An observability form for linear systems with unknown inputs

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Abstract

In this paper is given a constructive algorithm that transforms a linear system with unknown inputs into a novel observability form. This form is useful to derive finite time observers even if the system does not satisfy some matching conditions usually required for the design of unknown input observers. The practical example of an aircraft subject to actuator faults is provided to show the efficiency of the approach.

Keywords: Linear Systems, Unknown Input Observers, Sliding Mode

1 Introduction

It is of interest in many applications to design observers that provide both estimation of the state and the unknown inputs. Some of these applications are classical ones as fault detection and isolation (Chen *et al.* (1996); Floquet *et al.* (2004)), whereas other ones are more 'exotic' such as for example decoding an encrypted message (Barbot *et al.* (2003)). The application domains have an important consequence on the assumptions claimed on the unknown inputs. For example in fault detection, they are considered as piecewise constants, while in decoding an encrypted message they are assumed to be piecewise constant or continuous functions.

This paper deals with the design of a state observer and input estimator for a linear time-invariant system subject to unknown inputs:

$$\dot{x} = Ax + Bu + Dw \tag{1}$$

$$y = Cx \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^{p_1}$ is the output vector, $u \in \mathbb{R}^q$ represents the known inputs and $w \in \mathbb{R}^m$ stands for the unknown inputs. A, B, C and D are known constant matrices of appropriate dimension. It is assumed that the pair (C, A) is observable, that $m \leq p_1$ and, without loss of generality, that rank $C = p_1$ and rank $D = m$.

Many works in the literature have provided various techniques for designing unknown input observers. Some of them are based on linear methods (namely Luenberger like observers) and a necessary condition for the existence of such observers was given (see e.g. Kudva *et al.* (1980); Hou and Müller (1992); Darouach *et al.* (1994)):

$$\text{rank } CD = \text{rank } D = m. \tag{3}$$

Other ways of investigation for robust state reconstruction are based on the use of sliding mode observers. Actually, condition (3) has also to be satisfied in order to design such observers (see Utkin (1992), Edwards and Spurgeon (1994), Edwards and Spurgeon (1998) and the references therein). In this case, it is called *observer matching condition*, and it is the analogue of the well-known *matching condition* for a sliding mode controller to be insensitive to matched perturbations.

In both approaches, if the unknown input observer exists, it is straightforward under mild assumptions to obtain an estimation of the unknown inputs (see Hou and Müller (1992) for the linear observer case). In particular, the unknown inputs can be explicitly reconstructed using sliding mode observers by considering the so-called *equivalent output injection* (which is the counterpart of the equivalent control in the design of sliding mode control). This method has been used for fault detection and isolation in Edwards *et al.* (2000).

In this paper, it is aimed at deriving a new observability form well suited for the design of unknown input observers (that also gives an estimation of the unknown inputs) even if the condition (3) is not fulfilled. By the means of an algorithm, a structural analysis of the system allows to conclude at the possibility to estimate in finite time both the state and the unknown inputs. This estimation can be achieved using some existing finite time observers such as, for instance, numerical (Diop *et al.* (1999, 2000)) or sliding mode differentiators (Barbot *et al.* (1996), Levant (1998)). Finite time convergence property is often desirable in the framework of observation, fault detection or identification problems, and in general to solve in finite time the problem of left inversion and state observation.

This work is organized as follows. In the next Section is given the constructive algorithm that gives rise to a change of coordinates that put the system into a suitable observable form. In Section 3 is detailed how to recover the state and the unknown inputs by the means of finite time observers. Lastly, the algorithm is applied to the real problem of state estimation for an aircraft subject to actuator faults.

2 Output Information Algorithm

The aim of this algorithm is to find, by introducing suitable auxiliary outputs, a change of coordinates such that the system (1)-(2) is transformed in a set of block observable triangular forms.

Iteration 1: Consider the vector of outputs $y^1 \triangleq Cx$.

a. Without loss of generalities, reorder the components of y^1 as following

$$y^1 = \begin{bmatrix} C_1^T & \cdots & C_{\eta_1}^T & C_{\eta_1+1}^T & \cdots & C_{p_1}^T \end{bmatrix}^T x$$

where C_1,..., C_{η_1} are satisfying for all $j \leq \eta_1$

$$C_j A^k D = 0, \text{ for all } k \in \mathbb{N} \tag{4}$$

and where C_{η_1+1},..., C_{p_1} are such that for $1 \leq j \leq p_1 - \eta_1$, there exists an integer r_j^1 such that:

$$\begin{array}{rcl} C_{\eta_1+j} A^k D & = & 0, \text{ for all } k < r_j^1 - 1 \\ C_{\eta_1+j} A^{r_j^1-1} D & \neq & 0. \end{array} \tag{5}$$

Note that the outputs $y_j^1 = C_j x$, $j \leq \eta_1$, are not affected by the unknown inputs and that the $p_1 - \eta_1$ remaining outputs are affected by the disturbance vector.

b. Compute the set of row vectors

$$\Phi^1 = \text{span}\left\{C_1, ..., C_1 A^{n-1}, C_2, ..., C_2 A^{n-1}, ..., C_{\eta_1}, ..., C_{\eta_1} A^{n-1}\right\}$$

and note $\varphi^1 = \text{rank } \Phi^1$.

Find η_1 integers $\varphi_1^1, \ldots, \varphi_{\eta_1}^1$ such that

$$I_1 = \left\{C_1, ..., C_1 A^{\varphi_1^1-1}, C_2, ..., C_2 A^{\varphi_2^1-1}, ..., C_{\eta_1}, ..., C_{\eta_1} A^{\varphi_{\eta_1}^1-1}\right\}$$

is a basis of Φ^1. One has $\varphi^1 = \varphi_1^1 + \ldots + \varphi_{\eta_1}^1$. If $\varphi^1 = n$, the algorithm is stopped. Actually, this is the case when the state is not affected by any disturbance, i.e. $D = 0$.

c. Compute the set of row vectors

$$\Upsilon^1 = \text{span}\left\{C_{\eta_1+1}, ..., C_{\eta_1+1} A^{r_1^1-1}, ..., C_{p_1}, ..., C_{p_1} A^{r_{p_1-\eta_1}^1-1}\right\}$$

and note ρ^1 the integer such that $\varphi^1 + \rho^1 = \text{rank}\,(\Phi^1 \cup \Upsilon^1)$.

Find $p_1 - \eta_1$ integers $\rho_1^1, ..., \rho_{p_1-\eta_1}^1$ such that the set $I_1 \cup D_1$, where

$$D_1 = \left\{C_{\eta_1+1}, ..., C_{\eta_1+1} A^{\rho_1^1-1}, ..., C_{p_1}, ..., C_{p_1} A^{\rho_{p_1-\eta_1}^1-1}\right\}$$

is a basis of $\Phi^1 \cup \Upsilon^1$. One has $\rho_1^1 + .. + \rho_{p_1-\eta_1}^1 = \rho^1$.

If $\varphi^1 + \rho^1 = n$, quit the algorithm.

d. Define the matrix

$$\Gamma_1 = \begin{bmatrix} C_{\eta_1+1} A^{r_1^1-1} D \\ \vdots \\ C_{p_1} A^{r_{p_1-\eta_1}^1-1} D \end{bmatrix}$$

and note $d_1 = \text{rank } \Gamma_1$. If $d_1 < p_1 - \eta_1$, there exists a matrix $\Lambda_1 \in \mathbb{R}^{p_2 \times (p_1-\eta_1)}$, where $p_2 = p_1 - \eta_1 - d_1$, such that $\Lambda_1 \Gamma_1 = 0$.

Define the auxiliary variable (or fictitious output) $y^2 = \Lambda_1 \begin{bmatrix} C_{\eta_1+1} A^{r_1^1} \\ \vdots \\ C_{p_1} A^{r_{p_1-\eta_1}^1} \end{bmatrix} x \triangleq C^2 x$, $C^2 = \begin{bmatrix} C_1^2 \\ \vdots \\ C_{p_2}^2 \end{bmatrix}$. Note that C^2 is not necessarily full rank.

Iteration 2: The Output Information Algorithm is applied to the new vector of fictitious outputs $y^2 \in \mathbb{R}^{p_2}$.

a. After possible reordering of the components of y^2, by analogy with Step 1.a, define the integers η_2 and r_j^2, $1 \leq j \leq p_2 - \eta_2$.

b. Write $\varphi^1 + \rho^1 + \varphi^2 = \text{rank}\,(\Phi^1 \cup \Upsilon^1 \cup \Phi^2)$ where

$$\Phi^2 = \text{span}\left\{C_1^2, ..., C_1^2 A^{n-1}, C_2^2, ..., C_2^2 A^{n-1}, ..., C_{\eta_2}^2, ..., C_{\eta_2}^2 A^{n-1}\right\}.$$

Then define the integers φ_j^2, $1 \leq j \leq \eta_2$ and the related set I_2 such that $I_1 \cup D_1 \cup I_2$ is a basis of $\Phi^1 \cup \Upsilon^1 \cup \Phi^2$.

If $\varphi^1 + \rho^1 + \varphi^2 = n$, stop the algorithm.

c. By analogy with Step 1.c, define the sets Υ^2 and D_2 and the related integers ρ^2 and $(\rho_1^2, ..., \rho_{p_2-\eta_2}^2)$. If $\varphi^1 + \rho^1 + \varphi^2 + \rho^2 = n$, the algorithm stops. If $\varphi^1 + \rho^1 + \varphi^2 + \rho^2 < n$ and $D_2 = \emptyset$, the algorithm is also stopped.

d. Define the matrix

$$\Gamma_2 = \begin{bmatrix} \Gamma_1 \\ C_{\eta_2+1}^2 A^{r_1^2-1} D \\ \vdots \\ C_{p_2}^2 A^{r_{p_2-\eta_2}^2-1} D \end{bmatrix}$$

and note $d_2 = \text{rank}\ \Gamma_2$. If $d_2 < (p_1 - \eta_1) + (p_2 - \eta_2)$, one can find a matrix $\Lambda_2 \in \mathbb{R}^{p_3 \times ((p_1-\eta_1)+(p_2-\eta_2))}$, where $p_3 = (p_1 - \eta_1) + (p_2 - \eta_2) - d_2$, such that $\Lambda_2 \Gamma_2 = 0$. Then the Output Information Algorithm is applied to the new fictitious outputs

$$y^3 = \Lambda_2 \begin{bmatrix} C_{\eta_1+1} A^{r_1^1} \\ \vdots \\ C_{p_1} A^{r_{p_1-\eta_1}^1} \\ C_{\eta_2+1}^2 A^{r_1^2} \\ \vdots \\ C_{p_2}^2 A^{r_{p_2-\eta_2}^2} \end{bmatrix} x \triangleq C^3 x.$$

Repeating this procedure, one has:

Iteration k: The fictitious output $y^k \in \mathbb{R}^{p_k}$, that has been defined in step $k-1$, is considered.

a. Set the integers η_k and r_j^k, $1 \leq j \leq p_k - \eta_k$.

b. Compute the set of row vectors

$$\Phi^k = \text{span}\left\{C_1^k, ..., C_1^k A^{n-1}, C_2^k, ..., C_2^k A^{n-1}, ..., C_{\eta_k}^k, ..., C_{\eta_k}^k A^{n-1}\right\}$$

and write

$$\sum_{i=1}^{k-1}\left(\varphi^i+\rho^i\right)+\varphi^k=\operatorname{rank}\left(\left(\bigcup_{i=1}^{k-1}\Phi^i\cup\Upsilon^i\right)\cup\Phi^k\right).$$

Find η_k integers $\varphi_1^k,\ldots,\varphi_{\eta_k}^1$ such that $\left(\bigcup_{i=1}^{k-1}I_i\cup D_i\right)\cup I_k$, where

$$I_k=\left\{C_1^k,...,C_1^kA^{\varphi_1^k-1},...,C_{\eta_k}^k,...,C_{\eta_k}^kA^{\varphi_{\eta_k}^k-1}\right\},$$

is a basis of $\left(\bigcup_{i=1}^{k-1}\Phi^i\cup\Upsilon^i\right)\cup\Phi^k$.

c. Compute the set of row vectors

$$\Upsilon^k=\operatorname{span}\left\{C_{\eta_k+1}^k,...,C_{\eta_k+1}^kA^{r_1^k-1},...,C_{p_k}^k,...,C_{p_k}^kA^{r_{p_k-\eta_k}^k-1}\right\}$$

and write $\sum_{i=1}^{k}\left(\varphi^i+\rho^i\right)=\operatorname{rank}\left(\bigcup_{i=1}^{k}\Phi^i\cup\Upsilon^i\right)$.

Find $p_k-\eta_k$ integers $\rho_1^k,...,\rho_{p_k-\eta_k}^k$ such that $\bigcup_{i=1}^{k}\left(I_i\cup D_i\right)$, where

$$D_k=\left\{C_{\eta_k+1}^k,...,C_{\eta_k+1}^kA^{\rho_1^k-1},...,C_{p_k}^k,...,C_{p_k}^kA^{\rho_{p_k-\eta_k}^k-1}\right\},$$

is a basis of $\bigcup_{i=1}^{k}\left(\Phi^i\cup\Upsilon^i\right)$.

d. Define

$$\Gamma_k=\begin{bmatrix}\Gamma_{k-1}\\ C_{\eta_k+1}^kA^{r_1^k-1}D\\ \vdots\\ C_{p_k}^kA^{r_{p_k-\eta_k}^k-1}D\end{bmatrix}$$

and assume that $d_k=\operatorname{rank}\Gamma_k$. If $d_k<\sum_{s=1}^{k}(p_s-\eta_s)$, let us set $p_{k+1}=\sum_{s=1}^{k}(p_s-\eta_s)-d_k$.

There exists a matrix $\Lambda_k\in\mathbb{R}^{p_{k+1}\times(\sum_{s=1}^{k}(p_s-\eta_s))}$ such that $\Lambda_k\Gamma_k=0$.

Define a new fictitious output:

$$y^{k+1}=\Lambda_k\begin{bmatrix}C_{\eta_1+1}A^{r_1^1}\\ \vdots\\ C_{p_1}A^{r_{p_1-\eta_1}^1}\\ \vdots\\ C_{\eta_k+1}^kA^{r_1^k}\\ \vdots\\ C_{p_k}^kA^{r_{p_k-\eta_k}^k}\end{bmatrix}x\triangleq C^{k+1}x.$$

Stop the algorithm if:

1. there exists a $\mu \in \mathbb{N}$, such that $\varphi^1 + \rho^1 + \ldots + \varphi^\mu + \rho^\mu < n$ and
$$\left\{ D_\mu = \emptyset \text{ or } d_\mu = \sum_{s=1}^{\mu} (p_s - \eta_s) \right\},$$

2. there exists a $k^\star \in \mathbb{N}$ such that $\sum_{i=1}^{k^\star} (\varphi^i + \rho^i) = n$.

The number of iterations is finite ($\leq n - p_1$). In case 1, it is not possible to estimate the state of system (1-2) with the method described in this work. In case 2, one can define the following nonsingular $(n \times n)$ matrix

$$T = \begin{bmatrix} \bar{I}_1 \\ \bar{D}_1 \\ \vdots \\ \bar{I}_{k^\star} \\ \bar{D}_{k^\star} \end{bmatrix}$$

where $\bar{I}_i = \begin{bmatrix} C_1^i \\ \vdots \\ C_1^i A^{\varphi_1^i - 1} \\ C_2^i \\ \vdots \\ C_2^i A^{\varphi_2^i - 1} \\ \vdots \\ C_{\eta_i}^i \\ \vdots \\ C_{\eta i}^i A^{\varphi_{\eta_i}^i - 1} \end{bmatrix}$ and $\bar{D}_i = \begin{bmatrix} C_{\eta_i+1}^i \\ \vdots \\ C_{\eta_i+1}^i A^{\rho_1^i - 1} \\ C_{\eta_i+2}^i \\ \vdots \\ C_{\eta_i+2}^i A^{\rho_2^i - 1} \\ \vdots \\ C_{p_i}^i \\ \vdots \\ C_{p_i}^i A^{\rho_{p_i-\eta_i}^i - 1} \end{bmatrix}$. Then, let us set the change of coordinates

$$x = T^{-1} \begin{bmatrix} \sigma^1 \\ \chi^1 \\ \vdots \\ \sigma^{k^\star} \\ \chi^{k^\star} \end{bmatrix}$$

where, for $1 \leq i \leq k^\star$, $\sigma^i = \begin{bmatrix} \sigma_1^i \\ \vdots \\ \sigma_{\eta_i}^i \end{bmatrix}$, $\sigma_j^i \in \mathbb{R}^{\varphi_j^i}$, $1 \leq j \leq \eta_i$, and where $\chi^i = \begin{bmatrix} \chi_1^i \\ \vdots \\ \chi_{p_i - \eta_i}^i \end{bmatrix}$, $\chi_j^i \in \mathbb{R}^{\rho_j^i}$, $1 \leq j \leq p_i - \eta_i$.

Then the system (1-2) becomes, for $1 \leq i \leq k^{\star}$:

$$\begin{aligned} \dot{\sigma}_j^i &= \Delta_{i,j}^{\sigma}\sigma_j^i + \Xi_{i,j}^{\sigma}x + B_{i,j}^{\sigma}u, \; 1 \leq j \leq \eta_i & (6)\\ \dot{\chi}_j^i &= \Delta_{i,j}^{\chi}\chi_j^i + \Xi_{i,j}^{\chi}x + \Theta_{i,j}^{\chi}w + B_{i,j}^{\chi}u, \; 1 \leq j \leq p_i - \eta_i & (7) \end{aligned}$$

$$\Delta_{i,j}^{\sigma} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{\varphi_j^i \times \varphi_j^i}, \; \Xi_{i,j}^{\sigma} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ C_j A^{\varphi_j^i} \end{bmatrix}_{\varphi_j^i \times n}$$

$$\Delta_{i,j}^{\chi} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{\rho_j^i \times \rho_j^i}, \; \Xi_{i,j}^{\chi} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ C_{\eta_i+j} A^{\rho_j^i} \end{bmatrix}_{\rho_j^i \times n}, \; \Theta_{i,j}^{\chi} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ C_{\eta_i+j} A^{\rho_j^i - 1} D \end{bmatrix}_{\rho_j^i \times m}.$$

$B_{i,j}^{\sigma}$ and $B_{i,j}^{\chi}$ are a $\left(\varphi_j^i \times q\right)$ and a $\left(\rho_j^i \times q\right)$-matrix, respectively. The system is put in a set of block triangular observable forms.

The following proposition summarizes the main result of the paper.

Proposition 1 *Consider the system (1)-(2) and apply the Output Information Algorithm. If there exists a $k^{\star} \in \mathbb{N}$ such that $\sum_{i=1}^{k^{\star}} (\varphi^i + \rho^i) = n$, then:*

(i) $\operatorname{rank} \Gamma_{k^*} = m$*;*

(ii) the state and the unknown inputs can be estimated in finite time.

Proof: (i) From the definitions of the matrices $\bar{I}_i$ and $\bar{D}_i$, and since $\sum_{i=1}^{k^{\star}} (\varphi^i + \rho^i) = n$:

$$\begin{aligned} \operatorname{rank} \Gamma_{k^{\star}} &= \operatorname{rank} \begin{bmatrix} C_{\eta_1+1} A^{r_1^1 - 1} \\ \vdots \\ C_{p_1} A^{r_{p_1-\eta_1}^1 - 1} \\ \vdots \\ C_{\eta_{k^\star}+1}^{k^\star} A^{r_1^{k^\star} - 1} \\ \vdots \\ C_{p_{k^\star}}^{k^\star} A^{r_{p_{k^\star}-\eta_{k^\star}}^{k^\star} - 1} \end{bmatrix} D \\ &= \operatorname{rank} \begin{bmatrix} \bar{I}_1 \\ \bar{D}_1 \\ \vdots \\ \bar{I}_{k^\star} \\ \bar{D}_{k^\star} \end{bmatrix} D = m. \end{aligned}$$

(ii) This point is shown in the next Section.

3 State and unknown input estimation

In this section, it is shown how to recover in finite time the state and the unknown inputs using the observability form defined in (6-7).

It can be noticed that, for $1 \le i \le k^\star$, each subsystem of (6-7) is in a form similar to the following so-called triangular observable form:

$$\begin{aligned} \dot{\xi} &= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{l \times l} \xi + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \Theta \end{bmatrix} z + B_\xi u, & (8) \\ y &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \xi & (9) \end{aligned}$$

where $\xi = \begin{bmatrix} \xi_1 & \cdots & \xi_l \end{bmatrix}^T \in \mathbb{R}^l$ is the state vector, $y \in \mathbb{R}$ is the output vector, u is the known output and $z \in \mathbb{R}^m$ stands for some unknown inputs or uncertainties and where $\Theta \in \mathbb{R}^{1\times m}$. Several finite time observers for system (8-9) can be found in the literature. For instance, one can cite design methods based on step-by-step sliding mode techniques (Barbot *et al.* (1996); Drakunov (1992); Floret-Pontet and Lamnabhi-Lagarrigue (2001); Utkin (1992); Drakunov and Utkin (1995)), higher order sliding modes (Levant (1998)) or numerical issues (Diop *et al.* (1999), Diop *et al.* (2000)). Such observers allows for estimating the state ξ but also the last component in (8). This implies that one can also recover $\dot{\xi}_l = \Theta z$ in finite time.

The design of such observers is left to the reader, since they are straightforward applications of existing results and since it is beyond the scope of this paper (which is dedicated to the obtention of an observability form). Thus the observation method is not detailed here but it is worth mentioning that, depending on the choice of the observer, some assumptions have to be introduced. For instance, in all the previously mentioned works, the unknown input w has to be at least bounded.

3.1 Convergence of the state variables

We shall start with the first subsystem in (6-7), that is to say $i = 1$, where the available measurements appear. Using a finite time observer, one gets a finite time estimation of the states σ^1 and χ^1, and the last component of each subsystem in (7). Thus, for $1 \le j \le p_1 - \eta_1$, the following terms V_j^1 are also known:

$$V_j^1 = C_{\eta_1+j} A^{\rho_j^1} x + C_{\eta_1+j} A^{\rho_j^1 - 1} D w. \tag{10}$$

It can be noted that in the subsequent subsystems ($2 \le i \le k^*$), the information injections are not directly available since they are linear combination of some unknown states. Nevertheless, one can obtain the fictitious output y^2 in the following way. Let us introduce the auxiliary variable:

$$\tilde{y}^1 = \Lambda_1 \begin{bmatrix} C_{\eta_1+1}A^{r_1^1-1} \\ \vdots \\ C_{p_1}A^{r^1_{p_1-\eta_1}-1} \end{bmatrix} x. \tag{11}$$

Note that, from the construction of the set $\{I_1 \cup D_1\}$, one can write that

$$\begin{bmatrix} C_{\eta_1+1}A^{r_1^1-1} \\ \vdots \\ C_{p_1}A^{r^1_{p_1-\eta_1}-1} \end{bmatrix} = G_1 \begin{bmatrix} \bar{I}_1 \\ \bar{D}_1 \end{bmatrix}$$

where G_1 is a matrix of appropriate dimension. Thus $\tilde{y}^1$ is given by

$$\tilde{y}^1 = \Lambda_1 G_1 \begin{bmatrix} \bar{I}_1 \\ \bar{D}_1 \end{bmatrix} x = \Lambda_1 G_1 \begin{bmatrix} \sigma^1 \\ \chi^1 \end{bmatrix}$$

and is an available information. Its dynamics is given by:

$$\frac{d\tilde{y}^1}{dt} = \Lambda_1 \begin{bmatrix} C_{\eta_1+1}A^{r_1^1-1} \\ \vdots \\ C_{p_1}A^{r^1_{p_1-\eta_1}-1} \end{bmatrix} (Ax + Dw) = C^2 x = y^2.$$

Again, a finite time observer can be designed in order to obtain y^2.

Remark 2 *If $\rho^1_j = r^1_j$ for all $1 \leq j \leq p_1 - \eta_1$, y^2 can be obtained without any additional observer. Indeed, Equations (10) become:*

$$\Lambda_1 \begin{bmatrix} V_1^1 \\ \vdots \\ V^1_{p_1-\eta_1} \end{bmatrix} = \Lambda_1 \begin{bmatrix} C_{\eta_1+1}A^{r_1^1} \\ \vdots \\ C_{p_1}A^{r^1_{p_1-\eta_1}} \end{bmatrix} x + \Lambda_1 \Gamma_1 w = y^2.$$

The fictitious output y^2 is henceforth available and, repeating the same procedure for $i = 2$ in the observer, one obtains in finite time σ^2 and χ^2. Following the same scheme step-by-step, one gets all the state σ^i and χ^i, $1 \leq i \leq k^\star$, in a finite time $t_{k^\star}$.

3.2 Estimation of the unknown inputs

Since, after $t_{k^\star}$, the state x has been estimated, one has the knowledge of:

$$S^i_j = V^i_j - C_{\eta_i+j}A^{\rho^i_j}x = C_{\eta_i+j}A^{\rho^i_j-1}Dw \tag{12}$$

for $1 \leq i \leq k^\star$ and $1 \leq j \leq p_i - \eta_i$. Those equations can be written in compact form

$$S = \Theta^D w,$$

where $S = \mathrm{col}\left(S_j^i\right) \in \mathbb{R}^{\sum_{i=1}^{k^\star}(p_i-\eta_i)}$ is an available information that is computable online, and where $\Theta^D \in \mathbb{R}^{\left(\sum_{i=1}^{k^\star}(p_i-\eta_i)\right)\times m}$ is given by:

$$\Theta^D = \begin{bmatrix} \Theta_1^D \\ \vdots \\ \Theta_{k^\star}^D \end{bmatrix}, \; \Theta_i^D = \begin{bmatrix} C_{\eta_i+1}A^{\rho_1^i-1}D \\ \vdots \\ C_{p_i}A^{\rho_{p_i-\eta_i}^i-1}D \end{bmatrix}.$$

Following the same arguments as in Proposition 1, one has rank $\Theta^D = m$. Thus, the relations (12) provide a finite time estimation $\hat{w}$ of the unknown inputs w:

$$\hat{w} = \left(\Theta^D\right)^+ S_{\mathrm{eq}}$$

where $\left(\Theta^D\right)^+$ is the pseudo-inverse of Θ^D.

Remark 3 *In this paper, for a sake of simplicity, the outputs are not directly subject to unknown inputs (i.e. as in the case of noise measurement or sensor faults). Nevertheless, if it is the case, it is possible, as in Tan and Edwards (2003), to introduce appropriate filters of the outputs that lead to an augmented state space where the original sensor faults can be considered as actuator faults. Then, the algorithm given in this work can be applied.*

4 Example

To illustrate the effectiveness of the proposed technique, the example, taken from Mudge and Patton (1988), of the lateral motion of an aircraft is examined. A seven-state model of the linearized lateral dynamics is considered:

$$\dot{x} = Ax + Bu + Dw \tag{13}$$
$$y = Cx \tag{14}$$

where $x = [\upsilon, p, r, \phi, \psi, \zeta, \xi]$ is the state with υ being the sideslip velocity, p the roll rate, r the yaw rate, ϕ the roll angle, ψ the yaw angle, ζ the rudder angle and ξ the aileron angle.

u is the control input with the rudder angle and the aileron angle demand. The measurement outputs y are the roll rate and the yaw angle. The matrices A, B, C, D are given by:

$$A = \begin{bmatrix} -0.3 & 0 & -33 & 9.81 & 0 & -5.4 & 0 \\ -0.1 & -8.3 & 3.75 & 0 & 0 & 0 & -28.6 \\ 0.37 & 0 & -0.64 & 0 & 0 & -9.5 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 20 \end{bmatrix}^T .$$

An actuator fault in the rudder is considered, i.e. with $D = -\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 20 & 0 \end{bmatrix}^T$ and $w = f(t)u$. $f(t) = 0$ denotes the actuator fault-free case, while $0 < f(t) < 1$ represents the loss of actuator effectiveness in the rudder. It can be checked that $CD = 0$. This means that rank $CD \neq$ rank D and, thus, classical observers can not be designed for this application.

Let us show that the algorithm given in Section 2 can be useful to estimate not only the state but also the occurrence of the fault.

Iteration 1.a: since $CAD \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $CA^2D = \begin{bmatrix} 701.7 \\ 190 \end{bmatrix}$, one has $r_1^1 = r_2^1 = 3$.

Iteration 1.b and c: since all the outputs are affected by the unknown inputs, $I_1 = \emptyset$ and $D_1 = \{C_1, C_1A, C_1A^2, C_2, C_2A, C_2A^2\}$.

Iteration 1.d: The matrix

$$\Gamma_1 = \begin{bmatrix} C_1A^2D \\ C_2A^2D \end{bmatrix} = \begin{bmatrix} 701.7 \\ 190 \end{bmatrix}$$

leads to the following choice of the fictitious output:

$$y^2 = \left(\frac{190}{701.7} C_1A^2 - C_2A^2 \right) Ax = C^2x$$

where

$$C^2 = \begin{bmatrix} -4.728 & -155.09 & 66.904 & 2.340\,3 & 0 & 70.381 & -1048.5 \end{bmatrix}.$$

Then, it can be checked that the matrix

$$T = \begin{bmatrix} C_1 \\ C_1A \\ C_1A^2 \\ C_2 \\ C_2A \\ C_2A^2 \\ C^2 \end{bmatrix}$$

is nonsingular. After the change of coordinates $z = Tx$, the system is transformed in a set of 3 triangular observable forms:

$$\dot{z} = A_z z + B_z u + D_z w \tag{15}$$

$$y = C_z z \tag{16}$$

where

$$A_z = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1135 & -237.55 & 13.3 & 0 & 490.57 & -3.95 & 15 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -307.04 & -64.30 & 3.6 & 0 & 132.7 & -1 & 3.06 \\ 2276.5 & 470.9 & -39.79 & 0 & -984.6 & 14.82 & -36.51 \end{bmatrix}$$

$$D_z = \begin{bmatrix} 0 & 0 & 701.7 & 0 & 0 & 190 & -1407.6 \end{bmatrix}^T$$

$$C_z = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

It can then be concluded that the state and the fault can be estimated in finite time.

5 Conclusion

In this paper, the problem of the state and unknown input estimation for linear systems has been considered. The main result of the paper is the obtention of a novel observability form through a constructive algorithm. The interest of this form is twofold: (i) both the state and the unknown input can be estimated in finite time, and (ii) this method can deal with some class of systems that do not satisfy the observer matching condition required when designing classical unknown input observers or sliding mode observers. Further research are concerned with the practical realization of finite time observers for systems that can be transformed in such a form, and with the application of the algorithm in the fields of fault detection (a simple but illustrative example has been given in this paper) or parameter identification.

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