Analyse théorique et numérique du modèle de Webster Lokshin

Houssem Haddar, Denis Matignon

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Theoretical and numerical analysis of the Webster Lokshin model

H. Haddar — D. Matignon

N° 6558

Juin 2008

Thème NUM

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--6558--FR+ENG

INSTITUT NATIONAL
DE RECHERCHE
EN INFORMATIQUE
ET EN AUTOMATIQUE

INRIA

centre de recherche
SACLAY - ÎLE-DE-FRANCE

Theoretical and numerical analysis of the Webster Lokshin model

H. Haddar* , D. Matignon†

Thème NUM — Systèmes numériques
Équipe-Projet DeFI

Rapport de recherche n° 6558 — Juin 2008 — 29 pages

Abstract: Acoustic waves travelling in a duct with viscothermal losses at the wall and radiating conditions at both ends obey a Webster-Lokshin model that involves fractional time-derivatives in the domain and dynamical boundary conditions. This system can be interpreted as the coupling of three subsystems: a wave equation, a diffusive realization of the pseudo-differential time-operator and a dissipative realization of the impedance, thanks to the Kalman-Yakubovich-Popov lemma.

Existence and uniqueness of strong solutions of the system are proved, using the Hille-Yosida theorem.

Moreover, numerical schemes are derived and their stability is analyzed using energy methods; many simulation results are presented, which describe the behaviour of the model for different values of the parameters.

Key-words:

* INRIA Saclay Ile de France & CMAP, Ecole Polytechnique, Route de Saclay, 91128 Palaiseau.

† Université de Toulouse; ISAE; 10, av. E. Belin, F-31055 Toulouse.

Analyse thorique et numrique du modle de Webster Lokshin

Résumé : Les ondes acoustiques qui se propagent dans un pavillon dont la paroi est le sige de pertes visco-thermiques et dont les deux extremits sont sujettes des conditions de rayonnement obissent un modle de Webster-Lokshin, lequel fait intervenir des drives fractionnaires en temps dans le milieu et des conditions aux limites dynamiques. Ce systme peut s'interprter comme le couplage de trois sous-systmes : une quation des ondes, une ralisation diffusive de l'oprateur pseudo-diffrentiel en temps, et une ralisation dissipative de l'impdance par le lemme de Kalman-Yakubovich-Popov.

En utilisant le thorme de Hille-Yosida, l'existence et l'unicit des solutions fortes de ce systme sont tablies.

De plus, des schmas numriques sont proposs et leur stabilit est analyse en utilisant des techniques d'nergie ; de nombreuses simulations numriques viennent illustrer le comportement du modle pour diverses valeurs des paramtres.

Mots-clés :

Chapter 1

Introduction

The Lokshin model originally presented in [10, 11] and referred to in [5] in a half-space has then been derived in a bounded space in [17], and modified in [8].

In the case of constant-coefficients, it has been solved analytically and analyzed in both time-domain and frequency-domain in [13], while the principle of an energy analysis has been given in [14].

The problem at stake here is in a bounded domain, with non-constant coefficients (due to Webster equation for horns and space-varying coefficients for the viscothermal effects). Existence and uniqueness of strong solutions of the free evolution problem is proved in an energy space (see also [20]), and the coupling between passive subsystems is used as main method of analysis, as in [12, ch. 5].

In chapter 2, we begin the theoretical study with the formulation of the problem in §2.1; a key point is the reformulation as coupled first-order systems in §2.1.3, thanks to diffusive realizations of fractional differential operators; the analysis of the global system follows in §2.2.

The following slight extensions or wider perspectives are in view:

- use some infinite-dimensional analogue of the KYP lemma for some more realistic impedances, as in [3];
- study of the boundary-controlled equation;
- proof of the asymptotic stability of the global system, using a spectral condition on the generator of the semigroup, as in [16], rather than LaSalle's invariance principle, following [12, ch. 3], which does require the precompactness of strong trajectories in the energy space.

In chapter 3, the numerical analysis of this system is presented, based on finite difference methods with technical specificities due to diffusive representations. The properties of the continuous model are displayed and explored in the discrete domaine, and illustrated on many simulation examples in § 3.3.

Chapter 2

Theoretical analysis

2.1 Mathematical formulation of the Webster Lokshin model

Consider an axi-symmetric duct between $z = 0$ and $z = 1$ with cross section radius $r(z)$ (satisfying $r \geq r_0 > 0$), then the velocity potential ϕ (with appropriate scaling) satisfies the following equation:

$$\partial_t^2 \phi + (\eta(z)\, \partial_t^\alpha + \varepsilon(z)\, \partial_t^{-\beta})\, \partial_t \phi - \frac{1}{r^2(z)} \partial_z (r^2(z)\, \partial_z \phi) = 0, \tag{2.1}$$

for some $\alpha, \beta \in]0, 1[$ and $\varepsilon, \eta \in L^\infty(0, 1; \mathbb{R}^+)$. The terms in ∂_t^α and $\partial_t^{-\beta}$ model the effect of viscous and thermal losses at the lateral walls.

We can reformulate (2.1) as a first order system in the $(p,\, v)$ variables, where $p = \partial_t \phi$ is the pressure, and $v = r^2\, \partial_z \phi$ is the volume velocity:

$$\begin{aligned} \partial_t p &= -\frac{1}{r^2} \partial_z v - \varepsilon\, \partial_t^{-\beta} p - \eta\, \partial_t^\alpha p\,, && (2.2) \\ \partial_t v &= -r^2\, \partial_z p\,, && (2.3) \end{aligned}$$

To take into account the interaction with the exterior domain, one can add dynamical boundary conditions at $z = 0, 1$ that are of the type

$$\hat{p}_i(s) = \mp \mathcal{Z}_i(s)\, \hat{v}_i(s) \quad \text{for } i = 0, 1\,. \tag{2.4}$$

Conditions (2.4) are formulated in the Laplace domain, with shorthand notation $p_i(t) = p(z = i,\, t)$ for $i = 0, 1$; the *acoustic impedances* $\mathcal{Z}_i(s)$ are *strictly positive real* in the sense of [12, ch. 5] , that is $\Re e(\mathcal{Z}_i(s)) > 0$, $\forall s$, $\Re e(s) \geq 0$.

The system (2.2)-(2.3)-(2.4) can be transformed into a first order system in time, using appropriate realizations for the pseudo-differential operators involved in this model:

- dissipative realizations for the positive-real impedances, using Kalman-Yakubovich-Popov lemma in finite dimension, are recalled in § 2.1.1,

- dissipative realizations for positive pseudo-differential time-operators of diffusive type, such as $\partial_t^{-\beta}$ and ∂_t^{α}, are presented in § 2.1.2.

2.1.1 Dissipative realizations for positive-real impedances (Kalman-Yakubovich-Popov lemma)

We restrict ourselves to impedances $\mathcal{Z}_i(s)$ of *rational* type. Thus, for a strictly positive real impedance of rational type, one can choose a *minimal* realization $(A_i,\, B_i,\, C_i,\, d_i)$ with state x_i of *finite* dimension n_i ($A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times 1}$, $C_i \in \mathbb{R}^{1 \times n_i}$ and $d_i \in \mathbb{R}$), such that:

$$\begin{aligned} \frac{d}{dt} x_i(t) &= A_i\, x_i + B_i\, v_i(t), \quad x_i(0) = 0, && (2.5)\\ \mp p_i(t) &= C_i\, x_i(t) + d_i\, v_i(t). && (2.6) \end{aligned}$$

Then (following e.g. [1, 18]), there exists $P_i \in \mathbb{R}^{n_i \times n_i}$, $P_i = P_i^T > 0$, such that the following energy balance holds:

$$\mp \int_0^T p_i(t)\, v_i(t)\, dt = \frac{1}{2}(x_i^T(T)\, P_i\, x_i(T)) + \frac{1}{2}\int_0^T \begin{pmatrix} x_i^T(t) & v_i(t) \end{pmatrix} \mathcal{M}_i \begin{pmatrix} x_i(t) \\ v_i(t) \end{pmatrix} dt. \tag{2.7}$$

with $\mathcal{M}_i = \begin{pmatrix} -A_i^T P_i - P_i A_i & C_i^T - P_i B_i \\ C_i - B_i^T P_i & 2d_i \end{pmatrix} \geq 0$.

The right hand side of (2.7) is split into two terms, a storage function evaluated at time T only, proportional to $\|x_i(T)\|^2 = x_i^T(T) P_i x_i(T)$, and a dissipated energy on the time interval $(0, T)$, which involves the non-negative symmetric matrix $\mathcal{M}_i \in \mathbb{R}^{(n_i+1)\times(n_i+1)}$. We denote $E_x(t) := \frac{1}{2}\|x_0(t)\|^2 + \frac{1}{2}\|x_1(t)\|^2 = \frac{1}{2} x_0^T(t) P_0 x_0(t) + \frac{1}{2} x_1^T(t) P_1 x_1(t)$.

2.1.2 Dissipative realizations for positive pseudo-differential time-operators of diffusive type

Let us recall that the operator $\partial_t^{-\beta}$ is the causal Riemann-Liouville fractional integral of order β; it is a pseudo-differential time-operator, the symbol of which is $s^{-\beta}$, with $0 < \beta < 1$. It can also be seen as a convolution operator $\partial_t^{-\beta} p(t) = (h_\beta \star p)(t)$ with causal kernel defined by:

$$h_\beta(t) := \frac{1}{\Gamma(\beta)} t^{\beta-1} \quad \text{for} \quad t > 0,$$

which involves the Euler Γ function. This latter kernel can be decomposed onto purely decaying exponentials as

$$h_\beta(t) = \int_0^{+\infty} e^{-\xi t} dM_\beta(\xi) \quad \text{for} \quad t > 0,$$

where $dM_\beta(\xi) = \mu_\beta(\xi)\, d\xi$ with density $\mu_\beta(\xi) = \frac{\sin(\beta\pi)}{\pi}\, \xi^{-\beta}$.

Now ∂_t^α is the fractional derivative of order α, it is a pseudo-differential time-operator, the symbol of which is s^α, with $0 < \alpha < 1$. It can also be defined as the causal convolutive inverse of $\partial_t^{-\alpha}$; hence $\partial_t^\alpha p(t) = \frac{d}{dt}(h_{1-\alpha} \star p)(t)$, where the time derivative must be understood in the sense of distributions.

We shall now introduce the diffusive realization of the operator $\partial_t^{-\beta}$. This realization also applies to any convolution operator with kernel of the form $h(t) = \int_0^{+\infty} e^{-\xi t}\, dM$, where M is a positive measure on $\mathbb{R}^+$, such that $\int_0^{+\infty} \frac{dM(\xi)}{1+\xi} < +\infty$. We refer to [20, § 5.] for the treatment of completely monotone kernels, and [15] and references therein for links between diffusive representations and fractional differential operators.

The following functional spaces will be of interest in the sequel: $H_\beta = L^2(\mathbb{R}^+, dM_\beta)$, $V_\beta = L^2\left(\mathbb{R}^+, (1+\xi)\, dM_\beta\right)$, and $\widetilde{V}_\beta = L^2(\mathbb{R}^+, \xi\, dM_\beta)$. We also introduce the notation $c_\beta = \int_0^\infty \frac{dM_\beta}{1+\xi} < +\infty$.

First diffusive representations.

Consider the dynamical system with input $p \in L^2(0,T)$ and output $\theta \in L^2(0,T)$:

$$\begin{aligned} \partial_t \varphi(\xi,t) &= -\xi\, \varphi(\xi,t) + p(t) \quad \text{with} \quad \varphi(\xi,0) = 0 \quad \forall\, \xi \in \mathbb{R}^+ , \qquad (2.8) \\ \theta(t) &= \int_0^{+\infty} \varphi(\xi,t)\, dM_\beta(\xi) \, . \qquad (2.9) \end{aligned}$$

Then, it can easily be checked that $\theta(t) = \partial_t^{-\beta} p(t)$. The following energy balance can be proved:

$$\int_0^T p(t)\, \theta(t)\, dt = \frac{1}{2} \int_0^{+\infty} \varphi(\xi,T)^2\, dM_\beta + \int_0^T \int_0^{+\infty} \xi\, \varphi(\xi,t)^2\, dM_\beta\, dt \, . \qquad (2.10)$$

Similarly to (2.7), the right hand side of (2.10) is split into two terms, a storage function evaluated at time T only, $E_\varphi(T) := \frac{1}{2}\|\varphi(T)\|^2_{H_\beta}$, and a dissipated energy on the time interval $(0,T)$.

Extended diffusive representations.

Consider now the dynamical system with input $p \in H^1(0,T)$ and output $\widetilde{\theta} \in L^2(0,T)$:

$$\begin{aligned} \partial_t \widetilde{\varphi}(\xi,t) &= -\xi\, \widetilde{\varphi}(\xi,t) + p(t) \quad \text{with} \quad \widetilde{\varphi}(\xi,0) = 0 \quad \forall\, \xi \in \mathbb{R}^+ , \qquad (2.11) \\ \widetilde{\theta}(t) &= \int_0^{+\infty} \partial_t \widetilde{\varphi}(\xi,t)\, dM_{1-\alpha}(\xi) = \int_0^{+\infty} \left[p(t) - \xi\, \widetilde{\varphi}(\xi,t)\right] dM_{1-\alpha}(\xi) \qquad (2.12) \end{aligned}$$

Then, it can easily be checked that $\widetilde{\theta}(t) = \partial_t^\alpha p(t)$. The following energy balance can be proved:

$$\int_0^T p(t)\, \widetilde{\theta}(t)\, dt = \frac{1}{2} \int_0^{+\infty} \xi\, \widetilde{\varphi}(\xi,T)^2\, dM_{1-\alpha} + \int_0^T \int_0^{+\infty} (p - \xi\, \widetilde{\varphi})^2\, dM_{1-\alpha}\, dt \, . \qquad (2.13)$$

Again, the right hand side of (2.13) is split into two terms, a storage function evaluated at time T only, $\widetilde{E}_{\widetilde{\varphi}}(T) := \frac{1}{2}\|\widetilde{\varphi}(T)\|^2_{\widetilde{V}_{1-\alpha}}$, and a dissipated energy on the time interval $(0, T)$.

2.1.3 An abstract formulation

Now, using representations (2.5)-(2.8)-(2.11), the global system (2.2)-(2.3)-(2.4) can be transformed into the first order differential equation in time

$$\frac{d}{dt}X + \mathcal{A}\,X = 0, \tag{2.14}$$

where $X = (x_0, x_1, p, v, \varphi, \widetilde{\varphi})^T$ and

$$\mathcal{A}\begin{pmatrix} x_0 \\ x_1 \\ p \\ v \\ \varphi \\ \widetilde{\varphi} \end{pmatrix} = \begin{pmatrix} -A_0 x_0 - B_0\, v(z=0) \\ -A_1 x_1 - B_1\, v(z=1) \\ \frac{1}{r^2}\partial_z v + \varepsilon \int_0^{+\infty} \varphi\, dM_\beta + \eta \int_0^{+\infty} [p - \xi\,\widetilde{\varphi}]\; dM_{1-\alpha} \\ r^2\,\partial_z p \\ \xi\varphi - p \\ \xi\widetilde{\varphi} - p \end{pmatrix}. \tag{2.15}$$

The boundary conditions $p(z=0) = -C_0 x_0 - d_0\, v(z=0)$ and $p(z=1) = C_1 x_1 + d_1\, v(z=1)$ must be taken into account in the functional spaces of the solutions. In the sequel, we shall analyze the well-posedness of this system. This analysis is based on the following energy balance

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\int_0^1 |p(z,t)|^2 r^2(z)\, dz + \frac{1}{2}\int_0^1 |v(z,t)|^2 r^{-2}(z)\, dz\right) \\ &+ \frac{d}{dt}\left(E_x(t) \;+\; \int_0^1 \varepsilon(z) E_\varphi(z,t)\, r^2(z)\, dz \;+\; \int_0^1 \eta(z) \widetilde{E}_{\widetilde{\varphi}}(z,t)\, r^2(z)\, dz\right) \\ &\qquad\qquad = \;\; \frac{1}{2}\big(x_0^T \quad v(0)\big)\mathcal{M}_0\begin{pmatrix} x_0 \\ v(0)\end{pmatrix} + \frac{1}{2}\big(x_1^T \quad v(1)\big)\mathcal{M}_1\begin{pmatrix} x_1 \\ v(1)\end{pmatrix} \\ &\qquad\qquad\quad + \int_0^1 \|\varphi\|^2_{\widetilde{V}_\beta}\, \varepsilon\, r^2\, dz + \int_0^1 \|p - \xi\,\widetilde{\varphi}\|^2_{H_{1-\alpha}}\, \eta\, r^2\, dz, \end{aligned} \tag{2.16}$$

that will be proved in Theorem 2.2.1 below.

2.2 Well-posedness of the global system

We shall apply Hille-Yosida theorem in order to show existence and uniqueness of solutions to (2.14).

According to identity (2.16), the natural *energy space* for the solution X would be the following Hilbert space:

$$\mathcal{H} := \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \times L^2_{r^2} \times L^2_{r^{-2}} \times L^2(0,1; H_\beta; \varepsilon\, r^2\, dz) \times L^2(0,1; \widetilde{V}_{1-\alpha}; \eta\, r^2\, dz)\;,$$

where $L^2_{r^2} := L^2(0,1; r^2(z)\,dz)$, $L^2_{r^{-2}} := L^2(0,1; r^{-2}(z)\,dz)$ with scalar product for $X = (x_0, x_1, p, v, \varphi, \widetilde{\varphi})^T$ and $Y = (y_0, y_1, q, w, \psi, \widetilde{\psi})^T$ defined by:

$$
\begin{aligned}
(X,\, Y)_{\mathcal{H}} &= x_0^T P_0 y_0 + x_1^T P_1 y_1 + (p,\, q)_{L^2_{r^2}} + (v,\, w)_{L^2_{r^{-2}}} \\
&\quad + \int_0^1 (\varphi,\, \psi)_{H_\beta}\, \varepsilon(z)\, r^2(z)\, dz + \int_0^1 (\widetilde{\varphi},\, \widetilde{\psi})_{\widetilde{V}_{1-\alpha}}\, \eta(z)\, r^2(z)\, dz \qquad (2.17)
\end{aligned}
$$

We define the Hilbert space $\mathcal{V}$ as:

$$
\mathcal{V} := \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \times H^1_p \times H^1_v \times L^2(0,1; V_\beta; \varepsilon\, r^2\, dz) \times L^2(0,1; \widetilde{V}_{1-\alpha}; \eta\, r^2\, dz)\ ,
$$

where

$$
\begin{aligned}
H^1_p &:= \left\{ p \in L^2_{r^2},\ \int_0^1 \left[p^2 + (\partial_z p)^2\right]\, r^2(z)\, dz < +\infty \right\}, \\
H^1_v &:= \left\{ v \in L^2_{r^{-2}},\ \int_0^1 [v^2 + (\partial_z v)^2]\, r^{-2}(z)\, dz < +\infty \right\}.
\end{aligned}
$$

We set as domain of the operator $\mathcal{A}$, the space defined by:

$$
D(\mathcal{A}) := \left\{ (x_0, x_1, p, v, \varphi, \widetilde{\varphi})^T \in \mathcal{V},\ \left|\ \begin{array}{l} p(z=0) = -C_0 x_0 - d_0\, v(z=0) \\ p(z=1) = C_1 x_1 + d_1\, v(z=1) \\ (p - \xi\varphi) \in L^2(0,1; H_\beta; \varepsilon\, r^2\, dz) \\ (p - \xi\widetilde{\varphi}) \in L^2(0,1; V_{1-\alpha}; \eta\, r^2\, dz) \end{array} \right. \right\}.
$$

The operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is well defined, it is a bounded operator, namely:

- for the first two components of equation (2.15), from trace theorem $|v(z = i)| \le c_0\, \|v\|_{H^1}$ for some positive constant c_0, therefore $|v(z = i)| \le c_0\, \|r^2\|_{L^\infty} \|v\|_{H^1_v}$;
- for the third component, obviously $\|r^{-2}\, \partial_z v\|_{L^2_{r^2}} \le \|v\|_{H^1_v}$; then using Schwarz inequality

$$
\left(\int_0^\infty \varphi\, dM_\beta \right)^2 \le c_\beta \int_0^\infty (1 + \xi)\varphi^2\, dM_\beta\ ;
$$

  hence

$$
\left\| \varepsilon \int_0^\infty \varphi\, dM_\beta \right\|^2_{L^2_{r^2}} \le c_\beta\, \|\varepsilon\|_{L^\infty}\, \|\varphi\|^2_{L^2(0,1;V_\beta;\varepsilon r^2 dz)};
$$

  finally, in a similar way,

$$
\left\| \eta \int_0^\infty (p - \xi\widetilde{\varphi})\, dM_{1-\alpha} \right\|^2_{L^2_{r^2}} \le c_{1-\alpha}\, \|\eta\|_{L^\infty}\, \|p - \xi\widetilde{\varphi}\|^2_{L^2(0,1;V_{1-\alpha};\eta\, r^2\, dz)};
$$

- for the fourth component, obviously, $\|r^2 \partial_z p\|_{L^2_{r^{-2}}} \le \|p\|_{H^1_p}$;

- for the fifth component, there is nothing to prove: $(p - \xi\varphi) \in L^2(0,1; H_\beta; \varepsilon\, r^2\, dz)$;
- for the sixth component, since $V_{1-\alpha} \subset \widetilde{V}_{1-\alpha}$, we simply have

$$\|p - \xi\widetilde{\varphi}\|_{L^2(0,1;\widetilde{V}_{1-\alpha};\eta\, r^2\, dz)} \leq \|p - \xi\widetilde{\varphi}\|_{L^2(0,1;V_{1-\alpha};\eta\, r^2\, dz)}.$$

Theorem 2.2.1 *For all initial condition $X_0 \in D(\mathcal{A})$, there exists a unique solution $X \in C^1([0,+\infty[;\mathcal{H}) \cap C^0([0,+\infty[;D(\mathcal{A}))$ to*

$$\left\{ \begin{array}{l} \dfrac{d}{dt}X(t) + \mathcal{A}\,X(t) = 0 \quad \forall t > 0, \\ X(0) = X_0. \end{array} \right.$$

This solution satisfies

$$\frac{d}{dt}\left\{\frac{1}{2}\|X(t)\|^2_{\mathcal{H}}\right\} = -(\mathcal{A}X(t),\, X(t))_{\mathcal{H}} \leq 0\,. \tag{2.18}$$

Proof. We shall first prove the *monotonicity* of the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$. Let $X = (x_0, x_1, p, v, \varphi, \widetilde{\varphi})^T \in D(\mathcal{A})$. Integrating by parts the term $(\frac{1}{r^2}\partial_z v, v)_{L^2_{r^2}}$ in the scalar product $(\mathcal{A}X,\, X)_{\mathcal{H}}$, then using the boundary conditions (2.6) yields, after some algebraïc manipulations on the matrices $\mathcal{M}_i$:

$$\begin{aligned} (\mathcal{A}X,\, X)_{\mathcal{H}} \quad &= \quad \frac{1}{2}\big(x_0^T \quad v(0)\big)\mathcal{M}_0\begin{pmatrix} x_0 \\ v(0)\end{pmatrix} + \frac{1}{2}\big(x_1^T \quad v(1)\big)\mathcal{M}_1\begin{pmatrix} x_1 \\ v(1)\end{pmatrix} \\ &\quad + \int_0^1 \|\varphi\|^2_{\widetilde{V}_\beta}\, \varepsilon\, r^2\, dz + \int_0^1 \|p - \xi\,\widetilde{\varphi}\|^2_{H_{1-\alpha}}\, \eta\, r^2\, dz\,. \end{aligned} \tag{2.19}$$

therefore $(\mathcal{A}X,\, X)_{\mathcal{H}} \geq 0$, $\forall X \in D(\mathcal{A})$ and the inequality in equation (2.18) will be fulfilled.

Now let $Y = (y_0, y_1, f, g, \chi, \widetilde{\chi})^T \in \mathcal{H}$, we shall prove the existence of $X = (x_0, x_1, p, v, \varphi, \widetilde{\varphi})^T \in D(\mathcal{A})$ such that $(I + \mathcal{A})X = Y$, which proves the *maximality* of $\mathcal{A}$. The set of equations to solve is

(i) $x_0 - A_0 x_0 - B_0 v(0) = y_0$,

(ii) $x_1 - A_1 x_1 - B_1 v(1) = y_1$,

(iii) $p + \frac{1}{r^2}\partial_z v + \varepsilon \int_0^{+\infty} \varphi\, dM_\beta + \eta \int_0^{+\infty} [p\, -\, \xi\,\widetilde{\varphi}]\; dM_{1-\alpha} = f$,

(iv) $v + r^2 \partial_z p = g$,

(v) $\varphi + \xi\varphi - p = \chi$,

(vi) $\widetilde{\varphi} + \xi\widetilde{\varphi} - p = \widetilde{\chi}$.

We remark that one can solve the first two algebraïc equations with respect to x_0, x_1,

$$x_i = (I_{n_i} - A_i)^{-1}(y_i + B_i v(i))$$

for $i = 0, 1$, which requires $s = 1 \notin \mathrm{spec} A_i$. But this is indeed the case since $\mathcal{Z}_i(s) = d_i + C_i(s\, I_{n_i} - A_i)^{-1} B_i$ is strictly positive real and the realization is minimal: thus, all eigenvalues of A_i are poles of $\mathcal{Z}_i(s)$, with negative real parts (see e.g. [19]).

We also can rewrite the last two algebraïc equations equivalently as follows

$$\begin{aligned} \varphi &= \tfrac{1}{1+\xi} p, + \tfrac{1}{1+\xi} \chi, \\ \widetilde{\varphi} &= \tfrac{1}{1+\xi} p + \tfrac{1}{1+\xi} \widetilde{\chi}. \end{aligned} \tag{2.20}$$

System of equations (i)-(vi) can therefore be reduced to seeking $(p, v) \in H^1_p \times H^1_v$ such that

$$\left\{ \begin{array}{l} (1 + c_\beta\, \varepsilon + c_{1-\alpha}\, \eta)\, p + \frac{1}{r^2} \partial_z v = h \quad \text{ on } (0,1), \\ v + r^2 \partial_z p = g \quad \text{ on } (0,1), \\ p(i) = \mp C_i (I_{n_i} - A_i)^{-1} y_i \mp \mathcal{Z}_i(s=1)\, v(i) \quad \text{ for } i = 0, 1; \end{array} \right. \tag{2.21}$$

where we have set

$$h(z) := f(z) - \varepsilon(z) \int_0^\infty \frac{1}{1+\xi}\, \chi(z,\xi)\, dM_\beta + \eta(z) \int_0^\infty \frac{1}{1+\xi}\, \widetilde{\chi}(z,\xi)\, \xi\, dM_{1-\alpha}.$$

Using Cauchy-Schwarz inequality (with $\chi \in H_\beta$ locally, and $\widetilde{\chi} \in \widetilde{V}_{1-\alpha}$ locally) and using the boundness of ε and η, one can easily check that $h \in L^2_{r^2}$.

We now solve (2.21) with respect to (p, v) thanks to a variational formulation in p that is derived as follows:

1. take the $L^2_{r^2}$ inner product of the first equation of (2.21) with some $q \in H^1_p$,
2. take the $L^2_{r^{-2}}$ inner product of the second equation of (2.21) with some $w \in L^2_{r^{-2}}$,
3. set $w := r^2\, \partial_z q$, add the results of the preceding steps, (note that $\int_0^1 (q\, \partial_z v + v\, \partial_z q)\, dz = v(1)\, q(1) - v(0)\, q(0)$, since $q \in H^1_p \subset H^1$, $v \in H^1_v \subset H^1$), and use the third equation of (2.21) to write the problem in the standard form:

$$\left\{ \begin{array}{l} \text{Find } p \in H^1_p \text{ such that:} \\ a(p,q) = l(q), \quad \forall q \in H^1_p, \end{array} \right. \tag{2.22}$$

where

$$
\begin{aligned}
a(p,q) &:= \int_0^1 \left[\omega(z) p\, q + \partial_z p\, \partial_z q\right] r^2\, dz + \frac{1}{\mathcal{Z}_1(1)} p(1)\, q(1) + \frac{1}{\mathcal{Z}_0(1)} p(0)\, q(0), \\
l(q) &:= \int_0^1 h\, q\, r^2\, dz + \int_0^1 g\, \partial_z q\, dz + \lambda_1\, q(1) - \lambda_0\, q(0);
\end{aligned}
$$

with $\omega(z) := 1 + c_\beta\, \varepsilon(z) + c_{1-\alpha}\, \eta(z) \geq 1 > 0$ and $\lambda_i := \frac{1}{\mathcal{Z}_i(1)} C_i (I_{n_i} - A_i)^{-1} y_i$.

Note that $\mathcal{Z}_i(s = 1) > 0$ since the acoustic impedances are stricly positive real, hence, together with standard trace theorem, it is straighforward to prove that the bilinear form a is continuous and coercive on H^1_p. On the other hand, thanks to $h \in L^2_{r^2}$, $g \in L^2_{r^{-2}} (\Leftrightarrow r^{-2}\, g \in L^2_{r^2})$, and the same trace theorem, it is straightforward to see that the linear form l is continuous on H^1_p. We therefore can apply Lax-Milgram theorem (see e.g. [2, ch. VIII]) to prove existence and uniqueness of $p \in H^1_p$, such that (2.22) holds, $\forall q \in H^1_p$.

We finally conclude the proof of maximality by showing that this unique $p \in H^1_p$, solution of (2.22), enables to define all the state variables $(x_0, x_1, p, v, \varphi, \widetilde{\varphi})$ which are needed, and that the state X belongs to $D(\mathcal{A})$. This will be done in four steps.

1. Set $v := g - r^2\, \partial_z p$. It belongs to $L^2_{r^{-2}}$ a priori. The variational formulation (2.22) yields,
$$\partial_z v = (h - \omega\, p) r^2$$
in the distribution sense. Thanks to $p \in H^1_p \subset L^2_{r^2}$ and $h \in L^2_{r^2}$, we get $\partial_z v \in L^2_{r^{-2}}$, and therefore $v \in H^1_v$. The first equation of (2.21) is thus recovered, together with the second.

2. Now, in order to recover the boundary conditions, choose any $q \in H^1_p$, then use (2.22) to compute:
$$
\begin{aligned}
\left[\lambda_1 - \frac{1}{\mathcal{Z}_1(1)} p(1)\right] q(1) \quad &- \quad \left[\lambda_0 - \frac{1}{\mathcal{Z}_0(1)} p(0)\right] q(0) \\
&= \int_0^1 (\omega\, p - h)\, q\, r^2\, dz - \int_0^1 (g - r^2\, \partial_z p)\, \partial_z q\, dz, \\
&= -\int_0^1 \partial_z v\, q\, dz - \int_0^1 v\, \partial_z q\, dz, \\
&= v(0)\, q(0) - v(1)\, q(1).
\end{aligned}
$$
Since this equality is valid for any $q(0)$, $q(1)$, recalling the value of λ_i, we get for $i = 0, 1$, $p(i) = \mp C_i (I_{n_i} - A_i)^{-1} y_i \mp \mathcal{Z}_i(1)\, v(i)$, which is the third equation of (2.21).

3. In order to check that the unique solution X belongs to $\mathcal{V}$, we need to prove that, in (2.20), $\varphi \in L^2(0,1;V_\beta;\varepsilon\, r^2\, dz)$ and $\widetilde{\varphi} \in L^2(0,1;\widetilde{V}_{1-\alpha};\eta\, r^2\, dz)$, using $p \in H^1_p \subset L^2_{r^2}$, $\chi \in L^2(0,1;H_\beta;\varepsilon\, r^2\, dz)$ and $\widetilde{\chi} \in L^2(0,1;\widetilde{V}_{1-\alpha};\eta\, r^2\, dz)$.

   Recall $\varphi = \frac{1}{1+\xi}\, p + \frac{1}{1+\xi}\, \chi$. On the first hand, since, $\|\frac{1}{1+\xi}\|^2_{V_\beta} = c_\beta$ one has
$$\|\frac{1}{1+\xi}\, p\|^2_{L^2(0,1;V_\beta;\varepsilon\, r^2\, dz)} \leq c_\beta\, \|\varepsilon\|_\infty\, \|p\|^2_{L^2_{r^2}};$$
   on the other hand, since, $\|\frac{1}{1+\xi}\, \chi\|^2_{V_\beta} = \|\frac{1}{\sqrt{1+\xi}}\, \chi\|^2_{H_\beta}$, then one has
$$\|\frac{1}{1+\xi}\, \chi\|^2_{L^2(0,1;V_\beta;\varepsilon\, r^2\, dz)} \leq \|\chi\|^2_{L^2(0,1;H_\beta;\varepsilon\, r^2\, dz)}.$$
   Similar considerations apply to $\widetilde{\varphi} = \frac{1}{1+\xi}\, p + \frac{1}{1+\xi}\, \widetilde{\chi}$: $\|\frac{1}{1+\xi}\|^2_{\widetilde{V}_{1-\alpha}} \leq c_{1-\alpha}$ implies
$$\|\frac{1}{1+\xi}\, p\|^2_{L^2(0,1;\widetilde{V}_{1-\alpha};\eta\, r^2\, dz)} \leq c_{1-\alpha}\, \|\eta\|_\infty\, \|p\|^2_{L^2_{r^2}};$$
   whereas we trivially have
$$\|\frac{1}{1+\xi}\, \widetilde{\chi}\|^2_{L^2(0,1;\widetilde{V}_{1-\alpha};\eta\, r^2\, dz)} \leq \|\widetilde{\chi}\|^2_{L^2(0,1;\widetilde{V}_{1-\alpha};\eta\, r^2\, dz)}.$$

4. Finally, in order to check that the unique solution X belongs to $D(\mathcal{A})$, we need to prove that $p(z=0) = -C_0 x_0 - d_0\, v(z=0)$, $p(z=1) = C_1 x_1 + d_1\, v(z=1)$, $(p - \xi\varphi) \in L^2(0,1;H_\beta;\varepsilon\, r^2\, dz)$ and $(p - \xi\widetilde{\varphi}) \in L^2(0,1;V_{1-\alpha};\eta\, r^2\, dz)$.

   Since $p(i) = \mp C_i (I_{n_i} - A_i)^{-1} y_i \mp \mathcal{Z}_i(1)\, v(i)$, $x_i = (I_{n_i} - A_i)^{-1}(y_i + B_i v(i))$ and recalling $\mathcal{Z}_i(1) = d_i + C_i (I_{n_i} - A_i)^{-1} B_i$, we easily get: $p(i) = \mp C_i x_i \mp d_i\, v(i)$.

   From $\xi\,\varphi - p = -\frac{1}{1+\xi}\, p + \frac{\xi}{1+\xi}\, \chi$, one easily deduces that $(p - \xi\varphi) \in L^2(0,1;H_\beta;\varepsilon\, r^2\, dz)$, since
$$\|\frac{1}{1+\xi}\, p\|^2_{L^2(0,1;H_\beta;\varepsilon\, r^2\, dz)} \leq c_\beta\, \|\varepsilon\|_\infty\, \|p\|^2_{L^2_{r^2}},$$
   and using $\frac{\xi}{1+\xi} \leq 1$,
$$\|\frac{\xi}{1+\xi}\, \chi\|^2_{L^2(0,1;H_\beta;\varepsilon\, r^2\, dz)} \leq \|\chi\|^2_{L^2(0,1;H_\beta;\varepsilon\, r^2\, dz)}.$$
   One checks that $\xi\,\widetilde{\varphi} - p = -\frac{1}{1+\xi}\, p + \frac{\xi}{1+\xi}\, \widetilde{\chi} \in L^2(0,1;V_{1-\alpha};\eta\, r^2\, dz)$ by firstly noting that
$$\|\frac{1}{1+\xi}\, p\|^2_{L^2(0,1;V_{1-\alpha};\eta\, r^2\, dz)} \leq c_{1-\alpha}\, \|\eta\|_\infty\, \|p\|^2_{L^2_{r^2}},$$

and secondly using $\|\frac{\xi}{1+\xi}\,\widetilde{\chi}\|^2_{V_{1-\alpha}} = \|\sqrt{\frac{\xi}{1+\xi}}\,\widetilde{\chi}\|^2_{\widetilde{V}_{1-\alpha}} \leq \|\widetilde{\chi}\|^2_{\widetilde{V}_{1-\alpha}}$ to deduce

$$\|\frac{\xi}{1+\xi}\,\widetilde{\chi}\|^2_{L^2(0,1;V_{1-\alpha};\eta\, r^2\, dz)} \leq \|\widetilde{\chi}\|^2_{L^2(0,1;\widetilde{V}_{1-\alpha};\eta\, r^2\, dz)}.$$

From the monotonicity and maximality of operator $\mathcal{A}$, one concludes by applying Lümer-Phillips theorem (see e.g. [12, theorem 2.27]). □

Remark 2.2.1 *For the diffusive realizations of $\partial_t^{-\beta} p$ and $\partial_t^{\alpha} p$, we have introduced* two *state variables, namely φ and $\widetilde{\varphi}$; but it is easy to notice that they both fulfill $\varphi(\xi,z,t) = \widetilde{\varphi}(\xi,z,t) = \int_0^t e^{-\xi\,\tau} p(t-\tau,z)\,d\tau$: only the functional spaces are different. Hence, only* one *state variable is needed, which belongs to a smaller functional space as follows:*

$$\varphi = \widetilde{\varphi} \in L^2(0,1;H_\beta;\varepsilon\, r^2\, dz) \cap L^2(0,1;\widetilde{V}_{1-\alpha};\eta\, r^2\, dz)\ .$$

This point can be used in deriving a numerical scheme to reduce the number of state variables, and it is all the more useful from a computational point of view, than these unknowns depend upon three variables!

Chapter 3

Numerical analysis

3.1 A finite differences scheme to solve the Webster Lokshin equations

We recall here the equations of this model, where $(z, t, \xi) \in [0, 1] \times (0, \infty) \times (0, \infty)$,

(i) $\partial_t p(z,t) = -\dfrac{1}{r^2(z)} \partial_z v(z,t) - \varepsilon\, \theta(z,t) - \eta\, \tilde{\theta}(z,t)$,

(ii) $\partial_t v(z,t) = -r^2(z)\, \partial_z p(z,t)$,

(iii) $\theta(z,t) = \displaystyle\int_0^{+\infty} \varphi(\xi, z, t)\, dM_\beta(\xi)$ and $\tilde{\theta}(z,t) = \displaystyle\int_0^{+\infty} [p(z,t) - \xi\, \varphi(\xi, z, t)]\, dM_{1-\alpha}(\xi)$,

(iv) $\partial_t \varphi(\xi, z, t) = -\xi \varphi(\xi, z, t) + p(z,t)$,

(v) $p(0,t) = 0$,

(vi) $p(1,t) = Cx(t) + dv(1,t)$,

(vii) $\dfrac{dx}{dt}(t) = Ax(t) + Bv(1,t)$,

coupled with initial conditions

$$p(z,0) = p_0(z), \quad v(z,0) = v_0(z), \quad \varphi(\xi, z, 0) = 0, \quad x(0) = 0.$$

Remark that the impedance boundary condition at $z = 0$ has been replaced by a Dirichlet boundary condition. This does not affect the generality of our results since the treatment of an impedance boundary condition will be explained at $z = 1$. We recall that $x(t) \in \mathbb{R}^{n_1}$, $A \in \mathbb{R}^{n_1 \times n_1}$, $B \in \mathbb{R}^{n_1 \times 1}$, $C \in \mathbb{R}^{1 \times n_1}$ and $d \in \mathbb{R}$ for some integer n_1.

Let Δt and $h = 1/N$ respectively be the time and space steps where N is the number of discretization points of $[0,1]$. We set $z_i = ih$, $z_{i+\frac{1}{2}} = (i+\frac{1}{2})h$ and denote

$$p_i^n \approx p(ih, n\Delta t);\; v_{i+\frac{1}{2}}^{n+\frac{1}{2}} \approx v((i+\tfrac{1}{2})h, (n+\tfrac{1}{2})\Delta t);\; \theta_i^n \approx \theta(ih, n\Delta t);\; \tilde{\theta}_i^n \approx \tilde{\theta}(ih, n\Delta t)$$

Then a second order centered explicit scheme associated with (i) and (ii) can be written as

$$\frac{p_i^{n+1} - p_i^n}{\Delta t} = -\frac{1}{r^2(z_i)} \frac{v_{i+\frac{1}{2}}^{n+\frac{1}{2}} - v_{i-\frac{1}{2}}^{n+\frac{1}{2}}}{h} - \varepsilon(z_i)\, \frac{\theta_i^{n+1} + \theta_i^n}{2} - \eta(z_i)\, \frac{\tilde{\theta}_i^{n+1} + \tilde{\theta}_i^n}{2} \tag{3.1}$$

for $n > 0$ and $0 < i \leq N$ and

$$\frac{v_{i+\frac{1}{2}}^{n+\frac{1}{2}} - v_{i+\frac{1}{2}}^{n-\frac{1}{2}}}{\Delta t} = -r^2(z_{i+\frac{1}{2}}) \frac{p_{i+1}^n - p_i^n}{h} \tag{3.2}$$

for $n > 0$ and $0 \leq i < N$. The discrete version of (vi) and (vii) can be written as follows

$$\begin{cases} \dfrac{p_N^{n+1} + p_N^n}{2} = C \dfrac{x^{n+1} + x^n}{2} + d \dfrac{v_{N-\frac{1}{2}}^{n+\frac{1}{2}} + v_{N+\frac{1}{2}}^{n+\frac{1}{2}}}{2} \\[2ex] \dfrac{x^{n+1} - x^n}{\Delta t} = A \dfrac{x^{n+1} + x^n}{2} + B \dfrac{v_{N-\frac{1}{2}}^{n+\frac{1}{2}} + v_{N+\frac{1}{2}}^{n+\frac{1}{2}}}{2}. \end{cases} \tag{3.3}$$

Note that we have introduced in (3.1) and (3.3) (as it is classically done for convenience in writing the discretized scheme at the boundary) a fictitious node at $(N+\frac{1}{2})h$ for the variable v.

We remark that the scheme (3.3) is implicit for x^n. The evaluation of θ_i^n and $\tilde{\theta}_i^n$ requires the evaluation of the integrals in (iv). We introduce geometric grid of the ξ axis defined by lower bound ξ_m, upper bound ξ_M and the number of points N_ξ. We define

$$\xi_j = \left(\frac{\xi_M}{\xi_m}\right)^{\frac{j-1}{N_\xi - 1}} \xi_{m,};\; j = 1, \cdots, N_\xi,$$

and denote $\varphi_{i,j}^n \approx \varphi(\xi_j, z_i, n\Delta t)$. Then, the differential equation for φ is discretized, using the following unconditionally stable and second order explicit scheme

$$\varphi_{i,j}^{n+1} = e^{-\xi_j \Delta t} \varphi_{i,j}^n + \frac{1 - e^{-\xi_j \Delta t}}{\xi_j} \frac{p_i^{n+1} + p_i^n}{2} \tag{3.4}$$

which has been derived from the expression

$$\varphi(\xi, z, t) = \int_0^t e^{-\xi(t-s)} p(z,s)\, ds$$

by exact integration of the right hand side between $n\Delta t$ and $(n+1)\Delta t$. We deduce the expressions of θ_i^n and $\tilde{\theta}_i^n$ through an exact evaluation of

$$\theta_i^n = \int_0^{\xi_M} \sum_{j=1}^{N_\xi} \varphi_{i,j}^n \lambda_j(\xi)\, dM_\beta(\xi) \tag{3.5}$$

$$\tilde{\theta}_i^n = \int_0^{\xi_M} \sum_{j=1}^{N_\xi} (p_i^n - \xi_j \varphi_{i,j}^n)\, \lambda_j(\xi)\, dM_{1-\alpha}(\xi) \tag{3.6}$$

where λ_j is continuous, piecewise linear function that satisfies $\lambda_j(\xi_i) = \delta_{i,j}$ with $\delta_{i,j}$ denoting the Kronecker symbol. Hence

$$\begin{aligned} \theta_i^n &= \sum_{j=1}^{N_\xi} \rho_j(\beta) \varphi_{i,j}^n \\ \tilde{\theta}_i^n &= \sum_{j=1}^{N_\xi} \rho_j(1-\alpha)(p_i^n - \xi_j \varphi_{i,j}^n) \end{aligned} \tag{3.7}$$

for some non-negative quadrature weights ρ_j. Let us quote that the following analysis holds for any choice of quadrature of the form (3.7). In the present study, the choice of the nodes ξ_j is made the same for the evaluation of θ_i^n and $\tilde{\theta}_i^n$ and is independent of α. Of course, one may think that a more accurate evaluation of these quatities may require a dependence of the nodes ξ_j on the order of derivative (α or β). This change of nodes ξ_j has no influence on the validity of subsequent results as long as the form of the quadrature rule (3.7) is respected.

Remark 3.1.1 *Remark that another possibility to approximate $\tilde{\theta}$ would have been to replace (3.6) with*

$$\tilde{\theta}_i^n = \int_0^{\xi_M} \sum_{j=1}^{N_\xi} (p_i^n - \xi \varphi_{i,j}^n)\, \lambda_j(\xi)\, dM_{1-\alpha}(\xi).$$

This will lead to a quadrature formula that is not of the form (3.7) and therefore the following analysis does not include this case. Moreover, it turns out that this discritization does not respect the discrete energy balance and seems to be numerically less accurate (especially for long time behaviour).

3.2 Stability study of the discretized model

We introduce a so-called discrete wave energy associated with our scheme as being defined by

$$E^n = \frac{h}{2}\left(\sum_{i=0}^{N-1} |r(z_i)\, p_i^n|^2 + \frac{1}{r(z_{i+\frac{1}{2}})^2}\, v_{i+\frac{1}{2}}^{n+\frac{1}{2}}\, v_{i+\frac{1}{2}}^{n-\frac{1}{2}} \right) + \frac{h}{4}\, |r(z_N)\, p_N^n|^2\,.$$

Let

$$\gamma = \max \left\{ \frac{r(z_{i+\frac{1}{2}})}{r(z_{i+1})}, \frac{r(z_{i+\frac{1}{2}})}{r(z_i)}, \;\; i = 1, \cdots, N-1 \right\},$$

we first show that this energy is a norm under the CFL stability condition $\gamma \Delta t < h$.

Lemma 3.2.1 *Assume that $\gamma \Delta t < h$, then there exists a constant $C > 0$, independent of n, such that*

$$E^n \geq C \sum_{i=1}^{N} |r(z_i)\, p_i^n|^2 + \left| (v_{i-\frac{1}{2}}^{n+\frac{1}{2}} + v_{i-\frac{1}{2}}^{n-\frac{1}{2}})/2r(z_{i-\frac{1}{2}}) \right|^2.$$

Proof. The proof of this lemma is rather classical. We give it here for the sake of completeness. From equation (3.2) and the definition of γ we get

$$\frac{1}{r(z_{i+\frac{1}{2}})^2} \, |v_{i+\frac{1}{2}}^{n+\frac{1}{2}} - v_{i+\frac{1}{2}}^{n-\frac{1}{2}}|^2 \leq 2\gamma^2 \frac{\Delta t^2}{h^2} (|r(z_{i+1}) p_{i+1}^n|^2 + |r(z_i) p_i^n|^2)$$

for $i = 0, \cdots, N-1$. Using the identity

$$v_{i+\frac{1}{2}}^{n+\frac{1}{2}} \, v_{i+\frac{1}{2}}^{n-\frac{1}{2}} = \frac{1}{4}(v_{i+\frac{1}{2}}^{n+\frac{1}{2}} + v_{i+\frac{1}{2}}^{n-\frac{1}{2}})^2 - \frac{1}{4}(v_{i+\frac{1}{2}}^{n+\frac{1}{2}} - v_{i+\frac{1}{2}}^{n-\frac{1}{2}})^2$$

and the previous estimate yields (using $p_0^n = 0$)

$$E^n \geq \frac{h}{2} \left(\sum_{i=0}^{N-1} (1 - \frac{\gamma^2 \Delta t^2}{h^2}) \, |r(z_i)\, p_i^n|^2 + \left| \frac{v_{i+\frac{1}{2}}^{n+\frac{1}{2}} + v_{i+\frac{1}{2}}^{n-\frac{1}{2}}}{2r(z_{i+\frac{1}{2}})} \right|^2 \right) + \frac{h}{4}(1 - \frac{\gamma^2 \Delta t^2}{h^2}) \left| r(z_N)\, p_i^N \right|^2,$$

whence the desired estimate under the CFL condition. □

We shall prove that this energy is uniformly bounded with respect to n which yields the stability of our scheme under the CFL condition of Lemma 3.2.1. For the sake of clarity we will study first the three cases

1. $(\varepsilon, \eta) = (0, 0)$: stability of the coupling with the discretized impedance boundary condition,
2. $\eta = 0, \varepsilon > 0$: stability of the coupling with the discretized $\partial_t^{-\beta}$,
3. $\varepsilon = 0, \eta > 0$: stability of the coupling with the discretized ∂_t^{α},

separately then deduce the stability result for the general case.

3.2.1 Case $(\varepsilon, \eta) = (0, 0)$

We shall concentrate in this section on the stability of the coupling between the interior scheme with the discretized impedance boundary condition. We recall that there exists $P \in \mathbb{R}^{n_1 \times n_1}$, $P = P^T > 0$, such that

$$\mathcal{M} = \begin{pmatrix} -A^T P - PA & C^T - PB \\ C - B^T P & 2d \end{pmatrix}$$

is a non-negative matrix of $\mathbb{R}^{(n_1+1)\times(n_1+1)}$.

Proof of the energy balance: Multiplying equation (3.1) by $h\frac{r(z_i)^2}{2}(p_i^{n+1} + p_i^n)$ yields

$$\frac{h}{2\Delta t}(|r(z_i)p_i^{n+1}|^2 - |r(z_i)p_i^n|^2) = -\frac{1}{2}(v_{i+\frac{1}{2}}^{n+\frac{1}{2}} - v_{i-\frac{1}{2}}^{n+\frac{1}{2}})(p_i^{n+1} + p_i^n). \tag{3.8}$$

for $i = 1, \cdots, N$. Taking the difference between equation (3.2) at step $n + \frac{3}{2}$ and the same equation at step $n + \frac{1}{2}$ yields, after multiplication by $h\frac{1}{2r(z_i)^2}v_{i+\frac{1}{2}}^{n+\frac{1}{2}}$,

$$\frac{h}{2\Delta t}\left(\frac{v_{i+\frac{1}{2}}^{n+\frac{3}{2}}\, v_{i+\frac{1}{2}}^{n+\frac{1}{2}}}{r(z_{i+\frac{1}{2}})^2} - \frac{v_{i+\frac{1}{2}}^{n+\frac{1}{2}}\, v_{i+\frac{1}{2}}^{n-\frac{1}{2}}}{r(z_{i+\frac{1}{2}})^2}\right) = -\frac{1}{2}v_{i+\frac{1}{2}}^{n+\frac{1}{2}}((p_{i+1}^{n+1} + p_{i+1}^n) - (p_i^{n+1} + p_i^n)). \tag{3.9}$$

Using (3.8) and (3.9) we easily deduce the balance

$$\frac{E^{n+1} - E^n}{\Delta t} = -\frac{v_{N+\frac{1}{2}}^{n+\frac{1}{2}} + v_{N-\frac{1}{2}}^{n+\frac{1}{2}}}{2}\,\frac{p_N^{n+1} + p_N^n}{2}$$

Its very convenient to use the notation

$$v_N^{n+\frac{1}{2}} = \frac{v_{N+\frac{1}{2}}^{n+\frac{1}{2}} + v_{N-\frac{1}{2}}^{n+\frac{1}{2}}}{2} \text{ and } p_N^{n+\frac{1}{2}} = \frac{p_N^{n+1} + p_N^n}{2} \tag{3.10}$$

so that the previous balance can be written in the form

$$\frac{E^{n+1} - E^n}{\Delta t} = -v_N^{n+\frac{1}{2}} p_N^{n+\frac{1}{2}}. \tag{3.11}$$

From the first equation of (3.3) it is clear that

$$v_N^{n+\frac{1}{2}} p_N^{n+\frac{1}{2}} = v_N^{n+\frac{1}{2}} C x^{n+\frac{1}{2}} + d|v_N^{n+\frac{1}{2}}|^2 \tag{3.12}$$

where $x^{n+\frac{1}{2}} := \dfrac{x^{n+1} + x^n}{2}$. Let us set

$$E_x^n = \frac{1}{2}x^n \cdot P x^n \tag{3.13}$$

as being the energy associated with the boundary variable x^n. The second equation of (3.3) implies after taking the sum of a multiplication by $Px^{n+\frac{1}{2}}$ and a multiplication by $(x^{n+\frac{1}{2}})^T P$,

$$2\frac{E_x^{n+1}-E_x^n}{\Delta t} = x^{n+\frac{1}{2}}\cdot(A^TP+PA)x^{n+\frac{1}{2}}+v_N^{n+\frac{1}{2}}(Px^{n+\frac{1}{2}}\cdot B+x^{n+\frac{1}{2}}\cdot PB). \quad (3.14)$$

Combining (3.12) and (3.14) yields

$$v_N^{n+\frac{1}{2}}p_N^{n+\frac{1}{2}} = \frac{E_x^{n+1}-E_x^n}{\Delta t} + \frac{1}{2}\begin{pmatrix} x^{n+\frac{1}{2}} \\ v_N^{n+\frac{1}{2}}\end{pmatrix}\cdot\mathcal{M}\begin{pmatrix} x^{n+\frac{1}{2}} \\ v_N^{n+\frac{1}{2}}\end{pmatrix} \quad (3.15)$$

Let us define the total energy as

$$\mathcal{E}^n = E^n + E_x^n, \quad (3.16)$$

we deduce from (3.11) and (3.15) that

$$\frac{\mathcal{E}^{n+1}-\mathcal{E}^n}{\Delta t} = -\frac{1}{2}\begin{pmatrix} x^{n+\frac{1}{2}} \\ v_N^{n+\frac{1}{2}}\end{pmatrix}\cdot\mathcal{M}\begin{pmatrix} x^{n+\frac{1}{2}} \\ v_N^{n+\frac{1}{2}}\end{pmatrix}$$

which shows that the this energy is non increasing with n. Hence it is uniformly bounded with respect to n. Combining this fact with the result of Lemma 3.2.1 proves that our scheme is L^2 stable.

3.2.2 Case $\eta = 0$

We start by rewriting in a suitable form equation (3.4). Introducing

$$\omega_j = \frac{\xi_j\Delta t}{2}\,\frac{1+e^{-\xi_j\Delta t}}{1-e^{-\xi_j\Delta t}}$$

we see that (3.4) is equivalent to

$$\omega_j\frac{\varphi_{i,j}^{n+1}-\varphi_{i,j}^n}{\Delta t} = -\xi_j\frac{\varphi_{i,j}^{n+1}+\varphi_{i,j}^n}{2} + \frac{p_i^{n+1}+p_i^n}{2}. \quad (3.17)$$

Multiplying this equation by $\frac{1}{2}\left(\varphi_{i,j}^{n+1}+\varphi_{i,j}^n\right)$ yields

$$\frac{\varphi_{i,j}^{n+1}+\varphi_{i,j}^n}{2}\,\frac{p_i^{n+1}+p_i^n}{2} = \frac{1}{2}\omega_j\frac{|\varphi_{i,j}^{n+1}|^2-|\varphi_{i,j}^n|^2}{\Delta t} + \xi_j\left|\frac{\varphi_{i,j}^{n+1}+\varphi_{i,j}^n}{2}\right|^2.$$

Therefore, from (3.7),

$$\frac{\theta_i^{n+1}+\theta_i^n}{2}\,\frac{p_i^{n+1}+p_i^n}{2} = \sum_{j=1}^{N_\xi}\frac{1}{2}\rho_j(\beta)\omega_j\frac{|\varphi_{i,j}^{n+1}|^2-|\varphi_{i,j}^n|^2}{\Delta t}+\rho_j(\beta)\xi_j\left|\frac{\varphi_{i,j}^{n+1}+\varphi_{i,j}^n}{2}\right|^2. \quad (3.18)$$

Following the same steps as in the proof of (3.11) yields in the present case to

$$\frac{E^{n+1}-E^n}{\Delta t} = -v_N^{n+\frac{1}{2}} p_N^{n+\frac{1}{2}} - \sum_{i=1}^{N-1} \varepsilon(z_i)\, \theta_i^{n+\frac{1}{2}}\, p_i^{n+\frac{1}{2}} - \frac{1}{2}\varepsilon(z_N)\, \theta_N^{n+\frac{1}{2}} p_N^{n+\frac{1}{2}} \tag{3.19}$$

with short-hand notation

$$\theta_i^{n+\frac{1}{2}} := \frac{\theta_i^{n+1}+\theta_i^n}{2} \text{ and } p_i^{n+\frac{1}{2}} := \frac{p_i^{n+1}+p_i^n}{2}. \tag{3.20}$$

Let us introduce now

$$E_\varphi^n := \frac{1}{2}\sum_{j=1}^{N_\xi} \rho_j(\beta)\omega_j \left(\sum_{i=1}^{N-1} \varepsilon(z_i)|\varphi_{i,j}^n|^2 + \frac{1}{2}\varepsilon(z_N)|\varphi_{N,j}^n|^2 \right) \tag{3.21}$$

as being the energy associated with the antiderivative term $\partial_t^{-\beta}$. It is clear from (3.19) and (3.18) that we have the following balance

$$\frac{E^{n+1}-E^n}{\Delta t} = -v_N^{n+\frac{1}{2}} p_N^{n+\frac{1}{2}} - \frac{E_\varphi^{n+1}-E_\varphi^n}{\Delta t} - \sum_{j=1}^{N_\xi} \rho_j(\beta)\xi_j \left(\sum_{i=1}^{N-1} \varepsilon(z_i) \left| \varphi_{i,j}^{n+\frac{1}{2}} \right|^2 + \frac{1}{2}\varepsilon(z_N) \left| \varphi_{N,j}^{n+\frac{1}{2}} \right|^2 \right) \tag{3.22}$$

where we have set

$$\varphi_{i,j}^{n+\frac{1}{2}} := \frac{\varphi_{i,j}^{n+1}+\varphi_{i,j}^n}{2}. \tag{3.23}$$

Define the total energy in the present case as

$$\mathcal{E}^n = E^n + E_x^n + E_\varphi^n \tag{3.24}$$

we deduce from (3.15) and (3.22) that

$$\begin{aligned} \frac{\mathcal{E}^{n+1}-\mathcal{E}^n}{\Delta t} &= -\frac{1}{2}\begin{pmatrix} x^{n+\frac{1}{2}} \\ v_N^{n+\frac{1}{2}} \end{pmatrix} \cdot \mathcal{M} \begin{pmatrix} x^{n+\frac{1}{2}} \\ v_N^{n+\frac{1}{2}} \end{pmatrix} \\ &\quad - \sum_{j=1}^{N_\xi} \rho_j(\beta)\xi_j \left(\sum_{i=1}^{N-1} \varepsilon(z_i) \left| \varphi_{i,j}^{n+\frac{1}{2}} \right|^2 + \frac{1}{2}\varepsilon(z_N) \left| \varphi_{N,j}^{n+\frac{1}{2}} \right|^2 \right) \end{aligned}$$

which shows that the this energy is non increasing with n. Hence it is uniformly bounded with respect to n. Combining this fact with the result of Lemma 3.2.1 proves that our scheme is L^2 stable.

3.2.3 Case $\varepsilon = 0$

We study here the stability of the coupling with the discretization of the fractional derivative ∂_t^α. Using equation (3.17) and shorthand notation (3.20) and (3.23), we have the identity,

$$p_i^{n+\frac{1}{2}} \left(p_i^{n+\frac{1}{2}} - \xi_j \varphi_{i,j}^{n+\frac{1}{2}} \right) = \omega_j^2 \left| \frac{\varphi_{i,j}^{n+1}-\varphi_{i,j}^n}{\Delta t} \right|^2 + \frac{1}{2}\omega_j\xi_j \frac{|\varphi_{i,j}^{n+1}|^2 - |\varphi_{i,j}^n|^2}{\Delta t}$$

Therefore, from (3.7),

$$\tilde{\theta}_i^{n+\frac{1}{2}}\, p_i^{n+\frac{1}{2}} = \sum_{j=1}^{N_\xi} \frac{1}{2}\rho_j(1-\alpha)\omega_j\xi_j \frac{|\varphi_{i,j}^{n+1}|^2 - |\varphi_{i,j}^{n}|^2}{\Delta t} + \rho_j(1-\alpha)\left|\omega_j \dot{\varphi}_{i,j}^{n+\frac{1}{2}}\right|^2. \quad (3.25)$$

with short-hand notation

$$\tilde{\theta}_i^{n+\frac{1}{2}} := \frac{\tilde{\theta}_i^{n+1} + \tilde{\theta}_i^{n}}{2} \ \text{ and } \ \dot{\varphi}_{i,j}^{n+\frac{1}{2}} := \frac{\varphi_{i,j}^{n+1} - \varphi_{i,j}^{n}}{\Delta t}.$$

On the other hand, following the same steps as in the proof of (3.11) yields in the present case to

$$\frac{E^{n+1} - E^n}{\Delta t} = -v_N^{n+\frac{1}{2}} p_N^{n+\frac{1}{2}} - \sum_{i=1}^{N-1} \eta(z_i)\, \tilde{\theta}_i^{n+\frac{1}{2}}\, p_i^{n+\frac{1}{2}} - \frac{1}{2}\eta(z_N)\, \tilde{\theta}_N^{n+\frac{1}{2}} p_N^{n+\frac{1}{2}}. \quad (3.26)$$

Let us introduce now

$$\widetilde{E}_\varphi^n := \frac{1}{2}\sum_{j=1}^{N_\xi} \rho_j(1-\alpha)\omega_j\xi_j \left(\sum_{i=1}^{N-1} \eta(z_i)|\varphi_{i,j}^n|^2 + \frac{1}{2}\eta(z_N)|\varphi_{N,j}^n|^2\right) \quad (3.27)$$

as being the energy associated with the derivative term ∂_t^α. It is clear from (3.26) and (3.25) that we have the following balance

$$\begin{aligned} \frac{E^{n+1} - E^n}{\Delta t} &= -v_N^{n+\frac{1}{2}} p_N^{n+\frac{1}{2}} - \frac{\widetilde{E}_\varphi^{n+1} - \widetilde{E}_\varphi^n}{\Delta t} \\ &\quad - \sum_{j=1}^{N_\xi} \rho_j(1-\alpha)\omega_j^2 \left(\sum_{i=1}^{N-1} \eta(z_i)\left|\dot{\varphi}_{i,j}^{n+\frac{1}{2}}\right|^2 + \frac{1}{2}\eta(z_N)\left|\dot{\varphi}_{N,j}^{n+\frac{1}{2}}\right|^2\right). \end{aligned} \quad (3.28)$$

Define the total energy in the present case as

$$\mathcal{E}^n = E^n + E_x^n + \widetilde{E}_\varphi^n, \quad (3.29)$$

then it follows from (3.15) and (3.28) that

$$\begin{aligned} \frac{\mathcal{E}^{n+1} - \mathcal{E}^n}{\Delta t} &= -\frac{1}{2}\begin{pmatrix} x^{n+\frac{1}{2}} \\ v_N^{n+\frac{1}{2}} \end{pmatrix} \cdot \mathcal{M} \begin{pmatrix} x^{n+\frac{1}{2}} \\ v_N^{n+\frac{1}{2}} \end{pmatrix} \\ &\quad - \sum_{j=1}^{N_\xi} \rho_j(1-\alpha)\omega_j^2 \left(\sum_{i=1}^{N-1} \eta(z_i)\left|\dot{\varphi}_{i,j}^{n+\frac{1}{2}}\right|^2 + \frac{1}{2}\eta(z_N)\left|\dot{\varphi}_{N,j}^{n+\frac{1}{2}}\right|^2\right) \end{aligned}$$

which shows that this energy is non increasing with n. Hence it is uniformly bounded with respect to n. Combining this fact with the result of Lemma 3.2.1 proves that our scheme is L^2 stable.

3.2.4 The general case

From the results of previous subsections it is clear now that if we define the total energy as

$$\mathcal{E}^n = E^n + E_x^n + E_\varphi^n + \widetilde{E}_\varphi^n,$$

then one has the following balance,

$$\begin{aligned}
\frac{\mathcal{E}^{n+1} - \mathcal{E}^n}{\Delta t} &= -\frac{1}{2}\begin{pmatrix} x^{n+\frac{1}{2}} \\ v_N^{n+\frac{1}{2}} \end{pmatrix} \cdot \mathcal{M}\begin{pmatrix} x^{n+\frac{1}{2}} \\ v_N^{n+\frac{1}{2}} \end{pmatrix} \\
&\quad - \sum_{j=1}^{N_\xi} \rho_j(\beta)\xi_j \left(\sum_{i=1}^{N-1} \varepsilon(z_i) \left| \varphi_{i,j}^{n+\frac{1}{2}} \right|^2 + \frac{1}{2}\varepsilon(z_N) \left| \varphi_{N,j}^{n+\frac{1}{2}} \right|^2 \right) \\
&\quad - \sum_{j=1}^{N_\xi} \rho_j(1-\alpha)\omega_j^2 \left(\sum_{i=1}^{N-1} \eta(z_i) \left| \dot{\varphi}_{i,j}^{n+\frac{1}{2}} \right|^2 + \frac{1}{2}\eta(z_N) \left| \dot{\varphi}_{N,j}^{n+\frac{1}{2}} \right|^2 \right).
\end{aligned}$$

Combining this with Lemma 3.2.1 we deduce:

Lemma 3.2.2 *Assume that $\gamma\Delta t < h$, then there exists a constant $C > 0$, independent of n, such that*

$$\sum_{i=1}^{N} |r(z_i)\, p_i^n|^2 + \left| (v_{i-\frac{1}{2}}^{n+\frac{1}{2}} + v_{i-\frac{1}{2}}^{n-\frac{1}{2}})/2r(z_{i-\frac{1}{2}}) \right|^2 + E_x^n + E_\varphi^n + \widetilde{E}_\varphi^n \leq C.$$

This shows the L^2 stability of our scheme.

3.3 Some numerical results

We shall give hereafter some numerical experiments that illustrate the behaviour of the discretized model and mimicks the energy balance. They also aim at giving some ideas on the effect of the damping introduced by fractional integral and derivative terms. Let us point out that these experiments are simply synthetic ones; the numerical values of the physical constants are completely arbitrary.

In order to produce initial conditions we put at $z = 0$ a source term

$$p(0,t) = s(t)$$

where $s(t)$ is one prescribed signal that vanishes after some time T. The signal and the time T are chosen so that an initial pulse has been produced inside the spatial domain $(0,1)$. To concentrate on the effect of diffusive terms, we take a special case of the impedance boundary condition at $z = 1$, which produces a perfectly transparent boundary condition: it corresponds to

$$A = 0,\ B = 0,\ C = 0 \text{ and } d = 1.$$

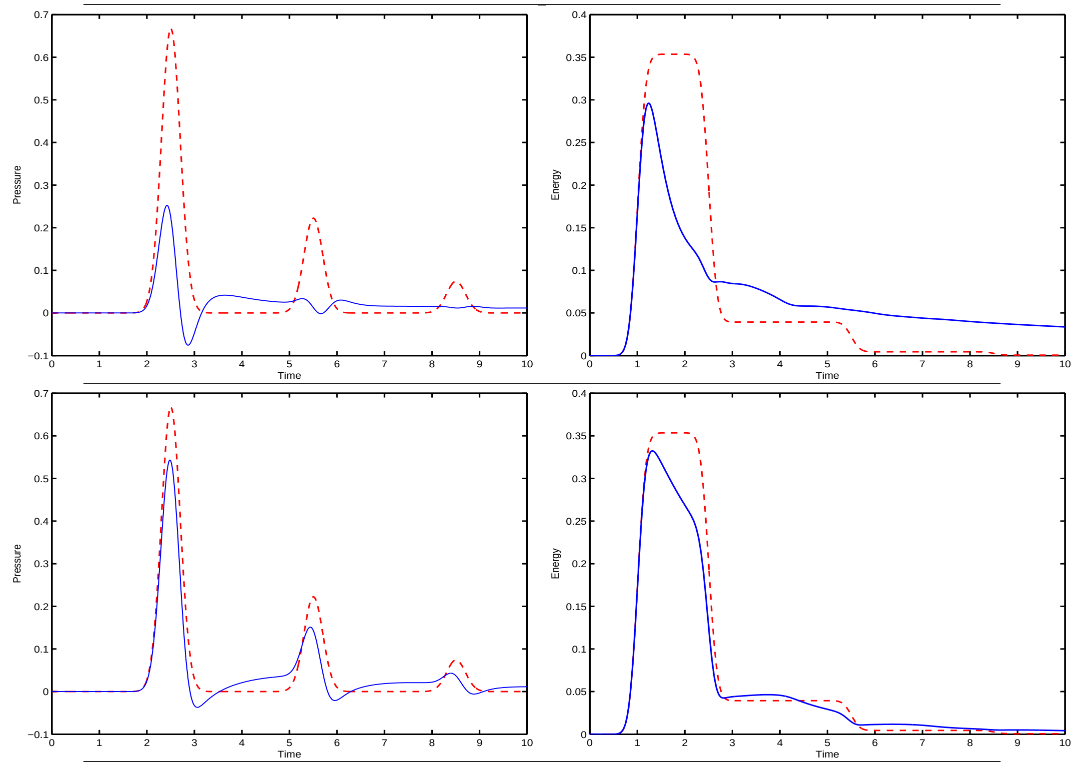

Figure 3.1: Influence of the parameter ε in a cylinder. Top: $\varepsilon = 1$ in blue plain line, compared to the reference $\varepsilon = 0$ in red dotted line. Bottom: $\varepsilon = 0.2$ in blue plain line, compared to the reference $\varepsilon = 0$ in red dotted line. The pressure signals at the output versus time are shown on the left, whereas the wave energy in the cylinder is plotted versus time on the right.

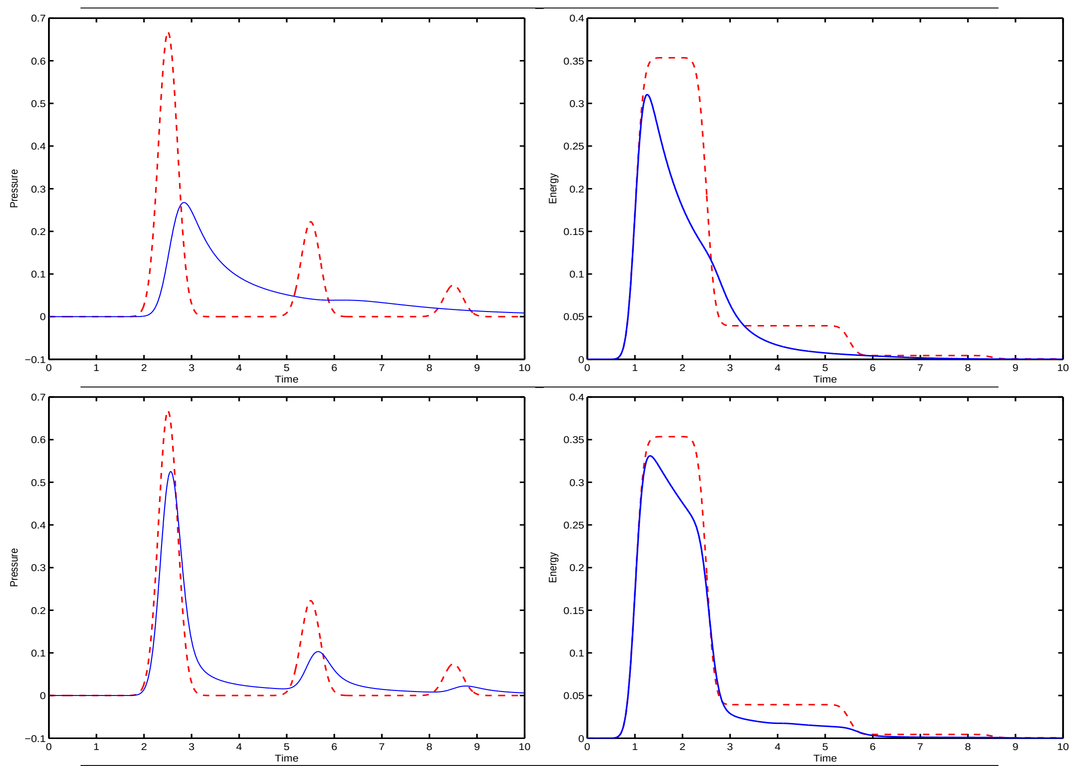

Figure 3.2: Influence of the parameter η in a cylinder. Top: $\eta = 0.5$ in blue plain line, compared to the reference $\eta = 0$ in red dotted line. Bottom: $\eta = 0.11$ in blue plain line, compared to the reference $\eta = 0$ in red dotted line. The pressure signals at the output versus time are shown on the left, whereas the wave energy in the cylinder is plotted versus time on the right.

Bibliography

[1] S. Boyd, L. El Ghaoui, E. Féron, and V. Balakrishnan. *Linear matrix inequalities in systems and control theory*, volume 15 of *Studies in Applied Mathematics*. SIAM, 1994.

[2] H. Brézis. *Analyse fonctionnelle. Théorie et applications*. Masson, 1992.

[3] R. F. Curtain. Old and new perspectives on the positive–real lemma in systems and control theory. *Z. Angew. Math. Mech.*, 79(9):579–590, 1999.

[4] R. F. Curtain and H. J. Zwart. *An introduction to infinite–dimensional linear systems theory*. Vol. 21 of Texts in Applied Mathematics. Springer Verlag, 1995.

[5] R. Dautray and J.-L. Lions. *Mathematical analysis and numerical methods for science and technology*, volume 5, chapter XVI, pages 286–290. Springer, 1984.

[6] B. Gustafsson, H.-O. Kreiss and J. Oliger. *Time dependent problems and difference methods*. Pure and Applied Mathematics. Wiley-Interscience, 1995.

[7] H. Haddar, Th. Hélie and D. Matignon. A Webster-Lokshin model for waves with viscothermal losses and impedance boundary conditions: strong solutions. In *Sixth int. conf. on math. and num. aspects of wave propagation phenomena*, p. 66–71, July 2003.

[8] Th. Hélie. Unidimensional models of the acoustic propagation in axisymmetric waveguides. *J. Acoust. Soc. Amer.*, 114(5):2633-47, November 2003.

[9] Th. Hélie and D. Matignon. Numerical simulation of acoustic waveguides for Webster-Lokshin model using diffusive representations. In *Sixth int. conf. on math. and num. aspects of wave propagation phenomena*, p. 72–77, July 2003.

[10] A. A. Lokshin. Wave equation with singular retarded time. *Dokl. Akad. Nauk SSSR*, 240:43–46, 1978. (in Russian).

[11] A. A. Lokshin and V. E. Rok. Fundamental solutions of the wave equation with retarded time. *Dokl. Akad. Nauk SSSR*, 239:1305–1308, 1978. (in Russian).

[12] Z. H. Luo, B. Z. Guo, and O. Morgul. *Stability and stabilization of infinite dimensional systems with applications*. Communications and Control Engineering. Springer Verlag, 1999.

[13] D. Matignon and B. d'Andréa-Novel. Spectral and time-domain consequences of an integro-differential perturbation of the wave PDE. In *Third int. conf. on math. and num. aspects of wave propagation phenomena*, p. 769–771, April 1995.

[14] D. Matignon, J. Audounet, and G. Montseny. Energy decay for wave equations with damping of fractional order. In *Fourth int. conf. on math. and num. aspects of wave propagation phenomena*, p. 638–640, June 1998.

[15] D. Matignon and G. Montseny, editors. *Fractional Differential Systems: models, methods and applications*, volume 5 of *ESAIM: Proceedings*, December 1998. SMAI. URL: `http://www.edpsciences.org/articlesproc/Vol.5/`

[16] D. Matignon and Ch. Prieur Asymptotic stability of linear conservative systems when coupled with diffusive systems. In *Mathematical Theory of Networks and Systems*, 8 p., July 2004.

[17] J.-D. Polack. Time domain solution of Kirchhoff's equation for sound propagation in viscothermal gases: a diffusion process. *J. Acoustique*, 4:47–67, Feb. 1991.

[18] A. Rantzer. On the Kalman–Yakubovich–Popov lemma. *Systems & Control Letters*, 28:7–10, 1996.

[19] E. D. Sontag. *Mathematical Control Theory. Deterministic Finite Dimensional Systems*. Vol. 6 of Texts in Applied Mathematics. Springer Verlag, 1990.

[20] O. J. Staffans. Well-posedness and stabilizability of a viscoelastic equation in energy space. *Trans. Amer. Math. Soc.*, 345(2):527–575, October 1994.

[21] J. C. Strikwerda. *Finite difference schemes and partial differential equations*. Mathematics series. Wadsworth and Brooks/Cole, 1989.

Contents

Centre de recherche INRIA Saclay – Île-de-France
Parc Orsay Université - ZAC des Vignes
4, rue Jacques Monod - 91893 Orsay Cedex (France)

Centre de recherche INRIA Bordeaux – Sud Ouest : Domaine Universitaire - 351, cours de la Libération - 33405 Talence Cedex
Centre de recherche INRIA Grenoble – Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier
Centre de recherche INRIA Lille – Nord Europe : Parc Scientifique de la Haute Borne - 40, avenue Halley - 59650 Villeneuve d'Ascq
Centre de recherche INRIA Nancy – Grand Est : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex
Centre de recherche INRIA Paris – Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex
Centre de recherche INRIA Rennes – Bretagne Atlantique : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex
Centre de recherche INRIA Sophia Antipolis – Méditerranée : 2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399