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PSEUDO-CONFORMING POLYNOMIAL LAGRANGE FINITE ELEMENTS ON QUADRILATERALS AND HEXAHEDRA

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This paper is dedicated to Philippe G. CIARLET
on the occasion of his 70th birthday

Abstract. The aim of this paper is to develop a new class of finite elements on quadrilaterals and hexahedra. The degrees of freedom are the values at the vertices and the approximation is polynomial on each element K. In general, with this kind of finite elements, the resolution of second order elliptic problems leads to non-conforming approximations.Degrees of freedom are the same than those of isoparametric finite elements. The convergence of the method is analyzed and the theory is confirmed by some numerical results. Note that in the particular case when the finite elements are parallelotopes, the method is conforming and coincides with the classical finite elements on structured meshes.

1. **Introduction.** Quadrilaterals and hexahedra are often used in meshers particularly in geophysical applications and in fluids mecanics. When the geometry and the medium are structured, regular rectangular or parallelipipedic meshes are used as far as possible. Otherwise general convex quadrilaterals or hexahedra are used. Then, with isoparametric Lagrange finite elements([5],[6],[13], see also [3],[4]), the basis functions are built by using multilinear mappings on a reference square or a reference cube; for an element which is not a parallelotope, it is well known that these basis functions are not polynomial.

One way for obtaining polynomial basis functions is to cut the quadrilaterals into triangles (or hexahedra into tetrahedra) and work with macro-elements ([9], [10], [11]). It is not our process. We choose to build finite elements by considering quadrilaterals and hexahedra as distortions of parallelograms and parallelepipeds. It is important to note here that the reserved vocabulary is the one of mathematicians; therefore an hexahedron is an example of polyhedron and its faces are plane. In the literature of the mechanics, usually an hexahedron denotes the image of a cube by a Q_1 transformation; commonly, the faces of a "trilinear hexahedral element" (for instance, see [7]) are not plane; they are nappes of hyperbolic paraboloids. Note that the inversibility of the transformation of a biunit cube into an hexahedron is still open ([8], [14]). The generalization of the forthcoming analysis to "trilinear hexahedra" shall be not tackled in this paper.

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In the presented method, the Lagrange basis functions are built under conditions of weak-continuity of the unknowns between the elements. In the general case, the resulting finite element is not conforming but the conditions of weak-continuity are sufficient to ensure the expected order of convergence. In the particular case of a parallelotope, the resulting finite element is conforming and coincides with the most classical finite element on a parallelotope. Returning to the general case, we call pseudo-conforming such a finite element.

The rest of the paper is organized as follow. The section 2 of the paper is devoted to the finite elements geometry. The choosen approach allows us to describe jointly quadrilaterals and hexahedra. The section 3 deals with some general results on local error estimates for quadrilateral and hexahedral Lagrange finite elements. These estimations are essential to obtain the expected order of convergence of our pseudo-conforming finite elements. In the section 4, the elliptic model problem of order 2 is presented and sufficient conditions to establish the convergence of the method are given. The finite elements are built in the section 5 and numerical results are presented in the Section 6. We end in section 7 with some remarks for extensions of the method.

In this paper we use the following notations:
For a vector $\mathbf{v} \in \mathbb{R}^\mathbf{n}$, $|\mathbf{v}|$ is the lenght of the vector $\mathbf{v}$; in matrix notation $\mathbf{v}$ is represented by the column vector $(v_1, ..., v_n)^T$ and then $|\mathbf{v}| = \left\{\sum_{1\le j\le n} |v_j|^2\right\}^{1/2}$, the Euclidean norm of the associated column vector. And for a square matrix B, $\|B\|$ is the spectral norm.

For a triangle or a quadrilateral K, $|K|$ is the area of K and if γ is a side of K, $|\gamma|$ is the lenght of γ; for a tetrahedron or an hexahedron K, $|K|$ is the volume of K and if γ is a face of K, $|\gamma|$ is the area of γ.

For a polyhedral domain K, we note

$$H^m(K) = \left\{v \in L^2(K); \partial^\alpha v \in L^2(K), \text{ for all } \alpha \text{ with } |\alpha| \le m\right\}$$

equipped with the norm and the semi-norm

$$\|v\|_{m,K} = \left(\sum_{|\alpha|\le m} \int_K |\partial^\alpha v|^2 dx\right)^{1/2}, \quad |v|_{m,K} = \left(\sum_{|\alpha|= m} \int_K |\partial^\alpha v|^2 dx\right)^{1/2}.$$

We consider also the following norm and semi-norm

$$\|v\|_{m,\infty,K} = \max_{|\alpha|\le m}\left\{\operatorname*{ess\,sup}_{x\in K} |\partial^\alpha v|\right\}, \quad |v|_{m,\infty,K} = \max_{|\alpha|= m}\left\{\operatorname*{ess\,sup}_{x\in K} |\partial^\alpha v|\right\}.$$

$P(K)$ is the vectorial space $\{\mathbf{x} \in K \mapsto p(\mathbf{x});\ p \in P\}$, where P is a N variables polynomial space and K is a domain in $\mathbb{R}^N$. For any integer k, P_k is the space of polynomial functions of degree $\le k$, while Q_k is the space of polyomial functions of degree $\le k$ in each variable.

For each polyhedral K, h_K denotes the diameter of K and ρ_K denotes the diameter of the largest ball contained in K.

2. **The finite element geometry.**

2.1. **The geometry; vertex and face numbering.** In $\mathbb{R}^N$ with $N = 2$ or 3, let K be a convex nondegenerated quadrilateral when $N = 2$, a convex nondegenerated hexahedron when $N = 3$. Let $\left\{\mathbf{a}_i \in \mathbb{R}^N,\ 1 \le i \le 2^N\right\}$ be the vertices of K. We use hereafter the word "face" for 2D and 3D geometries with the following vocabulary convention: for $N = 2$, a face of a quadrilateral K designates a side of K. We designate by "edge" of K a side of K when $N = 2$, the intersection of two adjacent quadrangular faces of K when $N = 3$.

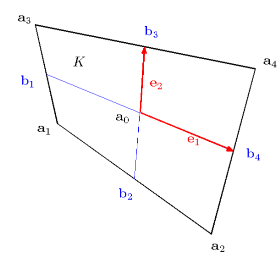


FIGURE 1. Numerotation ($N = 2$)

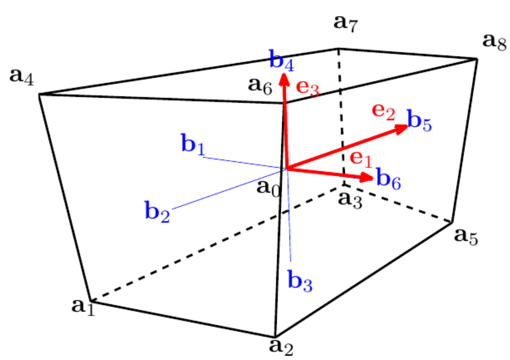


FIGURE 2. Numerotation ($N = 3$)

Two vertices which do not belong to a same face of K are said opposite vertices. The numbering of the vertices is shown on Figure 1 and Figure 2. Note that this vertex numbering is such that

$$\forall\ i = 2, ..., 2^{N-1}, \quad [\mathbf{a}_1, \mathbf{a}_i] \text{ is an edge of } K;$$
$$\forall\ i = 1, ..., 2^{N-1}, \quad \mathbf{a}_i \text{ and } \mathbf{a}_{2^N+1-i} \text{ are opposite vertices of } K.$$

The center of a polyhedral is the isobarycenter of its vertices; we note $\mathbf{a}_0$ the center of K:

$$\mathbf{a}_0 = \frac{1}{2^N} \sum_{1 \le i \le 2^N} \mathbf{a}_i.$$

Let now $\left\{\gamma_m \subset \mathbb{R}^N,\ 1 \le m \le 2N\right\}$ be the set of the faces of K. Two faces without common vertex are said opposite faces. The face numbering is shown on Figure 1 and Figure 2. This numbering is such that

$$\bigcap_{1 \le i \le N} \gamma_i = \mathbf{a}_1;$$
$$\forall\ m = 1, ..., N-1, \quad \mathbf{a}_{m+1} \notin \gamma_m;$$
$$\forall\ m = 1, ..., N, \quad \gamma_m \text{ and } \gamma_{2N+1-m} \text{ are opposite faces of } K.$$

Last, let $\mathbf{b}_m$ be the center of the face γ_m, for $m = 1, ..., 2N$, and let us introduce the vectors $\mathbf{e}_m \in \mathbb{R}^N$ defined by

$$\forall\ m = 1, ..., N, \quad \mathbf{e}_m = \mathbf{a}_0 - \mathbf{b}_m\ \left(= \mathbf{b}_{2N+1-m} - \mathbf{a}_0\right).$$

Since K is assumed to be a nondegenerated polyhedron, $(\mathbf{e}_1, ..., \mathbf{e}_N)$ is a basis of $\mathbb{R}^N$.

2.2. **Affine-equivalent elements.** Let $\widehat{K} = [-1, +1]^N$ be the reference square when $N = 2$, the reference cube when $N = 3$. The vertices of $\widehat{K}$ are denoted by $\widehat{\mathbf{a}}_i,\ 1 \le i \le 2^N$ and the faces are denoted by $\widehat{\gamma}_m,\ 1 \le m \le 2N$.

We choose the following vertex numbering

$$\text{for } N = 2: \quad \widehat{\mathbf{a}}_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix},\ \widehat{\mathbf{a}}_2 = \begin{pmatrix} +1 \\ -1 \end{pmatrix}$$

$$\text{for } N = 3: \quad \widehat{\mathbf{a}}_1 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix},\ \widehat{\mathbf{a}}_2 = \begin{pmatrix} +1 \\ -1 \\ -1 \end{pmatrix},\ \widehat{\mathbf{a}}_3 = \begin{pmatrix} -1 \\ +1 \\ -1 \end{pmatrix},\ \widehat{\mathbf{a}}_4 = \begin{pmatrix} -1 \\ -1 \\ +1 \end{pmatrix}$$

and

$$\text{for } N = 2 \text{ and } N = 3: \quad \widehat{\mathbf{a}}_i = -\widehat{\mathbf{a}}_{1+2^N-i}, \qquad 1 + 2^{N-1} \le i \le 2^N.$$

The face numbering is defined by

$$\begin{aligned} &\forall\ m = 1,\ ...,\ N, \quad \widehat{\gamma}_m = \left\{\widehat{\mathbf{x}} = (\widehat{x}_1, ..., \widehat{x}_N)^T \in \widehat{K};\ \widehat{x}_m = -1\right\}; \\ &\forall\ m = 1,\ ...,\ N, \quad \widehat{\gamma}_{2N+1-m} = \widehat{\gamma}_m. \end{aligned}$$

Let $\widehat{\mathbf{b}}_m$ be the center of the face $\widehat{\gamma}_m$, for $m = 1, ..., 2N$. The canonical basis $(\widehat{\mathbf{e}}_1, ..., \widehat{\mathbf{e}}_N)$ of $\mathbf{R}^N$ can be simply express with the vectors $\widehat{\mathbf{b}}_m$

$$\widehat{\mathbf{e}}_m = -\widehat{\mathbf{b}}_m\ \left(= \widehat{\mathbf{b}}_{2N+1-m}\right), \quad 1 \le m \le N.$$

Let B_K be the change of basis matrix given by

$$B_K \widehat{\mathbf{e}}_m = \mathbf{e}_m, \quad 1 \le m \le N..$$

and $F_K^\sharp$ be the invertible affine mapping

$$F_K^\sharp: \quad \widehat{\mathbf{x}} \in \mathbf{R}^N \to F_K^\sharp\left(\widehat{\mathbf{x}}\right) = \mathbf{a}_0 + B_K \widehat{\mathbf{x}}$$

This mapping $F_K^\sharp$ is the unique affine mapping such that

$$F_K^\sharp(\widehat{\mathbf{b}}_m) = \mathbf{b}_m, \quad 1 \le m \le N.$$

It is a bijection between $\widehat{K}$ and its image

$$K^\sharp = F_K^\sharp\left(\widehat{K}\right).$$

As image of the reference parallelotope by an inversible affine mapping, $K^\sharp$ is a parallelotope. The associated parallelotope of K being by definition the parallelotope which has the same face centers than K. We see that $K^\sharp$ is the associated parallelotope of K and we have $K^\sharp = K$ if and only if K is a parallelotope. Let

$$K^\vee = (F_K^\sharp)^{-1}(K).$$

The parallelotope associated to the polyhedron $K^\vee$ is the reference parallelotope $\widehat{K}$. For the analysis of quadrangular and hexahedral finite element, it is useful to precise how the element K is distorted.

2.3. **Distortion parameters.** When $N = 2$, let $\mathbf{d}$ be the vector of $\mathbb{R}^2$ given by

$$\mathbf{d} = \frac{1}{4}\left(\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3 + \mathbf{a}_4\right). \tag{1}$$

We can interpret $2\mathbf{d}$ as the vector whose extremities are the mid-points of the diagonals of the quadrilateral K. This means that the quadrilateral K is a parallelogram if and only if $\mathbf{d} = 0$. It is easy to see that the vertices of $K^\sharp$ (the parallelogram associated to the quadrilateral K), are given by

$$\mathbf{a}_i^\sharp = \mathbf{a}_i - s_i\ \mathbf{d}, \quad 1 \le i \le 4$$

where

$$s_1 = s_4 = +1, \qquad s_2 = s_3 = -1. \tag{2}$$

When $N = 3$, let $\mathbf{d}_0,\ \mathbf{d}_1,\ \mathbf{d}_2$ and $\mathbf{d}_3$ be the vectors of $\mathbb{R}^3$ given by

$$\left\{\begin{aligned}
\mathbf{d}_0 &= \frac{1}{8}\left(\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3 - \mathbf{a}_4 + \mathbf{a}_5 + \mathbf{a}_6 + \mathbf{a}_7 - \mathbf{a}_8\right),\\
\mathbf{d}_1 &= \frac{1}{8}\left(\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3 - \mathbf{a}_4 - \mathbf{a}_5 - \mathbf{a}_6 + \mathbf{a}_7 + \mathbf{a}_8\right),\\
\mathbf{d}_2 &= \frac{1}{8}\left(\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 - \mathbf{a}_4 - \mathbf{a}_5 + \mathbf{a}_6 - \mathbf{a}_7 + \mathbf{a}_8\right),\\
\mathbf{d}_3 &= \frac{1}{8}\left(\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3 + \mathbf{a}_4 + \mathbf{a}_5 - \mathbf{a}_6 - \mathbf{a}_7 + \mathbf{a}_8\right).
\end{aligned}\right. \tag{3}$$

These four vectors $\mathbf{d}_m$ are chosen for the hexahedron K to be a parallelepiped if and only if $\mathbf{d}_0 = \mathbf{d}_1 = \mathbf{d}_2 = \mathbf{d}_3 = 0$. The vertices of $K^\sharp$ (the parallelepiped associated to the hexahedron K) are

$$\mathbf{a}_i^\sharp = \mathbf{a}_i - \sum_{0\le m\le 3} s_{i,m}\ \mathbf{d}_m, \quad \text{for } 1 \le i \le 8$$

where

$$(s_{i,m}) = \begin{pmatrix}
+1 & +1 & +1 & +1\\
-1 & +1 & -1 & -1\\
-1 & -1 & +1 & -1\\
-1 & -1 & -1 & +1\\
+1 & -1 & -1 & +1\\
+1 & -1 & +1 & -1\\
+1 & +1 & -1 & -1\\
-1 & +1 & +1 & +1
\end{pmatrix} \tag{4}$$

From the equality

$$\sum_{1\le i\le 8} \Big|\sum_{0\le m\le 3} s_{i,m}\mathbf{d}_m\Big|^2 = 8 \sum_{0\le m\le 3} |\mathbf{d}_m|^2\ ,$$

we deduce that:

$$\sum_{0\le m\le 3} s_{i,m}\mathbf{d}_m = 0 \text{ for } 1 \le i \le 8 \Longrightarrow \mathbf{d}_m = 0 \text{ for } 0 \le m \le 3.$$

Thus as announced, the hexadron K is a parallelepiped if and only if $\mathbf{d}_m = 0$ for $0 \le m \le 3$. We resume the results for 2D and 3D geometries in the following proposition:

Proposition 1. *Let $N^* = \left(2^N - N - 1\right) N$; there exist a vector $\mathbf{d} \in \mathbb{R}^{N^*}$ and matrices S_i, $1 \le i \le 2^N$, with N rows and N^* columns and which entries are ± 1, such that the vertex $\mathbf{a}_i$ of K and the vertex $\mathbf{a}_i^\sharp$ of $K^\sharp$ are linked by the relation*

$$\mathbf{a}_i = \mathbf{a}_i^\sharp + S_i\ \mathbf{d}, \quad 1 \le i \le 2^N. \tag{5}$$

More precisely, for $N = 2$, $\mathbf{d} \in \mathbb{R}^2$ is given by (1) and the matrices S_i are square matrices of order 2 satisfying $S_i = s_i I$, where s_i are scalars given by (2). For $N = 3$, $\mathbf{d} = \left(\mathbf{d}_0^T,\ \mathbf{d}_1^T,\ \mathbf{d}_2^T,\ \mathbf{d}_3^T\right)^T$ is identified to a vector of $\mathbb{R}^{12}$, its coordinates $\mathbf{d}_m$ are given by (3) and the matrices $S_i := (s_{0,i}I,\ s_{1,i}I,\ s_{2,i}I,\ s_{3,i}I)$ are 3 rows and 12 columns matrices. The scalars $s_{i,m}$ are given by (4).

From (5) we deduce

$$\mathbf{d} = \frac{1}{4^{N-2}}\ {S_i}^T(\mathbf{a}_i - \mathbf{a}_i^\sharp), \quad 1 \le i \le 2^N.$$

We have $K = K^\sharp$ if and only if $\mathbf{d} = 0$ in $\mathbb{R}^{N^*}$.

Definition 1. *The vector $\mathbf{d}$ is named the distortion vector of K.*

Let us now introduce the N^* numbers δ_m defined by:

$$\begin{array}{lll} \mathbf{d} = \sum\limits_{1\le m\le 2} \delta_m\ \mathbf{e}_m & & \text{when } N = 2, \\ \mathbf{d}_l = \sum\limits_{1\le m\le 3} \delta_{m+3l}\ \mathbf{e}_m, & 0 \le l \le 3 & \text{when } N = 3. \end{array} \tag{6}$$

where we recall that $\mathbf{e}_m$ is given by $\mathbf{e}_m = \mathbf{a}_0 - \mathbf{b}_m$. These parameters are invariant by affine mapping; in particular for the distortion vector $\mathbf{d}^\vee$ of $K^\vee$ we have if $N = 2$

$$\mathbf{d}^\vee = \sum_{1\le m\le 2} \delta_m\ \widehat{\mathbf{e}}_m$$

and if $N = 3$

$$\mathbf{d}_l^\vee = \sum_{1\le m\le 3} \delta_{m+3l}\ \widehat{\mathbf{e}}_m, \quad 0 \le l \le 3$$

Definition 2. *The numbers $(\delta_m)_{1\le m\le N^*}$ are said the distortion parameters of K.*

Since the mapping $F_K^\sharp$ is invertible affine, K is a convex polyhedron if and only if $K^\vee$ is a convex polyhedron. So we see that the convexity of K and the face planarity when $N = 3$ can be expressed by a set of constraints on the distortion parameters only. For $N = 2$, it is easy to show that K is a convex quadrilateral if and only if we have

$$|\delta_1| + |\delta_2| < 1.$$

For $N = 3$, we can write a set of 6 equations and 18 inequations on the 12 distortion parameters which means that K is a convex hexahedron; but we cannot use this set of nonlinear constraints.

From now on, we shall assume for $N = 3$ as for $N = 2$ that

$$\sum_{1\le m\le N^*} |\delta_m| < 1 \tag{7}$$

holds . Then $K^\vee$ contains $B(\mathbf{0}, 1/\sqrt{N})$ the ball centered at the origin and of radius $1/\sqrt{N}$ and $K^\vee$ is contained in the cube $[-2, +2]^N$. The polyhedron K is contained in the parallelotope

$$K^{2\sharp} = F_K^\sharp\left([-2, +2]^N\right)$$

This element $K^{2\sharp}$ is homothetic to $K^\sharp$ with a ratio equal to 2. Then, we have the inequality

$$h_K \le 2h_{K^\sharp}\ .$$

Last, we note that the Euclidean norm of the distortion vector of K satisfies

$$\frac{1}{2N}\left(\sum_{1\le m\le N^\sharp}|\delta_m|\right)\rho_{K^\sharp}\le|\mathbf{d}|\le\frac{1}{2}\left(\sum_{1\le m\le N^\sharp}|\delta_m|\right)h_{K^\sharp}.$$

3. **Local error estimates.**

3.1. **Lagrange interpolation error estimates.** Let P_K be a finite dimensional vectorial space of polynomial functions defined over the quadrilateral or hexahedron K. We assume that the set $\left\{\mathbf{a}_i\ ,\ 1\le i\le 2^N\right\}$ is P_K-unisolvent. Then necessarily, $\dim\left(P_K\right)=2^N$. The basis functions of the Lagrange finite element $\left(K,\ P_K,\ \left\{\mathbf{a}_i, 1\le i\le 2^N\right\}\right)$ are noted $p_{i,K}$ and the P_K-Lagrange interpolation operator is noted Π_K: for every function u defined on the vertices of K,

$$\Pi_K u=\sum_{1\le i\le 2^N}u\left(\mathbf{a}_i\right)\ p_{i,K}.$$

The basis functions $p_{i,K}$ are functions defined by definition on K; in fact, since they are polynomial, we consider them as functions defined on $K^{2\sharp}$.

Proposition 2. *Let us assume that the distortion parameters of K satisfy (7), that the set $\left\{\mathbf{a}_i\ ,\ 1\le i\le 2^N\right\}$ is P_K-unisolvent and that the inclusion $P_1(K)\subset P_K$ holds. Then, for every $u\in H^2\left(K\right)$,*

$$\|u-\Pi_K u\|_{0,K}\le 4\ h_{K^\sharp}^2\left(\sum_{1\le i\le 2^N}\|p_{i,K}\|_{0,\infty,K^{2\sharp}}\right)\ |u|_{2,K}\,. \tag{8}$$

Moreover, let r be an integer sufficiently large for the the inclusion $P_K\subseteq P_r(K)$ to hold. Then there exists a constant c_r, which depends only on r, such that for every $u\in H^2\left(K\right)$,

$$\|u-\Pi_K u\|_{1,K}\le c_r\ \frac{h_{K^\sharp}^2}{\rho_{K^\sharp}}\ \left(\sum_{1\le i\le 2^N}\|p_{i,K}\|_{0,\infty,K^{2\sharp}}\right)\ |u|_{2,K}\,. \tag{9}$$

Proof. By the Taylor's formula with integral remainder, we have for all $\mathbf{x}\in K$,

$$\begin{aligned}u(\mathbf{a}_i)=\ & u(\mathbf{x})+\left(\mathbf{a}_i-\mathbf{x}\right)^T\ \mathbf{grad}\ u(\mathbf{x})\\ &+\int_0^1(1-\theta)\ \left(\mathbf{a}_i-\mathbf{x}\right)^T\ D^2u\left(\mathbf{x}+\theta\left(\mathbf{a}_i-\mathbf{x}\right)\right)\ \left(\mathbf{a}_i-\mathbf{x}\right)\ d\theta\end{aligned}$$

where D^2u denotes the hessian matrix of u. Reporting that in the expression of $\Pi_K u$, and using the assumption that the inclusion $P_1(K)\subset P\left(K\right)$ holds, we obtain

$$\begin{aligned}&\left(u-\Pi_K u\right)(\mathbf{x})=\\ &\quad-\sum_{1\le i\le 2^N}\left\{\int_0^1(1-\theta)\ \left(\mathbf{a}_i-\mathbf{x}\right)^T\ D^2u\left(\mathbf{x}+\theta\left(\mathbf{a}_i-\mathbf{x}\right)\right)\ \left(\mathbf{a}_i-\mathbf{x}\right)\ d\theta\right\}p_{i,K}\left(\mathbf{x}\right)\end{aligned}$$

and then

$$\begin{aligned}&\left|\left(u-\Pi_K u\right)(\mathbf{x})\right|\le\\ &\quad h_K^2\sum_{1\le i\le 2^N}\left\{\int_0^1(1-\theta)\ \left\|D^2u\left(\mathbf{x}+\theta\left(\mathbf{a}_i-\mathbf{x}\right)\right)\right\|\ d\theta\right\}\left|p_{i,K}\left(\mathbf{x}\right)\right|\end{aligned}$$

where $\left\|D^2u\right\|$ is the spectral norm of the matrix D^2u.

Now, we note that on a domain K which is star-shaped with respect to $\mathbf{a}_i$, if $|g(\mathbf{x})| \leq \int_0^1 (1-\theta)\ G(\mathbf{x}+\theta(\mathbf{a}_i-\mathbf{x}))\ d\theta$ holds $\mathbf{x}$ a.e. in K, then $\|g\|_{0,K} \leq \|G\|_{0,K}$. Therefore, we obtain

$$\|u-\Pi_K u\|_{0,K} \leq h_K^2 \left(\sum_{1\leq i\leq 2^N} \|p_{i,K}\|_{0,\infty,K} \right) |u|_{2,K}\,.$$

and a fortiori (8).

In a same way, we obtain for $\mathbf{x}$ a.e. in K,

$$\mathbf{grad}\ \Pi_K u(\mathbf{x}) = \sum_{1\leq i\leq 2^N} u(\mathbf{a}_i)\,\mathbf{grad}\ p_{i,K}(\mathbf{x})$$

$$\mathbf{grad}\ (u-\Pi_K\ u)(\mathbf{x}) =$$
$$-\sum_{1\leq i\leq 2^N} \left\{ \int_0^1 (1-\theta)\ \ (\mathbf{a}_i-\mathbf{x})^T\ \ D^2 u(\mathbf{x}+\theta(\mathbf{a}_i-\mathbf{x}))\ \ (\mathbf{a}_i-\mathbf{x})\ \ d\theta \right\} \mathbf{grad}\ p_{i,K}(\mathbf{x})$$

and as before, we deduce from that

$$|u-\Pi_K u|_{1,K} \leq h_K^2 \left(\sum_{1\leq i\leq 2^N} |p_{i,K}|_{1,\infty,K} \right) |u|_{2,K}$$

and a fortiori

$$|u-\Pi_K u|_{1,K} \leq 4h_{K^\sharp}^2 \left(\sum_{1\leq i\leq 2^N} |p_{i,K}|_{1,\infty,K^{2\sharp}} \right) |u|_{2,K}\ \ .$$

Now, we can note that there exists for every integer r a constant $\widehat{c}_r$ which depends only on r, such that

$$\forall p \in P_r\left(\widehat{K}\right), \quad |p|_{1,\infty,\widehat{K}} \leq \widehat{c}_r\ \ \|p\|_{0,\infty,\widehat{K}}\ \ ,$$

which is an "inverse inequality" on the biunit cube. Using an affine invertible mapping of $\widehat{K}$ onto $K^{2\sharp}$, we obtain the inequality

$$\forall p \in P_r\left(K^{2\sharp}\right), \quad |p|_{1,\infty,K^{2\sharp}} \leq \widehat{c}_r\ \frac{2\sqrt{N}}{\rho_{K^\sharp}}\ \ \|p\|_{0,\infty,K^{2\sharp}}\ \ .$$

Since by assumption the basis functions $p_{i,K}$ belong to $P_r\left(K^{2\sharp}\right)$, that leads to (9) with a constant $c_r = 8\sqrt{N}\ \widehat{c}_r$. □

3.2. **Face error estimates.** Let $\gamma_m^\vee = (F_K^\sharp)^{-1}(\gamma_m)$ be a face of $K^\vee$. We begin by bounding, independently of the distortion parameters, the continuous linear trace operator which sends $H^1(K^\vee)$ into $L^2\left(\gamma_m^\vee\right)$.

Lemma 1. *Assume (7); then there exists a constant C, independent of the distortion parameters, such that for every $u \in H^1(K^\vee))$ and every m with $1 \leq m \leq 2N$*

$$\|u\|_{0,\gamma_m^\vee} \leq C\ \ \|u\|_{1,K^\vee}\,.$$

Proof. Consider first the case $N=2$: let $T_m^\vee$ be the triangle which vertices are the two extremities of $\gamma^\vee$ and $(0,0)^T$, the center of $K^\vee$. Let now $F_m^\vee$ be an invertible linear mapping such that the reciprocal image of the triangle $T_m^\vee$ by $F_m^\vee$ is the unit triangle $\widehat{T}$ which vertices are $(0,1)^T, (1,0)^T$ and $(0,0)^T$. As we have already noticed,

(7) implies that the quadrilateral $K^\vee$ contains a ball of radius greater than $1/\sqrt{2}$, from where we deduce

$$|T_m^\vee| \geq \frac{1}{2\sqrt{2}} \, |\gamma_m^\vee| \, .$$

Moreover, the diameter of $T_m^\vee$ is bounded by 4. $\widehat{\gamma}$ denoting the side of $\widehat{T}$ which extremities are $(0,1)^T$ and $(1,0)^T$, we obtain for some constants c_1, c_2 and c_3 which are independent of the geometry

$$\begin{aligned} \|u\|_{0,\gamma_m^\vee} &\leq c_1 \, |\gamma_m^\vee|^{1/2} \, \|u \circ F_m^\vee\|_{0,\widehat{\gamma}} \\ &\leq c_1 c_2 \, |\gamma_m^\vee|^{1/2} \, \|u \circ F_m^\vee\|_{1,\widehat{T}} \\ &\leq c_1 c_2 c_3 \left(\frac{|\gamma_m^\vee|}{|T_m^\vee|} \right)^{1/2} \|u \circ F_m^\vee\|_{1,T_m^\vee} \end{aligned}$$

which proves the Lemma with $C = \left(2\sqrt{2}\right)^{1/2} c_1 c_2 c_3$.

For the case $N = 3$, we begin by decomposing a quadrilateral face $\gamma_m^\vee$ in two triangles $\gamma_{1,m}^\vee$ and $\gamma_{2,m}^\vee$; in the same way as in the case $N = 2$, we obtain for $i = 1$ and $i = 2$ the inequality

$$\|u\|_{0,\gamma_{i,m}^\vee} \leq C \;\; \|u\|_{1,T_{i,m}^\vee}$$

and we conclude easily. □

Proposition 3. *Assume that the distortion parameters of K satisfy (7). Then there exists a constant C, independent of the geometry of K, such that: $\forall u \in H^1(K)$ and $\forall m$ with $1 \leq m \leq 2N$, we have*

$$\|u - \overline{u}^{\gamma_m}\|_{0,\gamma_m} \leq C \; h_{K^\sharp}^{1/2} \left(\frac{h_{K^\sharp}}{\rho_{K^\sharp}} \right)^{(N-1)/2} |u|_{1,K} \tag{10}$$

where $\overline{u}^{\gamma_m}$ denotes the mean value of u on the face γ_m of K.

Proof. Let u be in $H^1(K)$ and γ_m be a face of K. The best approximation of the trace on γ_m of u in $L^2(\gamma_m)$ is the mean value of u on γ_m:

$$\|u - \overline{u}^{\gamma_m}\|_{0,\gamma_m} = \inf_{\chi \in \mathbb{R}} \|u - \chi\|_{0,\gamma_m} \, .$$

We note $\gamma_m^\vee$ the reciprocal image of γ_m by the application $F_K^\sharp$. Since for each constant χ we have $(u - \chi) \circ F_K^\sharp = u \circ F_K^\sharp - \chi$, we deduce that

$$\|u - \chi\|_{0,\gamma_m} \leq \left\|B_{K^\sharp}^{-1}\right\|^{(N-1)/2} \left\|u \circ F_K^\sharp - \chi\right\|_{0,\gamma_m^\vee} .$$

Using Lemma 1, there exists a constant C such that

$$\left\|u \circ F_K^\sharp - \chi\right\|_{0,\gamma_m^\vee} \leq C \left\|u \circ F_K^\sharp - \chi\right\|_{1,K^\vee} .$$

Therefore, we have

$$\inf_{\chi \in \mathbb{R}} \left\|u \circ F_K^\sharp - \chi\right\|_{0,\gamma_m^\vee} \leq C \inf_{\chi \in \mathbb{R}} \left\|u \circ F_K^\sharp - \chi\right\|_{1,K^\vee}$$

and

$$\inf_{\chi \in \mathbb{R}} \left\|u \circ F_K^\sharp - \chi\right\|_{0,\gamma_m^\vee} \leq C \left\|u \circ F_K^\sharp - \overline{u}^K\right\|_{1,K^\vee}$$

where $\overline{u}^K$ is the mean value of u on K and also the mean value of $u \circ F_K^\sharp$ on $K^\vee$. Moreover, we have for almost every $\mathbf{x}$ in $K^\vee$

$$\left(u \circ F_K^\sharp - \overline{u}^K\right)(\mathbf{x}) = \frac{1}{|K^\vee|} \int_{K^\vee} \left\{ \int_0^1 (\mathbf{x}-\mathbf{y})^T \mathbf{grad}(u(\mathbf{x}+\theta(\mathbf{y}-\mathbf{x}))d\theta \right\} d\mathbf{y}.$$

Consequently

$$\left\| u \circ F_K^\sharp - \overline{u}^K \right\|_{0,K^\vee} \leq 4\sqrt{2} |u \circ F_K^\sharp|_{1,K^\vee}$$

$$\left\| u \circ F_K^\sharp - \overline{u}^K \right\|_{1,K^\vee} \leq \sqrt{33} |u \circ F_K^\sharp|_{1,K^\vee}$$

and

$$|u \circ F_K^\sharp|_{1,K^\vee} \leq \|B_K\|^{N/2} |u|_{1,K}.$$

From all the previous inequalities, we have

$$\|u - \overline{u}^{\gamma_m}\|_{0,\gamma_m} \leq C\sqrt{33} \left\|B_K^{-1}\right\|^{(N-1)/2} \|B_K\|^{N/2} |u|_{1,K^\vee}.$$

Using (7), it remains to notice that

$$\|B_K\| \leq \frac{1}{2} h_{K^\sharp} \qquad \text{and} \qquad \left\|B_K^{-1}\right\| \leq \frac{2\sqrt{N}}{\rho_{K^\sharp}}$$

and the proof of the proposition is achieved. □

4. **The model problem.** We consider the second order elliptic model problem:

$$\begin{aligned} -div(A\nabla u) &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \Gamma \end{aligned} \tag{11}$$

where $A = (a_{i,j})$ a symmetric matrix satisfying

$$\forall \mathbf{x} \in \bar{\Omega}, \forall \xi \in \mathbb{R}^N, \quad c\sum_{i=1}^N \xi_i^2 \leq \sum_{i,j=1}^N a_{i,j}(\mathbf{x})\xi_i\xi_j \leq c^{-1}\sum_{i=1}^N \xi_i^2$$

and $\Gamma := \partial\Omega$ is the boundary of a polyhedral domain $\Omega \subset \mathbb{R}^N$ $(N = 2, 3)$.

The variational problem associated to (11) is: find $u \in H_0^1(\Omega)$ such that

$$\forall v \in H_0^1(\Omega) \qquad \int_\Omega A\nabla u \nabla v dx = \int_\Omega f v dx \tag{12}$$

Let $\mathcal{T}_h$ be a triangulation of Ω into quadrilaterals ($N = 2$) or hexahedra ($N = 3$). Let $\partial\mathcal{T}_h$ denotes the set of the faces of the elements of $\mathcal{T}_h$ and $\partial\mathcal{T}_h \backslash \partial\Omega$ denotes the set of interior faces. For each element γ of $\partial\mathcal{T}_h \backslash \partial\Omega$, there exist K^+ and K^- in $\mathcal{T}_h$ such that $\bar{K}^+ \cap \bar{K}^- = \gamma$. The unitary outward normal of K^+ is noted $\mathbf{n}^+$ and the normal of a face is defined by $\mathbf{n} = \mathbf{n}^+$. For each subset γ of $\partial\Omega$, $\mathbf{n}$ denotes the unitary outward normal of Ω.

We consider the spaces

$$V_{\mathcal{T}_h} = \left\{ v \in L^2(\Omega);\, v|_K \in H^1(K) \text{ for each } K \in \mathcal{T}_h \right\}$$

and

$$V_h = \left\{ v_h \in L^2(\Omega);\, v_h|_K \in P_K \text{ for each } K \in \mathcal{T}_h \right\}$$

where P_K is a polynomial space. A non conforming finite element method for problem (12) is: find $u_h \in V_h$ such that

$$\forall v_h \in V_h, \qquad \sum_{K \in \mathcal{T}_h} \int_K A\nabla u_h \nabla v_h dx = \int_\Omega f v_h dx. \tag{13}$$

For $v, w \in H_0^1(\Omega) + V_{\mathcal{T}_h}$ we define

$$a_h(v,w) = \sum_{K \in \mathcal{T}_h} \int_K A \nabla v \nabla w dx$$

and

$$\|v\|_{1,h} = \left(\sum_{K \in \mathcal{T}_h} \|v\|_{1,K}^2 \right)^{1/2}, \quad |v|_{1,h} = \left(\sum_{K \in \mathcal{T}_h} |v|_{1,K}^2 \right)^{1/2}.$$

Let us define the jump of $w_h \in V_h$ on γ. If γ is an interior face, $[w_h] = w_h^+ - w_h^-$ where $w_h^\pm$ is the trace on γ of $w_h^\pm \in H^1(K^\pm)$; otherwise $[w_h]$ represents the trace on γ of $w_h \in H^1(K)$.

Since $a_h(.,.)$ is uniformly $V_{\mathcal{T}_h}$ elliptic, the following basic error estimate (approximation and consistency error) hold (see [5], [13], [4]).

$$\|u - u_h\|_{1,h} \leq c \left(\inf_{w_h \in V_h} \|u - w_h\|_{1,h} + \sup_{0 \neq w_h \in V_h} \frac{|a_h(u,w_h) - (f,w_h)|}{\|w_h\|_{1,h}} \right) \tag{14}$$

with

$$a_h(u,w_h) - (f,w_h) = \sum_{\gamma \in \mathcal{T}_h} \int_\gamma \frac{\partial u}{\partial n} [w_h] ds.$$

Now, we give the definition of a family of regular meshes.

Definition 3. *A family of quadrangular or hexahedral meshes is said regular if and only if for each K of $\mathcal{T}_h$*

- *the distorsion parameters satisfy (7)*

and

- $\exists \sigma > 0 \quad \frac{h_{K\#}}{\rho_{K\#}} \leq \sigma$.

We can notice that this definition corresponds to the classical definition given by P.G. Ciarlet ([5]) when the K's are parallelotopes.

Proposition 4 (Convergence of the method)**.** *We suppose that the solution u of (11) is in $H^2(\Omega)$ and the family of meshes is regular. We assume moreover that the following properties hold:*

i) Approximation properties:

- $\forall K,$ *the set $\{\mathbf{a}_i, 1 \leq i \leq 4\}$ is P_K-unisolvent.*
- $\exists r > 0$, $\forall K$, $P_1(K) \subset P_K \subset P_r(K)$.
- $\exists C > 0$ *such that the basis functions $p_{i,K}$ of P_K satisfy* $\|p_{i,K}\|_{0,\infty,K^{2\#}} < C$.

ii) Patch test

- $\forall w_h \in V_h, \forall \gamma \in \partial \mathcal{T}_h, \quad \int_\gamma [w_h] ds = 0$.

Under all these assumptions we have

$$\|u - u_h\|_{1,h} \leq Ch|u|_{2,\Omega}$$

with a constant C independent of the mesh.

Proof. Using the assumed approximation properties and Proposition 2 , we have

$$\left(\sum_{K \in \mathcal{T}_h} \|u - \Pi_K u\|_{1,h} \right)^{1/2} \leq Ch|u|_{2,\Omega}$$

where C is a constant independent of the mesh.

So, it remains to prove that

$$\forall w_h \in V_h, \quad |a_h(u,w_h) - (f,w_h)| \leq Ch|u|_{2,\Omega}\|w_h\|_{1,h}.$$

Let γ be in $\partial\mathcal{T}_h\backslash\partial\Omega$ and w_h in V_h. The assumed patch test means that $\int_\gamma w_h^+ d\sigma = \int_\gamma w_h^- d\sigma$. Therefore, dividing by $|\gamma|$, the mean values over the face γ of w_h satisfy $\overline{w_h^+}^\gamma = \overline{w_h^-}^\gamma$. Thus, we obtain

$$\int_\gamma \frac{\partial u}{\partial \mathbf{n}}\left[w_h\right]\ d\sigma = \int_\gamma \frac{\partial u}{\partial \mathbf{n}^+}(w_h^+ - \overline{w_h^+}^\gamma)d\sigma - \int_\gamma \frac{\partial u}{\partial \mathbf{n}^-}(w_h^- - \overline{w_h^-}^\gamma)d\sigma.$$

Obviously, for each constant c we have

$$\int_\gamma \frac{\partial u}{\partial \mathbf{n}^+}(w_h^+ - \overline{w_h^+}^\gamma)d\sigma = \int_\gamma (\mathbf{grad}\ u\cdot\mathbf{n}^+ - c)\ (w_h^+ - \overline{w_h^+}^\gamma)d\sigma.$$

As γ is flat face, $\mathbf{n}^+$ is a constant vector on γ and it can be extented to K^+. Since $u \in H^2(\Omega)$, $\mathbf{grad}\ u\cdot\mathbf{n}^+ \in H^1(K^+)$. From (10) and with $c = \overline{\mathbf{grad}\ u\cdot\mathbf{n}^+}^\gamma$ we have

$$\left|\int_\gamma (\mathbf{grad}\ u\cdot\mathbf{n}^+ - c)\ (w_h^+ - \overline{w_h^+}^\gamma)d\sigma\right|$$
$$\leq C\ h\ \left|\mathbf{grad}\ u\cdot\mathbf{n}^+\right|_{1,K^+}\ |w_h|_{1,K^+} \leq C\ h\ |u|_{2,K^+}\ |w_h|_{1,K^+}$$

and finally

$$\left|\int_\gamma \frac{\partial u}{\partial \mathbf{n}}\left[w_h\right]\ d\sigma\right| \leq C\ h\ \left(|u|_{2,K^+}\ |w_h|_{1,K^+} + |u|_{2,K^-}\ |w_h|_{1,K^-}\right).$$

The result, for each $\gamma \subset \partial\Omega$, is similar

$$\left|\int_\gamma \frac{\partial u}{\partial \mathbf{n}}\left[w_h\right]\ d\sigma\right| \leq C\ h\ |u|_{2,K}\,|w_h|_{1,K}\,.$$

We sum on all the faces γ_m. In the right hand side of the inequality an element K appears at most 4 times when $N=2$ and 6 times when $N=3$. So the expected result holds. $\square$

5. **Lagrange polynomial finite element.** This section is devoted to building of finite elements (K, P_K, S_K) where the set of degrees of freedom S_K is the vertices set of K, and P_K is a polynomial space.

Note that $P_k = \left\{q = q^\vee \circ \left(F_K^\sharp\right)^{-1}; q^\vee \in P_k\right\}$. The same property is not true for the space Q_k. Therefore we introduce the space:

$$Q_k^K = \left\{q^\vee \circ \left(F_K^\sharp\right)^{-1}; q^\vee \in Q_k\right\}$$

which is a subspace of P_{Nk}.

If we choose $P_K = Q_1^K$ then (K, P_K, S_K) is a finite element, but the approximation u_h of the solution of (11) obtained with this element does not converge without additional assumptions (see the numerical results in the next section). Indeed the basic functions of the space $V_{\mathcal{T}_h}$ are discontinuous on the faces of the elements, and we loose the order of convergence on the consistancy error term.

Therefore our goal is to build polynomial finite elements on quadrilaterals and hexahedra satisfying the assumptions of Proposition 4.

5.1. **The quadrilateral case.** The set of degrees of freedom is $S_K = \{\mathbf{a}_i; 1 \le i \le 4\}$

Since the trapezoidal formula is exact for each polynomial of order 1, we have

$$\forall q \in P_1, \int_{\gamma_m} q\, d\sigma = \frac{1}{2}|\gamma_m| \sum_{\mathbf{a}_i \in \gamma_m} q(\mathbf{a}_i), \text{ for all } m = 1, ..., 4 \tag{15}$$

where $|\gamma_m|$ is the length of the edge γ_m.

Let P_K be the following polynomial space:

$$P_K = \left\{ q \in Q_2^K \cap P_3;\ \int_{\gamma_m} q\, d\sigma = \frac{1}{2}|\gamma_m| \sum_{\mathbf{a}_i \in \gamma_m} q(\mathbf{a}_i), \text{ for all } m = 1, ..., 4 \right\} \tag{16}$$

The Simpson formula integrates exactly the cubic functions on each edge and consequently the space P_K can be defined as well as

$$P_K = \left\{ q \in Q_2^K \cap P_3;\ q(\mathbf{b}_m) = \frac{1}{2} \sum_{\mathbf{a}_i \in \gamma_m} q(\mathbf{a}_i), \text{ for all } m = 1, ..., 4 \right\}$$

Proposition 5. *For any convex quadrilateral K, the triad (K, P_K, S_K) is a Lagrange finite element.*

Proof. Let us introduce $\mathbf{a}_i^{\vee} = \mathbf{b}_{i-4}^{\vee}$ and $\widehat{\mathbf{a}}_i = \widehat{\mathbf{b}}_{i-4}$ for $i = 5, ..., 8$. Using the invertible affine mapping $F_K^{\sharp}$, we only need to prove that for each distortion parameters $\delta = (\delta_1, \delta_2)$ such that $|\delta_1| + |\delta_2| < 1$, the unique function $q \in Q_2^K \cap P_3$ satisfying $(q^{\vee}(\mathbf{a}_i^{\vee}) = 0;\ 1 \le i \le 8)$ is $q \equiv 0$.

Let us introduce the polynomials $r_j \in Q_2 \cap P_3$ satisfying

$$r_j(\widehat{\mathbf{a}}_i) = \delta_{i,j},$$

and the square matrix R of order 8 defined by $R_{i,j} = r_j(\mathbf{a} =_i^{\vee})$. By using symbolic calculus, we obtain

$$detR = \left(1 - {\delta_1}^2\right)\left(1 - {\delta_2}^2\right)\left(1 - (\delta_1 + \delta_2)^2\right)\left(1 - (\delta_1 - \delta_2)^2\right).$$

Since $|\delta_1| + |\delta_2| < 1$ then $detR > 0$. Therefore R is invertible and $q \equiv 0$.

Note that if $\delta = 0$, the r_j's correspond to the basis of the serendipity finite element and $P_K \equiv Q_1$. □

Remarks.

- The finite element basis depends on δ.
- if $\delta = 0$ (i.e. K is a parallelogram), then (K, P_K, S_K) coincides with the classical bilinear finite element.
- $P_K = span\left(1, x_1, x_2, F_K^{\sharp}(\omega^{K^{\vee}})\right)$, where $\omega^{K^{\vee}}$ can be expressed explicitly.
- Numericaly, it is more efficient to calculate the finite element basis by solving a linear system of order 8 than to the explicit basis of P_K.

Proposition 6. *We assume that there exists a positive number α such that for each $K \in \mathcal{T}_h$, $|\delta_1| + |\delta_2| \le 1 - \alpha$. Then the assumptions of Proposition 4 are satisfied.*

Proof. By Proposition 5, (K, P_K, S_K) is a finite element. The inclusions $P_1 \subseteq P_K \subseteq P_3$ are obvious. On the compact $|\delta_1| + |\delta_2| \le 1 - \alpha$, the function $\frac{1}{|\det R|}$ is bounded, and consequently the $P_{i,K}$'s are bounded on $K^{2\#}$. Finally, by construction, the patch test is satisfied. □

5.2. **The hexahedron case.** In this subsection P_K has to be a polynomial space of dimension 6. In order to control the mean value of the functions on each face of the hexahedron, P_K must be built from a polynomial space of dimension 14 (8 vertices +6 faces). Noting that the dimension of $Q_2 \cap P_3$ is 17, we propose to consider the subspace Z of $Q_2 \cap P_3$ defined by

$$Z = span\Big\{1, x_1^\vee, x_2^\vee, x_3^\vee, x_1^\vee x_2^\vee, x_1^\vee x_3^\vee, x_2^\vee x_3^\vee, x_1^\vee x_2^\vee x_3^\vee, {x_1^\vee}^2, {x_2^\vee}^2, {x_3^\vee}^2,$$
$$x_1^\vee({x_2^\vee}^2 + {x_3^\vee}^2), x_2^\vee({x_1^\vee}^2 + {x_3^\vee}^2), x_3^\vee({x_1^\vee}^2 + {x_2^\vee}^2)\Big\}$$

which has the advantage of being invariant under any permutation of $x_1^\vee, x_2^\vee, x_3^\vee$.

Therefore we introduce:

$$Z^K = \Big\{q^\vee \circ (F_K^\sharp)^{-1}; q^\vee \in Z\Big\}$$

Z^K is a subspace of $Q_2^K \cap P_3$. The generalization to the 3-D case of formula (15) is to find $\omega_{m,i}$ such that for all $q \in P_1$ we have

$$\int_{\gamma_m} q\, d\sigma = \frac{1}{|\gamma_m|}\sum_{i=1}^{4}\omega_{m,i}q(\mathbf{a}_{m,i}), \text{ for all } m = 1, ..., 6$$

where $\{\mathbf{a}_{m,i}, i = 1, ..., 4\}$ are the vertices of the face γ_m and of aera $|T_{m,i}| = \theta_{m,i}|\gamma_m|$

It is enough to restrict ourselves to $q \in P_1(\gamma_m)$. However the coefficients $\omega_{m,i}$ are not unique. A solution consists in cutting the face γ_m into triangles, see Figure 3. To obtain a symetric formula, we consider the four triangles $\{T_i, i = 1, ..., 4\}$ constituted with the vertices $\{\mathbf{a}_{m,j}, j = 1, ..., 4, j \neq i\}$.

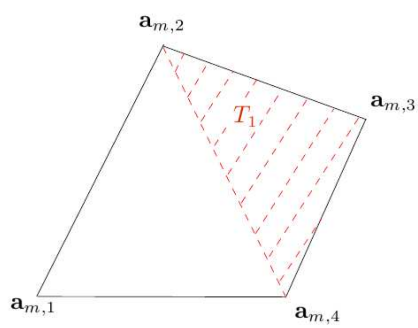


FIGURE 3

$$\int_{\gamma_m} q\, d\sigma = \frac{1}{2}\sum_{i=1}^{4}\int_{T_i} q\, d\sigma = \frac{|\gamma_m|}{6}\sum_{i=1}^{4}\left(\theta_{m,i}\sum_{j=1, j\neq i}^{4} q(\mathbf{a}_{m,j})\right).$$

Therefore

$$\omega_{m,i} = \frac{1}{6}\sum_{j=1,j\neq i}^{4}\theta_{m,j} = \frac{1}{6}(2 - \theta_{m,i}).$$

Finally we let

$$P_K = \left\{q \in Z^K;\ \int_{\gamma_m} q\, d\sigma = \frac{|\gamma_m|}{6}\sum_{i=1}^{4}(2 - \theta_{m,i})q(\mathbf{a}_{m,i}),\ \forall\, m = 1, ..., 6\right\}.$$

It seems difficult to prove the analogous of Proposition 5 in the hexahedron case. However we formulate a sufficient condition to obtain a finite element. For any polynomial p we let :

$$\widehat{I}_i(p) = p(\widehat{\mathbf{a}}_i) \quad 1 \leq i \leq 8, \qquad \widehat{I}_{8+m}(p) = \int_{\widehat{\gamma}_m} p\, d\sigma \quad 1 \leq m \leq 6,$$

and

$$I_i^{\vee}(p) = p(\mathbf{a}_i^{\vee}) \quad 1 \leq i \leq 8, \qquad I_{8+m}^{\vee}(p) = \int_{\gamma_m^{\vee}} p\, d\sigma \quad 1 \leq m \leq 6.$$

In a first step, we assume that $K^{\vee} = \widehat{K}$. And we define

$$\Sigma = \left\{ \widehat{\mathbf{a}}_i; 1 \leq i \leq 8,\ v \rightarrow \int_{\widehat{\gamma}_m} v\, d\sigma; 1 \leq m \leq 6. \right\}.$$

It is easy to show that the triad $(\widehat{K}, Z, \Sigma)$ is a Lagrange finite element. We denote by r_j's the basis functions, they satisfy $\widehat{I}_i(r_j) = \delta_{i,j}, 1 \leq i, j \leq 14$.

We assume that $K^{\vee} \neq \widehat{K}$ and we define the matrix R by $R_{i,j} = I_i^{\vee}(r_j)$. So the matrix R can be written $R = I - B$ where B is a matrix depending continously on the δ_i's parameters and $B = 0$ if the δ_i's vanish.

Proposition 7. *For any hexahedron K such that $\|B\| < 1$, the triad (K, P_K, S_K) is a Lagrange finite element.*

Proof. If $\|B\| < 1$ the matrix R is invertible and thus (K, P_K, S_K) is a Lagrange finite element. □

In the previous Proposition the norm $\|.\|$ can be replace by any subordinate norm.

Remarks. It is possible to choose another example of space Z. For instance, we can use the hiearchical basis proposed by Solin (see [12]). In this case

$$\begin{aligned} Z = Q_1 + span \Big(&(1 - x_1^{\vee})(x_2^{\vee 2} - 1)(x_3^{\vee 2} - 1), (1 + x_1^{\vee})(x_2^{\vee 2} - 1)(x_3^{\vee 2} - 1), \\ &(1 - x_2^{\vee})(x_1^{\vee 2} - 1)(x_3^{\vee 2} - 1), (1 + x_2^{\vee})(x_1^{\vee 2} - 1)(x_3^{\vee 2} - 1), \\ &(1 - x_3^{\vee})(x_2^{\vee 2} - 1)(x_1^{\vee 2} - 1), (1 + x_3^{\vee})(x_2^{\vee 2} - 1)(x_1^{\vee 2} - 1) \Big), \end{aligned}$$

but the degree of the polynomials is higher.

Proposition 8. *We assume that there exists $\alpha > 0$ such that for each $K \in \mathcal{T}_h$, $\|B\| \leq 1 - \alpha$. Then the assumptions of Proposition 4 are satisfied.*

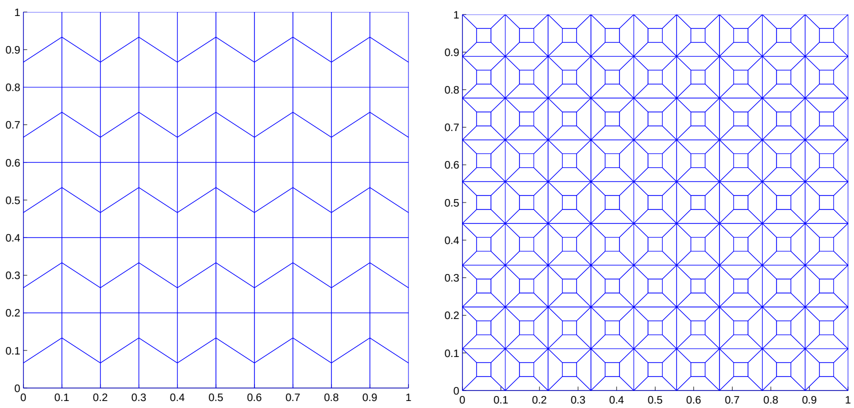
Proof. The inclusions $P_1 \subseteq P_K \subseteq P_3$ are obvious.

Since $\|B\| \leq 1 - \alpha$, $\|R^{-1}\|$ is bounded , and consequently the $P_{i,K}$'s are bounded on $K^{2\#}$. Finally, by construction, the patch test is satisfied. □

6. **Numerical results.**

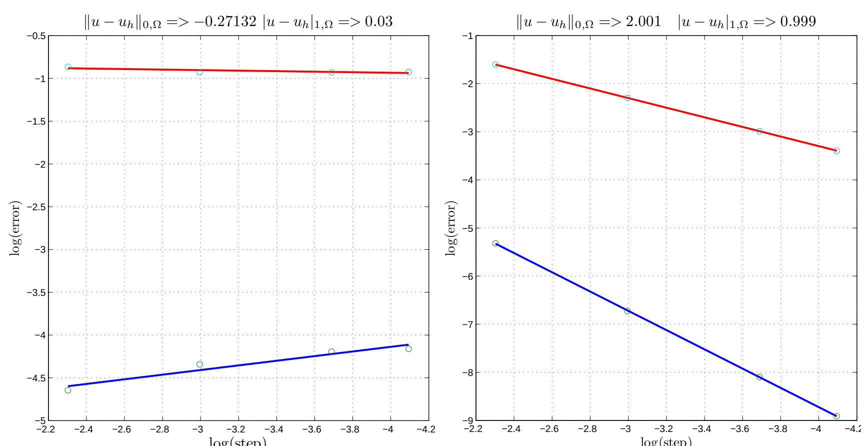
6.1. **Quadrilateral case.** We take $\Omega =]0,1[\times]0,1[$ and the exact solution is $u(x_1, x_2) = sin(\pi x_1) sin(\pi x_2)$.

We considere two types of mesh. They are composed of two patterns and their shapes are the same for each mesh used, see Figure 4. The first mesh is a mesh in chevron given in [1] and the second is a mesh in honeycomb. In the first test, we take $P_K = Q_1^K$ and as expected the method does not converge on (deformed) quadrilateral meshes but converges on meshes based on squares or parallelograms,



Mesh 1 (Chevron mesh) Mesh 2 (Honeycomb mesh)

FIGURE 4. Meshes 1 and 2



Non convergence on Chevron meshes Convergence on square meshes

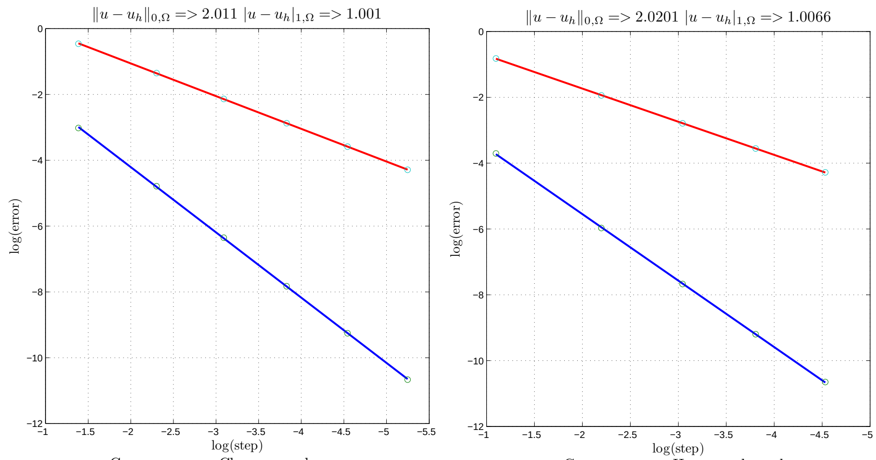
FIGURE 5. Convergence curves when $P_K = Q_1^K$

see Figure 5. In the second test, P_K is given by (16). For the two meshes proposed, we obtain the expected order of convergence, see Figure 6.

6.2. **Hexahedral case.** We take $\Omega =]0,1[\times]0,1[\times]0,1[$ and the exact solution is $u(x_1,x_2,x_3) = sin(\pi x_1)sin(\pi x_2)sin(\pi x_3)$.

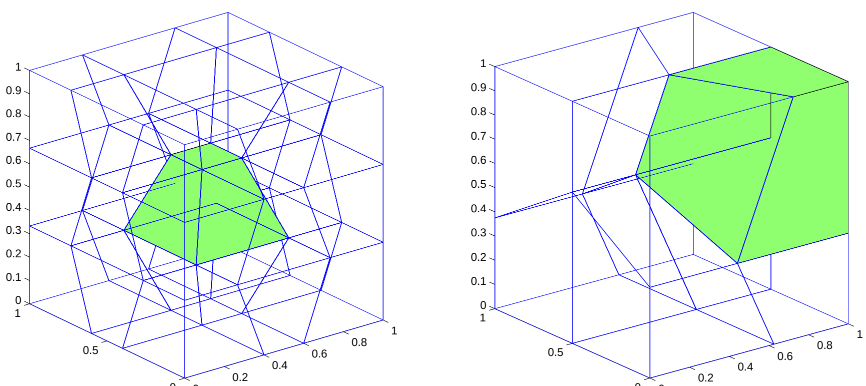
The first mesh used is based on truncated pyramids and the second is a generalization of the 2D Chevron mesh, see Figure 7. Figure 8 confirms the a priori error estimate.

7. **Prospects.** The theoretical part of this paper allows us to build pseudo-conforming finite elements of higher order without any particular difficulty.



Convergence on Chevron meshes　　Convergence on Honeycomb meshes

FIGURE 6. Convergence curves when P_K is given by (16)



Mesh 3 (Truncated pyramids mesh)　　Mesh 4 (3D Chevron mesh)

FIGURE 7. 3D meshes

Moreover we are able to adapt the process in the mixed finite elements context. The results will be presented in a companion paper. The loss of convergence problem when using classical mixed finite elements on quadrilaterals and hexahedra (see for instance [2], [11]) were in fact one of the motivations of the present work.

The extension to generalized hexahedra (i.e. hexahedra with non-plane faces) is a work in progress.

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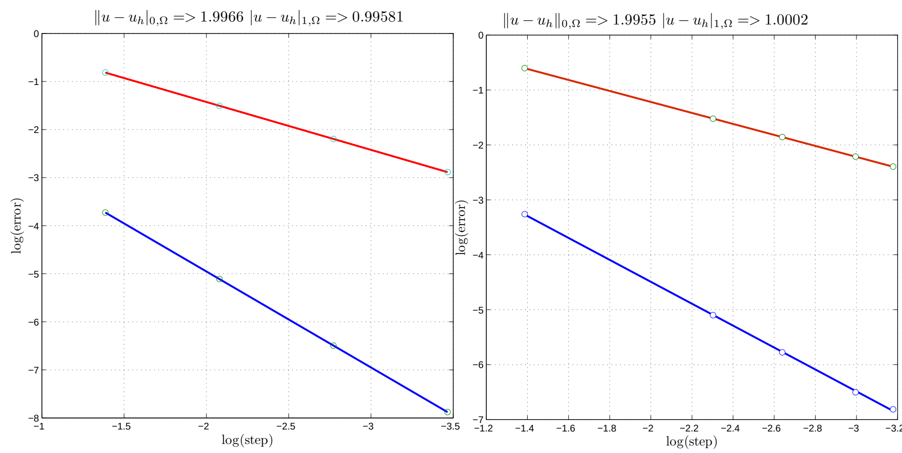

FIGURE 8. Convergence curves on mesh 3 and mesh 4

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