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Abstract:
Data Assimilation is a methodological framework used in environmental sciences to perform forecasts with complex systems such as meteorological, oceanographic and air quality models. Data Assimilation requires the resolution of a system with three components: one describing the temporal evolution of the state vector, one coupling the observations and the state vector, and one defining the initial condition. In this article we use this mathematical framework to study a class of ill-posed Image Processing problems, which are usually solved using regularization techniques. To this end, the ill-posed problem is formulated according to the three-component system of the Data Assimilation framework. To illustrate the method, an application for computing optical flow is described.

Key-words: data assimilation, iill-posed problem, regularization, image assimilation, optical flow, motion.

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Résolution des problèmes mal posés du traitement d'image par Assimilation de Données. Application au flot optique

Résumé : L'Assimilation de Données est un cadre méthodologique utilisé en sciences de l'environnement pour la prédiction des systèmes complexes, que sont les modèles météorologiques, océanographiques, ou encore de qualité de l'air. L'Assimilation de Données opère en résolvant un système à trois composantes: une première équation décrit l'évolution temporelle du vecteur d'état; une seconde équation décrit la relation entre le vecteur d'état et l'observation; enfin, une dernière équation décrit la condition initiale. Dans ce rapport, nous utilisons ce cadre mathématique pour l'étude des problèmes mal posés du traitement d'image, habituellement résolus par des techniques de régularisation. Pour cela, le problème est formulé au moyen du système à trois composantes de l'Assimilation de Données. Comme illustration, cette approche est appliquée au calcul du flot optique.

Mots-clés : assimilation de données, problème mal posés, régularisation, assimilation d'images, flot optique, mouvement.

1 Introduction

In the research field of Image Processing, most problems are ill-posed in the sense that it is not possible to provide a unique solution [Hadamard, 1923]. A first cause of ill-posedness is that the equations used to model image properties are under-determined. An example is the famous "aperture problem" occurring in the estimation of optical flow: a further constraint is required to compute a unique field of velocity vectors. As an image processing problem is usually represented by a system of equations to be solved, the so-called *Image Model*, this type of ill-posedness means that this Image Model is not invertible. A second cause of ill-posedness (and a common problem in Image Processing) is that the extraction of image features is rarely obtained in a unique way. For example, the computation of image gradient is ill-posed as it requires approximating a differential operator by a discrete one among several possible finite difference formulations; each one will give a different result. In such a case, the problem is ill-posed because the uniqueness of the solution is clearly unverified.

One possible strategy to solve ill-posed problems is to provide the Image Model with additional information. Two possibilities occur. 1) Explicit information: additional images are used to enlarge the set of data, but this is generally not possible because no further acquisition is available. 2) Implicit information: either hypotheses on image properties or constraints on the solution can be used. An example of a constraint is to restrict the dimension of the space of the admissible solutions: this can be done by searching for the result among the functions which have bounded spatial variations [Tikhonov, 1963]. In the general case, image properties and constraints are expressed as equations added to the Image Model in order to obtain a new one which will be invertible. This paper focuses on the methods constraining the function space, the so-called "Tikhonov's regularizing methods", and their formulation in a new methodological framework.

Nowadays, most acquisition systems produce sequences of images which provide information on the dynamic of the objects observed. This knowledge is then used to enhance the problem modelling. Let us consider the example of image segmentation. A regularization method, such as Shah-Mumford's functional [Mumford and Shah, 1989], produces a segmentation which is a compromise between the regularity of the solution and the low discrepancy with the input. The dynamic estimated from the sequence of images is then exploited to manage missing data. Indeed, a result in the areas where data are not available is estimated by linear interpolation of those obtained on the adjacent frames. Linear interpolation is, however, restrictive as it is only relevant for structures displaying a linear dynamic. To overcome this difficulty, segmentation could be performed directly on the whole sequence and not on a frame-by-frame basis; following Weickert, the solution [Weickert and Schnörr, 2001] is searched for as a function depending on the spatial and temporal coordinates. In this case, the Image Model will remain but will depend on the time coordinate and the regularization constraint will bound the variations of the solution simultaneously in space and time. Wieckert's method has three main drawbacks. First, spatio-temporal regularization correctly describes regular dynamics but a complex configuration, such as discontinuity or non-linearity, will not be handled correctly. Second, pixels with aberrant values introduce errors in their spatio-temporal neighbourhood because the method provides a smoothed result. Third, as the solution is searched for in the spatio-temporal domain, this leads to a significant increase in complexity, which is a limiting factor for operational applications. A model of the structures' temporal dynamics, called the "evolution model", has to be stated from the *a priori* knowledge on the data or on the physical properties of the structures. The evolution model will be used to obtain good results even if observations are missing.

In this paper, we propose to use the *Data Assimilation* framework as a generic tool to solve ill-posed problems in Image Processing. Although, this approach is not new, as some algorithms have already been proposed to perform curve tracking [Papadakis et al., 2005] and motion estimation using physical models [Papadakis and Mémin, 2008, Herlin et al., 2006], we, however, propose a general framework. Data Assimilation solves a system of three components with respect to a state vector representing the solution of the problem:

- one is an evolution equation describing the evolution of the state vector over time using an operator called the "evolution model". The evolution model provides the description of the images' temporal dynamics.
- another, called "observation equation", describes the link between the state vector and the observations included in the sequence of images,
- and the third describes the initial condition.

For each term, a description of the errors is stated as a covariance matrix. This matrix depicts the dependencies that exist between the components of the state vector on the one hand, and between two different locations in the space-time domain on the other hand.

Data Assimilation is a suitable approach to solve the three drawbacks of Weickert's method because: the evolution model provides an accurate description of images' temporal dynamics; the matrix associated to the observation equation overcomes the problem of missing data by weighting the importance of the observation equation in the computation of the solution; Data Assimilation is a frame-by-frame process reducing memory allocation.

This paper is organized as follow. Section 2 describes the variational Data Assimilation method known as the 4D-Var algorithm. How Data Assimilation can be used to solve ill-posed problems by assimilating image data within an appropriate evolution model is explained in Section 3. Section 4 is a direct application of Section 3 and describe a method to compute optical flow. The evolution model used in this case is the transport of velocity. Section 4 also presents and analyzes experimental results. We conclude in Section 5 and give some scientific perspectives to this work.

2 The Data Assimilation framework

The Data Assimilation framework aims to solve the system (1,2,3) with respect to a state vector $\mathbf{X}(\mathbf{x},t)$, depending on the spatial coordinate $\mathbf{x}$ and time t:

$$\frac{\partial \mathbf{X}}{\partial t}(\mathbf{x},t) + \mathbb{M}(\mathbf{X})(\mathbf{x},t) = \mathcal{E}_m(\mathbf{x},t) \tag{1}$$

$$\mathbf{X}(\mathbf{x},0) = \mathbf{X}_b(\mathbf{x}) + \mathcal{E}_b(\mathbf{x}) \tag{2}$$

$$\mathbf{Y}(\mathbf{x},t) = \mathbb{H}(\mathbf{X})(\mathbf{x},t) + \mathcal{E}_O(\mathbf{x},t) \tag{3}$$

Equation (1) describes the temporal evolution of $\mathbf{X}$. $\mathbb{M}$, called the *evolution model* is generally differential and possibly non linear. As $\mathbb{M}$ describes approximately the evolution of the state vector, the *model error* $\mathcal{E}_m$ is introduced to quantify the difference. Equation (2) states the initial condition of the state vector and the error is quantified by the *background error* $\mathcal{E}_b$. Equation (3), called the *observation equation*, describes the link between the observation vector, $\mathbf{Y}$, and the state vector. The observation may be an incomplete measure of the state vector, in which case $\mathbb{H}$ is a projection operator on a discrete and finite space. In other cases, the observation may be an indirect, and sometimes complex, measure of the state vector. Equation (3) is the standard form of the observation equation used in the Data Assimilation literature. This formulation is quite restrictive to describe the complex link between the observation and the state vector. To be more general, the following will be used in is article:

$$\mathbb{H}(\mathbf{Y},\mathbf{X})(\mathbf{x},t) = \mathcal{E}_O(\mathbf{x},t) \tag{4}$$

The *observation error* $\mathcal{E}_O$ represents the imperfection of $\mathbb{H}$ and the measurement errors. The errors $\mathcal{E}_m$, $\mathcal{E}_b$ and $\mathcal{E}_O$ are assumed to be Gaussian vectors and fully characterized by their covariance matrices Q, R and B.

Let X denote a Gaussian stochastic vector depending on a space-time coordinate $(\mathbf{x},t)$, $X = X(x,t)$ and $X' = X(x',t')$. As the covariance matrix Σ measures the dependency between X and X', it is defined by:

$$\Sigma(\mathbf{x},t,\mathbf{x}',t') = \iint (\mathbf{X} - \mathbb{E}\mathbf{X})^T(\mathbf{X}' - \mathbb{E}\mathbf{X}')dP_{\mathbf{X},\mathbf{X}'} \tag{5}$$

where $P_{\mathbf{X},\mathbf{X}'}$ is the joint distribution of $(\mathbf{X},\mathbf{X}')$ and $\mathbb{E}$ denotes the expectation. $P_{\mathbf{X},\mathbf{X}'}$ is a Gaussian distribution depending on variables $\mathbf{x}$, t, $\mathbf{x}'$ and t'. The inverse of a covariance matrix is formally and implicitly defined [Oliver, 1998] by:

$$\iint \Sigma^{-1}(\mathbf{x},t,\mathbf{x}'',t'')\Sigma(\mathbf{x}'',t'',\mathbf{x}',t')d\mathbf{x}''dt'' = \delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \tag{6}$$

In order to solve equations (1), (2) and (4), the following functional is defined:

$$\begin{aligned} E(\mathbf{X}) = & \int_A\int_A \big(\frac{\partial\mathbf{X}}{\partial t} + \mathbb{M}(\mathbf{X})\big)^T(\mathbf{x},t)Q^{-1}(\mathbf{x},t,\mathbf{x}',t')\big(\frac{\partial\mathbf{X}}{\partial t} + \mathbb{M}(\mathbf{X})\big)(\mathbf{x}',t')d\mathbf{x}dtd\mathbf{x}'dt' \\ & + \int_A\int_A \mathbb{H}(\mathbf{X},y)^T(\mathbf{x},t)R^{-1}(\mathbf{x},t,\mathbf{x}',t')\mathbb{H}(\mathbf{X},y)(\mathbf{x}',t')d\mathbf{x}dtd\mathbf{x}'dt' \\ & + \int_\Omega\int_\Omega \big(\mathbf{X}(\mathbf{x},0) - \mathbf{X}_b(\mathbf{x})\big)^T B^{-1}(\mathbf{x},\mathbf{x}')\big(\mathbf{X}(\mathbf{x}',0) - \mathbf{X}_b(\mathbf{x}')\big)d\mathbf{x}d\mathbf{x}' \end{aligned} \tag{7}$$

with $A = \Omega\times[0,\mathbf{T}]$, Ω being the spatial domain and $[0,\mathbf{T}]$ the temporal domain. If $\mathcal{E}_m$, $\mathcal{E}_b$ and $\mathcal{E}_O$ are assumed to be independent, the functional E represents the log-likelihood of $\mathbf{X}$. The function minimizing E is therefore a maximum likelihood estimator. This minimization is carried out using a calculus of variation: the differential of E with respect to $\mathbf{X}$, denoted $\frac{\partial E}{\partial\mathbf{X}}$, is calculated and the associated Euler-Lagrange equations are solved. $\frac{\partial E}{\partial\mathbf{X}}$ is obtained by computing the derivative of E with respect to $\mathbf{X}$ in the direction ψ:

$$\frac{\partial E}{\partial\mathbf{X}}(\psi) = \frac{\partial E}{\partial\mathbf{X}}^T\psi = \lim_{\gamma\to 0}\frac{d}{d\gamma}(E(\mathbf{X}+\gamma\psi)) \tag{8}$$

and by introducing an auxiliary variable λ, called the *adjoint variable* in the literature of Data Assimilation:

$$\lambda(\mathbf{x},t) = \int_A Q^{-1}(\mathbf{x},t,\mathbf{x}',t)\left(\frac{\partial \mathbf{X}}{\partial t} + \mathbb{M}(\mathbf{X})\right)(\mathbf{x}',t')d\mathbf{x}'dt \tag{9}$$

The reader is referred to [Valur Hólm, 2003] for details on the determination of the differential of E and the Euler-Lagrange equations. This process leads to the following system:

$$\lambda(\mathbf{x},\mathbf{T}) = 0 \tag{10}$$

$$-\frac{\partial \lambda}{\partial t} + \left(\frac{\partial \mathbb{M}}{\partial \mathbf{X}}\right)^* \lambda = -\int_A \left(\frac{\partial \mathbb{H}}{\partial \mathbf{X}}\right)^* (\mathbf{x},t)R^{-1}\mathbb{H}(\mathbf{X},\mathbf{Y})(\mathbf{x}',t')d\mathbf{x}'dt' \tag{11}$$

$$\mathbf{X}(\mathbf{x},0) = \int_\Omega B\lambda(\mathbf{x}',0)d\mathbf{x}' + \mathbf{X}_b(\mathbf{x}) \tag{12}$$

$$\frac{\partial \mathbf{X}}{\partial t} + \mathbb{M}(\mathbf{X}) = \int_A Q\lambda(\mathbf{x}',t')d\mathbf{x}'dt' \tag{13}$$

Equation (11) makes use of *adjoint operators* denoted $\left(\frac{\partial \cdot}{\partial \mathbf{X}}\right)^*$ and formally defined as the differential in the dual space of $\mathbf{X}$, by:

$$\int \left(\frac{\partial \mathbb{M}}{\partial \mathbf{X}}(\phi)\right)^T \psi d\mathbf{x}dt = \int \phi^T \left(\frac{\partial \mathbb{M}}{\partial \mathbf{X}}\right)^* (\psi)d\mathbf{x}dt \tag{14}$$

for all integrable functions ϕ and ψ. The adjoint operator represents a compact notation for integration by parts. Riesz's theorem assumes the existence and uniqueness of the adjoint operator.

A difficulty occurs: equations (10,11) allow the state vector to determine the adjoint variable, and equations (12,13) allow the adjoint variable to determine the state vector. To break this deadlock, an incremental algorithm is used: the state vector is written as $\mathbf{X}_b + \delta\mathbf{X}$ where $\mathbf{X}_b$ is a mean term, called the background variable in the Data Assimilation literature, and $\delta\mathbf{X}$ is an incremental term. $\mathbf{X}$ is then replaced by $\mathbf{X}_b + \delta\mathbf{X}$ in equations (11), (12) and (13). Then, the operators $\mathbb{M}$ and $\mathbb{H}$, not necessarily linear, have to be linearized using a first order Taylor expansion: $\mathbb{M}(\mathbf{X}) \simeq \mathbb{M}(\mathbf{X}_b) + \frac{\partial \mathbb{M}}{\partial \mathbf{X}_b}(\delta\mathbf{X})$. This leads to the following new system of equations:

$$\lambda(\mathbf{x},\mathbf{T}) = 0 \tag{15}$$

$$-\frac{\partial \lambda}{\partial t} + \left(\frac{\partial \mathbb{M}}{\partial \mathbf{X}_b}\right)^* \lambda = -\int_A \left(\frac{\partial \mathbb{H}}{\partial \mathbf{X}_b}\right)^* R^{-1}\left(\mathbb{H}(\mathbf{X}_b,\mathbf{Y}) + \frac{\partial \mathbb{H}}{\partial \mathbf{X}_b}(\delta\mathbf{X})\right)d\mathbf{x}'dt' \tag{16}$$

$$\mathbf{X}_b(\mathbf{x},0) = \mathbf{X}_b(\mathbf{x}) \tag{17}$$

$$\frac{\partial \mathbf{X}_b}{\partial t} + \mathbb{M}(\mathbf{X}_b) = 0 \tag{18}$$

$$\delta\mathbf{X}(\mathbf{x},t) = \int_\Omega B\lambda(\mathbf{x}',0)d\mathbf{x}' \tag{19}$$

$$\frac{\partial \delta\mathbf{X}}{\partial t} + \frac{\partial \mathbb{M}}{\partial \mathbf{X}_b}(\delta\mathbf{X}) = \int_A Q\lambda(\mathbf{x}',t')d\mathbf{x}'dt' \tag{20}$$

The adjoint variable is now obtained from the background variable $\mathbf{X}_b$ using equations (15) and (16). The background variable is calculated from equations (17) and (18) independently of the adjoint variable and the state vector. The incremental variable $\delta\mathbf{X}$ is obtained from the adjoint variable using equations (19) and (20). Due to the linearization of $\mathbb{M}$ and $\mathbb{H}$, equations (15,16,19,20) produce an approximated solution $\delta\mathbf{X} + \mathbf{X}_b$. The function minimizing E (equation (7)) is obtained using an iterative algorithm: at each iteration, the value $\delta\mathbf{X}$ is computed from the background variable $\mathbf{X}_b$ and the value $\delta\mathbf{X} + \mathbf{X}_b$ becomes the background variable for the next iteration.

3 Image Assimilation

This section explains how to solve ill-posed Image Processing problems using the image assimilation. Obviously, the observations to be used in the observation equation will be images, but the nature of the state vector will depend on the problem under study. For example, segmentation, denoising and restoration use a state vector which is an image. Tracking, image registration and motion estimation use a vector field. Active contours use a curve. To solve an Image Processing problem using the 4D-Var algorithm as described in Section 2, a suitable evolution model (describing the temporal evolution of the state vector) and its error should first be chosen to constrain the state vector. Next, an observation equation (describing the link between the state vector and the image) and its error should characterize the properties of observations. Lastly, the initial condition and its background error should be defined. Although these choices generally depend on the context, we will however describe some general principles in the next subsections.

3.1 The evolution model

One first choice is to impose a temporal regularity on the state vector. This property is modeled as the transport equation $\frac{d\mathbf{X}}{dt} = 0$, depending on the space-time coordinates. Using the chain rule, we obtain:

$$\frac{\partial \mathbf{X}}{\partial t} + \frac{\partial \mathbf{X}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial t} = 0 \tag{21}$$

The evolution model is then $\mathbb{M}(\mathbf{X})(\mathbf{x}, t) = \frac{\partial \mathbf{X}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial t}$. Usually $\frac{\partial \mathbf{x}}{\partial t} = 0$ and this model becomes unsuitable. However, in the case of determining optical flow, the spatial coordinates stand for a trajectory and $\frac{\partial \mathbf{x}}{\partial t}$ becomes a velocity vector. An example of using equation (21) as an evolution equation is given in Section 4.

Another choice of evolution model is to express the transport of the state vector as a diffusion, a physical law ruling the transport of chemical species or temperature. The general formulation is: $\frac{\partial \mathbf{X}}{\partial t} = \nabla^T (D \nabla \mathbf{X})$ with $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})^T$ the gradient operator of $\mathbb{R}^2$. The matrix D is a tensor characterizing the direction and the intensity of the diffusion. It is possible to drive the diffusion according to the image characteristics. A standard example is the Perona &

Malik diffusion: the tensor matrix is equal to $c(\|\nabla \mathbf{X}\|)Id$ with c a Gaussian function and Id the identity matrix. One of the properties of this diffusion is to smooth the image while preserving the contours. In this case, the evolution model is $\mathbb{M}(\mathbf{X}) = -\nabla^T(D\nabla\mathbf{X})$.

In addition to the evolution model, the matrix Q in equation (13) has to be specified and it is important to understand the action of Q^{-1} in the functional (7). Let us consider two possible choices for Q and define their impact in a functional $\iint F^T(\mathbf{X})Q^{-1}F(\mathbf{X})d\mathbf{x}d\mathbf{x}'$ which has to be minimized.
As a first example, let Q be the Dirac covariance defined by $Q(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')$. The Dirac function has the property of being its own functional inverse:

$$\int_\Omega \delta(x')\,\delta(x-x')\,dx' = \delta(x)$$

So, we have $Q^{-1}(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')$ and:

$$\begin{aligned} &\iint_{\Omega^2} F(\mathbf{X})^T(\mathbf{x})Q^{-1}(\mathbf{x},\mathbf{x}')F(\mathbf{X})(\mathbf{x}')d\mathbf{x}d\mathbf{x}' \qquad (22)\\ =& \int_\Omega F(\mathbf{X})^T(\mathbf{x})F(\mathbf{X})(\mathbf{x})d\mathbf{x} = \int_\Omega \|F(\mathbf{X})\|^2\,d\mathbf{x}\end{aligned}$$

A Dirac covariance corresponds to a L^2 regularization.
A second example is the exponential covariance defined by $Q(\mathbf{x},\mathbf{x}') = \exp(-\dfrac{\mathbf{x}-\mathbf{x}'}{\sigma})$.
Let us write $Q(\mathbf{x},\mathbf{x}') = q(\mathbf{x}-\mathbf{x}')$, then (6) becomes $q^{-1}\star q(\mathbf{x}-\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')$.
The inverse of the exponential covariance is obtained from the Fourier theorem:

$$\begin{aligned} \widehat{q^{-1}}(\omega)\widehat{q}(\omega) &= 1\\ \widehat{q^{-1}}(\omega) &= \frac{1}{\widehat{q}(\omega)} = \frac{2\sigma}{1+\sigma^2\omega^2}\\ q^{-1}(\mathbf{x}) &= \frac{1}{2\sigma}\left(\delta(\mathbf{x}) - \sigma^2\delta''(\mathbf{x})\right)\end{aligned}$$

Let us replace the expression of Q^{-1} in the functional (22):

$$\begin{aligned} &\iint_{\Omega^2} F(\mathbf{X})^T(\mathbf{x})Q^{-1}(\mathbf{x},\mathbf{x}')F(\mathbf{X})(\mathbf{x}')d\mathbf{x}d\mathbf{x}'\\ =& \frac{1}{2\sigma}\int_\Omega F(\mathbf{X})^T(\mathbf{x})\left(F(\mathbf{X}) - \sigma^2\frac{\partial^2 F(X)}{\partial \mathbf{x}^2}\right)d\mathbf{x}\\ =& \frac{1}{2\sigma}\int_\Omega \left(\|F(\mathbf{X})\|^2 + \sigma^2\left\|\frac{\partial F(X)}{\partial\mathbf{x}}\right\|^2\right)d\mathbf{x} \qquad (23)\end{aligned}$$

Line (23) is obtained from the previous line using integration by parts. The exponential covariance is then equivalent to a quadratic second-order regularization.
However, the inversion of a covariance matrix is often non-trivial and usually inaccessible. Restrictive choices have to be made such as Dirac or exponential covariance. For further details, the reader is referred to [Oliver, 1998, Tarantola, 2005].

3.2 The observation equation

As previously pointed out, the observation equation describes the link between the state vector and the observation. If the generic formulation of Tikhonov's approach is considered, it is clear that the Image Model corresponds exactly to the observation equation of the Data Assimilation framework because the regularization part of Tikhonov's method does not contain information about the input.

In addition to the observation equation, the observation error has to be stated and its matrix of covariance R must be specified. R is a good candidate to manage missing data. This matrix must also be inverted (see equation (16)) and, as explained before, determining the inverse is only possible in simple cases of covariance matrices. The Dirac covariance can be used to express a null interaction between two space-time locations. This choice is not restrictive because observation errors have only to be located and quantified in space-time. R is then written as:

$$R(\mathbf{x},t,\mathbf{x}',t') = r(\mathbf{x},t)\delta\left(\mathbf{x}-\mathbf{x}'\right)\delta\left(t-t'\right)$$

where r becomes a real matrix. The advantage is that determining the inverse is trivial when r is invertible:

$$R^{-1}(\mathbf{x},t,\mathbf{x}',t') = \delta\left(\mathbf{x}-\mathbf{x}'\right)\delta\left(t-t'\right)r^{-1}(\mathbf{x},t) \tag{24}$$

The function r must characterize the quality of the observation: a high value should indicate that the observation is relevant and a value close to zero indicates an irrelevant observation (like missing data), that should not be taken into account. One possible formulation of r is:

$$r(\mathbf{x},t) = r_0(1-f(\mathbf{Y}(\mathbf{x},t)) + r_1 f(\mathbf{Y}(\mathbf{x},t)) \tag{25}$$

If a real function $f \in [0,1]$ characterizing the relevance of the observation $\mathbf{Y}$ is defined, r will be close to a "minimal covariance" r_0 on missing or irrelevant data and r will be close to a "maximal covariance" r_1 on relevant data. r_0 and r_1 are chosen to be constant and invertible matrices (in order for r to be invertible).

4 Estimation of optical flow

Optical flow is the apparent motion in sequences of images and is defined as the transport of brightness over time [Horn and Schunk, 1981]. Let I be a sequence of images on a bounded domain of $\mathbb{R}^2$, denoted Ω. Let $\mathbf{W}(\mathbf{x},t)$ be the velocity vector of a point $\mathbf{x} \in \Omega$ between frame t and frame $t+\delta t$. The optical flow is the vector $\mathbf{W}(\mathbf{x},t)$ verifying:

$$I(\mathbf{x}+\mathbf{W}(\mathbf{x},t)\delta t, t+\delta t) = I(\mathbf{x},t) \quad \forall \mathbf{x} \in \Omega \tag{26}$$

As this equation is non linear with respect to $\mathbf{W}$, the left member of equation (26) is linearized using a first order Taylor development around $\delta t = 0$. This provides the optical flow constraint:

$$\nabla I^T(\mathbf{x},t)\mathbf{W}(\mathbf{x},t) + \frac{\partial I}{\partial t}(\mathbf{x},t) = 0 \quad \forall \mathbf{x} \in \Omega \tag{27}$$

This is an ill-posed problem: the velocity vector has two components — $\mathbf{W}(\mathbf{x},t) = (U(\mathbf{x},t), V(\mathbf{x},t))$ — and the optical flow constraint is not sufficient to compute this vector. A solution could be obtained in the Image Processing context using the Tikhonov's method or Weickert's method described in the Introduction and we study, in the following, the solution obtained in the Data Assimilation context.

4.1 Modelling in the Data Assimilation context

The field $\mathbf{W}$ of velocity vectors is now considered as the state vector and the sequence of images I constitutes the observation vector. The optical flow constraint is used as the observation equation:

$$\mathbb{H}(\mathbf{W}, I)(\mathbf{x},t) = \nabla I(\mathbf{x},t)^T \mathbf{W}(\mathbf{x},t) + I_t(\mathbf{x},t) \tag{28}$$

Equation $\mathbb{H} = 0$ means that the vector $(\mathbf{W}, 1)$ is orthogonal to the spatio-temporal gradient of the observation $\nabla_3 I = \left(\frac{\partial I}{\partial x}, \frac{\partial I}{\partial y}, \frac{\partial I}{\partial t}\right)$. The areas where the value of the spatio-temporal gradient are close to zero are not relevant observation because the equation (27) is always true. This information has to be taken into account by the observation error R. Applying the principle described in Subsection 3.2, the observation error is chosen as:

$$R(\mathbf{x},t,\mathbf{x}',t') = \left(r_0 + (r_1 - r_0)\exp(-\|\nabla_3 I(\mathbf{x},t)\|^2)\right)\delta\left(\mathbf{x}-\mathbf{x}'\right)\delta\left(t-t'\right) \tag{29}$$

The transport of the velocity is chosen as the evolution equation. In this case, equation (21) is rewritten as the two-component system:

$$\frac{\partial U}{\partial t} + UU_x + VU_y = 0 \tag{30}$$

$$\frac{\partial V}{\partial t} + UV_x + VV_y = 0 \tag{31}$$

and the evolution model is $\mathbb{M}(\mathbf{W}) = \begin{pmatrix}\mathbb{M}_1(\mathbf{W}) & \mathbb{M}_2(\mathbf{W})\end{pmatrix}$ with $\mathbb{M}_1(\mathbf{W}) = U_x U + U_y V$ and $\mathbb{M}_2(\mathbf{W}) = V_x U + V_y V$. The Q matrix is chosen as:

$$Q(\mathbf{x},t,\mathbf{x}',t') = \exp\left(-\frac{1}{\sigma}(\|\mathbf{x}-\mathbf{x}'\| - |t-t'|)\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{32}$$

to ensure a spatial second-order regularization. The initial condition of the state vector has also to be provided. We use Horn and Schunck's algorithm [Horn and Schunk, 1981] (Tikhonov's method with a second-order regularization) to compute the velocity field on the first image of the sequence. We assume the initial condition to be without errors and we set $B(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')$ as the background error.

4.2 Adjoint operators

In order to determine the adjoint operators of $\mathbb{M}$ and $\mathbb{H}$, the directional derivatives must first be established. Using the definition (8), we obtain:

$$\begin{aligned}
\frac{\partial \mathbb{M}_1}{\partial \mathbf{W}}(\eta) &= \frac{\partial \mathbb{M}_1}{\partial U}(\eta^1) + \frac{\partial \mathbb{M}_1}{\partial V}(\eta^2) = U\eta^1_x + U_x\eta^1 + V\eta^1_y + U_y\eta^2 \\
\frac{\partial \mathbb{M}_2}{\partial \mathbf{W}}(\eta) &= \frac{\partial \mathbb{M}_2}{\partial U}(\eta^1) + \frac{\partial \mathbb{M}_2}{\partial V}(\eta^2) = U\eta^2_x + V_y\eta^2 + V\eta^2_y + V_x\eta^1
\end{aligned}$$

with $\eta = \begin{pmatrix}\eta^1 & \eta^2\end{pmatrix}^T$. Using (14), integration by parts and considering the border terms to be equal to zero, the adjoint operator of $\mathbb{M}$ is :

$$\left(\frac{\partial \mathbb{M}_1}{\partial \mathbf{W}}\right)^* (\lambda) = -U\lambda^1_x - V_y\lambda^1 - V\lambda^1_y + U_y\lambda^2$$
$$\left(\frac{\partial \mathbb{M}_2}{\partial \mathbf{W}}\right)^* (\lambda) = -U_x\lambda^2 - U\lambda^2_x - V\lambda^2_y + V_x\lambda^1$$

with $\lambda = \begin{pmatrix}\lambda^1 & \lambda^2\end{pmatrix}^T$.
The directional derivative of the observation operator is:

$$\frac{\partial \mathbb{H}}{\partial \mathbf{W}}(\eta) = I_x\eta^1 + I_y\eta^2 = \nabla I^T\eta$$

and determining the adjoint operator is direct as it does not necessitate integration by parts:

$$\left(\frac{\partial \mathbb{H}}{\partial \mathbf{W}}\right)^* (\lambda) = \nabla I\lambda$$

4.3 Discretization

Using the choices made in Subsection 4.1, the three EDPs (11,13,16) become:

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{W}^T\nabla\mathbf{W} = 0 \tag{33}$$
$$\frac{\partial \lambda}{\partial t} - \nabla\lambda^T\mathbf{W} - \lambda^T\nabla^\perp\mathbf{W} = -\frac{1}{2\sigma}\nabla IR \star (I_t + \nabla I^T(\mathbf{W}+\delta\mathbf{W})) \tag{34}$$
$$\frac{\partial \delta\mathbf{W}}{\partial t} + \mathbf{W}^T\nabla\delta\mathbf{W} + \nabla\mathbf{W}^T\delta\mathbf{W} = Q \star \lambda \tag{35}$$

with $\nabla^\perp\mathbf{W} = \begin{pmatrix}\nabla^\perp U & \nabla^\perp V\end{pmatrix}$ and $\nabla^\perp U = \begin{pmatrix}U_y & -U_x\end{pmatrix}^T$. These three equations are approximated using the finite difference technique.

As $\mathbf{W}(\mathbf{x}, t)$ is a vector of $\mathbb{R}^2$, equation (33) has two components. The first component combines a term of linear advection in direction y and a term of non linear advection in direction x. It can be expressed as a two-equation system using a splitting method [Verwer and Sportisse, 1998]:

$$\frac{\partial U}{\partial t} + UU_x = 0 \tag{36}$$
$$\frac{\partial U}{\partial t} + VU_y = 0 \tag{37}$$

The equation (36) is efficiently approximated using the Lax-Friedrich method [Sethian, 1996] which rewrites it in the following form $\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0$ with $F(u) = \frac{1}{2}u^2$. This new equation is approximated by the explicit scheme:

$$U^{k+1}_{i,j} = \frac{1}{2}(U^k_{i+1,j} + U^k_{i-1,j}) - \frac{\triangle t}{2}(F^k_{i+1,j} - F^k_{i-1,j})$$

with $U^k_{i,j} = U(x_i, y_i, t_k)$, $F^k_{i,j} = F(U(x_i, y_i, t_k))$ and $\triangle t$ the time step. The term $\frac{1}{2}(U^k_{i+1,j} + U^k_{i-1,j})$ stabilizes the scheme (by adding a diffusive effect) when

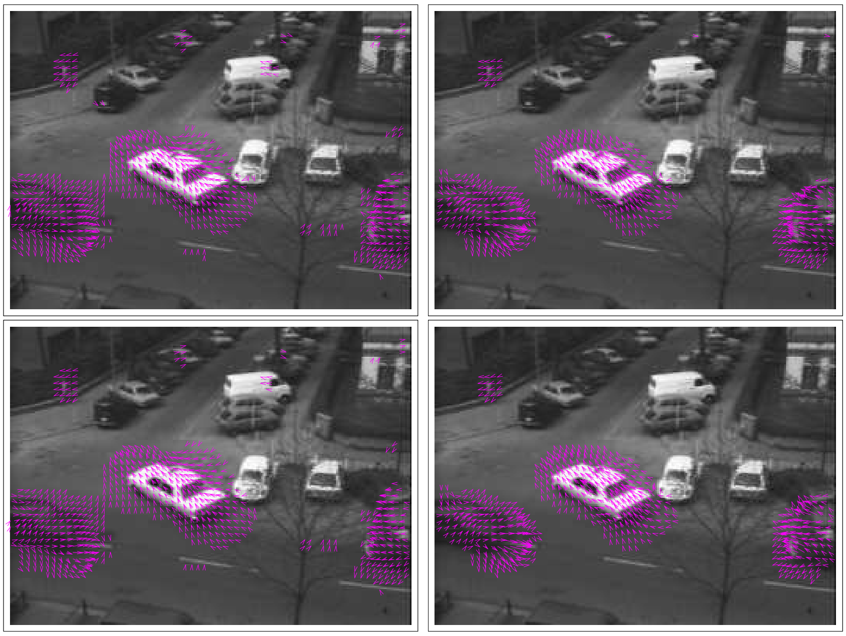

Figure 1: The taxi sequence. Top line: results on frame 3 for Data Assimilation and Horn & Schunk methods. Bottom line: results on frame 4 with missing data.

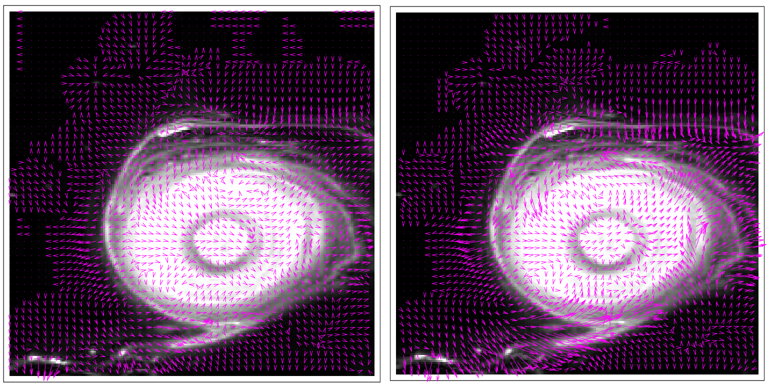

Figure 2: The Coriolis sequence: results on frame 2 for Data Assimilation and Horn & Schunk methods.

$\triangle t$ has a low value (the Courant-Friedrich-Levy condition). The linear advection (37) is approximated using a shock explicit scheme [Sethian, 1996]:

$$U_{i,j}^{k+1} = U_{i,j}^{k} - \triangle t \left(\max(V_{i,j}^{k}, 0) \left(U_{i,j}^{k} - U_{i-1,j}^{k}\right) + \min(V_{i,j}^{k}, 0) \left(U_{i+1,j}^{k} - U_{i,j}^{k}\right)\right)$$

Having defined the discretisation of the first component of (33), it can be seen that the second component contains a linear advection term in direction x and a non linear advection term in direction y. The same strategy is then used to perform the discretization process.

Equation (34) combines a linear advection, a term of reaction and a forcing term. It also has two components. The first one is $\dfrac{\partial \lambda^1}{\partial t} - U\lambda^1_x - V_y\lambda^1 - V\lambda^1_y + U_y\lambda^2 = I_x B$ with $B = -\frac{1}{2\sigma} R \star (I_t + \nabla I^T(\mathbf{W} + \delta\mathbf{W}))$. It is split into two parts. The first one contains the linear advection in direction x and the reaction term $\dfrac{\partial \lambda^1}{\partial t} - U\lambda^1_x - V_y\lambda^1 = 0$ and is approximated in the same way as (37) with an explicit shock scheme. However, the equation is retrograde as the initial condition is given at time T:

$$\begin{aligned}(\lambda^1)^{k-1}_{i,j} \quad = \quad & \left(1 - \frac{1}{2}(V^k_{i,j+1} - V^k_{i,j-1})\right)(\lambda^1)^k_{i,j} - \\ & \triangle t \left(\max(U^k_{i,j}, 0)((\lambda^1)^k_{i,j} - (\lambda^1)^k_{i-1,j}) + \min(U^k_{i,j}, 0)((\lambda^1)^k_{i+1,j} - (\lambda^1)^k_{i,j}))\right)\end{aligned}$$

The second part contains the linear advection term in direction y and the forcing term: $\dfrac{\partial \lambda^1}{\partial t} - V\lambda^1_y + U_y\lambda^2 = I_y B$. Again, a shock explicit scheme is used:

$$\begin{aligned}(\lambda^1)^{k-1}_{i,j} \quad = \quad & (\lambda^1)^k_{i,j} + \frac{\triangle t}{2}\left(U^k_{i,j+1} - U^k_{i,j-1}\right)(\lambda^2)^k_{i,j} - \triangle t B^k_{i,j} - \\ & \triangle t \left(\max(V^k_{i,j}, 0)((\lambda^1)^k_{i,j} - (\lambda^1)^k_{i,j-1}) + \min(V^k_{i,j}, 0)((\lambda^1)^k_{i,j+1} - (\lambda^1)^k_{i,j})\right)\end{aligned}$$

Having the same structure as the first component, the second component of (34) is approximated using the same method.

The last equation, (35), is similar to equation (34): a linear advection with a reaction term and a forcing term. We therefore use the same discretization technique.

4.4 Results

We tested the algorithm on the "taxi" sequence and one a sequence provided by the Coriolis team (LEGI, France) showing a simulation (in a water tank) of the oceanic circulation (Fig 2). For these two sequences, the initial condition is given by the Horn and Schunk method. The image gradients are computed using a convolution by a derivative Gaussian kernel. The parameter σ of the matrix Q and the Gaussian kernel variance are set to 1. The number of iterations is set to 10. We also force the image gradients to zero on the fourth frame of the taxi sequence to simulate missing data. Results are displayed in Figure 1 and confirm the capability of the algorithm to manage missing data.

5 Conclusion

In this paper we proposed a framework to solve ill-posed problems using Data Assimilation methods. This approach is an alternative to the Weickert's method, which constrains the solution's variations in space and time.

We gave general principles to choose a suitable evolution model, an observation equation, and their matrices of covariance. The observation equation corresponds to the Image Model of the Weickert's method. The evolution equation constrains the solution in time and the model error constrains it in space. As an example, we saw that an exponential covariance is equivalent to a second order Tikhonov regularization. The three drawbacks of the Weickert's method, pointed out in the Introduction, are then overcome:

- the dynamic of the vector state is accurately described by the suitable evolution model;
- missing data are managed by the observation error: the covariance evaluates the confidence of the observation data. Depending on the covariance value, the observation will or will not be taken into account in the computation of the solution. Typically, the covariance matrix must have values close to 0 where data are missing, leaving the evolution law determining the solution;
- the algorithm processes a sequence of images frame by frame, which allows an implementation with low memory requirements.

The choice of the evolution model remains the most important point as it greatly depends on the applicative context. We described the process of determining it to solve the optical flow problem: the image gradients (or the image's grey level values) are assimilated in an evolution model representing a transport equation of the velocity. This transport equation has however the drawback of being non linear and we described a stable numerical scheme.

The main perspective of this work is to define a generic architecture of the evolution model which will then be applicable to specific cases. The diffusion laws, which model the transport of the state vector using the physical conservative principles, are good candidates for a number of reasons. First, their approximation in a discrete space is robust as stable numerical schemes exist. Second, the adjoint of the model operator is known explicitly. Third, several types of diffusion are possible. It may be anisotropic and driven by specific configurations of the state vector or of the image values, as the diffusion has a strong impact on spatial regularization [Nielsen et al., 1994]. An adaptive diffusion can spatially regularize the state vector while preserving the discontinuities. Such a diffusion, used as an evolution equation, performs both the temporal and the spatial regularization of the state vector.

As a second perspective, we plan to apply this method to 3D reconstruction from a sequence of 2D images and one initial 3D image. The process is considered as an interpolation driven by the sequence of 2D images: these images are the observation and assimilated into an evolution model which is the transport of the voxel brightness.

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