilon_{42} c\alpha_4 X_3) \end{aligned}$$

and the concise notation $s\alpha = \sin(\alpha)$ and $c\alpha = \cos(\alpha)$. Using (9) and (17), this yields:

$$\begin{aligned} f(\alpha) &= \begin{pmatrix} \varepsilon_{41} s\alpha_4 \\ 0 \\ -\varepsilon_{42} c\alpha_4 \\ -\frac{\varepsilon_{41}\varepsilon_{42}}{4} s2\alpha_4 \end{pmatrix} \begin{pmatrix} \varepsilon_{31} s\alpha_3 \\ \varepsilon_{32} c\alpha_3 \\ \frac{\varepsilon_{31}\varepsilon_{32}}{4} s2\alpha_3 \\ \frac{\varepsilon_{31}^2\varepsilon_{32}}{6} (s\alpha_3)^2 c\alpha_3 \end{pmatrix} \\ &= \begin{pmatrix} \varepsilon_{31} s\alpha_3 + \varepsilon_{41} s\alpha_4 \\ \varepsilon_{32} c\alpha_3 \\ \frac{\varepsilon_{31}\varepsilon_{32}}{4} s2\alpha_3 - \varepsilon_{42} c\alpha_4 \\ \frac{\varepsilon_{31}^2\varepsilon_{32}}{6} (s\alpha_3)^2 c\alpha_3 - \frac{\varepsilon_{41}\varepsilon_{42}}{4} s2\alpha_4 - \varepsilon_{31}\varepsilon_{42} s\alpha_3 c\alpha_4 \end{pmatrix} \end{aligned} \tag{29}$$

We leave to the interested reader the task of verifying that, in this case,

$$\begin{aligned} A^1(\alpha) &= \begin{pmatrix} \varepsilon_{31} c\alpha_3 & \varepsilon_{41} c\alpha_4 \\ -\varepsilon_{32} s\alpha_3 & 0 \end{pmatrix} \\ A^2(\alpha) &= \begin{pmatrix} \frac{\varepsilon_{31}\varepsilon_{32}}{2} & -\varepsilon_{42} s\alpha_4 + \varepsilon_{32}\varepsilon_{41} c\alpha_3 c\alpha_4 \\ -\frac{\varepsilon_{31}^2\varepsilon_{32}}{6} s\alpha_3 & \begin{array}{c} \frac{\varepsilon_{41}\varepsilon_{42}}{2} + \varepsilon_{31}\varepsilon_{42} s\alpha_3 s\alpha_4 \\ -\frac{\varepsilon_{31}\varepsilon_{32}\varepsilon_{41}}{2} s\alpha_3 c\alpha_3 c\alpha_4 \end{array} \end{pmatrix} \end{aligned} \tag{30}$$

and that sufficient conditions for the invertibility of $A^2(\alpha)$ are

$$|\varepsilon_{41}| > \frac{4}{3}|\varepsilon_{31}| > 0 \quad , \quad |\varepsilon_{42}| > \frac{\varepsilon_{32}}{2(\frac{3}{|\varepsilon_{31}|} - \frac{4}{|\varepsilon_{41}|})} > 0 \tag{31}$$

4.3 Transformation of a controllable nonholonomic system into an omnidirectional companion system

It is conceptually useful to view the t.f. control approach as a means to transform an initial controllable (left-invariant) system $\dot{g} = X(g)Cu$ into a *companion system* whose state is

$$\bar{g} := gf^{-1} \tag{32}$$

and whose equation of evolution, obtained for instance by setting $g_r = e$ and $u_r = 0$ in (23), is

$$\dot{\bar{g}} = X(\bar{g})w \tag{33}$$

with $w = \mathrm{Ad}^X(f)\bar{C}\bar{u}$. Since $\dim(w) = \dim(\bar{u})$, and since both matrices $\mathrm{Ad}^X(f)$ and $\bar{C}$ are invertible (provided that f is a transverse function), this equation indicates that the companion state can be directly modified along any direction of the tangent space. Being omnidirectional, the companion system is much more easily controlled than the original system. Moreover, thanks to the associativity of the group product, the modified tracking error $z = \tilde{g}f^{-1}$ may also be viewed as the tracking error $z = g_r^{-1}\bar{g}$ associated with the companion system. The corresponding equation, given by (23), should then be written as follows

$$\dot{z} = X(z)(w - \mathrm{Ad}^X(z^{-1})v_r)$$

4.4 Transverse function shaping for the asymptotic stabilization of feasible trajectories

Throughout this section it is assumed that the reference trajectory g_r is feasible, i.e. $v_r = Cu_r$. When $\bar{v}(z)$ is an exponential stabilizer of $z = e$ for System (26), $\tilde{g}(t)$ converges to the set $f(\mathbb{T}^{n-m})$ contained in a neighborhood of e. This convergence property is clearly a desirable feature, but in many cases one would like to guarantee the convergence of $\tilde{g}(t)$ to e. This convergence is possible in the first place only if there exists $\alpha \in \mathbb{T}^{n-m}$ such that $f(\alpha) = e$. For instance, one easily verifies that the latter equality cannot be satisfied in the case of the transverse functions (27) and (29) proposed previously. This in turn raises the question of the existence of transverse functions which admit e as an image point, and also, more generally, of criteria for the selection of an adequate function among all possibilities (since there are clearly infinitely many transverse functions). In the case of the functions (27) and (29), another matter related to this issue is the choice of the parameters ε_i $(i = 1, 2)$ and ε_{ij} $(i = 3, 4\ , j = 1, 2)$, knowing that large values for these parameters increase the maximal distance between $f(\alpha)$ and e, whereas small values render $\bar{C}(\alpha)$ close to singular, yielding large control gains and problems commonly associated with such gains. Note also that nothing forbids the use of time-varying parameters, provided that the property of transversality is preserved all the time. The design of transverse functions is still a largely open research domain and, in what follows, the present paper only explores the connection existing between the choice of a transverse function and the possibility of achieving asymptotic stabilization in the case of persistently exciting feasible trajectories, as a complement to the practical stabilization objective which, as explained above, is achieved whatever the chosen t.f. and whatever the reference trajectory.

We define a *generalized transverse function* as a smooth function $\bar{f} : (\alpha, \alpha_r) \in \mathbb{T}^{n-m} \times \mathbb{T}^{n-m} \mapsto \bar{f}(\alpha, \alpha_r) \in G$ such that

1. $\bar{f}$ is *transversal to* X^1 *w.r.t.* α, i.e. the matrix $A^2(\alpha, \alpha_r)$ defined by the relation $\dot{\bar{f}}(\alpha, \alpha_r) = X^1(\bar{f}(\alpha, \alpha_r))A^1(\alpha, \alpha_r)\dot{\alpha} + X^2(\bar{f}(\alpha, \alpha_r))A^2(\alpha, \alpha_r)\dot{\alpha}$, with α an arbitrary smooth curve and α_r constant, is invertible $\forall(\alpha, \alpha_r)$,
2. $\bar{f}(\alpha_r, \alpha_r) = e$, $\forall \alpha_r \in \mathbb{T}^{n-m}$.

In other words, a generalized transverse function is a function which, besides the variables needed for the satisfaction of the transversality property, depends on as many additional variables which, when equal to the first variables, "shrink" the image of this function to the unit element e. This feature may be thought of as a *phase synchronisation* property.

Given any transverse function f, it is not difficult to obtain a generalized transverse function. An example is the function $\bar{f}$ defined by

$$\bar{f}(\alpha, \alpha_r) := f(\alpha_r)^{-1} f(\alpha) \tag{34}$$

The conservation of the transversality property w.r.t. α comes from that, for any smooth curve $\alpha(.)$ and any constant α_r,

$$\begin{aligned}\dot{\bar{f}}(\alpha,\alpha_r) &= dL_{f(\alpha_r)^{-1}}(f(\alpha))\dot{f}(\alpha)\\ &= dL_{f(\alpha_r)^{-1}}(f(\alpha))X(f(\alpha))A(\alpha)\dot{\alpha}\\ &= X(\bar{f}(\alpha,\alpha_r))A(\alpha)\dot{\alpha}\end{aligned}$$

whereas the fact that $\bar{f}(\alpha_r,\alpha_r)=e$ is just a consequence of the definition of the inverse of an element of G. In [31], other generalized transverse functions are proposed to achieve the asymptotic stabilization of fixed equilibrium points, for the n-D chained system. When using such a function in the control law, the convergence of $\tilde{g}$ to e is then obtained when α converges to α_r. Are there "good" values of α_r for which this latter convergence can take place when tracking a feasible trajectory? This question is treated next.

Let us assume that the feedback control (25) is applied to the system with the transverse function (34) and with $\bar{v}(z)$ an exponential stabilizer of e for the system (26). Then $z=\tilde{g}\bar{f}(\alpha)^{-1}$ converges exponentially to e. The extinction of the transient phase of convergence of z to e, characterized by the equality $\tilde{g}=\bar{f}(\alpha)$, leaves us with a differential system in the variables α and α_r, the so-called *zero dynamics*. If $\alpha-\alpha_r=0$ is an asymptotically stable equilibrium of this system, then one can prove that $(\tilde{g},\alpha)=(e,\alpha_r)$ is asymptotically stable for the controlled system. Let us thus have a closer look at the system's zero dynamics.

Proposition 1 *Assume that the reference trajectory is feasible. Then, on the zero dynamics $z=e$ the variable $\bar{\alpha}:=A(\alpha_r)(\alpha-\alpha_r)$ satisfies the equation*

$$P\dot{\bar{\alpha}} = -P\mathrm{ad}^X(Cu_r)\bar{\alpha}+o(\bar{\alpha}) \tag{35}$$

with $P=(0_{m\times m}|I_{n-m})$ (i.e. such that $PC=0$) and $o(.)$ denoting a function such that $\lim_{|y|\to 0}\frac{|o(y)|}{|y|}=0$, uniformly w.r.t. $\dot{\alpha}_r$ and u_r in compacts sets.

The proof is given in Appendix B.

Eq. (35) is much related to the linearized equation (18) of the error system. Indeed, by pre-multiplying both sides of (18) by the matrix P, one obtains $P\dot{\tilde{\xi}}=-P\mathrm{ad}^X(Cu_r)\tilde{\xi}$.

Since $\bar{\alpha}$ is a n-dimensional vector and $\alpha-\alpha_r$ is only $(n-m)$-dimensional, the components of $\bar{\alpha}$ are not independent. Let $y:=P\bar{\alpha}=A^2(\alpha_r)(\alpha-\alpha_r)$. By the property of transversality $y=0$ if and only if $\alpha=\alpha_r$. Then Eq. (35) can be rewritten as

$$\begin{aligned}\dot{y} &= -P\mathrm{ad}^X(Cu_r)A(\alpha_r)(PA(\alpha_r))^{-1}y+o(|y|)\\ &= -P\mathrm{ad}^X(Cu_r)\begin{pmatrix}A^1(\alpha_r){A^2(\alpha_r)}^{-1}\\ I_{n-m}\end{pmatrix}y+o(|y|)\end{aligned} \tag{36}$$

Therefore, the linear approximation of the zero dynamics at the equilibrium $y=0$ is

$$\dot{y} = -\mathrm{ad}^X_{21}(Cu_r)A^1(\alpha_r){A^2(\alpha_r)}^{-1}y-\mathrm{ad}^X_{22}(Cu_r)y \tag{37}$$

where we have used the decomposition of ad^X into four blocks ad^X_{ij} $(i,j \in \{1,2\})$ of adequate dimensions. From the above equation, this equilibrium is (exponentially) stable if and only if the feedback control $v = A^1(\alpha_r)A^2(\alpha_r)^{-1}y$ (exponentially) stabilizes the origin of the linear system

$$\dot{y} = -\mathrm{ad}^X_{22}(Cu_r)y - \mathrm{ad}^X_{21}(Cu_r)v \tag{38}$$

Note that the linearized error system (18) can also be written as

$$\left\{ \begin{array}{rcl} \dot{\xi}_1 & = & w \\ \dot{\xi}_2 & = & -\mathrm{ad}^X_{22}(Cu_r)\xi_2 - \mathrm{ad}^X_{21}(Cu_r)\xi_1 \end{array} \right.$$

with $w = \tilde{u} - (ad^X_{11}(Cu_r)\xi_1 + ad^X_{21}(Cu_r)\xi_2)$. This is just a dynamic extension of (38) with integrators added at the input control level. To simplify the exposition of forthcoming results and put the focus on ideas, rather than on technical details, *we will assume from now on that u_r is constant*. However, most of these results find extensions when u_r is time-varying. Under this assumption, the above linear systems do not depend on time and the concept of (exponential) stabilizability for such systems is classical. It is also well known (and simple to verify) that the stabilizability of the latter system is equivalent to the stabilizability of (38). As a consequence, one can assert that

Lemma 1 *When u_r is constant, a necessary condition for the exponential stability of $\alpha - \alpha_r = 0$ and, subsequently, of $\tilde{x} = e$, is the stabilizability of the linearized error-system (18).*

For the 3-D (resp. 4-D) chained system, we have already seen that this condition is equivalent to $(u_{r,1}, u_{r,2}) \neq (0,0)$ (resp. $u_{r,1} \neq 0$).

Lemma 1 is conceptually interesting because it basically indicates that, as for the problem of asymptotic stabilization of feasible trajectories, the TF control approach applied with a "basic" transverse function (by opposition to a generalized one) cannot perform better than classical control methods. But it may perform as well (in the sense of achieving exponential stabilization) and, to this aim, the linear feedback $v = A^1(\alpha_r)A^2(\alpha_r)^{-1}y$ must asymptotically stabilize the origin of (38). For the 3-D (resp. 4-D) chained system and the t.f. $\bar{f}(\alpha) = f(\alpha_r)^{-1}f(\alpha)$ with f given by (27) (resp. (29)), we show below that the satisfaction of this condition itself depends on the choice of α_r in relation with the signs of the transverse function parameters ε_i (resp. ε_{ij}), $i,j \in \{1,2\}$.

4.4.1 3-D chained system

From (20), the system (38) specializes to $\dot{y} = \begin{pmatrix} u_{r,2} & -u_{r,1} \end{pmatrix} v$ and, from (28),

$$A^1(\alpha)A^2(\alpha)^{-1} = \begin{pmatrix} \frac{2\cos(\alpha)}{\varepsilon_2} \\ -\frac{2\sin(\alpha)}{\varepsilon_1} \end{pmatrix}$$

Therefore, the application of the feedback $v = A^1(\alpha_r)A^2(\alpha_r)^{-1}y$ to this system yields the closed-loop system

$$\dot{y} = 2(\frac{u_{r,1}\sin(\alpha_r)}{\varepsilon_1} + \frac{u_{r,2}\cos(\alpha_r)}{\varepsilon_2})y$$

whose origin is asymptotically stable iff $(\frac{u_{r,1}\sin(\alpha_r)}{\varepsilon_1} + \frac{u_{r,2}\cos(\alpha_r)}{\varepsilon_2}) < 0$. This yields the following result, with $sign(.)$ denoting the classical sign function and $sign(0)$ chosen equal to either 1 or -1 :

Lemma 2 *For the 3-D chained system, consider i) a feasible reference trajectory generated by a constant control input $u_r \neq 0$, and ii) the modified tracking error $z = \tilde{x}\bar{f}(\alpha)^{-1}$ with $\bar{f}(\alpha)$ given by (27), (34) and the following complementary specifications*

$$\begin{cases} \varepsilon_i = |\varepsilon_i| sign(u_{r,i}) \ , \quad i = 1,2 \\ -\pi < \alpha_r < -\frac{\pi}{2} \end{cases} \tag{39}$$

Then the control (25) with $\alpha(0) = \alpha_r$ and $\bar{v}(z)$ denoting an exponential stabilizer of $z = 0$ for the system $\dot{z} = X(z)\bar{v}$ –take, e.g., $\bar{v}(z) = (-k_1 z_1, -k_2 z_2, k_3 z_3)'$ with $k_{1,2,3} > 0$– (locally) exponentially stabilizes $\tilde{x} = 0$ for the closed-loop system.

Therefore, it suffices to choose α_r in the quadrant $(-\pi, -\frac{\pi}{2})$, when ε_i is chosen of the same sign as $u_{r,i}$ $(i = 1,2)$, to ensure the exponential stability of zero for the tracking error. Among all possible values in this range, the middle value $\alpha_r = -\frac{3\pi}{4}$ seems particularly appropriate. Note also that the (exponential) rate of convergence on the zero dynamics is proportional to $|u_{r,i}|$ and to the inverse of $|\varepsilon_i|$.

4.4.2 4-D chained system

In this case, the system (38) specializes to

$$\dot{y} = \begin{pmatrix} 0 & 0 \\ -u_{r,1} & 0 \end{pmatrix} y + \begin{pmatrix} u_{r,2} & -u_{r,1} \\ 0 & 0 \end{pmatrix} v$$

In view of (30), when setting $\alpha_r = (-\frac{\pi}{2}, -\frac{\pi}{2})'$ one obtains after elementary calculations

$$A^1(\alpha_r)A^2(\alpha_r)^{-1} = \frac{4}{\varepsilon_{31}(\varepsilon_{41} + \frac{4}{3}\varepsilon_{31})} \begin{pmatrix} 0 & 0 \\ (\varepsilon_{31} + \frac{\varepsilon_{41}}{2}) & -1 \end{pmatrix}$$

Therefore, the feedback $v = A^1(\alpha_r)A^2(\alpha_r)^{-1}y$ with $\alpha_r = (-\frac{\pi}{2}, -\frac{\pi}{2})$ yields the closed-loop system

$$\dot{y} = u_{r,1} \begin{pmatrix} -\frac{2(2\varepsilon_{31}+\varepsilon_{41})}{\varepsilon_{31}(\frac{4}{3}\varepsilon_{31}+\varepsilon_{41})} & \frac{4}{\varepsilon_{31}(\frac{4}{3}\varepsilon_{31}+\varepsilon_{41})} \\ -1 & 0 \end{pmatrix} y$$

whose origin is exponentially stable when ε_{31} and ε_{41} have the same sign as $u_{r,1}$. One deduces the following result :

Lemma 3 *For the 4-D chained system, consider i) a feasible reference trajectory generated by a constant control input u_r such that $u_{r,1} \neq 0$, and ii) the modified tracking error $z = \tilde{x}\bar{f}(\alpha)^{-1}$ with $\bar{f}(\alpha)$ given by (29), (34) and the following complementary specifications*

$$\begin{cases} \varepsilon_{i1} = |\varepsilon_{i1}| sign(u_{r,1}) \ , \quad i = 3,4 \\ \alpha_r = (-\frac{\pi}{2}, -\frac{\pi}{2}) \end{cases} \tag{40}$$

Then the control (25) with $\alpha(0) = \alpha_r$ and $\bar{v}(z)$ denoting an exponential stabilizer of $z = e$ for the system $\dot{z} = X(z)\bar{v}$ –take, e.g., $\bar{v}(z) = (-k_1 z_2, -k_2 z_2, k_3 z_3, -k_4 z_4)'$ with $k_{1,2,3,4} > 0$– (locally) exponentially stabilizes $\tilde{x} = 0$ for the closed-loop system.

As in the case of the 3-D chained system, other values of α_r also ensure the convergence of $\tilde{x}$ to zero. The important point here was to show that, by a proper choice of the transverse function used in the control law, perfect tracking of feasible reference trajectories can be achieved asymptotically, with the complementary insurance of global practical stabilization when the reference trajectory is not feasible, or when it is feasible but the linear approximation of the error system is not stabilizable. When u_r is not constant, it is possible to complement the above lemmas with complementary conditions the satisfaction of which ensures asymptotic stabilization. For instance, in the case of the 4-D chained system, it is sufficient that $|u_{r,1}(t)|$ remains larger than some positive number. Note also that the conditions (39) or (40) upon the parameters ε_i or ε_{ij} entering the expression of the transverse function render this function dependent upon the signs of the components of u_r and that they introduce discontinuities at the time-instants when one of these signs changes. Since all previously stated stability results rely on the differentiability of the transverse function, they do not apply *stricto sensu* in this case. However, it is not difficult to show that, provided that $u_r(t)$ remains either left or right-differentiable $\forall$ t, then *i)* the control expression remains well-defined $\forall(x, t)$, *ii)* practical stabilization of $\tilde{x} = e$ remains unconditionally granted, whatever the reference trajectory, and *iii)* $\mathrm{dist}(\tilde{x}, e)$ is still ultimately bounded by a value which can be rendered as small as desired by choosing the absolute values of the transverse function parameters small enough. The proofs of the last two points much rely on the fact that the distance between two modified tracking errors $z_1 = \tilde{x} f_1(\alpha_1)^{-1}$ and $z_2 = \tilde{x} f_2(\alpha_2)^{-1}$ associated with two different transverse functions, being equal to the distance between these two functions, is upper-bounded by a value depending only on the size of the parameters entering the expressions of the functions (but not on their signs).

5 Application to unicycle and car-like vehicles

In this section we apply the approach exposed above to the familiar examples of unicycle-type and car-like vehicles, chosen for their importance in robotic applications and also because they allow to illustrate several aspects of the approach. Besides the application concern, this section also contains methodological developments of practical importance. The first one is about the possibility of exploiting control degrees of freedom in order to limit the vehicle's velocity inputs and the number of transient maneuvers associated with the reduction of initially large tracking errors. The proposed method is reminiscent of control gain determination techniques for linear systems, and it relies on the constrained minimization, at each time-instant, of an adequate cost-function. A second original development concerns the application to the car-like example. The framework of invariant systems on a Lie group has been used in the previous sections. One of the reasons for choosing this framework is that the kinematics of unicycle-type vehicles form a control system having this property

–with the Lie group SE(2) in the present case. On the other hand, as explained in the first part, the kinematics of car-like vehicles form a control system which is not invariant on a Lie group. As a consequence, the TF approach does not apply directly in this case. A way to circumvent this difficulty consists in transforming the car's kinematic equations into the 4-D chained system. One of the drawbacks of this solution is that the involved transformation is only locally defined. The solution here chosen does not rely on this transformation and yields the largest possible domain of stability. It is also interesting conceptually because it illustrates that the assumption of invariance w.r.t. a Lie group is not an absolute pre-requisite for the application of the TF approach. For instance, the more general case of the standard N-trailer system (see e.g. [7, 42]) can be addressed via an extension of this solution. This case is not developed here for the sake of concentrating on the approach foundations, but it will be the subject of a future publication.

This section is organized as follows. Subsection 5.1 is dedicated to the application of the TF control approach to the unicycle example, whereas Subsection 5.2 describes how to adapt the approach to the car-example. In both cases, several simulation results illustrate various aspects of the controller's performance in relation to the reference trajectory properties: unconditional ultimate uniform boundedness of the tracking errors, maneuvers management during initial transient phases, convergence of the tracking errors to zero when the reference trajectory is admissible and persistently exciting. For the sake of legilibility, the figures of the simulation results have been placed at the end of the report.

5.1 Control of a unicycle-type vehicle

As explained in Section 2, the kinematic equations of a unicycle-type vehicle given by (2) define a (left-invariant) system on the Lie group $SE(2)$, with the group operation defined by (5), the unit element $e = (0,0,0)'$, and the inverse defined by (6). With $g = (x, y, \theta)'$ and $g_r = (x_r, y_r, \theta_r)'$ the tracking error $\tilde{g}$ is given by

$$\tilde{g} := g_r^{-1} g = \begin{pmatrix} R(-\theta_r) \begin{pmatrix} x - x_r \\ y - y_r \end{pmatrix} \\ \theta - \theta_r \end{pmatrix}$$

Note that the components of this tracking error vector are nothing else than the coordinates of the unicycle's situation with respect to the reference frame associated with g_r, expressed in the basis of this frame. One also deduces from the expression (5) of the group law that

$$dL_g(\bar{g}) = \begin{pmatrix} R(\theta) & 0 \\ 0 & 1 \end{pmatrix} \;,\; dR_{\bar{g}}(g) = \begin{pmatrix} I_2 & R(\theta) S \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \\ 0 & 1 \end{pmatrix}$$

with

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

the skew symmetric matrix such that $\dot{R}(\theta) = \dot{\theta}R(\theta)S = \dot{\theta}SR(\theta)$ along any smooth curve $\theta(.)$. Using the above relations, the "perturbation" term P in the error system equation (10) is defined by

$$P(\tilde{g}, g_r, \dot{g}_r) = -\begin{pmatrix} R(-\theta_r) & S\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} \\ 0 & 1 \end{pmatrix} \dot{g}_r$$

and, from (12),

$$\mathrm{Ad}(g) = \begin{pmatrix} R(\theta) & -S\begin{pmatrix}x\\ y\end{pmatrix} \\ 0 & 1 \end{pmatrix}$$

From now on, $X = \{X_1, X_2, X_3 = [X_1, X_2]\}$ is the Lie algebra basis that we choose. Then, the matrix-valued function $\mathrm{Ad}^X(.)$ in (13) is defined by

$$\mathrm{Ad}^X(g) = X(e)^{-1} \begin{pmatrix} R(\theta) & -S\begin{pmatrix}x\\ y\end{pmatrix} \\ 0 & 1 \end{pmatrix} X(e) \tag{41}$$

with

$$X(g) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ \sin\theta & 0 & -\cos\theta \\ 0 & 1 & 0 \end{pmatrix} \tag{42}$$

5.1.1 Transverse functions

There are many ways to derive transverse functions. One of them consists in using the general expression given in [30, Th. 1], as we did before for the 3-D and 4-D chained systems (relations (27) and (29) respectively). In the case of the kinematic model (2), another option consists in using the close kinship between this system and the 3-D chained system. Indeed, by setting

$$\begin{cases} \bar{x}_1 = x \\ \bar{x}_2 = \tan\theta \\ \bar{x}_3 = y \end{cases} \quad \text{and} \quad \begin{cases} v_1 = u_1 \cos\theta \\ v_2 = \frac{u_2}{(\cos\theta)^2} \end{cases}$$

System (2) is transformed into the 3-D chained system with state $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)'$ and input vector $v = (v_1, v_2)'$. This transformation involves both a change of state coordinates and a change of control inputs, and it is well-defined provided that $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Such a transformation is not unique. In fact, other transformations are more global in the sense that they are defined for all angles $\theta \neq \pm\pi$. But, this is not important here. Let ϕ denote the local diffeomorphism which relates $\bar{x}$ to g, i.e. such that $g = \phi(\bar{x}) = (\bar{x}_1, \bar{x}_3, \arctan(\bar{x}_2))'$. Then one can show that the function f defined by $f(\alpha) = \phi(\bar{f}^c(\alpha))$ is transversal to the v.f. X_1 and X_2 of System (2) provided that $\bar{f}^c$ is transversal to the v.f. of the 3-D chained system. For instance, one can take $\bar{f}^c(\alpha) := f^c(\alpha_r)^{-1} f^c(\alpha)$ with f^c the basic transverse function given by (27) associated with the 3-D chained system [2]. This yields:

$$\bar{f}^c(\alpha) = \begin{pmatrix} \varepsilon_1(s\alpha - s\alpha_r) \\ \varepsilon_2(c\alpha - c\alpha_r) \\ \frac{\varepsilon_1\varepsilon_2}{4}(s2\alpha + s2\alpha_r - 4s\alpha c\alpha_r) \end{pmatrix}$$

and

$$f(\alpha) = \begin{pmatrix} \varepsilon_1(s\alpha - s\alpha_r) \\ \frac{\varepsilon_1\varepsilon_2}{4}(s2\alpha + s2\alpha_r - 4s\alpha c\alpha_r) \\ \arctan(\varepsilon_2(c\alpha - c\alpha_r)) \end{pmatrix} \tag{43}$$

Differentiation w.r.t. α gives:

$$\frac{\partial f}{\partial \alpha}(\alpha) = \begin{pmatrix} \varepsilon_1 c\alpha \\ \varepsilon_1\varepsilon_2(c\alpha(c\alpha - c\alpha_r) - \frac{1}{2}) \\ -\dfrac{\varepsilon_2 s\alpha}{1 + \varepsilon_2^2(c\alpha - c\alpha_r)^2} \end{pmatrix} = X(f(\alpha))A(\alpha)$$

with

$$A(\alpha) = \begin{pmatrix} \cos(f_3(\alpha))(\varepsilon_1 c\alpha + \varepsilon_1\varepsilon_2^2(c\alpha(c\alpha - c\alpha_r)^2 - \frac{c\alpha - c\alpha_r}{2})) \\ -\dfrac{\varepsilon_2 s\alpha}{1 + \varepsilon_2^2(c\alpha - c\alpha_r)^2} \\ \frac{\varepsilon_1\varepsilon_2}{2}\cos(f_3(\alpha)) \end{pmatrix} \tag{44}$$

5.1.2 Control

To calculate the feedback control $\bar{u} = (u_1, u_2, \dot{\alpha})'$ defined by (25), there remains to determine *i)* the transverse function parameters ε_1, ε_2, and α_r, and *ii)* an asymptotic stabilizer $\bar{v}(z)$ of the origin of the system $\dot{z} = X(z)\bar{v}$, with $z := \tilde{g}f(\alpha)^{-1}$. Concerning the first issue, the transposition of the study performed for the 3-D chained system suggests to choose the transverse function parameters according to (39) in order to obtain a control which stabilizes admissible trajectories asymptotically. As for the second issue, a possibility consists in linearizing the closed-loop system (w.r.t. the chosen coordinates) by taking

$$\bar{v}(z) = X(z)^{-1}Kz \tag{45}$$

with K denoting a Hurwitz stable matrix. Indeed, this choice yields the linear closed-loop system $\dot{z} = Kz$ whose origin is exponentially stable. Another possibility, proposed in [2], arises from the concern of limiting the control energy during transient phases corresponding to the convergence of z to e. A way to address this issue consists in rewriting the error system's equation (23) as

$$\dot{z} = H(z, \alpha)\bar{\bar{u}}$$

with

$$H(z, \alpha) := X(z)Ad^X(f(\alpha))\bar{C}(\alpha)$$

[2] The group product here involved is the one associated with chained systems, i.e. the one defined by (9).

and

$$\bar{\bar{u}} := \bar{u} - \bar{C}(\alpha)^{-1}\mathrm{Ad}^X(\tilde{g}^{-1})v_r$$

and in determining the control $\bar{\bar{u}}$ which minimizes the cost function $\bar{\bar{u}}'W_1\bar{\bar{u}}$ under the constraint $z'H(z,\alpha)\bar{\bar{u}} + z'W_2z = 0$, with W_1 and W_2 denoting two symmetric positive definite (s.p.d.) matrices. The underlying idea is to select W_1 in order to penalize the physical entries of the control, i.e. the velocities u_1 and u_2, more than the virtual control input $\dot{\alpha}$. For instance, the fact that $\bar{\bar{u}} = \bar{u} = (u', \dot{\alpha})'$ when $v_r = 0$ suggests to choose W_1 diagonal with the first two elements on the diagonal significantly larger than the third one. As for the enforcement of the constraint equality, it yields the closed-loop equation $\frac{d}{dt}|z|^2 = -2z'W_2z$ and thus the exponential stabilization of $z = 0$. The solution to this simple constrained minimization problem is:

$$\bar{\bar{u}} = -\frac{z'W_2z}{z'HW_1^{-1}H'z}W_1^{-1}H'z \tag{46}$$

One easily verifies that this is the same as taking:

$$\bar{v}(z) = -\frac{z'W_2z}{z'HW_1^{-1}H'z}\mathrm{Ad}^X(f(\alpha))\bar{C}(\alpha)W_1^{-1}H'z \tag{47}$$

Intuitively, lateral motion of the vehicle can be performed via the execution of either frequent maneuvers involving large and rapidly changing velocity values or less frequent maneuvers involving smaller velocities. Therefore, by penalizing the size of these velocities one can expect to reduce the number of maneuvers during the transient phase of convergence of z to zero. This has been confirmed by many simulations.

5.1.3 Simulation results

For these simulations, the length and width of the unicycle represented on the figures are equal to 2 (meters). A single reference trajectory presenting different properties at different times is used. The time history of the associated reference frame velocity v_r is summarized in the following table.

$t \in (s)$	$v_r = (m/s, rad/s, m/s)'$	properties
$[0,5)$	$(0,0,0)'$	ad,npe
$[5,10)$	$(1,0,0)'$	ad,pe
$[10,20)$	$(-1,0,0)'$	ad,pe
$[20,25)$	$(1,0.314,0)'$	ad,pe
$[25,30)$	$(-1,-2\sin(2t),0)'$	ad,pe
$[30,35)$	$(0,0,-1)'$	nad
$[35,40)$	$(0,0,0)'$	ad,npe
$[40,45)$	$(2,-0.5\sin(3t),0.5)'$	nad
$[45,50)$	$(0,0,0)'$	ad,npe

In this table, the abbreviations used to describe the properties of each part of the reference trajectory are:

- *ad* and *nad* for admissible and non-admissible respectively, according to whether $v_{r,3}$ is or is not equal to zero;
- *pe* and *npe* for persistently exciting and non-persistently exciting respectively, according to whether $(v_{r,1}, v_{r,2})$ is or is not equal to zero.

Figure 1 shows the time evolution of the tracking position error $(\tilde{x}, \tilde{y})$ and tracking orientation error $\tilde{\theta}$ respectively. The vertical dotted lines separate the time-periods associated with the different parts and properties of the reference trajectory. From these figures, one can observe *i)* the uniform boundedness of the tracking errors whatever the properties of the reference trajectory, *ii)* the convergence of the tracking errors to zero when the reference trajectory is admissible and persistently exciting, *iii)* the automatic (resp. non-systematic) production of maneuvers when the reference trajectory is "strongly" (resp. "weakly") non-admissible.

Figure 2 shows the time-evolution of the three components of the modified tracking error z. One can observe that, besides the initial transient phase of convergence of z to zero, this error is also different from zero during the time-interval $[40s, 45s)$ (and a few seconds after corresponding to a final transient phase). This is due to periodic discontinuities of the transverse function which themselves result from the combination of two facts during this time-interval: *i)* α is different from α_r and $f(\alpha)$ is different from zero, because the reference trajectory is not-admissible, and *ii)* the sign of the reference input $v_{r,2}(t)$, and thus the sign of the transverse function parameter ε_2, are modified periodically. Nevertheless, $|z(t)|$ could still be maintained as small as desired by choosing $|\varepsilon_1|$ and $|\varepsilon_2|$ small enough, knowing that small values yielding small tracking errors also inevitably produce maneuvers when the reference trajectory is not admissible. The other figures are attempts to visualize the vehicle's motion in the plane during different phases of the reference trajectory.

Except for Figure 4, the feedback control (25) with $\bar{v}$ defined by (47), which includes a monitoring of the transient phase before the convergence of z to zero, has been used. The parameters chosen for this control are $W_1 = \text{diag}\{1, 1, 0.01\}$, $W_2 = \text{diag}\{1, 1, 1\}$. Figure 4 shows the stabilization of a fixed reference situation as obtained when applying the control (25) with $\bar{v}$ defined by (45) and $K = -\text{diag}\{1, 1, 1\}$ (i.e. this solution does not incorporate the transient phase monitoring). Comparison with Figure 3 illustrates the effect of this monitoring on the reduction of maneuvers. The parameters of the transverse function used in the control are $\varepsilon_1 = 0.8$, $\varepsilon_2 = 0.5$.

5.2 Control of a car-like vehicle

5.2.1 Kinematic model

The kinematic equations of a car-like vehicle, given by (4), do not define a left-invariant system on a Lie group. Nevertheless, we show below that, modulo minor adaptations, the control approach presented in Section 4 also applies to car-like vehicles.

To simplify the notation, let us rewrite the kinematic model (4) of a car-like vehicle as

$$\left\{\begin{array}{lcl}\dot{x} & = & u_1\cos\theta \\ \dot{y} & = & u_1\sin\theta \\ \dot{\theta} & = & u_1\eta \\ \dot{\eta} & = & u_\eta\end{array}\right. \tag{48}$$

with $\eta := (\tan\varphi)/L$ and $u_\eta := (1+(\tan\varphi)^2)/L$. This system can be rewritten as

$$\left\{\begin{array}{lcl}\dot{g} & = & X(g)C(\eta)u_1 \\ \dot{\eta} & = & u_\eta\end{array}\right. \tag{49}$$

with $g = (x, y, \theta)'$ and $X(g)$ (given by (42)), defined as for unicycle-type vehicles, and $C(\eta) = (1, \eta, 0)'$. Note that, if C was a constant vector, the above system would be left-invariant on $G = SE(2)\times\mathbb{R}$, with the group law inherited from the group law of $SE(2)$ and the addition on $\mathbb{R}$, i.e.

$$\begin{pmatrix} g_1 \\ \eta_1\end{pmatrix}\begin{pmatrix} g_2 \\ \eta_2\end{pmatrix} = \begin{pmatrix} g_1g_2 \\ \eta_1+\eta_2\end{pmatrix} \tag{50}$$

Let us now consider a reference trajectory $(g_r(t), \eta_r(t))$ for this system, and define the tracking error as $(\tilde{g}, \tilde{\eta}) := (g_r^{-1}g, \eta - \eta_r)$. This corresponds to the group product of the inverse of (g_r, η_r) by (g, η), for the group law (50). One deduces from this definition that (compare with (13)):

$$\left\{\begin{array}{lcl}\dot{\tilde{g}} & = & X(\tilde{g})(C(\eta)u_1 - \mathrm{Ad}^X(\tilde{g}^{-1})v_r) \\ \dot{\tilde{\eta}} & = & \tilde{u}_\eta := u_\eta - \dot{\eta}_r\end{array}\right. \tag{51}$$

with Ad^X defined by (41) and $\dot{g}_r = X(g_r)v_r$. Following the transverse function approach, let us consider a function $f = (f_g, f_\eta) \in SE(2)\times\mathbb{R}$, with the objective of stabilizing to zero the distance between the tracking error $(\tilde{g}, \tilde{\eta})$ and f. For reasons that will become clear later on we consider a function f which depends on both an element $\alpha\in\mathbb{T}^2$ and the independent time-variable t, i.e. $f(\alpha, t)$. Define the "modified" tracking error

$$z := \begin{pmatrix} z_g \\ z_\eta\end{pmatrix} := \begin{pmatrix} \tilde{g}f_g^{-1} \\ \tilde{\eta} - f_\eta\end{pmatrix}$$

It follows from relation (77) in Appendix A that

$$\left\{\begin{array}{l}\dot{z}_g = X(z_g)\mathrm{Ad}^X(f_g(\alpha,t))\big(C(\eta)u_1 - A_\alpha(\alpha,t)\dot{\alpha} \\ \qquad\qquad\qquad\qquad -A_t(\alpha,t) - \mathrm{Ad}^X(\tilde{g}^{-1})v_r\big) \\ \dot{z}_\eta = \tilde{u}_\eta - \dot{f}_\eta\end{array}\right. \tag{52}$$

with A_α and A_t defined by the relation $\dot{f}_g = X(f_g(\alpha,t))(A_\alpha(\alpha,t)\dot{\alpha} + A_t(\alpha,t))$. Exponential stabilization of z_η to zero is simply achieved by setting

$$\tilde{u}_\eta = \dot{f}_\eta - k_\eta z_\eta$$

with $k_\eta > 0$ a control gain. To simplify the exposition, we will assume from now on that the convergence of z_η to zero has taken place. In doing so we thus neglect transient effects associated with this phase and concentrate on the stabilization of z_g to the origin when $z_\eta = 0$. Since $z_\eta = \eta - \eta_r - f_\eta$, the first equation of (52) then becomes

$$\dot{z}_g = X(z_g)\mathrm{Ad}^X(f_g(\alpha,t))\left(\bar{C}(\alpha,t)\bar{u} - A_t(\alpha,t) - \mathrm{Ad}^X(\tilde{g}^{-1})v_r\right) \tag{53}$$

with

$$\bar{C}(\alpha,t) := \left(C(\eta_r(t) + f_\eta(\alpha,t)) \mid\ - A_\alpha(\alpha,t)\right) \tag{54}$$

and $\bar{u}' := (u_1, \dot{\alpha}')$. If $\bar{C}(\alpha,t)$ is invertible for any (α,t), then the feedback law

$$\bar{u} = \bar{C}(\alpha,t)^{-1}\left(A_t(\alpha,t) - \mathrm{Ad}^X(\tilde{g}^{-1})v_r + \mathrm{Ad}^X(f_g(\alpha,t)^{-1})\bar{v}\right) \tag{55}$$

transforms System (53) into $\dot{z}_g = X(z_g)\bar{v}$. It is now simple to asymptotically stabilize z_g to the origin via the choice of $\bar{v}(z_g)$ (see Section 5.2.4).

5.2.2 Transverse functions

Let us now address the design of f in order to ensure the invertibility of the matrix $\bar{C}(\alpha,t)$ for any (α,t). Since $X(z_g)$ is an invertible matrix for any z_g, this is equivalent to finding f such that the matrix

$$X(f_g)\bar{C}(\alpha,t) = \begin{pmatrix} \cos(f_\theta) & & \\ \sin(f_\theta) & -\dfrac{\partial f_g}{\partial \alpha_1} & -\dfrac{\partial f_g}{\partial \alpha_2} \\ \eta_r(t) + f_\eta & & \end{pmatrix}$$

is invertible for any (α,t), with f_θ the third component of f_g. The argument (α,t) of f_g, f_θ, and f_η is omitted for legibility. An equivalent condition is the invertibility of the matrix

$$H(\alpha,t) = \left(X_{1,\eta_r(t)}(f)\ X_2\ -\frac{\partial f}{\partial \alpha_1}\ -\frac{\partial f}{\partial \alpha_2}\right) \tag{56}$$

with

$$\begin{aligned} X_{1,\eta_r}(g,\eta) &= (\cos\theta, \sin\theta, \eta_r + \eta, 0)' \\ X_2 &= (0,0,0,1)' \end{aligned} \tag{57}$$

This corresponds to the property of transversality of f w.r.t. the v.f. X_{1,η_r} and X_2 –compare with the control v.f. of System (48)–, for any value $\eta_r(t)$.

Lemma 4 *Let $\phi_{\eta_r} : (g,\eta) \longmapsto \bar{x} = \phi_{\eta_r}(g,\eta)$ denote the mapping defined by*

$$\left\{\begin{array}{l} \bar{x}_1 = x \\ \bar{x}_2 = (\eta + \eta_r(1-\cos^3\theta))/(\cos^3\theta) \\ \bar{x}_3 = \tan\theta - \eta_r x \\ \bar{x}_4 = y - \eta_r\frac{x^2}{2} \end{array}\right. \tag{58}$$

with η_r an arbitrary constant. Then,

1. *ϕ_{η_r} defines a diffeomorphism from $\mathbb{R}^2 \times (-\pi/2, \pi/2) \times \mathbb{R}$ to $\mathbb{R}^4$,*
2. *$\phi_{\eta_r}(0,0) = 0$,*
3. *if $\bar{f}^c$ is transverse to the v.f. of the 4-D chained system, then $f = \phi_{\eta_r}^{-1}(\bar{f}^c)$ is transverse to the v.f. X_{1,η_r} and X_2.*

The third property implies that the matrix $H(\alpha, t)$ of relation (56) is invertible for any (α, t). The proof of this lemma, given in Appendix B, relies on the possibility of transforming, via a change of state and control variables, the kinematic equations of a car-like vehicle into a 4-D chained system. This is a particular case of a more general result which will be presented in a future publication and used for the control of a vehicle with multiple trailers.

From the above lemma, the design of a function f such that the matrix $\bar{C}(\alpha, t)$ defined by (54) is invertible reduces essentially to the design of a transverse function for the 4-D chained system. For instance, one can take the function $\bar{f}^c(\alpha) = f^c(\alpha_r)^{-1} f^c(\alpha)$ with f^c given by (29) –the product here involved is the group operation associated with the 4-D chained system. Moreover, by choosing $\alpha_r = (-\frac{\pi}{2}, -\frac{\pi}{2})$ and ε_{i1} $(i = 3, 4)$ as specified in Lemma (3), one allows for the asymptotic stabilization of admissible reference trajectories. In this respect, Properties 1-2 in Lemma 4 are important because they ensure that $f = \phi_{\eta_r}^{-1}(\bar{f}_c)$ vanishes when $\bar{f}_c$ vanishes. With these choices for f^c and α_r one obtains:

$$\bar{f}^c(\alpha) = \begin{pmatrix} \varepsilon_{31}(s\alpha_3 + 1) + \varepsilon_{41}(s\alpha_4 + 1) \\ \varepsilon_{32} c\alpha_3 \\ \frac{\varepsilon_{31}\varepsilon_{32}}{4} s2\alpha_3 - \varepsilon_{42} c\alpha_4 \\ \frac{\varepsilon_{31}^2 \varepsilon_{32}}{6} (s\alpha_3)^2 c\alpha_3 - \frac{\varepsilon_{41}\varepsilon_{42}}{4} s2\alpha_4 - \varepsilon_{31}\varepsilon_{42} s\alpha_3 c\alpha_4 \end{pmatrix}$$

Recall that the parameters $\varepsilon_{i,j}$ $(i, j = 3, 4)$ should also satisfy the inequalities (31). The corresponding function f to be used in the control expression is thus

$$f(\alpha, t) = \phi_{\eta_r(t)}^{-1}(\bar{f}_c(\alpha)) \tag{59}$$

with $\phi_{\eta_r}^{-1}$, the inverse of ϕ_{η_r}, given by

$$\phi_{\eta_r}^{-1}(\bar{x}) = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_4 + \eta_r \bar{x}_1^2/2 \\ \arctan(\bar{x}_3 + \eta_r \bar{x}_1) \\ \dfrac{\bar{x}_2 + \eta_r}{\left(\sqrt{1 + (\bar{x}_3 + \eta_r \bar{x}_1)^2}\right)^3} - \eta_r \end{pmatrix} \tag{60}$$

From there the calculation of A_α and A_t in (54) and (55) can be performed by using the relations

$$\begin{aligned} A_\alpha(\alpha, t) &= X(f_g(\alpha, t))^{-1} \frac{\partial f_g}{\partial \alpha}(\alpha, t) \\ &= X(f_g(\alpha, t))^{-1} \frac{\partial}{\partial \bar{x}} \phi_{\eta_r(t)}^{-1}(\bar{f}_c(\alpha))_{1,2,3} \frac{\partial \bar{f}^c}{\partial \alpha}(\alpha) \end{aligned}$$

with

$$\frac{\partial}{\partial \bar{x}}\phi^{-1}_{\eta_r}(\bar{x})_{1,2,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \eta_r \bar{x}_1 & 0 & 0 & 1 \\ \eta_r/d(\bar{x},\eta_r) & 0 & 1/d(\bar{x},\eta_r) & 0 \end{pmatrix}, \quad d(\bar{x},\eta_r) = 1 + (\bar{x}_3 + \eta_r \bar{x}_1)^2$$

and

$$A_t(\alpha,t) = X(f(\alpha,t))^{-1} \frac{\partial}{\partial \eta_r}\phi^{-1}_{\eta_r(t)}(\bar{f}_c(\alpha))_{1,2,3}\ \dot{\eta}_r(t)$$

with

$$\frac{\partial}{\partial \eta_r}\phi^{-1}_{\eta_r}(\bar{x})_{1,2,3} = \left(0, \frac{\bar{x}_1^2}{2}, \frac{\bar{x}_1}{d(\bar{x},\eta_r)}\right)'$$

5.2.3 Determination of η_r

When addressing trajectory stabilization problems, it is usually assumed that all reference trajectory components (the functions of time g_r and η_r in the present case) are specified. However, in the case of mobile robot applications, it is often convenient to only specify the reference pose g_r which corresponds to the desired situation of the vehicle's main body. An issue then is the determination of η_r. For admissible trajectories, provided that $u_{r,1} = \dot{x}_r \cos\theta_r + \dot{y}_r \sin\theta_r$ is different from zero, one has $\eta_r = \frac{\dot{\theta}_r}{u_{r,1}}$ (see Eq. (48)). This suggests, among other possibilities, the following choice

$$\eta_r = \frac{\dot{\theta}_r u_{r,1}}{u_{r,1}^2 + \varepsilon} \tag{61}$$

with ε a small positive number whose role is to ensure that *i)* η_r is always well defined, in particular when the longitudinal velocity $u_{r,1}$ vanishes or when the motion of g_r is not admissible for a car, and *ii)* η_r is close to the ideal desired value $\frac{\dot{\theta}_r}{u_{r,1}}$ when the reference trajectory is admissible and $u_{r,1} \neq 0$.

5.2.4 Control

To calculate the control (55) there remains to determine an auxiliary control vector $\bar{v}(z_g)$ which asymptotically stabilizes $z_g = e$ for the control system $\dot{z}_g = X(z_g)\bar{v}$. A possible choice yielding exponential stabilization is

$$\bar{v}(z_g) = X(z_g)^{-1} K z_g \tag{62}$$

with K a Hurwitz-stable matrix. Another possibility, as in the unicycle case, arises from the concern of limiting the control energy during the transient phase when z_g converges to e and, at the same time, of limiting the number of car maneuvers during this phase. As in the unicycle case, let us rewrite the error system's equation (53) as

$$\dot{z}_g = H(z_g,\alpha,t)\bar{\bar{u}}$$

with

$$H(z_g, \alpha, t) = X(z_g)\mathrm{Ad}^X(f_g(\alpha, t))\bar{C}(\alpha, t)$$

and

$$\bar{\bar{u}} = \bar{u} - \bar{C}(\alpha, t)^{-1}(A_t(\alpha, t) + \mathrm{Ad}^X(\tilde{g}^{-1})v_r) \tag{63}$$

The idea is again to determine $\bar{\bar{u}}$ which minimizes at every time-instant the quadratic cost $\bar{\bar{u}}'W_1\bar{\bar{u}}$ under the constraint $z_g'H\bar{\bar{u}} + z_g'W_2z_g = 0$, with W_1 and W_2 denoting two s.p.d. matrices. The fact that $\bar{\bar{u}}_1 = u_1$ when $v_r \equiv 0$ and $A_t \equiv 0$ suggests to choose W_1 diagonal with the first diagonal entry larger than the others. The solution to this simple problem, previously derived in the unicycle case, is given by the relation (46) with z_g replacing z, i.e.

$$\bar{\bar{u}} = -\frac{z_g'W_2z_g}{z_g'HW_1^{-1}H'z_g}W_1^{-1}H'z_g \tag{64}$$

The control input vector $\bar{u} = (u_1, \dot{\alpha}')'$ is then calculated by using (63).

5.2.5 Simulation results

For these simulations, the car is represented as a tricycle whose length (distance between front and rear wheels) and width (distance between the two rear wheels) are equal to 2 (meters). The same reference trajectory as for the unicycle simulations is used. Note that, for this particular trajectory, the phases when it is either persistently exciting (*pe*) or not persistently exciting (*npe*) are the same as in the unicycle case. The reason is that $v_{r,2}$ is equal to zero only when $v_{r,1}$ is itself equal to zero, when the trajectory is admissible, i.e. when $v_{r,3} = 0$.

Figure 9 shows the time evolution of the tracking position error $(\tilde{x}_1, \tilde{x}_2)$ and tracking orientation error $\tilde{x}_3$ respectively. The same control properties as in the unicycle case can be observed from these figures, namely i) the uniform boundedness of the tracking errors whatever the properties of the reference trajectory, ii) the convergence of the tracking errors to zero when the reference trajectory is admissible and persistently exciting, iii) the automatic (resp. non-systematic) production of maneuvers when the reference trajectory is "strongly" (resp. "weakly") non-admissible.

Figure 10 shows the time-evolution of the four components of the modified tracking error z. One can observe that, besides the initial transient phase of convergence of z to zero, this error is also different from zero during short time-intervals. This is due to discontinuities of the transverse function which, in this case, result from discontinuities of the term $\eta_r(t)$ involved in the transverse function calculation, themselves induced by discontinuities of the reference velocity $\dot{x}_r(t)$. The other figures are attempts to visualize the vehicle's motion in the plane during the different phases of the reference trajectory.

Except for the Figure 12, the feedback control (63,64) which includes a monitoring of the transient phase (before the convergence of z to zero) has been used. The parameters chosen for this control are $W_1 = diag\{1, 0.01, 0.01\}$, $W_2 = diag\{1, 1, 1\}$, $k_\eta = 5$. Figure 12 shows the stabilization of a fixed reference situation as obtained with the control (62) –with

$K = -diag\{1, 1, 1\}$– which does not incorporate such a monitoring. Comparison with Figure 11 illustrates the effect of this monitoring on the reduction of intermediary maneuvers. The parameters of the transverse function used in the control are $\varepsilon_{31} = 0.14$, $\varepsilon_{32} = 1.8$, $\varepsilon_{41} = 0.8$, $\varepsilon_{42} = 0.64$.

Conclusion

The stabilization of trajectories for nonholonomic systems has been addressed. For this study, the general framework of systems on Lie groups is particularly well adapted to the treatment of mechanical systems and their symmetries. The first sections of this report provide an overview of theoretical issues associated with the problem, and recalls the basic elements of the Transverse Function (TF) control approach developed by the authors. In contrast with other methods dedicated to the stabilization of particular trajectories –like fixed-points, or persistently exciting admissible trajectories–, this approach aims in the first place at achieving the *practical* –by opposition to asymptotic– stabilization of reference trajectories regardless of their admissibility and other specific properties. From there, complementary properties can be considered. The asymptotic stabilization of persistently exciting admissible trajectories is one of them, and an original contribution of the present study was to show that it can be achieved via a proper choice of the transverse function involved in the control law. Another one is the asymptotic stabilization of fixed-points. A preliminary study of this issue in [31], limited to the case of chained systems, shows that solutions can again be obtained via the search for adequate *generalized* transverse functions. When addressing these complementary issues, it matters to keep in mind that the "perfect" controller capable of stabilizing any admissible reference trajectory asymptotically probably does not exist [22]. The last section of this report has focused on the application of the TF approach to the control of unicycle and car-like vehicles, with the complementary concern of limiting the size of the velocity inputs and the number of maneuvers involved during the initial tracking error reduction phase.

They are numerous possible extensions to the present study. One of them concerns experimental testing and validation. Whereas the TF control approach has already been experimented on a unicycle-type vehicle [1, 2], no experimentation on a car-like vehicle has been reported so far. Then, as mentioned above, the fine tuning of the properties of a TF controller much depends on the selected transverse function. The exploration of the possibilities offered via the choice of this function is a research avenue by itself, still largely open. Concerning nonholonomic systems other than unicycles and cars, the application and adaptation of the approach to systems like the rolling sphere [11, 35, 38], the general N-trailer [24, 45], and snake-like robots [15, 39] constitute, in our eyes, interesting and challenging research topics. In the case of the rolling sphere, the solution proposed by the authors in [34] can probably be refined in order to improve the closed loop system's performance. The control of underactuated mechanical systems is also a domain for which encouraging initial results [25, 32] have been obtained and which calls for new developments.

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A Recalls of differential relations on Lie groups

Let g, h, σ denote elements of a Lie group G.

$$
\begin{aligned}
dL_{gh}(\tau) &= dL_g(h\tau)dL_h(\tau) && (65)\\
dR_{gh}(\tau) &= dR_h(\tau g)dR_g(\tau) && (66)\\
(dL_g(\tau))^{-1} &= dL_{g^{-1}}(g\tau) && (67)\\
(dR_g(\tau))^{-1} &= dR_{g^{-1}}(\tau g) && (68)\\
\mathrm{Ad}(gh) &= \mathrm{Ad}(g)\mathrm{Ad}(h) && (69)\\
\mathrm{Ad}(g)^{-1} &= \mathrm{Ad}(g^{-1}) && (70)
\end{aligned}
$$

Relations (65) and (66) are obtained by application of the chain rule to the relations $L_{gh} = L_g \circ L_h$ and $R_{gh} = R_h \circ R_g$. Relations (67) and (68) are then deduced from (65) and (66) by setting $h = g^{-1}$ and using the fact that L_e and R_e are the identity operator on G. Relation (69) is deduced from the fact that, by (12) and the definition of J_σ,

$$
\begin{aligned}
\mathrm{Ad}(gh) &= dJ_{gh}(e) = d(J_g \circ J_h)(e)\\
&= dJ_g(e)dJ_h(e) = \mathrm{Ad}(g)\mathrm{Ad}(h)
\end{aligned}
$$

Relation (70) is deduced from (69) by setting $h = g^{-1}$ and using the fact that, by definition, $\mathrm{Ad}(e)$ is the identity operator.

Let g_i $(i = 1, 2)$ denote two smooth curves on a Lie group G, and $v_i = (v_{i,1}, \ldots, v_{i,n})'$ denote the decomposition of $\dot{g}_i$ on a basis of the group's Lie algebra $\mathfrak{g}$, i.e.

$$
\dot{g}_i = X(g_i)v_i := \sum_{k=1}^{n} X_k(g_i)v_{i,k}
$$

with $X_1, \ldots, X_n$ a basis of left-invariant v.f. on G. Then,

$$
\begin{aligned}
\frac{d}{dt}(g_1^{-1}) &= -dL_{g_1^{-1}}(e)dR_{g_1^{-1}}(g_1)\dot{g}_1 && (71)\\
&= -dR_{g_1^{-1}}(e)dL_{g_1^{-1}}(g_1)\dot{g}_1 && (72)\\
&= -dR_{g_1^{-1}}(e)X(e)v_1 && (73)\\
\frac{d}{dt}(g_1^{-1}g_2) &= X(g_1^{-1}g_2)v_2 - dR_{g_1^{-1}g_2}(e)X(e)v_1 && (74)\\
&= X(g_1^{-1}g_2)v_2 - dL_{g_1^{-1}g_2}(e)\mathrm{Ad}(g_2^{-1}g_1)X(e)v_1 && (75)\\
\frac{d}{dt}(g_1g_2^{-1}) &= dR_{g_2^{-1}}(g_1)dL_{g_1}(e)X(e)(v_1 - v_2) && (76)\\
\frac{d}{dt}(g_1g_2^{-1}) &= dL_{g_1g_2^{-1}}(e)\mathrm{Ad}(g_2)X(e)(v_1 - v_2) && (77)
\end{aligned}
$$

Relations (71) and (72) are obtained by differentiating the relation $g_1g_1^{-1} = g_1^{-1}g_1 = e$ and using (67) and (68). Relation (73) is directly deduced from (72) and the fact that

$\dot{g}_1 = X(g_1)v_1$, with $X_1, \ldots, X_n$ left-invariant. Relation (74) is then deduced from (73) and (66). Relation (75) is deduced from (74) and (12). Relation (76) is obtained by differentiating the equality $g_1 = (g_1 g_2^{-1})g_2$ and using (68). Finally, Relation (77) is deduced from (76), the fact that $\mathrm{Ad}(g_2) = \mathrm{Ad}(g_2 g_1^{-1})\mathrm{Ad}(g_1)$, (by (69)), and also the fact that

$$\begin{aligned}\mathrm{Ad}(g_1) &= dR_{g_1^{-1}}(g_1)dL_{g_1}(e)\\ &= dR_{g_2 g_1^{-1}}(g_1 g_2^{-1})dR_{g_2^{-1}}(g_1)dL_{g_1}(e)\end{aligned}$$

where the first equality comes from (12) and the second one from (66).

B Proofs

Proof of Proposition 1

Let $\tilde{\alpha} := \alpha - \alpha_r$. The function $\bar{f}$ can be expressed as a function of $\tilde{\alpha}$ and α_r, i.e.

$$\bar{f}(\alpha, \alpha_r) = \bar{f}(\tilde{\alpha} + \alpha_r, \alpha_r) = \tilde{f}(\tilde{\alpha}, \alpha_r)$$

From the definition of $\bar{f}$, $\tilde{f}(0, \alpha_r) = e$, $\forall \alpha_r$. Therefore, there exists a smooth function $\tilde{\gamma}$, defined for any α_r and for $\tilde{\alpha}$ in the neighborhood of $\tilde{\alpha} = 0$, such that

$$\tilde{f}(\tilde{\alpha}, \alpha_r) = \exp(X\tilde{\gamma}(\tilde{\alpha}, \alpha_r)) \tag{78}$$

Note that $\tilde{\gamma}(\tilde{\alpha}, \alpha_r) \in \mathbb{R}^n$ is the vector of coordinates of the first kind of the point $\tilde{f}(\tilde{\alpha}, \alpha_r) \in G$ (see Section 2.2.3). Since $\tilde{f}(0, \alpha_r) = e$, one has that $\tilde{\gamma}(0, \alpha_r) = 0$, $\forall \alpha_r$, so that

$$\tilde{\gamma}(\tilde{\alpha}, \alpha_r) = \frac{\partial \tilde{\gamma}}{\partial \tilde{\alpha}}(0, \alpha_r)\tilde{\alpha} + o_{\alpha_r}(\tilde{\alpha}) \tag{79}$$

Let us show that

$$\frac{\partial \tilde{\gamma}}{\partial \tilde{\alpha}}(0, \alpha_r) = A(\alpha_r) \tag{80}$$

By differentiating the equality (78), and using the usual identification of $\mathfrak{g}$ with its tangent space, (see e.g. [44, Sec. 2.14]), one obtains that

$$\begin{aligned} \dot{\tilde{f}} &= (d\exp)(X\tilde{\gamma}).X\dot{\tilde{\gamma}} \\ &= (d\exp)(X\tilde{\gamma}).X\left(\frac{\partial \tilde{\gamma}}{\partial \tilde{\alpha}}\dot{\tilde{\alpha}} + \frac{\partial \tilde{\gamma}}{\partial \alpha_r}\dot{\alpha}_r\right) \end{aligned} \tag{81}$$

On the other hand,

$$\begin{aligned} \dot{\tilde{f}} &= \dot{\bar{f}} \\ &= X(\bar{f})\left(A(\alpha)\dot{\alpha} - \mathrm{Ad}^X(\bar{f}^{-1})A(\alpha_r)\dot{\alpha}_r\right) \\ &= X(\tilde{f})\left(A(\alpha)\dot{\tilde{\alpha}} + (I - \mathrm{Ad}^X(\bar{f}^{-1}))A(\alpha_r)\dot{\alpha}_r\right) \end{aligned} \tag{82}$$

with the second equality deduced from (34), (75), and (21). Since $\tilde{\gamma}(0, \alpha_r) = 0$, $\tilde{f}(0, \alpha_r) = e$, and $(d\exp)(0)$ is the identity operator, it comes from (81) and (82) that

$$X(e)\left(\frac{\partial \tilde{\gamma}}{\partial \tilde{\alpha}}\dot{\tilde{\alpha}} + \frac{\partial \tilde{\gamma}}{\partial \alpha_r}\dot{\alpha}_r\right)_{|\tilde{\alpha}=0} = X(e)\left(A(\alpha)\dot{\tilde{\alpha}}\right)_{|\tilde{\alpha}=0}$$

and (80) follows.

When using the coordinates of the first kind ξ, the linearization of System (15) at $(\tilde{g}, \tilde{u}) = (e, 0)$ is given by equation (18). Therefore, there exists a function o such that System (15) satisfies, in these coordinates,

$$\dot{\tilde{\xi}} = -\mathrm{ad}^X(Cu_r)\tilde{\xi} + C\tilde{u} + o(\tilde{\xi}, \tilde{u}) \tag{83}$$

On the zero dynamics $z = e$, the control $\bar{u}$ defined by (25) reduces to

$$\begin{aligned}\bar{u} &= \bar{C}(\alpha)^{-1}\mathrm{Ad}^X(\tilde{g}^{-1})Cu_r \\ &= \bar{C}(\alpha)^{-1}Cu_r + \bar{C}(\alpha)^{-1}(\mathrm{Ad}^X(\tilde{g}^{-1}) - I_n)Cu_r\end{aligned}$$

because $v(e) = 0$ and $v_r = Cu_r$. Therefore,

$$\begin{aligned}\tilde{u} &= u - u_r \\ &= (I_m\ |0)\bar{u} - u_r \\ &= (I_m \mid 0)\bar{C}(\alpha)^{-1}Cu_r - u_r \\ &+ (I_m \mid 0)\bar{C}(\alpha)^{-1}(\mathrm{Ad}^X(\tilde{g}^{-1}) - I_n)Cu_r\end{aligned}$$

From the expression (24) of $\bar{C}$, one easily verifies that $(I_m \mid 0)\bar{C}(\alpha)^{-1}Cu_r - u_r = 0$. Since $\mathrm{Ad}^X(e) = I_n$, it follows from the above equation that, in the coordinates $\tilde{\xi}$ and on the zero dynamics $z = e$, one has $\tilde{u} = O(\tilde{\xi})$. More precisely, $|\tilde{u}| \leq K|u_r||\tilde{\xi}|$ in the neighborhood of $\tilde{\xi} = 0$, with K a constant. Eq. (83) can then be rewritten as

$$\dot{\tilde{\xi}} = -\mathrm{ad}^X(Cu_r)\tilde{\xi} + C\tilde{u} + o(\tilde{\xi}) \tag{84}$$

Now, on the zero dynamics, $\tilde{g} = \bar{f} = \tilde{f}$. Therefore, since $\tilde{\gamma}$ is the vector of coordinates of the first kind at the point $\tilde{f}$, $\tilde{\gamma}$ must also satisfy Eq. (84), i.e.

$$\dot{\tilde{\gamma}} = -\mathrm{ad}^X(Cu_r)\tilde{\gamma} + C\tilde{u} + o(\tilde{\gamma})$$

Pre-multiplying both sides of this equality by P yields

$$P\dot{\tilde{\gamma}} = -P\mathrm{ad}^X(Cu_r)\tilde{\gamma} + Po(\tilde{\gamma}) \tag{85}$$

From (79), (80), and the definition of $\bar{\alpha}$,

$$\tilde{\gamma} = \bar{\alpha} + o_{\alpha_r}(\tilde{\alpha}) = \bar{\alpha} + o_{\alpha_r}(\bar{\alpha}) \tag{86}$$

with the second equality resulting from the fact that

$$\tilde{\alpha} = (PA(\alpha_r))^{-1}PA(\alpha_r)\tilde{\alpha} = (PA(\alpha_r))^{-1}P\bar{\alpha}$$

Eq. (35) then follows from (85) and (86). ∎

Proof of Lemma 4

The bijectivity of ϕ_{η_r} can be verified directly from (58). The expression of $\phi_{\eta_r}^{-1}$ is given by Eq. (60). Since ϕ_{η_r} is differentiable with invertible Jacobian matrix, Property 1 follows.

The verification of Property 2 is straightforward.

Let us now establish Property 3. Using the fact that

$$\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$$

one verifies by a simple calculation that the expression of the v.f. X_{1,η_r} and X_2 in the coordinates $\bar{x}$ is

$$\bar{X}_1(\bar{x}) = ct(\bar{x}) \begin{pmatrix} 1 \\ \psi_{\eta_r}(\bar{x}) \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix}, \quad \bar{X}_2(\bar{x}) = \frac{1}{ct(\bar{x})^3} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

with ψ_{η_r} some smooth function, and

$$ct(\bar{x}) = \frac{1}{\sqrt{1+(\bar{x}_3+\eta_r\bar{x}_1)^2}}$$

the expression of $\cos\theta$ in the coordinates $\bar{x}$. Let $\bar{f}^c : \mathbb{T}^2 \longrightarrow \mathbb{R}^4$ denote a function transverse to the v.f. of the 4-D chained system. By the property of transversality, the matrix

$$\begin{pmatrix} 1 & 0 & & \\ 0 & 1 & -\dfrac{\partial \bar{f}^c}{\partial \alpha_1} & -\dfrac{\partial \bar{f}^c}{\partial \alpha_2} \\ \bar{f}_2^c & 0 & & \\ \bar{f}_3^c & 0 & & \end{pmatrix}$$

is invertible for any α. Since $ct(\bar{x}) \neq 0 \; \forall \bar{x}$, it is straightforward to verify that the matrix

$$\begin{pmatrix} \bar{X}_1(\bar{f}_c) & \bar{X}_2(\bar{f}_c) & -\dfrac{\partial \bar{f}^c}{\partial \alpha_1} & -\dfrac{\partial \bar{f}^c}{\partial \alpha_2} \end{pmatrix}$$

is also invertible for any α. From there, the invertibility of the matrix H in (56) follows from the fact that the property of transversality is independent of the system of coordinates and that $\bar{f}^c = \phi_{\eta_r}(f)$ is the expression of f in the coordinates $\bar{x}$. ∎

C Figures

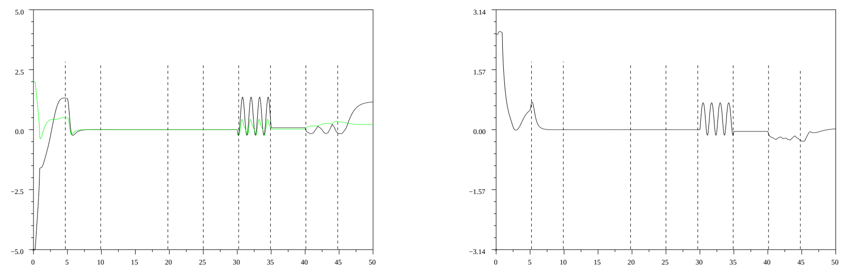

Figure 1: Unicycle: Tracking errors vs. time. Left: position $\tilde{x}$, $\tilde{y}$; Right: orientation $\tilde{\theta}$.

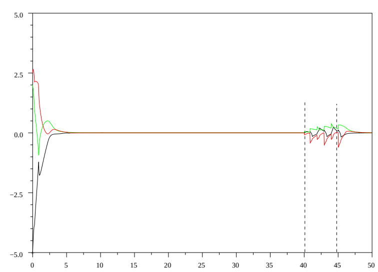

Figure 2: Unicycle: $z_{1,2,3}$ vs. time

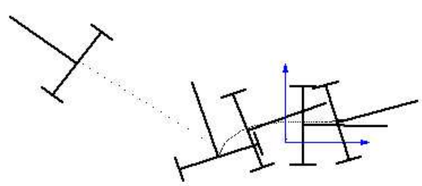

Figure 3: Unicycle: Fixed reference **with** transient monitoring $t \in [0s, 5s)$

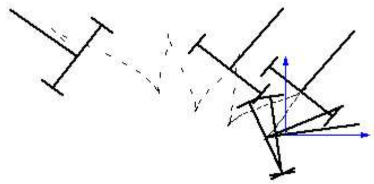

Figure 4: Unicycle: Fixed reference **without** transient monitoring $t \in [0s, 5s)$

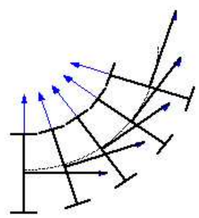

Figure 5: Unicycle: Admissible arc of circle $t \in [20s, 25s)$

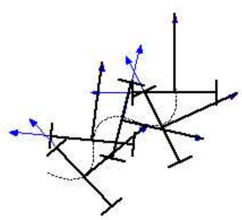

Figure 6: Unicycle: Admissible trajectory with rapidly changing curvature $t \in [25s, 30s)$

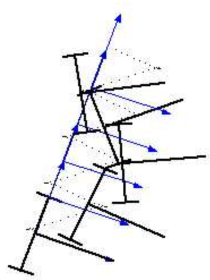

Figure 7: Unicycle: Non-admissible lateral motion inducing maneuvers $t \in [30s, 35s)$

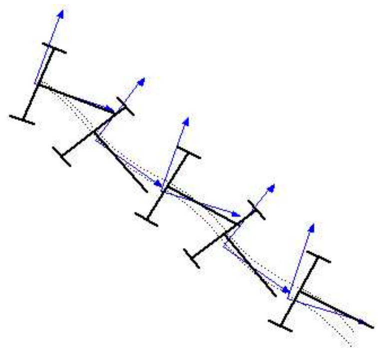

Figure 8: Unicycle: Non-admissible motion not inducing maneuvers $t \in [40s, 45s)$

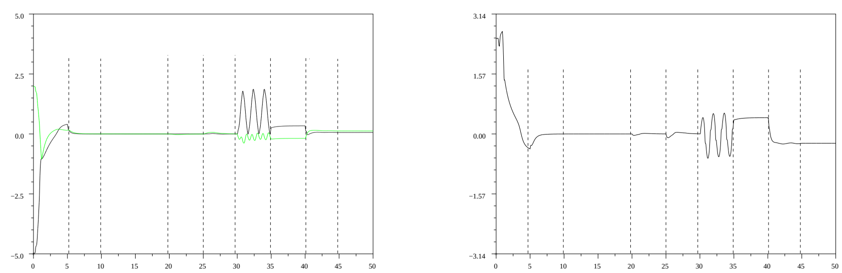

Figure 9: Car: Tracking errors vs. time. Left: position $\tilde{x}_1$, $\tilde{x}_2$; Right: orientation $\tilde{x}_3$.

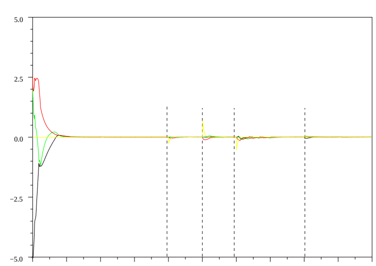

Figure 10: Car: $z_{1,2,3,4}$ vs. time

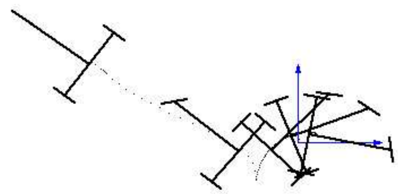

Figure 11: Car: Fixed reference **with** transient monitoring $t \in [0s, 5s)$

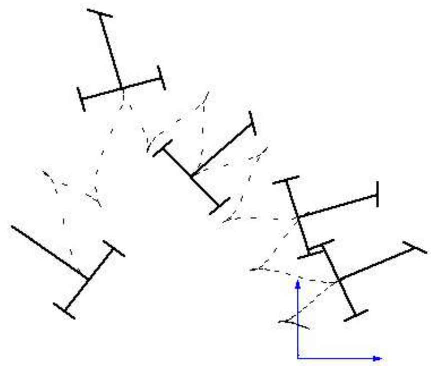

Figure 12: Car: Fixed reference **without** transient monitoring $t \in [0s, 5s)$

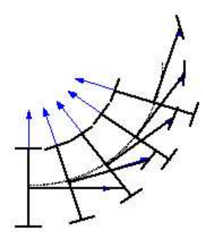

Figure 13: Car: Admissible arc of circle $t \in [20s, 25s)$

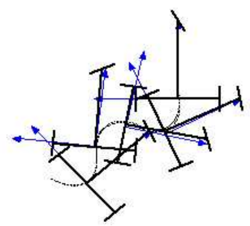

Figure 14: Car: Admissible trajectory with rapidly changing curvature $t \in [25s, 30s)$

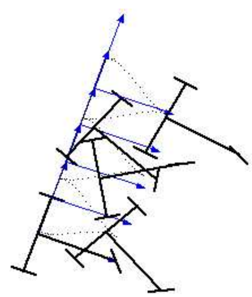

Figure 15: Car: Non-admissible lateral motion inducing maneuvers $t \in [30s, 35s)$

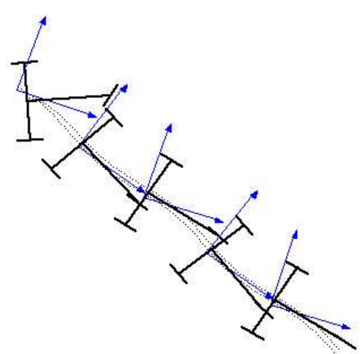

Figure 16: Car: Non-admissible motion not inducing maneuvers $t \in [40s, 45s)$

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Unité de recherche INRIA Futurs : Parc Club Orsay Université - ZAC des Vignes
4, rue Jacques Monod - 91893 ORSAY Cedex (France)
Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
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Unité de recherche INRIA Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399