CONVERGENCE OF A SIMPLE ADAPTIVE FINITE ELEMENT METHOD FOR OPTIMAL CONTROL

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Abstract. We prove convergence and optimal complexity of an adaptive finite element algorithm for a model problem of optimal control. Following previous work, our algorithm is based on an adaptive marking strategy which compares a simple edge estimator with an oscillation term in each step of the algorithm in order to adapt the marking of cells.

Key words. Optimal control, adaptive finite elements, convergence of adaptive algorithms, complexity estimates

AMS subject classifications. 49J20, 49M15, 49N10, 65K10, 65N12, 65N15, 65N30, 65N50

1. Introduction. We consider a model problem for optimal control, the solution of which is approximated by an adaptive finite element method. The analysis of adaptive finite element methods has made important progress in recent years. Based on classical residual-based a posteriori error estimators [1, 14, 21] it has been shown by Dörfler [13] and Morin, Nochetto, and Siebert [19] that an adaptive mesh refinement algorithm converges towards the solution of the Poisson equation. An important further result is the estimation of the dimension of the adaptively constructed discrete spaces by Binev, Dahmen, and DeVore in [6], and Stevenson [20]. The importance of these contributions lays in the fact that they show optimal complexity of certain adaptive algorithms: if the solution of the problem can be approximated by a given discretization method on a given family of meshes at a certain rate (quotient of accuracy to number of unknowns), the iteratively constructed sequence of meshes will realize this rate up to a constant factor.

In this work, we present an adaptive finite element method for an optimal control problem. Our approach is based on the Courant finite element on locally refined meshes obtained by hierarchical bisection. Following the idea of [4], we use an adaptive markings strategy which either performs the refinement according to a simple edge-based estimator or according to a so-called oscillation term. From a computational point of view, the resulting algorithm is therefore simpler than the MNS algorithm [19], which is underlaying the work of [15] for an optimal control problem similar to the one considered here. Improving upon the known results in the literature, we prove convergence and optimal complexity of the adaptive algorithm. In order to treat the non-orthogonality of the error resulting from the coupling of the system to be solved, we make an assumption on the fineness of the first mesh, see hypothesis (5.7). Such an assumption seems to be natural, since the considered coupled system is equivalent to an indefinite scalar equation.

The paper is organized as follows: After introduction of the optimal control problem and its discretization on a single mesh in Sections 2 and 3, in Section 4 we define the adaptive algorithm. In Section 5 we prove some lemmata concerning lower/upper

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local/global bounds which are used lateron. In Section 6 we prove geometrical convergence of the error of the adaptive algorithm, under natural assumptions. In Section 7 we prove an asymptotic estimate for the complexity of the sequence of the meshes. Finally, we report on some numerical experiments in Section 8.

Throughout the paper we use the following notation. For the norm of the standard Sobolev space $H_0^1(\Omega)$ we write $|u|_1 := \left(\int_\Omega |\nabla u|^2\,dx\right)^{1/2}$. The $L^2(A)$-scalar product and norm are denoted by $\langle\cdot,\cdot\rangle_A$ and $|\cdot|_A$, respectively, omitting the subscript in case $A=\Omega$, for either a measurable subset $A\subset\Omega$ or for an edge A of a finite element mesh (with obvious modification of the measure).

We work with families of shape regular triangular meshes in the sense of [11]. In order to deviate as less as possible from standard notation, we denote by h a mesh of a family $\mathcal{H}$, and by u_h the corresponding finite element solution. The set of cells of mesh h is denoted by $\mathcal{K}_h$, the set of edges by $\mathcal{E}_h$, and the subset of interior edges by $\mathcal{E}_h^{int}$. In addition, the set of nodes is $\mathcal{N}_h$. The diameter of $K\in\mathcal{K}_h$ is denoted by h_K and in addition we define $h_{max}(h) := \max_{K\in\mathcal{K}_h} h_K$. As compared to standard notation in finite element literature, h denotes a mesh in a family of meshes $\mathcal{H}$ and *not* a global maximal cell width.

2. The optimal control problem. Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with polygonal boundary $\partial\Omega$. Let $\Omega_B\subset\Omega$ and $\Omega_C\subset\Omega$ be polygonal subdomains. Further let $f\in L^2(\Omega)$ and $\alpha>0$ be given. We consider the following optimization problem:

$$\left\{\begin{array}{r} \displaystyle\inf_{q\in L^2(B),u\in H_0^1(\Omega)} \frac{\alpha}{2}|q|^2_{\Omega_B} + \frac{1}{2}|u|^2_{\Omega_C} \quad \text{subject to:} \\ -\Delta u = f+q \quad \text{in } \Omega_B, \quad -\Delta u = f \quad \text{in } \Omega\setminus\Omega_B, \quad u=0 \quad \text{on } \partial\Omega, \\ q\geq 0 \quad \text{a.e. } \Omega_B. \end{array}\right. \tag{2.1}$$

This is a linear-quadratic problem. Introducing the operator $B: L^2(\Omega_B)\to L^2(\Omega)$ by $q\mapsto \chi_{\Omega_B} q$, we may alternatively write the state equation as

$$-\Delta u = f + Bq \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

The state equation can be used to eliminate the state variable, such that we end up with the minimization of a quadratic functional in the control variable q alone. Although conceptionally important, such a formulation hides the main difficulty inherent to optimization problems containing a partial differential equation as constraint: the discretization of the state equation.

We remark that $\alpha>0$ is necessary for well-posedness. However, in case of finite-dimensional controls (where q is sought in the linear space spanned by given functions $\psi_i\in L^2(\Omega_B), i=1,\dots m$), $\alpha=0$ may lead to a well-posed minization problem. This case will be addressed elsewhere. Here we suppose $\alpha>0$.

More general linear-quadratic optimal control problems involving non-zero observations $u_d\in L^2(\Omega_C)$, a reference control value $q_d\in L^2(\Omega_B)$, and inhomogenous Dirichlet boundary data can be directly reduced to (2.1).

Writing $u(q)$ for the unique solution of the state equation for given control, the reduced functional is defined as

$$j(q) := \frac{\alpha}{2}|q|^2_{\Omega_B} + \frac{1}{2}|u(q)|^2_{\Omega_C}$$

We denote by $L^2_+(\Omega)$ the cone of positive square-integrable functions. Let $Q = L^2_+(\Omega_B)$ be the set of admissible controls.

In terms of the reduced functional, the optimization problem (2.1) simply reads

$$\inf_{q\in Q} j(q). \tag{2.2}$$

We note that j is quadratic since $q \mapsto u(q)$ is linear. Its first- and second-order derivatives are given by

$$j'(q)(p) = \alpha\langle q,p\rangle_{\Omega_B} + \langle u(q),u(p)\rangle_{\Omega_C},\ j''(p_1,p_2) = \alpha\langle p_1,p_2\rangle_{\Omega_B} + \langle u(p_1),u(p_2)\rangle_{\Omega_C} \tag{2.3}$$

We observe that $j''(p,p) \geq \alpha|p|^2_{\Omega_B}$ such that j is strictly convex and the minimization problem (2.2) admits a unique solution which is characterized by the variational inequality

$$j'(q)(p) = \langle \nabla j(q), p\rangle_{\Omega_B} \geq 0 \qquad \forall p \in Q. \tag{2.4}$$

Next we define the Lagrange functional by

$$\mathcal{L}(q,u,z) := \frac{\alpha}{2}|q|^2_{\Omega_B} + \frac{1}{2}|u|^2_{\Omega_C} + \langle f,z\rangle + \langle q,z\rangle_{\Omega_B} - \langle \nabla u, \nabla z\rangle.$$

The first-order necessary conditions, which we also call optimality system, is the variational system

$$\begin{aligned}
\langle \nabla u, \nabla v\rangle - \langle q,v\rangle_{\Omega_B} &= \langle f,v\rangle \quad \forall v \in H^1_0(\Omega), & (2.5)\\
\langle \nabla v, \nabla z\rangle - \langle u,v\rangle_{\Omega_C} &= 0 \quad \forall v \in H^1_0(\Omega), & (2.6)\\
\alpha\langle q,p\rangle + \langle z,p\rangle_{\Omega_B} &\geq 0 \quad \forall p \in Q. & (2.7)
\end{aligned}$$

Let (q,u,z) be the solution of (2.5-2.7). In addition it holds that $\alpha|q|^2_{\Omega_B} + \langle z,q\rangle_{\Omega_B} = 0$. Note that equation (2.7) simply translates the inequality $j'(q)(p) = \mathcal{L}_q(q,u,z)(p) \geq 0$. We have that $q \geq 0$ and $\alpha q + z \geq 0$ almost everywhere. The variational inequality also implies that

$$\alpha q + z^- = 0, \tag{2.8}$$

with $x^+ := \max(0,x)$ and $x^- := x - x^+$. We can use (2.8) in order to eliminate the control variable from the system, leading to the nonlinear system of partial differential equations

$$-\Delta u = f - \chi_B z^-, \quad -\Delta z = u. \tag{2.9}$$

3. Discretization of the optimal control problem. In this section we consider discretization of the optimal control problem on a fixed mesh $h \in \mathcal{H}$. We decide to directly discretize the system (eq:CouplesSystem), since it is conceptually simpler than introducing an additional control space. However, our results carry over to the last case.

Let h be a shape-regular partition of Ω into triangles verifying the standard assumptions [11]. A partition h consists of cells K and edges E, the set of all cells is denotes by $\mathcal{K}_h$, the set of all edges being denoted by $\mathcal{E}_h$, and the set of interior edges by $\mathcal{E}_h^{int}$. We suppose for simplicity that $\bigcup_{K\in\mathcal{K}_h} K$ covers Ω accurately, and that there exist subsets $\mathcal{K}_h^B \subset \mathcal{K}_h$ and $\mathcal{K}_h^C \subset \mathcal{K}_h$ such that $\bigcup_{K\in\mathcal{K}_h^B} K = \Omega_B$ and $\bigcup_{K\in\mathcal{K}_h^C} K = \Omega_C$, respectively.

The finite-element spaces $V_h \subset H^1_0(\Omega)$ is defined in standard way:

$$V_h := \{v \in C(\Omega) : \; v|_K \in P^1 \text{ for all } K \in h\}.$$

We denote by π_h the L^2-projection on the discrete space of piecewise constant functions: for $K \in \mathcal{K}_h$ we have $\pi_h w|_K = 1/|K| \int_K w\,dx$. We note that $|\pi_h w| \le |w|$.

Next we introduce the discrete system to be solved : Find $u_h \in V_h$ and $z_h \in V_h$ such that

$$\langle \nabla u_h, \nabla v_h\rangle + \langle z_h^-, v_h\rangle_{\Omega_B} = \langle f, v_h\rangle \quad \forall v_h \in V_h, \tag{3.1}$$

$$\langle \nabla v_h, \nabla z_h\rangle - \langle u_h, v_h\rangle_{\Omega_C} = 0 \quad \forall v_h \in V_h, \tag{3.2}$$

$$\tag{3.3}$$

The corresponding control q_h is obtained by $q_h := -z_h^-$, which is not a member of V_h. This approach of indirectly discretizing the control variable has been used in [17] in order to derive a prior error estimates. If we use instead a discretization of the control space by piecewise constants, we would end up with the relation $q_h = (\pi_h z_h)^-$ instead.

The coupled system of equations (3.1-3.2) can be solved efficiently by the primal-dual active-set strategy combined with multigrid, see [5, 7, 16].

4. Definition of the adaptive algorithm. We define the family of admissible meshes $\mathcal{H}$ in the following recursive way. Starting from an initial mesh h_0, we denote by $\mathcal{R}_{loc}(h,\mathcal{F})$ with $\mathcal{F} \subset \mathcal{E}_h$ the mesh resulting from a local mesh refinement algorithm such as the newest vertex bisection algorithm, see [18, 6] for details. In this article, we use the following properties of the local mesh refinement algorithm.

Lemma 4.1. *Let $h_k, k = 0,\dots n$ be a sequence of locally refined triangulations created by the newest vertex algorithm, starting from the initial mesh h_0. Let $\mathcal{F}_k \subset \mathcal{E}_{h_k}, k = 0,\dots n-1$ be the collection of all marked edges in step k. Then h_n is uniformly shape regular and the shape regularity of $\mathcal{T}_{h_n}$ only depends on that of $\mathcal{T}_{h_0}$. Furthermore, we have*

$$N_{h_n} \le N_{h_0} + C_0 \sum_{k=0}^{n-1} \#\mathcal{F}_k. \tag{4.1}$$

Lemma 4.1 and especially the complexity estimate (4.1) are known to be true for the newest vertex bisection algorithm, see Theorem 2.4 of [6] (where the set of marked cells instead of the set of marked edges is used).

Let ω_ξ be the set of cells joining a node $\xi \in \mathcal{N}_H$ and denote by π_ω the mean-value operator ($\pi_\omega(f) := \int_\omega f\,dx/|\omega|$). We define for given $\xi \in \mathcal{N}_H$ and $\mathcal{P} \subset \mathcal{N}_H$ an oscillation term

$$\operatorname{osc}_\xi(f) := |\omega_\xi|^{1/2} \, |f - \pi_{\omega_\xi} f|_{\omega_\xi}, \quad \operatorname{osc}_H(f,\mathcal{P}) := \left(\sum_{z\in\mathcal{P}} \operatorname{osc}^2_\xi(f)\right)^{1/2} \tag{4.2}$$

Next we define for $E \in \mathcal{E}_h$ and any given subset $\mathcal{F} \subseteq \mathcal{E}_H$ edge residuals of a given a function $v_h \in V_h$ by

$$J_E(v_H) := |E| \, \|[\frac{\partial v_H}{\partial n}]\|_E, \quad J_H(v_H,\mathcal{F}) := \left(\sum_{E\in\mathcal{F}} J^2_E(v_H)\right)^{1/2}. \tag{4.3}$$

We set for brevity $\mathrm{osc}_H(f) := \mathrm{osc}_H(f, \mathcal{N}_H)$ and $J_H(v_H) := J_H(v_H, \mathcal{E}_H)$.

The purpose of this article is to analyze the following adaptive finite element algorithm:

Algorithm 1 $\mathcal{AFEM}$

(0) Choose parameters $0 < \theta, \sigma < 1$, $\gamma > 0$, and an initial mesh h_0, and set $k = 0$.
(1) Solve the discrete optimization problem (3.1-3.2) with h replaced by h_k to obtain the finite element solutions u_k, z_k.
(2) Compute $J_{h_k}(u_k)$, $J_{h_k}(z_k)$, and oscillation term $\mathrm{osc}_{h_k}(f)$.
(3) – If $\mathrm{osc}^2_{h_k}(f) \leq \gamma \left(J^2_{h_k}(u_k) + J^2_{h_k}(z_k)\right)$ then mark a set $\mathcal{F} \subset \mathcal{E}_{h_k}$ with minimal cardinality such that

$$J^2_{h_k}(u_k, \mathcal{F}) + J^2_{h_k}(z_k, \mathcal{F}) \geq \theta \left(J^2_{h_k}(u_k) + J^2_{h_k}(z_k)\right). \tag{4.4}$$

– else find a set $\mathcal{P} \subset \mathcal{N}_{h_k}$ with minimal cardinality such that

$$\mathrm{osc}^2_{h_k}(f, \mathcal{P}) \geq \sigma \, \mathrm{osc}^2_{h_k}(f). \tag{4.5}$$

and define $\mathcal{F}$ to be the set of edges containing at least one node in $\mathcal{P}$.
(4) Adapt the mesh : $h_{k+1} := \mathcal{R}_{loc}(h_k, \mathcal{F})$.
(5) Set $k := k + 1$ and go to step (1).

REMARK 4.2. *The refinement is only determined by the oscillation term, if it is big compared to the estimator, following the idea of [3]. Therefore, in most practical cases, the edge residuals alone dominate the error estimation, such as suggested in the work of Carstensen and Verfürth [10].*

REMARK 4.3. *The choice of parameters can be guided by our theoretical results. The parameters θ, σ, and γ are arbitrary. The fact that γ is arbitrary for our convergence result indicates that the edge residuals play the dominant role in the overall refinement.*

In order to achieve optimal complexity, the marking parameter θ has to be small enough, as known from other complexity estimates [6, 20], and γ has to satisfy a condition, whereas σ is free.

5. Upper and lower bounds of the error. For the purpose of the later convergence and complexity proofs, we collect here some lemmata concerning upper and lower bounds of the estimators.

We start with upper and lower bounds between the edge estimator and the energy-error for the Poisson equation with right-hand side $g \in L^2(\Omega)$:

$$-\Delta w = g \text{ in } \Omega, \qquad w = 0 \text{ on } \partial\Omega. \tag{5.1}$$

These results form the basis of our following bounds for the optimal control problem. Whereas the upper and global bounds have a standard form, the local lower bound is different from the one commonly used for convergence proofs, since the inner node property is not assumed here.

LEMMA 5.1. ***(bounds state equation)*** *Consider the solution w of the Poisson equation (5.1) and its Ritz projection w_H on V_H. There exists a constant $\tilde{C}_1 > 0$ depending only on the minimum angle of h_0 such that*

$$|w - w_H|_1^2 \leq \tilde{C}_1 \left(J^2_H(w_H) + osc^2_H(g)\right) \tag{5.2}$$

and such that for a subset $\mathcal{F} \subset \mathcal{E}_H$ and $h = \mathcal{R}_{loc}(H, \mathcal{F})$ with $\mathcal{P}$ the set of nodes included in $\mathcal{F}$

$$|w_h - w_H|_1^2 \leq \tilde{C}_1 \left(J_H^2(w_H, \mathcal{F}) + osc_H^2(g, \mathcal{P}) \right), \tag{5.3}$$

and

$$\# \mathcal{F} \leq C_3 \, (N_h - N_H). \tag{5.4}$$

Furthermore there exists constants $\tilde{C}_2 > 0$ and $C_4 > 0$ depending only on the minimum angle of h_0 such that

$$J_H^2(w_H) \leq \tilde{C}_2 \left(|w - w_H|_1^2 + osc_H^2(g) \right), \tag{5.5}$$

and such that for $\mathcal{F} \subset \mathcal{E}_h$, $h = \mathcal{R}_{loc}(H, \mathcal{F})$ and arbitrary $\delta > 0$

$$J_h^2(w_h) \leq (1+\delta) J_H^2(w_H) - \frac{1+\delta}{2} J_H^2(w_H, \mathcal{F}) + C_4(1 + 1/\delta)|w_h - w_H|_1^2. \tag{5.6}$$

Proof. The proof of the upper bound (5.2) can be found in [9, 10], whereas the other results of the Lemma are contained in [2]. □

We now turn back to the optimal control problem. The next lemma provides global bounds of the error. We make the following hypothesis concerning the starting mesh h_0:

$$\max_{K \in \mathcal{K}_{h_0}} h_K \ \text{ is small enough.} \tag{5.7}$$

From now on, we suppose that hypothesis (5.7) is verified.

Lemma 5.2. ***(upper bounds)*** *There exists a constant $C_1 > 0$ depending only on the minimum angle of h_0 such that*

$$|u - u_H|_1^2 + \alpha^{-1}|z - z_H|_1^2 \leq C_1 \left(J_H^2(u_H) + \alpha^{-1} J_H^2(z_H) + osc_H^2(f) \right), \tag{5.8}$$

and such that for a subset $\mathcal{F} \subset \mathcal{E}_H$ and $h = \mathcal{R}_{loc}(H, \mathcal{F})$ with $\mathcal{P}$ the set of nodes included in $\mathcal{F}$

$$|u_h - u_H|_1^2 + \alpha^{-1}|z_h - z_H|_1^2 \leq C_1 \left(J_H^2(u_H, \mathcal{F}) + \alpha^{-1} J_H^2(z_H, \mathcal{F}) + osc_H^2(f, \mathcal{P}) \right) \tag{5.9}$$

and

$$\# \mathcal{F} \leq C_3 \, (N_h - N_H). \tag{5.10}$$

Proof. The proof follows basically from Lemma 5.1. In addition, we have to bound the terms $\mathrm{osc}_H(u - u_h)$ and $\mathrm{osc}_H(z^- - z_h^-)$. For the first term, we have

$$\begin{aligned} \mathrm{osc}_H(u - u_h)^2 &= \sum_{\xi \in \mathcal{N}_H} |\omega_\xi| |(u - u_h) - \pi_{\omega_\xi}(u - u_h)|_{\omega_\xi}^2 \leq \sum_{\xi \in \mathcal{N}_H} |\omega_\xi|^2 |u - u_h|_{1,\omega_\xi}^2 \\ &\leq C h_{max}(h)^4 |u - u_h|_1^2. \end{aligned}$$

For the second term, we remark that $z^- \in H^1(\Omega)$ and therefore as before

$$|(z^- - z_h^-) - \pi_{\omega_\xi}(z^- - z_h^-)|_{\omega_\xi}^2 \leq |\omega_\xi| |z^- - z_h^-|_{1,\omega_\xi}^2 \leq |\omega_\xi| |z - z_h|_{1,\omega_\xi}^2,$$

from which it follows that $\mathrm{osc}_H(z^- - z_h^-) \le h_{max}(h)^4 |z - z_h|_1^2$. We conclude with (5.7). □

The next lemma concerns lower bounds of the error for the optimal control system, which it follows from Lemma 5.1.

LEMMA 5.3. ***(lower bounds)*** *There exists a constant $C_2 > 0$ depending only on the minimum angle of h_0 such that for all $v_H \in V_H$*

$$J_H^2(u_H) + \alpha^{-1} J_H^2(z_H) \le C_2 \left(|u - u_H|_1^2 + \alpha^{-1}|z - z_H|_1^2 + osc_H^2(f) \right). \tag{5.11}$$

There exists a constant $C_4 > 0$ depending only on the minimum angle of h_0 such that for $\mathcal{F} \subset \mathcal{E}_h$, $h = \mathcal{R}_{loc}(H, \mathcal{F})$ and arbitrary $\delta > 0$

$$\begin{aligned} J_h^2(u_h) + \alpha^{-1} J_h^2(z_h) &\le (1+\delta) \left(J_H^2(u_H) + \alpha^{-1} J_H^2(z_H) \right) \\ &- \frac{1+\delta}{2} \left(J_H^2(u_H, \mathcal{F}) + \alpha^{-1} J_H^2(z_H, \mathcal{F}) \right) + C_4 (1 + 1/\delta) \left(|u_h - u_H|_1^2 + |z_h - z_H|_1^2 \right). \end{aligned} \tag{5.12}$$

Our last Lemma deals with the coupling due to control.

LEMMA 5.4. ***(coupling)*** *Under our hypothesis, there exists $0 < \kappa < 1$ such that for any $\varepsilon > 0$ there holds*

$$\begin{aligned} (1-\varepsilon) \left(|u - u_h|_1^2 + \alpha^{-1} |z - z_h|_1^2 \right) &\le \left(|u - u_H|_1^2 + \alpha^{-1} |z - z_H|_1^2 \right) - \\ &(1 - \frac{\kappa^2}{\varepsilon}) \left(|u_h - u_H|_1^2 + \alpha^{-1} |z_h - z_H|_1^2 \right). \end{aligned} \tag{5.13}$$

In addition, there holds

$$|u - u_H|_1^2 + \alpha^{-1}|z - z_H|_1^2 \le (1+\kappa^2) \left(|u - u_h|_1^2 + \alpha^{-1}|z - z_h|_1^2 \right) + (1+\kappa^2) \left(|u_h - u_H|_1^2 + \alpha^{-1} |z_h - z_H|_1^2 \right). \tag{5.14}$$

Proof. Hypothesis (5.7) implies the existence of $0 < \kappa < 1$ such that $\|I - R_h\|_{H^1 \to L^2} \le \kappa$. By the discrete optimality system, we have

$$\begin{aligned} |u - u_h|_1^2 + \alpha^{-1} |z - z_h|_1^2 &= \left(|u - u_H|_1^2 + \alpha^{-1} |z - z_H|_1^2 \right) - \left(|u_h - u_H|_1^2 + \alpha^{-1} |z_h - z_H|_1^2 \right) \\ &+ \frac{2}{\alpha} \int_{\Omega_B} (z^- - z_h^-)(u_h - u_H) + \frac{2}{\alpha} \int_{\Omega_C} (u - u_h)(z_h - z_H). \end{aligned}$$

We next pretend that

$$|z^- - z_h^-|_{\Omega_B} \le |z - z_h|_{\Omega_B}. \tag{5.15}$$

(5.15) implies that by Young's inequality

$$\begin{aligned} \frac{2}{\alpha} \int_{\Omega_B} (z^- - z_h^-)(u_h - u_H) &\le \frac{2}{\alpha} |z - z_h|_{\Omega_B} \, |u_h - u_H|_{\Omega_B} \le \frac{2\kappa^2}{\alpha} |z - z_h|_1 \, |u_h - u_H|_1 \\ &\le \frac{\varepsilon}{\alpha} |z - z_h|_1^2 + \kappa^2 |u_h - u_H|_1^2. \end{aligned}$$

The second term $\frac{2}{\alpha} \int_{\Omega_C} (u - u_h)(z_h - z_H)$ is treated similarly.

It remains to prove (5.15). Define $\Omega_A := \{x \in \Omega_B : \ z(x) z_h(x) \ge 0\}$. Then

$$\begin{aligned} \int_{\Omega_B} |z^- - z_h^-|^2 &= \int_{\Omega_A} |z^- - z_h^-|^2 + \int_{\Omega_B \setminus \Omega_A} |z^- - z_h^-|^2 \\ &\le \int_{\Omega_A} |z - z_h|^2 + \int_{\Omega_B \setminus \Omega_A} |z - z_h|^2, \end{aligned}$$

since on $\Omega_B \setminus \Omega_A$ one term of z^- and z_h^- vanishes, and this implies, in case $z^- = 0$ that $|z^- - z_h^-| = |z_h| \le |z_h - z|$. □

We remark that κ in (5.13) can be made arbitrarily small, provided $h_{max}(h_0)$ is sufficiently small.

6. Convergence proof. We prove error reduction with respect to the following error measure:

$$e_h := \left(|u-u_h|_1^2 + \alpha^{-1}|z-z_h|_1^2\right) + \beta_1 \left(J_h^2(u_h) + \alpha^{-1} J_h^2(z_h)\right) + \beta_2 \operatorname{osc}_h^2(f) \qquad (6.1)$$

for some constants $\beta_1 > 0$ and $\beta_2 > 0$.

THEOREM 6.1. *Let $\{h_k\}_{k\ge 0}$ be a sequence of meshes generated by algorithm $\mathcal{AFEM}$and let $\{u_k, z_k\}_{k\ge 0}$ be the corresponding sequence of finite element solutions. Under the hypothesis*

$$\kappa^2 \le \frac{\theta}{16\, C_1 C_4 (1+\frac{4}{\theta})}, \qquad (6.2)$$

there exist constants $\beta_1 > 0$, $\beta_2 > 0$, and $\rho < 1$ such that for all $k = 1, 2, \ldots$

$$e_{h_{k+1}} \le \rho\, e_{h_k}. \qquad (6.3)$$

REMARK 6.2. *For the convergence result of Theorem 1, γ, $\theta < 1$, and $\sigma < 1$ can be chosen arbitrarily. In order to verify our hypothesis (6.2), we should assume the initial mesh is fine enough.*

Proof. We use Lemma 5.4 and (5.12) of Lemma 5.3 in order to obtain

$$\begin{aligned}
&(1-\varepsilon)\left(|u-u_h|_1^2 + \alpha^{-1}|z-z_h|_1^2\right) + \beta_1\left(J_h^2(u_h) + \alpha^{-1}J_h^2(z_h)\right) + \beta_2 \operatorname{osc}_h^2(f) \le \\
&\left(|u-u_H|_1^2 + \alpha^{-1}|z-z_H|_1^2\right) - \left(1 - \frac{\kappa^2}{\varepsilon} - \beta_1 C_4(1+1/\delta)\right)\left(|u_h-u_H|_1^2 + \alpha^{-1}|z_h-z_H|_1^2\right) \\
&+ \beta_1(1+\delta)\left(J_H^2(u_H) + \alpha^{-1}J_H^2(z_H)\right) - \beta_1\frac{1+\delta}{2}\left(J_H^2(u_H,\mathcal{F}) + \alpha^{-1}J_H^2(z_H,\mathcal{F})\right) + \beta_2 \operatorname{osc}_H^2(f).
\end{aligned} \qquad (6.4)$$

We now split the proof into two parts depending on the two cases of the algorithm.

In the first case we have $\operatorname{osc}_H^2(f) \le \left(J_H^2(u_H) + \alpha^{-1}J_H^2(z_H)\right)$ and the refinement is made such that $\left(J_H^2(u_H,\mathcal{F}) + \alpha^{-1}J_H^2(z_H,\mathcal{F})\right) \ge \theta\left(J_H^2(u_H) + \alpha^{-1}J_H^2(z_H)\right)$. Using in addition the monotonicity of the oscillation term, (6.4) becomes

$$\begin{aligned}
&(1-\varepsilon)\left(|u-u_h|_1^2 + \alpha^{-1}|z-z_h|_1^2\right) + \beta_1\left(J_h^2(u_h) + \alpha^{-1}J_h^2(z_h)\right) + \beta_2 \operatorname{osc}_h^2(f) \le \\
&\left(|u-u_H|_1^2 + \alpha^{-1}|z-z_H|_1^2\right) - \left(1 - \frac{\kappa^2}{\varepsilon} - \beta_1 C_4(1+1/\delta)\right)\left(|u_h-u_H|_1^2 + \alpha^{-1}|z_h-z_H|_1^2\right) \\
&+ \beta_1(1+\delta-\theta\frac{1+\delta}{2})\left(J_H^2(u_H) + \alpha^{-1}J_H^2(z_H)\right) + \beta_2 \operatorname{osc}_H^2(f).
\end{aligned}$$

Under the condition

$$1 - \frac{\kappa^2}{\varepsilon} - \beta_1 C_4(1+1/\delta) \ge 0, \qquad (6.5)$$

we find

$$
\begin{aligned}
&(1-\varepsilon)\left(|u-u_h|_1^2+\alpha^{-1}|z-z_h|_1^2\right)+\beta_1\left(J_h^2(u_h)+\alpha^{-1}J_h^2(z_h)\right)+\beta_2\,\mathrm{osc}_h^2(f)\leq\\
&(1-\rho_1)\left(|u-u_H|_1^2+\alpha^{-1}|z-z_H|_1^2\right)+(1-\rho_2)\beta_1\left(J_H^2(u_H)+\alpha^{-1}J_H^2(z_H)\right)+(1-\rho_3)\beta_2\mathrm{osc}_H^2(f)\\
&+\rho_1\left(|u-u_H|_1^2+\alpha^{-1}|z-z_H|_1^2\right)+\rho_3\beta_2\mathrm{osc}_H^2(f)-\frac{1}{2}\theta\frac{1+\delta}{2}\beta_1\left(J_H^2(u_H)+\alpha^{-1}J_H^2(z_H)\right),
\end{aligned}
\tag{6.6}
$$

with positive ρ_1, ρ_3 to be determined below, and

$$
\rho_2=\theta\frac{1+\delta}{4}-\delta. \tag{6.7}
$$

Let us denote the sum in last line of (6.6) by A. Using the upper bound of the error and the condition of case one, we get

$$
\begin{aligned}
A&\leq(\rho_1C_1+\rho_3\beta_2)\mathrm{osc}_H^2(f)+(\rho_1C_1-\beta_1\theta\frac{1+\delta}{4})\left(J_H^2(u_H)+\alpha^{-1}J_H^2(z_H)\right)\\
&\leq\left(\gamma(\rho_1C_1+\rho_3\beta_2)+(\rho_1C_1-\beta_1\theta\frac{1+\delta}{4})\right)\left(J_H^2(u_H)+\alpha^{-1}J_H^2(z_H)\right).
\end{aligned}
\tag{6.8}
$$

It remains to impose the following four conditions: the term in brackets in the last line of (6.8) has to be negative, ρ_2 has to be positive, (6.5), and $(1-\rho_1)<(1-\varepsilon)$, i.e.

$$
\varepsilon<\rho_1. \tag{6.9}
$$

We now use our hypothesis (6.2). Setting $\tilde{\theta}=(\theta(4+\theta))/16$ and $\tilde{C}_4:=C_4(1+4\theta)$, we have

$$
\kappa^2\leq\frac{\theta(4+\theta)}{64(1+\gamma)C_1\tilde{C}_4}=\frac{\tilde{\theta}}{4(1+\gamma)C_1\tilde{C}_4}.
$$

This implies that $B:=\tilde{\theta}^2/(4(1+\gamma)C_1\tilde{C}_4)-\tilde{\theta}\kappa^2\geq 0$. We then define $\varepsilon:=\tilde{\theta}/(2(1+\gamma)C_1\tilde{C}_4)+\frac{1}{2}\sqrt{B}/\sqrt{(1+\gamma)C_1\tilde{C}_4}>0$. From this definition, it follows that $(1+\gamma)C_1\tilde{C}_4\varepsilon^2-\tilde{\theta}\varepsilon<-\tilde{\theta}\kappa^2$, which implies

$$
(1+\gamma)C_1\tilde{C}_4\varepsilon<\tilde{\theta}(1-\frac{\kappa^2}{\varepsilon}).
$$

We can therefore choose ρ_1 such that

$$
(1+\gamma)C_1\tilde{C}_4\varepsilon<(1+\gamma)C_1\tilde{C}_4\rho_1<\tilde{\theta}(1-\frac{\kappa^2}{\varepsilon}), \tag{6.10}
$$

which implies (6.9). From (6.10) it also follows that we can chose β_1 such that

$$
(1+\gamma)C_1\tilde{C}_4\rho_1<\tilde{\theta}\tilde{C}_4\beta_1<\tilde{\theta}(1-\frac{\kappa^2}{\varepsilon}). \tag{6.11}
$$

The left inequality of (6.11) implies that

$$
(1+\gamma)C_1\rho_1-\frac{\theta(4+\theta)}{16}\beta_1<0.
$$

We can therefore choose for arbitrary $\beta_2 > 0$ ρ_2 sufficiently small such that

$$(1+\gamma)C_1\rho_1 - \frac{\theta(4+\theta)}{16}\beta_1 + \gamma\beta_2\rho_2 < 0,$$

which implies the positivity of A (last line in (6.8)). The fact that β_2 is arbitrary up to now will be used in the second part of the proof.

The right inequality of (6.11) implies that

$$C_4(1+\frac{4}{\theta})\beta_1 \leq 1 - \frac{\kappa^2}{\varepsilon},$$

which is (6.5). This concludes the convergence proof in the first case.

Now we consider the second case. We have the following property concerning the oscillation term involving a constant $0 < \mu < 1$:

$$\operatorname{osc}_H^2(f) - \operatorname{osc}_h^2(f) \geq \mu \operatorname{osc}_H^2(f, \mathcal{P}). \tag{6.12}$$

This implies

$$\operatorname{osc}_h^2(f) \leq (1-\mu\sigma)\operatorname{osc}_H^2(f). \tag{6.13}$$

We obtain therefore from (6.4)

$$\begin{aligned}
&(1-\varepsilon)\left(|u-u_h|_1^2 + \alpha^{-1}|z-z_h|_1^2\right) + \beta_1\left(J_h^2(u_h) + \alpha^{-1}J_h^2(z_h)\right) + \beta_2 \operatorname{osc}_h^2(f) \leq \\
&\left(|u-u_H|_1^2 + \alpha^{-1}|z-z_H|_1^2\right) - \left(1 - \frac{\kappa^2}{\varepsilon} - \beta_1 C_4(1+1/\delta)\right)\left(|u_h-u_H|_1^2 + \alpha^{-1}|z_h-z_H|_1^2\right) \\
&+ \beta_1(1+\delta)\left(J_H^2(u_H) + \alpha^{-1}J_H^2(z_H)\right) + \beta_2(1-\mu\sigma)\operatorname{osc}_H^2(f).
\end{aligned} \tag{6.14}$$

Under the condition

$$1 - \frac{\kappa^2}{\varepsilon} - \beta_1 C_4(1+1/\delta) \geq 0, \tag{6.15}$$

and introducing positive constants ρ_1 and ρ_3 we have

$$\begin{aligned}
&(1-\varepsilon)\left(|u-u_h|_1^2 + \alpha^{-1}|z-z_h|_1^2\right) + \beta_1\left(J_h^2(u_h) + \alpha^{-1}J_h^2(z_h)\right) + \beta_2 \operatorname{osc}_h^2(f) \leq \\
&(1-\rho_1)\left(|u-u_H|_1^2 + \alpha^{-1}|z-z_H|_1^2\right) + \beta_1(1-\rho_2)\left(J_H^2(u_H) + \alpha^{-1}J_H^2(z_H)\right) + \beta_2(1-\frac{1}{2}\mu\sigma)\operatorname{osc}_H^2(f) \\
&+ \rho_1\left(|u-u_H|_1^2 + \alpha^{-1}|z-z_H|_1^2\right) + \beta_1(\delta+\rho_2)\left(J_H^2(u_H) + \alpha^{-1}J_H^2(z_H)\right) - \frac{1}{2}\beta_2\mu\sigma\operatorname{osc}_H^2(f).
\end{aligned} \tag{6.16}$$

Denote the last line of (6.16) by A. Using the global upper bound and $\left(J_H^2(u_H) + \alpha^{-1}J_H^2(z_H)\right) \leq \gamma^{-1}\operatorname{osc}_H^2(f)$ yields

$$\begin{aligned}
A &= \rho_1\left(|u-u_H|_1^2 + \alpha^{-1}|z-z_H|_1^2\right) + \beta_1(\delta+\rho_2)\left(J_H^2(u_H) + \alpha^{-1}J_H^2(z_H)\right) - \frac{1}{2}\beta_2\mu\sigma\operatorname{osc}_H^2(f) \\
&\leq \left(\rho_1 C_1 + \beta_1(\delta+\rho_2)\right)\left(J_H^2(u_H) + \alpha^{-1}J_H^2(z_H)\right) + \left(\rho_1 C_1 - \frac{1}{2}\beta_2\mu\sigma\right)\operatorname{osc}_H^2(f) \\
&\leq \left(\gamma^{-1}\left(\rho_1 C_1 + \beta_1(\delta+\rho_2)\right) + \rho_1 C_1 - \frac{1}{2}\beta_2\mu\sigma\right)\operatorname{osc}_H^2(f).
\end{aligned}$$

In order to obtain convergence we have to choose the different parameters in such a way that (6.15) as well as the following three inequalities are satisfied:

$$\rho_1 > \varepsilon, \tag{6.17}$$

$$\frac{1}{2}\beta_2\mu\sigma \geq (1+\gamma^{-1})\rho_1 C_1 + \gamma^{-1}\beta_1(\delta+\rho_2). \tag{6.18}$$

With the same choice of δ, ε, and β_1 condition (6.15) is verified in connection with (6.17) as before. It remains to choose β_2 sufficiently large in order to ensure (6.18). This is possible since β_2 was arbitrary in the first part of the proof. □

7. Complexity estimate. In order to express the optimal complexity, we introduce some notation from nonlinear approximation theory, developed in [6, 12]. Let $\mathcal{H}_N$ be the set of all triangulations h which satisfy $N_h \leq N$.

Next we define the approximation class

$$\mathcal{W}^s := \Big\{(u,z,f) \in (H_0^1(\Omega), L^2(\Omega)) : \|(u,z,f)\|_{\mathcal{W}^s} < +\infty\Big\}. \tag{7.1}$$

with

$$\|(u,z,f)\|_{\mathcal{W}^s} := \sup_{N\geq N_0} N^s \inf_{h\in H_N}\Big(|u-u_h|_1^2 + \alpha^{-1}|z-z_h|_1^2 + \mathrm{osc}_h^2(f)\Big).$$

We say that an adaptive finite element method realizes optimal convergence rates if, whenever $(u,z,f)\in\mathcal{W}^s$, it produces a triangulation h_k with dimension N_k and corresponding approximations u_k, z_k such that

$$|u-u_k|_1^2 + \alpha^{-1}|z-z_k|_1^2 \leq C N_k^{-s}. \tag{7.2}$$

Theorem 7.1. *Suppose $(f,u,z)\in\mathcal{W}^s$. Let $\{h_k\}_{k\geq 0}$ be a sequence of meshes generated by algorithm $\mathcal{AFEM}$and let $\{V_k\}_{k\geq 0}$ and $\{u_k,z_k\}_{k\geq 0}$ be the corresponding sequences of finite element spaces and solutions. Let $\varepsilon_k := |u-u_k|_1^2 + \alpha^{-1}|z-z_k|_1^2 + osc_k^2(f)$ and $N_k = \dim(V_k)$. Assuming the parameters γ and θ to satisfy*

$$\gamma \leq \frac{1}{4}C_2^{-1}(1+C_1)^{-1}, \quad \theta < \frac{1}{4}C_1^{-1}C_2^{-1}, \tag{7.3}$$

we have the following estimate on the complexity of the algorithm: there exists a constant C such that

$$N_k - N_0 \leq C\,\varepsilon_k^{-1/s}. \tag{7.4}$$

Proof. We use the same notation as in the convergence proof. Let in addition $\varepsilon_{h^*} = |u-u_{h^*}|_1^2 + \alpha^{-1}|z-z_{h^*}|_1^2 + \mathrm{osc}_{h^*}^2(f)$ and $\varepsilon_H = |u-u_H|_1^2 + \alpha^{-1}|z-z_H|_1^2 + \mathrm{osc}_H^2(f)$. From the regularity assumption we have existence of a mesh $h^*\in\mathcal{H}$ such that for $\lambda > 0$ to be chosen below

$$\varepsilon_{h^*} \leq \lambda\,\varepsilon_H, \tag{7.5}$$

and

$$N_{h^*} \leq C\varepsilon_H^{-1/s}. \tag{7.6}$$

Following the proof of Stevenson [20] (proof of Lemma 5.2), we can suppose that h^* is a refinement of H, if we replace (7.6) by:

$$N_{h^*} - N_H \le C\, \varepsilon_H^{-1/s}. \tag{7.7}$$

Let $\mathcal{F}^* \subset \mathcal{E}_h$ be the set of refined edges and let $\mathcal{P}^*$ be the set of corresponding nodes. In additon, $\mathcal{F}_k$ denotes the marked set of edges in iteration k.

We will prove below the estimate

$$\#\mathcal{F}_k \le C\, \varepsilon_k^{-1/s}. \tag{7.8}$$

This implies the complexity estimate (7.4) as follows. Let $e_l := \left(|u - u_{h_l}|_1^2 + \alpha^{-1}|z - z_{h_l}|_1^2\right) + \beta_1 \left(J_{h_l}^2(u_{h_l}) + \alpha^{-1} J_{h_l}^2(z_{h_l})\right) + \beta_2 \operatorname{osc}_h^2(f)$. From Theorem 6.1 we know that for some constant $\rho < 1$

$$e_k \le \rho^{k-l}\, e_l, \quad 0 \le l \le k.$$

We obviously have $\varepsilon_l \le \max(1, \beta_2) e_l$. By the global lower bound (5.11) we also have $e_l \le C\, \varepsilon_l$ with an absolute constant C. This implies

$$\varepsilon_k \le C\, \rho^{k-l}\, \varepsilon_l, \quad 0 \le l \le k. \tag{7.9}$$

The bound (7.9) and Lemma 4.1 imply

$$\begin{aligned} N_{k+1} - N_0 &\le C \sum_{l=0}^{k} \#\mathcal{F}_k \le C \sum_{l=0}^{k} \varepsilon_l^{-1/s} \\ &\le C \left(\sum_{l=0}^{k} \rho_l^{(k-l)/s} \right) \varepsilon_k^{-1/s} \le \frac{C}{1 - \rho^{1/s}}\, \varepsilon_k^{-1/s}. \end{aligned}$$

yielding (7.4).

We now turn the proof of (7.8). As before, we consider the two cases of the algorithm separately.

In the first case we have

$$\operatorname{osc}_H^2(f) \le \gamma \left(J_H^2(u_H) + \alpha^{-1} J_H^2(z_H) \right). \tag{7.10}$$

We will prove below that

$$J_H^2(u_{h^*}, \mathcal{F}^*) + \alpha^{-1} J_H^2(z_{h^*}, \mathcal{F}^*) \ge \theta \left(J_H^2(u_H) + \alpha^{-1} J_H^2(z_H) \right). \tag{7.11}$$

This implies the estimate (7.8): Since $\mathcal{F}$ is chosen to be the set with minimal cardinality satisfying the bound (7.11), we find that

$$\#\mathcal{F}_k \le \#\mathcal{F}^* \le C(N_{h^*} - N_k) \le C\, \varepsilon_k^{-1/s}. \tag{7.12}$$

The proof of (7.11) is obtained as follows. We successively use (5.11), (5.14), (7.5), (5.9), and (7.10), introducing a parameter $a = (1 - (1 + \kappa^2)\lambda)^{-1}$, in order to obtain

$$
\begin{aligned}
&C_2^{-1}\left(J_H^2(u_H)+\alpha^{-1}J_H^2(z_H)\right) \le |u-u_H|_1^2+\alpha^{-1}|z-z_H|_1^2+\mathrm{osc}_H^2(f)\\
&= a\left(|u-u_H|_1^2+\alpha^{-1}|z-z_H|_1^2\right)+(1-a)\left(|u-u_H|_1^2+\alpha^{-1}|z-z_H|_1^2\right)+\mathrm{osc}_H^2(f)\\
&\le a(1+\kappa^2)\left(\left(|u-u_h^*|_1^2+\alpha^{-1}|z-z_h^*|_1^2\right)+\left(|u_{h^*}-u_H|_1^2+\alpha^{-1}|z_{h^*}-z_H|_1^2\right)\right)\\
&\quad +(1-a)\left(|u-u_H|_1^2+\alpha^{-1}|z-z_H|_1^2\right)+\mathrm{osc}_H^2(f)\\
&\le a(1+\kappa^2)\lambda\left(|u-u_H|_1^2+\alpha^{-1}|z-z_H|_1^2\right)+a(1+\kappa^2)\left(|u_{h^*}-u_H|_1^2+\alpha^{-1}|z_{h^*}-z_H|_1^2\right)\\
&\quad +(1-a)\left(|u-u_H|_1^2+\alpha^{-1}|z-z_H|_1^2\right)+(1+a(1+\kappa^2)\lambda)\mathrm{osc}_H^2(f)\\
&\le \left(a(1+\kappa^2)\lambda+1-a\right)\left(|u-u_H|_1^2+\alpha^{-1}|z-z_H|_1^2\right)+(1+a(1+\kappa^2)\lambda)\mathrm{osc}_H^2(f)\\
&\quad +a(1+\kappa^2)\left(|u_{h^*}-u_H|_1^2+\alpha^{-1}|z_{h^*}-z_H|_1^2\right)\\
&\le (1+a(1+\kappa^2)\lambda)\mathrm{osc}_H^2(f)\\
&\quad +a(1+\kappa^2)C_1\left(J_H^2(u_H,\mathcal{F}^*)+\alpha^{-1}J_H^2(z_H,\mathcal{F}^*)\right)+a(1+\kappa^2)C_1\mathrm{osc}_H^2(f,\mathcal{P}^*)\\
&\le (1+a(1+\kappa^2)(\lambda+C_1))\gamma\left(J_H^2(u_H)+\alpha^{-1}J_H^2(z_H)\right)+a(1+\kappa^2)C_1\left(J_H^2(u_H,\mathcal{F}^*)+\alpha^{-1}J_H^2(z_H,\mathcal{F}^*)\right)\\
&= (1+(\frac{1}{1+\kappa^2}-\lambda)^{-1}(\lambda+C_1))\gamma\left(J_H^2(u_H)+\alpha^{-1}J_H^2(z_H)\right)\\
&+(\frac{1}{1+\kappa^2}-\lambda)^{-1}C_1\left(J_H^2(u_H,\mathcal{F}^*)+\alpha^{-1}J_H^2(z_H,\mathcal{F}^*)\right).
\end{aligned}
$$

Choosing $\lambda=\frac{1}{1+\kappa^2}-\frac{1}{2}$, it follows that

$$
\left(C_2^{-1}-2\gamma(\lambda+C_1))\right)\left(J_H^2(u_H)+\alpha^{-1}J_H^2(z_H)\right)\le 2C_1\left(J_H^2(u_H,\mathcal{F}^*)+\alpha^{-1}J_H^2(z_H,\mathcal{F}^*)\right). \tag{7.13}
$$

By the assumption on γ $(7.3)_1$ it follows that

$$
\begin{aligned}
\frac{1}{2}C_1^{-1}\left(C_2^{-1}-2\gamma(\lambda+C_1))\right) &\ge \frac{1}{2}C_1^{-1}\left(C_2^{-1}-2\gamma(1+C_1))\right)\\
&\ge \frac{1}{4}C_1^{-1}C_2^{-1}\ge\theta.
\end{aligned}
$$

where have used the assumption on θ $(7.3)_2$ in the last inequality. This completes the proof in the first case.

Now we consider the second case. We thus have

$$
\left(J_H^2(u_H)+\alpha^{-1}J_H^2(z_H)\right)\le\gamma^{-1}\mathrm{osc}_H^2(f). \tag{7.14}
$$

We will prove that

$$
\mathrm{osc}_H^2(f,\mathcal{P}^*)\ge\sigma\mathrm{osc}_H^2(f). \tag{7.15}
$$

This implies (7.8) as before by the optimality of the choice of $\mathcal{P}$. First we note that by (5.11) and (7.15) we have

$$
\begin{aligned}
\left(|u-u_H|_1^2+\alpha^{-1}|z-z_H|_1^2\right) &\le C_1\left(J_H^2(u_H)+\alpha^{-1}J_H^2(z_H)+\mathrm{osc}_H^2(f)\right)\\
&\le C_1(1+\gamma^{-1})\mathrm{osc}_H^2(f).
\end{aligned}
$$

This implies together with (7.5) that

$$
\begin{aligned}
\mathrm{osc}_H^2(f)-\mathrm{osc}_H^2(f,\mathcal{P}^*) &\le \mathrm{osc}_{h^*}^2(f)\\
&\le \lambda\left(|u-u_H|_1^2+\alpha^{-1}|z-z_H|_1^2+\mathrm{osc}_H^2(f)\right)\\
&\le \lambda\left(1+C_1(1+\gamma^{-1})\right)\mathrm{osc}_H^2(f),
\end{aligned}
$$

and therefore with λ small enough

$$\sigma\, \mathrm{osc}_H^2(f) \leq \left(1 - \lambda\left(1 + C_1(1+\gamma^{-1})\right)\right) \mathrm{osc}_H^2(f) \leq \mathrm{osc}_H^2(f, \mathcal{P}^*)$$

This concludes the proof. ☐

COROLLARY 7.2. *The algorithm $\mathcal{AFEM}$, combined with multigrid iteration [8, 22], has optimal work count in the sense that for a given accuracy $\varepsilon > 0$, the algorithm provides a discrete solution u_h satisfying $|u - u_h|_1 \leq \varepsilon$ with a number of operations proportional to $\varepsilon^{-1/s}$. The combination of the adaptive algorithm with multigrid requires the introduction of a stopping criterion leading to an additional iteration error. Such an algorithm has been proposed and analyzed in [3].*

We finally remark that the regularity assumption $(f, u, z) \in \mathcal{W}^s$ is difficult to verify in practice. However, the a priori error analysis on meshes adapted to corner singularities suggests that $s = 1$ if $f \in L^2(\Omega)$ under mild restrictions on the domain, at least in the two-dimensional case.

8. Numerical experiments. In this section we report on two numerical experiments. The first one has an exact solution and is used in order to investigate the complexity of the sequence of meshes generated by the adaptive algorithm. The computational domain is $\Omega = (0,1)^2$ and the right-hand side is constructed in such a way that $u(x,y) = z(x,y) = \sin(\pi(x+2y))$. The parameter is α and $Q = \left\{q \in L^2(\Omega) :\ q_{min} \leq q \leq q_{max}\right\}$ with $q_{min} = -50 = -q_{max}$ such that the control has the appearance shown in Figure 8.1.

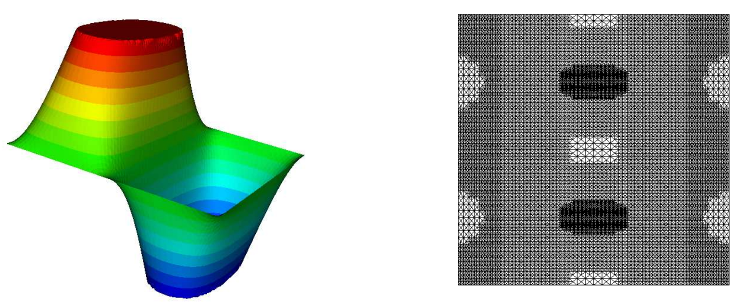


FIG. 8.1. *Control* $-\min(50, \max(-50, -z_h))/\alpha$ *and locally refined mesh.*

In Table 8 the value of the errors and estimators are given on a typical adaptive iteration. The oscillation term has the expected second-order behavior. The error is over-estimated by a factor of 3.5. This is due to the fact that we have arbitrarily set the constant of the interpolation error to 1.

The computational domain for the second example is the L-shaped domain $\Omega = (-1,1) \times (0,1) \cup (-1,0) \times (-1,0]$. The parameters are $\alpha = 10^{-4}$ and $q_{min} = -10$ $(q_{max} = \infty)$, $f = 0$, $u_d = -1$, $\Omega_B = \Omega \cap \{y \geq 0\}$, and $\Omega_C := \Omega \cap \{x \leq 0\}$. The discrete solutons u_h, z_h, corresponding control, and a typical mesh are shown in Figure 8.2. There is a strong refinement at the re-entrant corner and a long the boundary of $\partial\Omega_B \setminus \partial\Omega$. The first is however significantly stronger, which is due to the fact that it generates a stronger singularity.

N_h	e_h	J_h	osc_h	e_h/J_h
100	1.3686	0.9340	4.6864	0.292
364	0.7168	0.2609	2.5108	0.285
1244	0.4042	0.0844	1.4371	0.281
4091	0.2275	0.0453	0.8019	0.283
10752	0.1382	0.0215	0.4846	0.285
24280	0.0871	0.0069	0.3094	0.281
70468	0.0534	0.0026	0.1886	0.283
201632	0.0321	0.0010	0.1132	0.284
457800	0.0204	0.0005	0.0723	0.282

TABLE 8.1
Adaptive iteration with $\theta = 0.75$. $e_h := \sqrt{|u-u_h|_1 + |z-z_h|_1}$.

Finally, we make a comparison of the asymptotic behavior of J_h^2 for different refinement parameters θ. Note that $\theta = 1$ leads to uniform refinement, which is known to lead to a loss of convergence rate due to the corner singularity. It can be seen from Figure 8.3 that the adaptive algorithm is able to regain the convergence rate -1 (for the square of the energy error). This follows from Theorem 7.1, since the construction of meshes recovering this rate is well known, which implies that $u \in \mathcal{A}^s$ with $s = 1$.

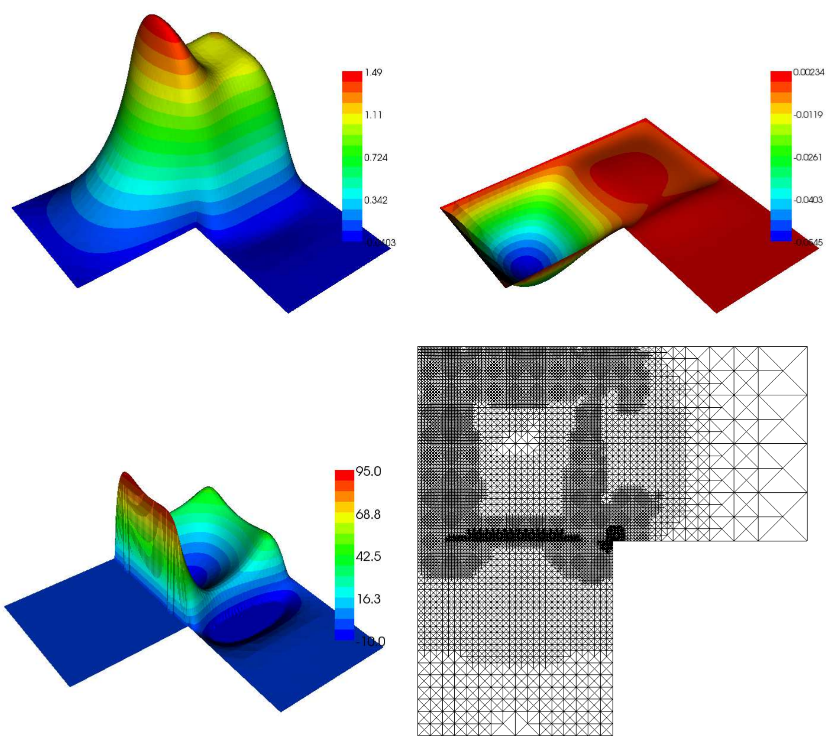
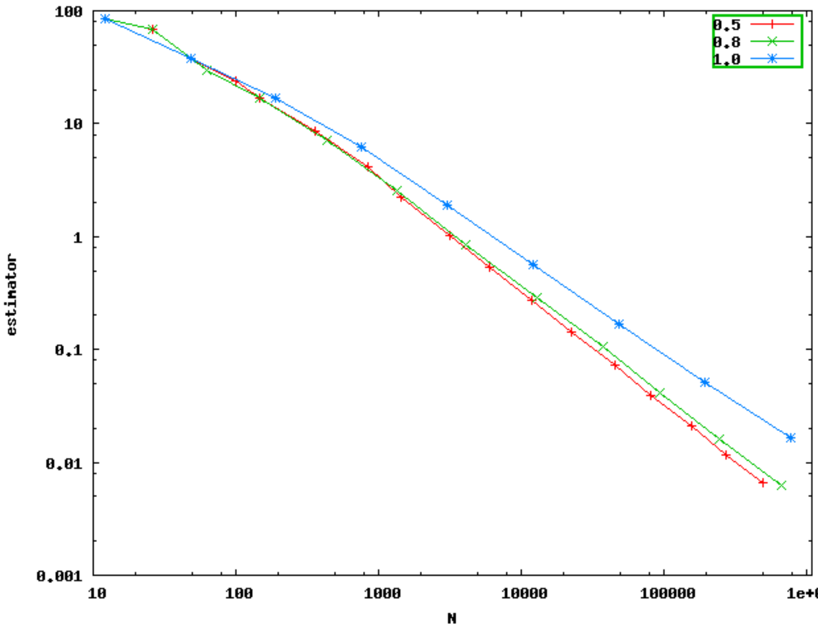


FIG. 8.2. *Second example: u_h, z_h (scaled by a factor of 10), contol (scaled by a factor 0.01) and locally refined mesh.*



2mm

FIG. 8.3. *Behavior of* J_h^2 *vs.* N_h *for* $\theta = 0.5, 0.8, 1.0$ *(log-log-scale).*

9. Conclusion. We have proposed a new adaptive algorithm for optimal control based on standard conforming finite elements with standard local mesh refinement. This algorithm uses an adaptive marking strategy.

We have carried out the proofs of geometric convergence of the error and asymptotic complexity of the resulting meshes in the case that the control variable is eliminated from the system. The generalization to the more commonly studied case of Galerkin discretization of all variables including control, at least if we follow the approach of [17] seems to be straightforward.

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