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Analytical Solution for Wave Propagation in Stratified Poroelastic Medium. Part I: the 2D Case

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Abstract: We are interested in the modeling of wave propagation in poroelastic media. We consider the biphasic Biot's model in an infinite bilayered medium, with a plane interface. We adopt the Cagniard-De Hoop's technique. This report is devoted to the calculation of analytical solutions in two dimensions. The solutions we present here have been used to validate numerical codes.

Key-words: Biot's model, poroelastic waves, analytical solution, Cagniard-De Hoop's technique.

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Solution analytique pour la propagation d'ondes en milieu poroélastique stratifié. Partie I : en dimension 2

Résumé : Nous nous intéressons à la modélisation de la propagation d'ondes dans les milieux infinis bicouches poroélastiques. Nous considérons ici le modèle bi-phasique de Biot. Cette partie est consacrée au calcul de la solution analytique en dimension deux à l'aide de la technique de Cagniard-De Hoop. Les solutions que nous présentons ici ont été utilisés pour la validation de codes numériques.

Mots-clés : Modèle de Biot, ondes poroélastiques, solution analytique, technique de Cagniard de Hoop.

Introduction

Many seismic materials cannot only be considered as solid materials. They are often porous media, i.e. media made of a solid fully saturated with a fluid: there are solid media perforated by a multitude of small holes (called pores) filled with a fluid. It is in particular often the case of the oil reservoirs. It is clear that the analysis of results by seismic methods of the exploration of such media must take to account the fact that a wave being propagated in such a medium meets a succession of phases solid and fluid: we speak about poroelastic media, and the more commonly used model is the Biot's model [1, 2, 3].

When the wavelength is large in comparison with the size of the pores, rather than regarding such a medium as an heterogeneous medium, it is legitimate to use, at least locally, the theory of homogeneization [4, 12]. This leads to the Biot's model [1, 2, 3] which involves as unknown not only the displacement field in the solid but also the displacement field in the fluid. The principal characteristic of this model is that in addition to the classical P and S waves in a solid one observes a P "slow" wave, which we could also call a "fluid" wave: the denomination "slow wave" refers to the fact that in practical applications, it is slower (and probably much slower) than the other two waves.

The computation of analytical solutions for wave propagation in poroelastic media is of high importance for the validation of numerical computational codes or for a better understanding of the reflexion/transmission properties of the media. Cagniard-de Hoop method [5, 7] is a useful tool to obtain such solutions and permits to compute each type of waves (P wave, S wave, head wave...) independently. Although it was originally dedicated to the solution to elastodynamic wave propagation, it can be applied to any transient wave propagation problem in stratified medium. However, as far as we know, few works have been dedicated to the application of this method to poroelastic medium. In [11] the analytical solution to poroelastic wave propagation in an homogeneous 2D medium is provided.

In order to validate computational codes of wave propagation in poroelastic media, we have implemented the codes Gar6more 2D [9] and Gar6more 3D [10] which provide the complete solution (reflected and transmitted waves) of the propagation of wave in stratified 2D or 3D media composed of acoustic/acoustic, acoustic/elastic, acoustic/poroelastic or poroelastic/poroelastic layers. The codes are freely downloadable at

`http://www.spice-rtn.org/library/software/Gar6more2D`

and

`http://www.spice-rtn.org/library/software/Gar6more3D`.

We will focus in this paper on the 2D poroelastic case, the 2D acoustic/poroelastic case is detailed in [8] and the three dimensional cases will be the object of forthcoming papers. The outline of the paper is as follows: we first present the model problem we want to solve and derive the Green problem from it (section 1). Then we present the analytical solution to the wave propagation problem in a two-layered 2D poroelastic (section 2). Finally we show how the analytical solution can be used to validate a numerical code (section 3).

1 The model problem

We consider an infinite two dimensional medium ($\Omega = \boldsymbol{R}^2$) composed of two homogeneous poroelastic layers $\Omega^+ = \boldsymbol{R}\times]-\infty, 0]$ and $\Omega^- = \boldsymbol{R} \times [0, +\infty[$ separated by an horizontal interface Γ (see Fig. 1). We first describe the equations in the two layers (§1.1) and the transmission conditions on the interface Γ (§1.2), then we present the Green problem from which we compute the analytical solution (§1.3).

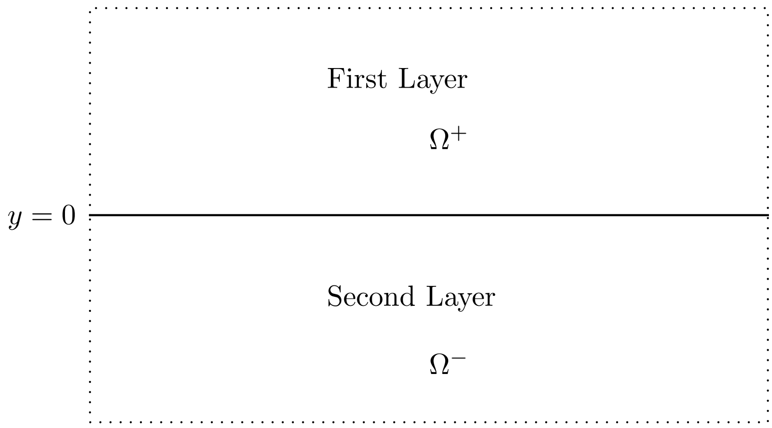

Figure 1: Configuration of the study

1.1 Poroelastic equations

We consider the second-order formulation of the poroelastic equations [1, 2, 3]:

$$\begin{cases} \rho\, \ddot{\boldsymbol{U}}_s + \rho_f\, \ddot{\boldsymbol{W}} - \boldsymbol{\nabla}\cdot\Sigma = \boldsymbol{F}_u, & \text{in } \Omega\times]0,T], \\ \rho_f\, \ddot{\boldsymbol{U}}_s + \rho_w\, \ddot{\boldsymbol{W}} + \dfrac{1}{\mathcal{K}}\,\dot{\boldsymbol{W}} + \nabla P = \boldsymbol{F}_w, & \text{in } \Omega\times]0,T], \\ \Sigma = \lambda \nabla\cdot\boldsymbol{U}_s\, \boldsymbol{I}_2 + 2\mu\varepsilon(\boldsymbol{U}_s) - \beta\, P\, \boldsymbol{I}_2, & \text{in } \Omega\times]0,T], \\ \dfrac{1}{m}\, P + \beta\, \nabla\cdot\boldsymbol{U}_s + \nabla\cdot\boldsymbol{W} = F_p, & \text{in } \Omega\times]0,T], \\ \boldsymbol{U}_s(x,0) = 0,\ \boldsymbol{W}(x,0) = 0, & \text{in } \Omega, \\ \dot{\boldsymbol{U}}_s(x,0) = 0,\ \dot{\boldsymbol{W}}(x,0) = 0, & \text{in } \Omega, \end{cases} \tag{1}$$

with

$$(\boldsymbol{\nabla}\cdot\Sigma)_i = \sum_{j=1}^{2} \frac{\partial \Sigma_{ij}}{\partial x_j}\ \ \forall\, i = 1, 2. \text{ As usual } \boldsymbol{I}_2 \text{ is the identity matrix of } \mathcal{M}_2(\mathbb{R}),$$

and $\varepsilon(\boldsymbol{U}_s)$ is the solid strain tensor defined by:

$$\varepsilon_{ij}(\boldsymbol{U}) = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right).$$

In (1), the unknowns are:

- $\boldsymbol{U}_s$ the displacement field of solid particles;
- $\boldsymbol{W} = \phi(\boldsymbol{U}_f - \boldsymbol{U}_s)$, the relative displacement, $\boldsymbol{U}_f$ being the displacement field of fluid particles and ϕ the porosity;
- P, the fluid pressure;
- Σ, the solid stress tensor.

The parameters describing the physical properties of the medium are given by:

- $\rho = \phi\,\rho_f + (1-\phi)\rho_s$ is the overall density of the saturated medium, with ρ_s the density of the solid and ρ_f the density of the fluid;
- $\rho_w = a\rho_f/\phi$, where a is the tortuosity of the solid matrix;
- $\mathcal{K} = \kappa/\eta$, where κ is the permeability of the solid matrix and η is the viscosity of the fluid;
- m and β are positive physical coefficients: $\beta = 1 - K_b/K_s$ and $m = [\phi/K_f + (\beta - \phi)/K_s]^{-1}$, where K_s is the bulk modulus of the solid, K_f is the bulk modulus of the fluid and K_b is the frame bulk modulus;
- μ is the frame shear modulus, and $\lambda = K_b - 2\mu/3$ is the Lamé constant.
- $\boldsymbol{F}_u$, $\boldsymbol{F}_w$ and F_p are the force densities.

To simplify this study, we consider only the case of a compression source

$$\boldsymbol{F}_u(x,y,t) = f_u \nabla(\delta_x\,\delta_{y-h})\,f(t) \text{ and } \boldsymbol{F}_w(x,y,t) = f_w \nabla(\delta_x\,\delta_{y-h})\,f(t)$$

and a pressure source $F_p = f_p \delta_x\,\delta_{y-h}\,f(t)$, where f_u, f_w and f_p are constant and f is a regular function source in time. We can generalize this approach for other types of punctual sources such as for instance

$$\boldsymbol{F}_u = f_u \nabla \times (\delta_x \delta_{y-h})\,f(t) \text{ and } \boldsymbol{F}_w = f_w \nabla \times (\delta_x \delta_{y-h})\,f(t).$$

1.2 Transmission conditions

Let $\boldsymbol{n}$ be the unitary normal vector of Γ outwardly directed to Ω^-. The transmission conditions on the interface Γ between the two poroelastic medium are [6]:

$$\begin{cases} \boldsymbol{U}_s^+ = \boldsymbol{U}_s^-, \\ \boldsymbol{W}^+ \cdot \boldsymbol{n} = \boldsymbol{W}^- \cdot \boldsymbol{n}, \\ P^+ = P^-, \\ \Sigma^+ \boldsymbol{n} = \Sigma^- \boldsymbol{n}. \end{cases} \tag{2}$$

1.3 The Green problem

We won't compute directly the solution to (2) but the solution to the following Green problem:

$$\rho^\pm \ddot{\boldsymbol{u}}_s^\pm + \rho_f^\pm \ddot{\boldsymbol{w}}^\pm - \boldsymbol{\nabla} \cdot \sigma^\pm = f_u \nabla(\delta_x \, \delta_{y-h})\delta_t, \qquad \text{in } \Omega^\pm \times]0,T], \tag{3a}$$

$$\rho_f^\pm \ddot{\boldsymbol{u}}_s^\pm + \rho_w^\pm \ddot{\boldsymbol{w}}^\pm + \frac{1}{\mathcal{K}^\pm} \dot{\boldsymbol{w}}^\pm + \nabla p^\pm = f_w \nabla(\delta_x \, \delta_{y-h})\delta_t, \qquad \text{in } \Omega^\pm \times]0,T], \tag{3b}$$

$$\sigma^\pm = \lambda^\pm \nabla \cdot \boldsymbol{u}_s^\pm \boldsymbol{I}_2 + 2\mu^\pm \varepsilon(\boldsymbol{u}_s^\pm) - \beta^\pm p^\pm \boldsymbol{I}_2, \qquad \text{in } \Omega^\pm \times]0,T], \tag{3c}$$

$$\frac{1}{m^\pm} p^\pm + \beta^\pm \nabla \cdot \boldsymbol{u}_s^\pm + \nabla \cdot \boldsymbol{w}^\pm = f_p \delta_x \, \delta_{y-h} \, f(t), \qquad \text{in } \Omega^\pm \times]0,T], \tag{3d}$$

$$\boldsymbol{u}_s^- = \boldsymbol{u}^+, \qquad \text{on } \Gamma \times]0,T] \tag{3e}$$

$$\boldsymbol{w}^- \cdot \boldsymbol{n} = \boldsymbol{w}^+ \cdot \boldsymbol{n}, \qquad \text{on } \Gamma \times]0,T] \tag{3f}$$

$$p^- = p^+, \qquad \text{on } \Gamma \times]0,T] \tag{3g}$$

$$\sigma^- \boldsymbol{n} = \sigma^+ \boldsymbol{n}, \qquad \text{on } \Gamma \times]0,T]. \tag{3h}$$

The solution to (1) is then computed from the solution to the Green Problem thanks to a convolution by the source function. For instance we have:

$$P^+(x,y,t) = p^+(x,y,.) * f(.) = \int_0^t p^+(x,y,\tau) f(t-\tau)\, d\tau$$

(we have similar relations for the other unknowns). We also suppose that the poroelastic medium is non dissipative, i.e the viscosity $\eta^\pm = 0$. Using the equations (3c,3d) we can eliminate $\sigma^\pm$ and $p^\pm$ in (3) and we obtain the equivalent system:

$$\begin{cases} \rho^\pm \ddot{\boldsymbol{u}}_s^\pm + \rho_f^\pm \ddot{\boldsymbol{w}}^\pm - \alpha^\pm \nabla(\nabla \cdot \boldsymbol{u}_s^\pm) + \mu^\pm \nabla \times (\nabla \times \boldsymbol{u}_s^\pm) - m^\pm \beta^\pm \nabla(\nabla \cdot \boldsymbol{w}^\pm) \\ = (f_u - \beta^+ m^+ f_p) \nabla(\delta_x \, \delta_{y-h}) \, \delta_t, \\ \rho_f^\pm \ddot{\boldsymbol{u}}_s^\pm + \rho_w^\pm \ddot{\boldsymbol{w}}^\pm - m^\pm \beta^\pm \nabla(\nabla \cdot \boldsymbol{u}_s^\pm) - m^\pm \nabla(\nabla \cdot \boldsymbol{w}^\pm) = (f_w - m^+ f_p) \nabla(\delta_x \, \delta_{y-h}) \, \delta_t, \end{cases} \tag{4}$$

with $\alpha^- = \lambda^- + 2\mu^- + m^- \beta^{-2}$.

And the transmission conditions on Γ are rewritten as:

$$\begin{cases} u_{sx}^{+} = u_{sx}^{-}, \\ u_{sy}^{+} = u_{sy}^{-}, \\ w_{y}^{-} = w_{y}^{+}, \\ m^{+}\beta^{+}\,\nabla\cdot\boldsymbol{u}_{s}^{+} + m^{+}\,\nabla\cdot\boldsymbol{w}^{+} = m^{-}\beta^{-}\,\nabla\cdot\boldsymbol{u}_{s}^{-} + m^{-}\,\nabla\cdot\boldsymbol{w}^{-}, \\ \mu^{+}(\partial_{y}u_{sx}^{+} + \partial_{x}u_{sy}^{+}) = \mu^{-}(\partial_{y}u_{sx}^{-} + \partial_{x}u_{sy}^{-}), \\ (\lambda^{-} + m^{+}{\beta^{+}}^{2})\nabla\cdot\boldsymbol{u}_{s}^{+} + 2\mu^{+}\partial_{y}u_{sy}^{+} + m^{+}\beta^{+}\nabla\cdot\boldsymbol{w}^{+} = \\ (\lambda^{-} + m^{-}{\beta^{-}}^{2})\nabla\cdot\boldsymbol{u}_{s}^{-} + 2\mu^{-}\partial_{y}u_{sy}^{-} + m^{-}\beta^{-}\nabla\cdot\boldsymbol{w}^{-}. \end{cases} \tag{5}$$

We split the displacement fields $\boldsymbol{u}_{s}^{\pm}$ and $\boldsymbol{w}^{\pm}$ on irrotational and isovolumic fields (P-wave and S-wave):

$$\boldsymbol{u}_{s}^{\pm} = \nabla\Theta_{u}^{-} + \nabla\times\Psi_{u}^{\pm} \;\; ; \;\; \boldsymbol{w}^{\pm} = \nabla\Theta_{w}^{\pm} + \nabla\times\Psi_{w}^{\pm}. \tag{6}$$

We can then rewrite system (4) in the following form:

$$\begin{cases} A^{+}\ddot{\Theta}^{+} - B^{+}\Delta\Theta^{+} = \delta_{x}\delta_{y-h}\delta_{t}\,\boldsymbol{F}, & \text{in } \Omega^{+}\times]0,T] \\ A^{-}\ddot{\Theta}^{-} - B^{-}\Delta\Theta^{-} = 0, & \text{in } \Omega^{-}\times]0,T] \\ \ddot{\Psi}_{u}^{\pm} - {V_{S}^{\pm}}^{2}\Delta\Psi_{u}^{\pm} = 0, & \text{in } \Omega^{\pm}\times]0,T] \\ \ddot{\Psi}_{w}^{\pm} = -\dfrac{\rho_{f}^{\pm}}{\rho_{w}^{\pm}}\ddot{\Psi}_{u}^{\pm}, & \text{in } \Omega^{\pm}\times]0,T] \end{cases} \tag{7}$$

where $\Theta^{\pm} = (\Theta_{u}^{-}, \Theta_{w}^{-})^{t}$, $\boldsymbol{F} = (f_{u} - \beta^{+}m^{+}f_{p}, f_{w} - m^{+}f_{p})^{t}$, $A^{\pm}$ and $B^{\pm}$ are 2×2 symmetrical matrices:

$$A^{\pm} = \begin{pmatrix} \rho^{\pm} & \rho_{f}^{\pm} \\ \rho_{f}^{\pm} & \rho_{w}^{\pm} \end{pmatrix} \;\; ; \;\; B^{\pm} = \begin{pmatrix} \lambda^{\pm} + 2\mu^{\pm} + m^{\pm}(\beta^{\pm})^{2} & m^{\pm}\beta^{\pm} \\ m^{\pm}\beta^{\pm} & m^{\pm} \end{pmatrix},$$

and

$$V_{S}^{\pm} = \sqrt{\frac{\mu\rho_{w}^{\pm}}{\rho^{\pm}\rho_{w}^{\pm} - {\rho_{f}^{\pm}}^{2}}}$$

is the S-wave velocity.

We multiply the first (resp. the second) equation of system (7) by the inverse of A^{+} (resp. A^{-}). The matrix ${A^{+}}^{-1}B^{+}$ (resp. ${A^{-}}^{-1}B^{-}$) is diagonalizable: ${A^{\pm}}^{-1}B^{\pm} = \mathcal{P}^{\pm}D^{\pm}{\mathcal{P}^{\pm}}^{-1}$, where $\mathcal{P}^{\pm}$ is the change-of-coordinates matrix, $D^{\pm} = diag({V_{Pf}^{\pm}}^{2}, {V_{Ps}^{\pm}}^{2})$ is the diagonal matrix similar to ${A^{\pm}}^{-1}B^{\pm}$, $V_{Pf}^{\pm}$ and $V_{Ps}^{\pm}$ are respectively the fast P-wave velocity and the slow P-wave velocity ($V_{Ps}^{\pm} < V_{Pf}^{\pm}$).

Using the change of variables

$$\Phi^{\pm} = (\Phi_{Pf}^{\pm}, \Phi_{Ps}^{\pm})^{t} = {\mathcal{P}^{\pm}}^{-1}\Theta^{\pm}, \tag{8}$$

we obtain the uncoupled system on fast P-waves, slow P-waves and S-waves:

$$\left\{\begin{array}{ll}\ddot{\Phi}^+ - D^+\Delta\Phi^+ = \delta_x\delta_{y-h}\delta_t\,\boldsymbol{F}^+, & \text{in } \Omega^+\times]0,T]\\ \ddot{\Phi}^- - D^-\Delta\Phi^- = 0, & \text{in } \Omega^-\times]0,T]\\ \ddot{\Psi}_u^\pm - {V_S^\pm}^2\Delta\Psi_u^\pm = 0, & \text{in } \Omega^\pm\times]0,T]\\ \Psi_w^\pm = -\dfrac{\rho_f^\pm}{\rho_w^\pm}\Psi_u^\pm, & \text{in } \Omega^\pm\times]0,T]\end{array}\right. \tag{9}$$

with $\boldsymbol{F}^+ = (A^+\mathcal{P}^+)^{-1}\boldsymbol{F} = (F_{Pf}^+, F_{Ps}^+)^t$.

Finally, we obtain the Green problem equivalent to (3):

$$\left\{\begin{array}{ll}\ddot{\Phi}_i^+ - {V_i^+}^2\Delta\Phi_i^+ = \delta_x\delta_{y-h}\delta_t\,F_i^+, \quad i\in\{Pf,Ps\} & y>0\\ \ddot{\Phi}_S^+ - {V_S^+}^2\Delta\Phi_S^+ = 0 & y>0\\ \ddot{\Phi}_i^- - {V_i^-}^2\Delta\Phi_i^- = 0, \quad i\in\{Pf,Ps,S\} & y<0\\ \mathcal{B}(\Phi_{Pf}^+,\Phi_{Ps}^+,\Phi_S^+,\Phi_{Pf}^-,\Phi_{Ps}^-,\Phi_S^-) = 0, & y=0\end{array}\right. \tag{10}$$

where we have set $\Phi_S^\pm = \Psi_u^\pm$ in order to have similar notations for the Pf, Ps and S waves. The operator $\mathcal{B}$ represents the transmission conditions on Γ:

$$\mathcal{B}\begin{pmatrix}\Phi_{Pf}^+\\ \Phi_{Ps}^+\\ \Phi_S^+\\ \Phi_{Pf}^-\\ \Phi_{Ps}^-\\ \Phi_S^-\end{pmatrix} = \begin{bmatrix}\mathcal{P}_{11}^+\,\partial_x & \mathcal{P}_{12}^+\,\partial_x & \partial_y & -\mathcal{P}_{11}^-\,\partial_x & -\mathcal{P}_{12}^-\,\partial_x & -\partial_y\\ \mathcal{P}_{11}^+\,\partial_y & \mathcal{P}_{12}^+\,\partial_y & -\partial_x & -\mathcal{P}_{11}^-\,\partial_y & -\mathcal{P}_{12}^-\,\partial_y & \partial_x\\ \mathcal{P}_{21}^+\,\partial_y & \mathcal{P}_{22}^+\,\partial_y & \dfrac{\rho_f^+}{\rho_w^+}\partial_x & -\mathcal{P}_{21}^-\,\partial_y & -\mathcal{P}_{22}^-\,\partial_y & -\dfrac{\rho_f^-}{\rho_w^-}\partial_x\\ \mathcal{B}_{41} & \mathcal{B}_{42} & 0 & \mathcal{B}_{44} & \mathcal{B}_{45} & 0\\ \mathcal{B}_{51} & \mathcal{B}_{52} & \mu^+(\partial_{yy}^2-\partial_{xx}^2) & \mathcal{B}_{54} & \mathcal{B}_{55} & -\mu^-(\partial_{yy}^2-\partial_{xx}^2)\\ \mathcal{B}_{61} & \mathcal{B}_{62} & -2\mu^+\partial_{xy}^2 & \mathcal{B}_{64} & \mathcal{B}_{65} & 2\mu^-\partial_{xy}^2\end{bmatrix}\begin{bmatrix}\Phi_{Pf}^+\\ \Phi_{Ps}^+\\ \Phi_S^+\\ \Phi_{Pf}^-\\ \Phi_{Ps}^-\\ \Phi_S^-\end{bmatrix}$$

where $\mathcal{P}^{\pm}_{ij}$, $i, j = 1, 2$ are the components of the change-of-coordinates matrix $\mathcal{P}^{\pm}$ and

$$\mathcal{B}_{41} = \frac{m^+(\beta^+\mathcal{P}^+_{11} + \mathcal{P}^+_{21})}{{V^+_{Pf}}^2}\,\partial^2_{tt} \;;\; \mathcal{B}_{42} = \frac{m^+(\beta^+\mathcal{P}^+_{12} + \mathcal{P}^+_{22})}{{V^+_{Ps}}^2}\,\partial^2_{tt};$$

$$\mathcal{B}_{44} = -\frac{m^-(\beta^-\mathcal{P}^-_{11} + \mathcal{P}^-_{21})}{{V^-_{Pf}}^2}\,\partial^2_{tt} \;;\; \mathcal{B}_{45} = -\frac{m^-(\beta^-\mathcal{P}^-_{12} + \mathcal{P}^-_{22})}{{V^-_{Ps}}^2}\,\partial^2_{tt};$$

$$\mathcal{B}_{51} = 2\mu^+\mathcal{P}^+_{11}\,\partial^2_{xy} \;;\; \mathcal{B}_{52} = 2\mu^+\mathcal{P}^+_{12}\,\partial^2_{xy} \;;\; \mathcal{B}_{54} = -2\mu^-\mathcal{P}^-_{11}\,\partial^2_{xy} \;;\; \mathcal{B}_{55} = -2\mu^-\mathcal{P}^-_{12}\,\partial^2_{xy};$$

$$\mathcal{B}_{61} = \frac{(\lambda^+ + m^+{\beta^+}^2)\mathcal{P}^+_{11} + m^+\beta^+\mathcal{P}^+_{21}}{{V^+_{Pf}}^2}\partial^2_{tt} + 2\mu^+\mathcal{P}^+_{11}\partial^2_{yy};$$

$$\mathcal{B}_{62} = \frac{(\lambda^+ + m^+{\beta^+}^2)\mathcal{P}^+_{12} + m^+\beta^+\mathcal{P}^+_{22}}{{V^+_{Ps}}^2}\,\partial^2_{tt} + 2\mu^+\mathcal{P}^+_{12}\partial^2_{yy};$$

$$\mathcal{B}_{64} = \frac{(\lambda^- + m^-{\beta^-}^2)\mathcal{P}^-_{11} + m^-\beta^-\mathcal{P}^-_{21}}{{V^-_{Pf}}^2}\partial^2_{tt} + 2\mu^-\mathcal{P}^-_{11}\partial^2_{yy};$$

$$\mathcal{B}_{65} = \frac{(\lambda^- + m^-{\beta^-}^2)\mathcal{P}^-_{12} + m^-\beta^-\mathcal{P}^-_{22}}{{V^-_{Ps}}^2}\,\partial^2_{tt} + 2\mu^-\mathcal{P}^-_{12}\partial^2_{yy}.$$

To obtain this operator we have used the transmission conditions (5), the change of variables (6)-(8) and the uncoupled system (9).

Moreover, from the unknowns $\Phi^{\pm}_{Pf}$, $\Phi^{\pm}_{Ps}$ and $\Phi^{\pm}_{S}$ we can determine the solid displacement $\boldsymbol{u}^{\pm}_s$ and the relative displacement $\boldsymbol{w}^{\pm}$ by using the change of variables presented below.

2 Expression of the analytical solution

To state our results, we need the following notations and definitions:

1. **Definition of the complex square root**. For $q \in \mathbb{C}\backslash\mathbb{R}^-$, we use the following definition of the square root $g(q) = q^{1/2}$:

$$g(q)^2 = q \quad \text{and} \quad \Re e[g(q)] > 0.$$

   The branch cut of $g(q)$ in the complex plane will thus be the half-line defined by $\{q \in \mathbb{R}^-\}$ (see Fig. 2). In the following, we use the abuse of notation $g(q) = \mathrm{i}\sqrt{-q}$ for $q \in \mathbb{R}^-$.

2. **Definition of the functions $\kappa^{\pm}_i$**. For $i \in \{Pf, Ps, S\}$ and $q \in \mathbb{C}$, we define the functions

$$\kappa^{\pm} := \kappa^{+}(q) = \left(\frac{1}{V_i^{\pm 2}} + q^2\right)^{1/2}.$$

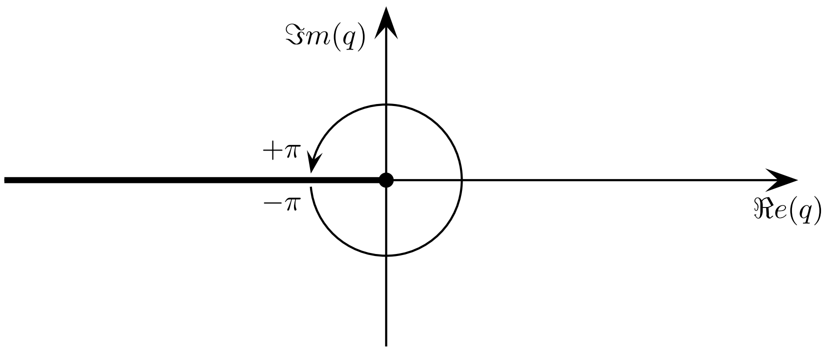

Figure 2: Definition of the function $x \mapsto (x)^{1/2}$

3. **Definition of the reflection and transmission coefficients.** For a given $q \in \mathbb{C}$, we denote by $\mathcal{R}_{PfPf}(q)$, $\mathcal{R}_{PfPs}(q)$, $\mathcal{R}_{PfS}(q)$, $\mathcal{T}_{PfPf}(q)$, $\mathcal{T}_{PfPs}(q)$, and $\mathcal{T}_{PfS}(q)$ the solution to the linear system

$$\mathcal{A}(q)\begin{bmatrix} \mathcal{R}_{PfPf}(q) \\ \mathcal{R}_{PfPs}(q) \\ \mathcal{R}_{PfS}(q) \\ \mathcal{T}_{PfPf}(q) \\ \mathcal{T}_{PfPs}(q) \\ \mathcal{T}_{PfS}(q) \end{bmatrix} = \frac{1}{2\kappa^+_{Pf}(q){V^+_{Pf}}^2}\begin{bmatrix} \mathrm{i}\, q\mathcal{P}^+_{11} \\ -\kappa^+_{Pf}(q)\mathcal{P}^+_{11} \\ -\kappa^+_{Pf}(q)\mathcal{P}^+_{21} \\ -\dfrac{m^+}{{V^+_{Pf}}^2}(\beta^+\mathcal{P}^+_{11}+\mathcal{P}^+_{21}) \\ 2\mathrm{i}\, q\mu^+\kappa^+_{Pf}(q)\mathcal{P}^+_{11} \\ -\dfrac{(\lambda^+ + m^+{\beta^+}^2)\mathcal{P}^+_{11} + m^+\beta^+\mathcal{P}^+_{21}}{{V^+_{Pf}}^2} - 2\mu^+\mathcal{P}^+_{11}\kappa^+_{Pf}(q) \end{bmatrix}$$

and by $\mathcal{R}_{PsPf}(q)$, $\mathcal{R}_{PsPs}(q)$, $\mathcal{R}_{PsS}(q)$, $\mathcal{T}_{PsPf}(q)$, $\mathcal{T}_{PsPs}(q)$ and $\mathcal{T}_{PsS}(q)$ the solution to the linear system

$$\mathcal{A}(q)\begin{bmatrix} \mathcal{R}_{PsPf}(q) \\ \mathcal{R}_{PsPs}(q) \\ \mathcal{R}_{PsS}(q) \\ \mathcal{T}_{PsPf}(q) \\ \mathcal{T}_{PsPs}(q) \\ \mathcal{T}_{PsS}(q) \end{bmatrix} = \frac{1}{2\kappa^+_{Ps}(q){V^+_{Ps}}^2}\begin{bmatrix} \mathrm{i}\, q\mathcal{P}^+_{12} \\ -\kappa^+_{Ps}(q)\mathcal{P}^+_{12} \\ -\kappa^+_{Ps}(q)\mathcal{P}^+_{22} \\ -\dfrac{m^+}{{V^+_{Ps}}^2}(\beta^+\mathcal{P}^+_{12}+\mathcal{P}^+_{22}) \\ 2\mathrm{i}\, q\mu^+\kappa^+_{Ps}(q)\mathcal{P}^+_{12} \\ -\dfrac{(\lambda^+ + m^+{\beta^+}^2)\mathcal{P}^+_{12} + m^+\beta^+\mathcal{P}^+_{22}}{{V^+_{Ps}}^2} - 2\mu^+\mathcal{P}^+_{12}\kappa^+_{Ps}(q) \end{bmatrix},$$

where the matrix $\mathcal{A}(q)$ is defined for $q \in \mathbb{C}$ by:

$$A(q) = \begin{bmatrix} -\mathrm{i}\,q\mathcal{P}_{11}^{+} & -\mathrm{i}\,q\mathcal{P}_{12}^{+} & -\kappa_S^{+}(q) & \mathrm{i}\,q\mathcal{P}_{11}^{-} & \mathrm{i}\,q\mathcal{P}_{12}^{-} & -\kappa_S^{-}(q) \\ -\kappa_{Pf}^{+}(q)\mathcal{P}_{11}^{+} & -\kappa_{Ps}^{+}(q)\mathcal{P}_{12}^{+} & \mathrm{i}\,q & -\kappa_{Pf}^{-}(q)\mathcal{P}_{11}^{-} & -\kappa_{Ps}^{-}(q)\mathcal{P}_{12}^{-} & -\mathrm{i}\,q \\ -\kappa_{Pf}^{+}(q)\mathcal{P}_{21}^{+} & -\kappa_{Ps}^{+}(q)\mathcal{P}_{22}^{+} & -\mathrm{i}\,q\frac{\rho_f^{+}}{\rho_w^{+}} & -\kappa_{Pf}^{-}(q)\mathcal{P}_{21}^{-} & -\kappa_{Ps}^{-}(q)P_{22}^{-} & \mathrm{i}\,q\frac{\rho_f^{-}}{\rho_w^{-}} \\ \mathcal{A}_{41}(q) & \mathcal{A}_{42}(q) & 0 & \mathcal{A}_{44}(q) & \mathcal{A}_{45}(q) & 0 \\ \mathcal{A}_{51}(q) & \mathcal{A}_{52}(q) & \mathcal{A}_{53}(q) & \mathcal{A}_{54}(q) & \mathcal{A}_{55}(q) & \mathcal{A}_{56}(q) \\ \mathcal{A}_{61}(q) & \mathcal{A}_{62}(q) & -2\mathrm{i}\,q\mu^{+}\kappa_S^{+}(q) & \mathcal{A}_{64}(q) & \mathcal{A}_{65}(q) & -2\mathrm{i}\,q\mu^{-}\kappa_S^{-}(q) \end{bmatrix},$$

with

$$\begin{aligned}
\mathcal{A}_{41}(q) &= \frac{m^{+}}{{V_{Pf}^{+}}^{2}}\left[\beta^{+}\mathcal{P}_{11}^{+} + \mathcal{P}_{21}^{+}\right]; \quad & \mathcal{A}_{42}(q) &= \frac{m^{+}}{{V_{Ps}^{+}}^{2}}\left[\beta^{+}\mathcal{P}_{12}^{+} + \mathcal{P}_{22}^{+}\right];\\
\mathcal{A}_{44}(q) &= -\frac{m^{-}}{{V_{Pf}^{-}}^{2}}\left[\beta^{-}\mathcal{P}_{11}^{-} + \mathcal{P}_{21}^{-}\right]; \quad & \mathcal{A}_{45}(q) &= -\frac{m^{-}}{{V_{Ps}^{-}}^{2}}\left[\beta^{-}\mathcal{P}_{12}^{-} + P_{22}^{-}\right];
\end{aligned}$$

$$\mathcal{A}_{51}(q) = 2\mathrm{i}\,q\mu^{+}\kappa_{Pf}^{+}(q)\mathcal{P}_{11}^{+};\quad \mathcal{A}_{52}(q) = 2\mathrm{i}\,q\mu^{+}\kappa_{Ps}^{+}(q)\mathcal{P}_{12}^{+};\quad \mathcal{A}_{53}(q) = \mu^{+}({\kappa_S^{+}}^{2}(q) + q^{2});$$

$$\mathcal{A}_{54}(q) = 2\mathrm{i}\,q\mu^{-}\kappa_{Pf}^{-}(q)\mathcal{P}_{11}^{-};\quad \mathcal{A}_{55}(q) = 2\mathrm{i}\,q\mu^{-}\kappa_{Ps}^{-}(q)P_{12}^{-};\quad \mathcal{A}_{56}(q) = -\mu^{-}({\kappa_S^{-}}^{2}(q) + q^{2});$$

$$\mathcal{A}_{61}(q) = \frac{(\lambda^{+} + m^{+}{\beta^{+}}^{2})\mathcal{P}_{11}^{+} + m^{+}\beta^{+}\mathcal{P}_{21}^{+}}{{V_{Pf}^{+}}^{2}} + 2\mu^{+}{\kappa_{Pf}^{+}}^{2}(q)\mathcal{P}_{11}^{+};$$

$$\mathcal{A}_{62}(q) = \frac{(\lambda^{+} + m^{+}{\beta^{+}}^{2})\mathcal{P}_{12}^{+} + m^{+}\beta^{+}\mathcal{P}_{22}^{+}}{{V_{Ps}^{+}}^{2}} + 2\mu^{+}\kappa_{Ps}^{+}(q)^{2}\mathcal{P}_{12}^{+};$$

$$\mathcal{A}_{64}(q) = -\frac{(\lambda^{-} + m^{-}{\beta^{-}}^{2})\mathcal{P}_{11}^{-} + m^{-}\beta^{-}P_{21}^{-}}{{V_{Pf}^{-}}^{2}} - 2\mu^{-}{\kappa_{Pf}^{-}}^{2}(q)\mathcal{P}_{11}^{-};$$

$$\mathcal{A}_{65}(q) = -\frac{(\lambda^{-} + m^{-}{\beta^{-}}^{2})\mathcal{P}_{12}^{-} - m^{-}\beta^{-}P_{22}^{-}}{{V_{Ps}^{-}}^{2}} + 2\mu^{-}{\kappa_{Ps}^{-}}^{2}(q)\mathcal{P}_{12}^{-}.$$

We also denote by $V_{\max}$ the greatest velocity in the medium:

$$V_{\max} = \max(V_{Pf}^{+}, V_{Ps}^{+}, V_S^{+}, V_{Pf}^{-}, V_{Ps}^{-}, V_S^{-}).$$

We can now present the expression of the solution to the Green Problem:

Theorem 2.1. *The solid displacement in the top medium is given by*

$$\boldsymbol{u}_s^{+}(x,y,t) = \int_0^t \boldsymbol{\nu}^{+}(x,y,\tau)\,d\tau,$$

with

$$\boldsymbol{\nu}^{+} = \boldsymbol{\nu}_{Pf}^{+} + \boldsymbol{\nu}_{Ps}^{+} + \boldsymbol{\nu}_{PfPf}^{+} + \boldsymbol{\nu}_{PfPs}^{+} + \boldsymbol{\nu}_{PfS}^{+} + \boldsymbol{\nu}_{PsPf}^{+} + \boldsymbol{\nu}_{PsPs}^{+} + \boldsymbol{\nu}_{PsS}^{+}$$

and the solid displacement in the bottom medium is given by

$$\boldsymbol{u}_s^-(x,y,t) = \int_0^t \boldsymbol{\nu}^-(x,y,\tau)\,d\tau$$

with

$$\boldsymbol{\nu}^- = \boldsymbol{\nu}_{PfPf}^+ + \boldsymbol{\nu}_{PfPs}^+ + \boldsymbol{\nu}_{PfS}^+ + \boldsymbol{\nu}_{PsPf}^+ + \boldsymbol{\nu}_{PsPs}^+ + \boldsymbol{\nu}_{PsS}^+,$$

where

- $\boldsymbol{\nu}_{Pf}^+$ *is the velocity of the incident* Pf *wave and satisfies:*

$$\begin{cases} \nu_{Pf,x}^+(x,y,t) = -\dfrac{\mathcal{P}_{11}^+ F_{Pf}^+}{{V_{Pf}^+}^2} \dfrac{tx}{2\pi r\sqrt{t^2-t_0^2}}, \\ \nu_{Pf,y}^+(x,y,t) = -\dfrac{\mathcal{P}_{11}^+ F_{Pf}^+}{{V_{Pf}^+}^2} \dfrac{t(y-h)}{2\pi r\sqrt{t^2-t_0^2}}, \end{cases} \quad \textit{if } t > t_0$$

$$\boldsymbol{\nu}_{Pf}^+(x,y,t) = 0. \qquad \textit{else}$$

We set here $r = (x^2+(y-h)^2)^{1/2}$ *and* $t_0 = r/V_{Pf}^+$ *denotes the arrival time of the incident* Pf *wave.*

- $\boldsymbol{\nu}_{Ps}^+$ *is the velocity of the incident* Ps *wave and satisfies:*

$$\begin{cases} \nu_{Ps,x}^+(x,y,t) = -\dfrac{\mathcal{P}_{12}^+ F_{Ps}^+}{{V_{Ps}^+}^2} \dfrac{tx}{2\pi r\sqrt{t^2-t_0^2}}, \\ \nu_{Ps,y}^+(x,y,t) = -\dfrac{\mathcal{P}_{12}^+ F_{Ps}^+}{{V_{Ps}^+}^2} \dfrac{t(y-h)}{2\pi r\sqrt{t^2-t_0^2}}, \end{cases} \quad \textit{if } t > t_0$$

$$\boldsymbol{\nu}_{Ps}^+(x,y,t) = 0. \qquad \textit{else}$$

We set here $r = (x^2+(y-h)^2)^{1/2}$ *and* $t_0 = r/V_{Ps}^+$ *denotes the arrival time of the incident* Ps *wave.*

- $\boldsymbol{\nu}_{PfPf}^+$ *is the velocity of the reflected* $PfPf$ *wave (the* Pf *reflected wave generated by the* Pf *incident wave) and satisfies:*

$$\begin{cases} \nu_{PfPf,x}^+(x,y,t) = \dfrac{\Im m\left[\mathrm{i}\,\upsilon(t)\kappa_{Pf}^+(\upsilon(t))\mathcal{R}_{PfPf}(\upsilon(t))\right]}{\pi\sqrt{t_0^2-t^2}} \mathcal{P}_{11}^+ F_{Pf}^+, \\ \nu_{PfPf,y}^+(x,y,t) = \dfrac{\Im m\left[{\kappa_{Pf}^+}^2(\upsilon(t))\mathcal{R}_{PfPf}(\upsilon(t))\right]}{\pi\sqrt{t_0^2-t^2}} \mathcal{P}_{11}^+ F_{Pf}^+, \end{cases} \quad \textit{if } t_h < t \le t_0 \textit{ and } \frac{x}{r} > \frac{V_{Pf}^+}{V_{\max}}$$

$$\left\{\begin{aligned} \nu^+_{PfPf,x}(x,y,t) &= -\frac{\Re e\left[\mathrm{i}\,\gamma(t)\kappa^+_{Pf}(\gamma(t))\mathcal{R}_{PfPf}(\gamma(t))\right]}{\pi\sqrt{t_0^2-t^2}}\mathcal{P}^+_{11}F^+_{Pf}, \\ \nu^+_{PfPf,y}(x,y,t) &= -\frac{\Re e\left[{\kappa^+_{Pf}}^2(\gamma(t))\mathcal{R}_{PfPf}(\gamma(t))\right]}{\pi\sqrt{t^2-t_0^2}}\mathcal{P}^+_{11}F^+_{Pf}, \end{aligned}\right. \qquad \textit{if } t > t_0$$

$$\boldsymbol{\nu}^+_{Pf}(x,y,t) = 0. \qquad \textit{else}$$

We set here $r = (x^2 + (y+h)^2)^{1/2}$ *and* $t_0 = r/V^+_{Pf}$ *denotes the arrival time of the reflected* $PfPf$ *volume wave,*

$$t_h = (y+h)\sqrt{\frac{1}{{V^+_{Pf}}^2} - \frac{1}{V^2_{\max}}} + \frac{|x|}{V_{\max}}$$

denotes the arrival time of the reflected $PfPf$ *head wave and the complex functions* $\upsilon := \upsilon(t)$ *and* $\gamma := \gamma(t)$ *are defined by*

$$\left\{\begin{aligned} \upsilon(t) &= -\mathrm{i}\left(\frac{y+h}{r}\sqrt{\frac{1}{{V^+_{Pf}}^2} - \frac{t^2}{r^2}} + \frac{xt}{r^2}\right) \qquad \textit{for } t_h < t \leq t_0 \textit{ and } x < 0, \\ \upsilon(t) &= \mathrm{i}\left(\frac{y+h}{r}\sqrt{\frac{1}{{V^+_{Pf}}^2} - \frac{t^2}{r^2}} - \frac{xt}{r^2}\right) \qquad \textit{for } t_h < t \leq t_0 \textit{ and } x \geq 0, \end{aligned}\right.$$

and

$$\gamma(t) = -\mathrm{i}\,\frac{x}{r^2}t + \frac{y+h}{r}\sqrt{\frac{t^2}{r^2} - \frac{1}{{V^+_{Pf}}^2}} \quad \textit{for } t > t_0.$$

- $\boldsymbol{\nu}^+_{PfPs}$ *is the velocity of the reflected* $PfPs$ *wave and satisfies:*

$$\left\{\begin{aligned} \nu^+_{PfPs,x}(x,y,t) &= -\frac{\mathcal{P}^+_{12}F^+_{Pf}}{\pi}\Re e\left[\mathrm{i}\,\upsilon(t)\mathcal{R}_{PfPs}(\upsilon(t))\frac{d\upsilon}{dt}(t)\right], \\ \nu^+_{PfPs,y}(x,y,t) &= -\frac{\mathcal{P}^+_{12}F^+_{Pf}}{\pi}\Re e\left[\kappa^+_{Ps}(\upsilon(t))\mathcal{R}_{PfPs}(\upsilon(t))\frac{d\upsilon}{dt}(t)\right], \end{aligned}\right. \qquad \begin{aligned} &\textit{if } t_h < t \leq t_0 \\ &\textit{and } |\Im m\,(\upsilon(t_0))| > \frac{1}{V_{\max}} \end{aligned}$$

$$\left\{\begin{aligned} \nu^+_{PfPs,x}(x,y,t) &= -\frac{\mathcal{P}^+_{12}F^+_{Pf}}{\pi}\Re e\left[\mathrm{i}\,\gamma(t)\mathcal{R}_{PfPs}(\gamma(t))\frac{d\gamma}{dt}(t)\right], \\ \nu^+_{PfPs,y}(x,y,t) &= -\frac{\mathcal{P}^+_{12}F^+_{Pf}}{\pi}\Re e\left[\kappa^+_{Ps}(\gamma(t))\mathcal{R}_{PfPs}(\gamma(t))\frac{d\gamma}{dt}(t)\right], \end{aligned}\right. \qquad \textit{if } t > t_0$$

$$\boldsymbol{\nu}^+_{PfPs}(x,y,t) = 0. \qquad \textit{else}$$

Here t_0 *denotes the arrival time of the reflected* $PfPs$ *wave (its calculation is similar to the calculation of the arrival time of the transmitted wave, see the appendix of [8])* *and t_h denotes the arrival time of the $PfPs$ head wave:*

$$t_h = y\sqrt{\frac{1}{{V_{Ps}^{+}}^2} - \frac{1}{V_{\max}^2}} + h\sqrt{\frac{1}{{V_{Pf}^{+}}^2} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}}.$$

For $t_h < t \leq t_0$, the function $\upsilon(t)$ is implicitly defined as the only root of

$$q \in \mathbb{C} \mapsto \mathcal{F}(q,t) = y\left(\frac{1}{{V_{Ps}^{+}}^2} + q^2\right)^{1/2} + h\left(\frac{1}{{V_{Pf}^{+}}^2} + q^2\right)^{1/2} + iqx - t,$$

such that $\Im m\left(x\frac{d\upsilon(t)}{dt}\right) \leq 0$.
For $t > t_0$, the function $\gamma(t)$ is defined as the only root of $q \in \mathbb{C} \mapsto \mathcal{F}(q,t)$ whose real part is positive.

- $\boldsymbol{\nu}_{PfS}^{+}$ *is the velocity of the reflected PfS wave and satisfies:*

$$\begin{cases} \nu_{PfS,x}^{+}(x,y,t) = -\dfrac{F_{Pf}^{+}}{\pi}\Re e\left[\kappa_S^{+}(\upsilon(t))\mathcal{R}_{PfS}(\upsilon(t))\dfrac{d\upsilon}{dt}(t)\right], \\ \nu_{PfS,y}^{+}(x,y,t) = \dfrac{F_{Pf}^{+}}{\pi}\Re e\left[\mathrm{i}\,\upsilon(t)\mathcal{R}_{PfS}(\upsilon(t))\dfrac{d\upsilon}{dt}(t)\right], \end{cases} \qquad \textit{if } t_h < t \leq t_0 \textit{ and } |\Im m\,(\upsilon(t_0))| > \frac{1}{V_{\max}}$$

$$\begin{cases} \nu_{PfS,x}^{+}(x,y,t) = -\dfrac{F_{Pf}^{+}}{\pi}\Re e\left[\kappa_S^{+}(\gamma(t))\mathcal{R}_{PfS}(\gamma(t))\dfrac{d\gamma}{dt}(t)\right], \\ \nu_{PfS,y}^{+}(x,y,t) = \dfrac{F_{Pf}^{+}}{\pi}\Re e\left[\mathrm{i}\,\gamma(t)\mathcal{R}_{PfS}(\gamma(t))\dfrac{d\gamma}{dt}(t)\right], \end{cases} \qquad \textit{if } t > t_0$$

$$\boldsymbol{\nu}_{PfS}^{+}(x,y,t) = 0. \qquad \textit{else}$$

Here t_0 denotes the arrival time of the reflected PfS wave and t_h denotes the arrival time of the reflected PfS head wave:

$$t_h = y\sqrt{\frac{1}{{V_{S}^{+}}^2} - \frac{1}{V_{\max}^2}} + h\sqrt{\frac{1}{{V_{Pf}^{+}}^2} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}}.$$

For $t_h < t \leq t_0$, the function $\upsilon(t)$ is implicitly defined as the only root of

$$q \in \mathbb{C} \mapsto \mathcal{F}(q,t) = y\left(\frac{1}{{V_{S}^{+}}^2} + q^2\right)^{1/2} + h\left(\frac{1}{{V_{Pf}^{+}}^2} + q^2\right)^{1/2} + iqx - t,$$

such that $\Im m\left(x\frac{d\upsilon(t)}{dt}\right) \leq 0$.
For $t > t_0$, the function $\gamma(t)$ is defined as the only root of $q \in \mathbb{C} \mapsto \mathcal{F}(q,t)$ whose real part is positive.

- $\boldsymbol{\nu}^+_{PsPf}$ *is the velocity of the reflected $PsPf$ wave and satisfies:*

$$\begin{cases} \nu^+_{PsPf,x}(x,y,t) = -\dfrac{\mathcal{P}^+_{11}F^+_{Ps}}{\pi}\Re e\left[\mathrm{i}\,\upsilon(t)\mathcal{R}_{PsPf}(\upsilon(t))\dfrac{d\upsilon}{dt}(t)\right], \\ \nu^+_{PsPf,y}(x,y,t) = -\dfrac{\mathcal{P}^+_{11}F^+_{Ps}}{\pi}\Re e\left[\kappa^+_{Pf}(\upsilon(t))\mathcal{R}_{PsPf}(\upsilon(t))\dfrac{d\upsilon}{dt}(t)\right], \end{cases} \quad \text{if } t_h < t \leq t_0 \text{ and } |\Im m\,(\upsilon(t_0))| > \frac{1}{V_{\max}}$$

$$\begin{cases} \nu^+_{PsPf,x}(x,y,t) = -\dfrac{\mathcal{P}^+_{11}F^+_{Ps}}{\pi}\Re e\left[\mathrm{i}\,\gamma(t)\mathcal{R}_{PsPf}(\gamma(t))\dfrac{d\gamma}{dt}(t)\right], \\ \nu^+_{PsPf,y}(x,y,t) = -\dfrac{\mathcal{P}^+_{11}F^+_{Ps}}{\pi}\Re e\left[\kappa^+_{Pf}(\gamma(t))\mathcal{R}_{PsPf}(\gamma(t))\dfrac{d\gamma}{dt}(t)\right], \end{cases} \quad \text{if } t > t_0$$

$$\boldsymbol{\nu}^+_{PsPf}(x,y,t) = 0. \qquad \text{else}$$

Here t_0 denotes the arrival time of the reflected $PsPf$ wave and t_h denotes the arrival time of the reflected $PsPf$ head wave:

$$t_h = y\sqrt{\frac{1}{{V^+_{Pf}}^2} - \frac{1}{V^2_{\max}}} + h\sqrt{\frac{1}{{V^+_{Ps}}^2} - \frac{1}{V^2_{\max}}} + \frac{|x|}{V_{\max}}.$$

For $t_h < t \leq t_0$, the function $\upsilon(t)$ is implicitly defined as the only root of

$$q \in \mathbb{C} \mapsto \mathcal{F}(q,t) = y\left(\frac{1}{{V^+_{Pf}}^2} + q^2\right)^{1/2} + h\left(\frac{1}{{V^+_{Ps}}^2} + q^2\right)^{1/2} + iqx - t,$$

such that $\Im m\left(x\frac{d\upsilon(t)}{dt}\right) \leq 0$.
For $t > t_0$, the function $\gamma(t)$ is defined as the only root of $q \in \mathbb{C} \mapsto \mathcal{F}(q,t)$ whose real part is positive.

- $\boldsymbol{\nu}^+_{PsPs}$ *is the velocity of the reflected $PsPs$ wave and satisfies:*

$$\begin{cases} \nu^+_{PsPs,x}(x,y,t) = \dfrac{\Im m\left[\mathrm{i}\,\upsilon(t)\kappa^+_{Pf}(\upsilon(t))\mathcal{R}_{PsPs}(\upsilon(t))\right]}{\pi\sqrt{t_0^2 - t^2}}\mathcal{P}^+_{12}F^+_{Ps}, \\ \nu^+_{PsPs,y}(x,y,t) = \dfrac{\Im m\left[{\kappa^+_{Ps}}^2(\upsilon(t))\mathcal{R}_{PsPs}(\upsilon(t))\right]}{\pi\sqrt{t_0^2 - t^2}}\mathcal{P}^+_{12}F^+_{Ps}, \end{cases} \quad \text{if } t_h < t \leq t_0 \text{ and } \frac{x}{r} > \frac{V^+_{Ps}}{V_{\max}}$$

$$\begin{cases} \nu^+_{PsPs,x}(x,y,t) = -\dfrac{\Re e\left[\mathrm{i}\,\gamma(t)\kappa^+_{Ps}(\gamma(t))\mathcal{R}_{PsPs}(\gamma(t))\right]}{\pi\sqrt{t_0^2 - t^2}}\mathcal{P}^+_{12}F^+_{Ps}, \\ \nu^+_{PsPs,y}(x,y,t) = -\dfrac{\Re e\left[{\kappa^+_{Ps}}^2(\gamma(t))\mathcal{R}_{PsPs}(\gamma(t))\right]}{\pi\sqrt{t^2 - t_0^2}}\mathcal{P}^+_{12}F^+_{Ps}, \end{cases} \quad \text{if } t > t_0$$

$$\boldsymbol{\nu}^+_{Ps}(x,y,t) = 0. \qquad \text{else}$$

We set here $r = (x^2 + (y+h)^2)^{1/2}$ *and* $t_0 = r/V_{Ps}^{+}$ *denotes the arrival time of the reflected* $PsPs$ *volume wave,*

$$t_h = (y+h)\sqrt{\frac{1}{{V_{Pf}^{+}}^2} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}}$$

denotes the arrival time of the reflected $PsPs$ *head wave and the complex functions* $\upsilon := \upsilon(t)$ *and* $\gamma := \gamma(t)$ *are defined by*

$$\begin{cases} \upsilon(t) = -\mathrm{i}\left(\dfrac{y+h}{r}\sqrt{\dfrac{1}{{V_{Ps}^{+}}^2} - \dfrac{t^2}{r^2}} + \dfrac{xt}{r^2}\right) & \text{for } t_h < t \leq t_0 \text{ and } x < 0, \\ \upsilon(t) = \mathrm{i}\left(\dfrac{y+h}{r}\sqrt{\dfrac{1}{{V_{Ps}^{+}}^2} - \dfrac{t^2}{r^2}} - \dfrac{xt}{r^2}\right) & \text{for } t_h < t \leq t_0 \text{ and } x \geq 0, \end{cases}$$

and

$$\gamma(t) = -\mathrm{i}\,\frac{x}{r^2}t + \frac{y+h}{r}\sqrt{\frac{t^2}{r^2} - \frac{1}{{V_{Ps}^{+}}^2}} \quad \text{for } t > t_0.$$

- $\boldsymbol{\nu}_{PsS}^{+}$ *is the velocity of the reflected* PsS *wave and satisfies:*

$$\begin{cases} \nu_{PsS,x}^{+}(x,y,t) = -\dfrac{F_{Ps}^{+}}{\pi}\Re e\left[\kappa_S^{+}(\upsilon(t))\mathcal{R}_{PsS}(\upsilon(t))\dfrac{d\upsilon}{dt}(t)\right], \\ \nu_{PsS,y}^{+}(x,y,t) = \dfrac{F_{Ps}^{+}}{\pi}\Re e\left[\mathrm{i}\,\upsilon(t)\mathcal{R}_{PsS}(\upsilon(t))\dfrac{d\upsilon}{dt}(t)\right], \end{cases} \quad \text{if } t_h < t \leq t_0 \text{ and } |\Im m\,(\upsilon(t_0))| > \frac{1}{V_{\max}}$$

$$\begin{cases} \nu_{PsS,x}^{+}(x,y,t) = -\dfrac{F_{Ps}^{+}}{\pi}\Re e\left[\kappa_S^{+}(\gamma(t))\mathcal{R}_{PsS}(\gamma(t))\dfrac{d\gamma}{dt}(t)\right], \\ \nu_{PsS,y}^{+}(x,y,t) = \dfrac{F_{Ps}^{+}}{\pi}\Re e\left[\mathrm{i}\,\gamma(t)\mathcal{R}_{PsS}(\gamma(t))\dfrac{d\gamma}{dt}(t)\right], \end{cases} \quad \text{if } t > t_0$$

$$\boldsymbol{\nu}_{PsS}^{+}(x,y,t) = 0. \qquad \text{else}$$

Here t_0 *denotes the arrival time of the reflected* PsS *wave and* t_h *denotes the arrival time of the reflected* PsS *head wave:*

$$t_h = y\sqrt{\frac{1}{{V_S^{+}}^2} - \frac{1}{V_{\max}^2}} + h\sqrt{\frac{1}{{V_{Ps}^{+}}^2} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}}.$$

For $t_h < t \leq t_0$*, the function* $\upsilon(t)$ *is implicitly defined as the only root of*

$$q \in \mathbb{C} \mapsto \mathcal{F}(q,t) = y\left(\frac{1}{{V_S^{+}}^2} + q^2\right)^{1/2} + h\left(\frac{1}{{V_{Ps}^{+}}^2} + q^2\right)^{1/2} + iqx - t,$$

such that $\Im m\left(x\frac{d\upsilon(t)}{dt}\right) \leq 0$.
For $t > t_0$*, the function* $\gamma(t)$ *is defined as the only root of* $q \in \mathbb{C} \mapsto \mathcal{F}(q,t)$ *whose real part is positive.*

- $\boldsymbol{\nu}^-_{PfPf}$ *is the velocity of the transmitted $PfPf$ wave (the Pf transmitted wave generated by the Pf incident wave) and satisfies:*

$$\begin{cases} \nu^-_{PfPf,x}(x,y,t) = -\dfrac{\mathcal{P}^-_{11}F^+_{Pf}}{\pi}\Re e\left[\mathrm{i}\,\upsilon(t)\mathcal{T}_{PfPf}(\upsilon(t))\dfrac{d\upsilon}{dt}(t)\right], \\ \\ \nu^-_{PfPf,y}(x,y,t) = \dfrac{\mathcal{P}^-_{11}F^+_{Pf}}{\pi}\Re e\left[\kappa^-_{Pf}(\upsilon(t))\mathcal{T}_{PfPf}(\upsilon(t))\dfrac{d\upsilon}{dt}(t)\right], \end{cases} \quad \textit{if } t_h < t \le t_0 \textit{ and } |\Im m\,(\upsilon(t_0))| > \frac{1}{V_{\max}}$$

$$\begin{cases} \nu^-_{PfPf,x}(x,y,t) = -\dfrac{\mathcal{P}^-_{11}F^+_{Pf}}{\pi}\Re e\left[\mathrm{i}\,\gamma(t)\mathcal{T}_{PfPf}(\gamma(t))\dfrac{d\gamma}{dt}(t)\right], \\ \\ \nu^-_{PfPf,y}(x,y,t) = \dfrac{\mathcal{P}^-_{11}F^+_{Pf}}{\pi}\Re e\left[\kappa^-_{Pf}(\gamma(t))\mathcal{T}_{PfPf}(\gamma(t))\dfrac{d\gamma}{dt}(t)\right], \end{cases} \quad \textit{if } t > t_0$$

$$\boldsymbol{\nu}^-_{PfPf}(x,y,t) = 0. \quad \textit{else}$$

Here t_0 denotes the arrival time of the transmitted $PfPf$ wave and t_h denotes the arrival time of the transmitted $PfPf$ head wave:

$$t_h = -y\sqrt{\frac{1}{{V^-_{Pf}}^2} - \frac{1}{V^2_{\max}}} + h\sqrt{\frac{1}{{V^+_{Pf}}^2} - \frac{1}{V^2_{\max}}} + \frac{|x|}{V_{\max}}.$$

For $t_h < t \le t_0$, the function $\upsilon(t)$ is implicitly defined as the only root of

$$q \in \mathbb{C} \mapsto \mathcal{F}(q,t) = -y\left(\frac{1}{{V^-_{Pf}}^2} + q^2\right)^{1/2} + h\left(\frac{1}{{V^+_{Pf}}^2} + q^2\right)^{1/2} + iqx - t,$$

such that $\Im m\left(x\frac{d\upsilon(t)}{dt}\right) \le 0$.
For $t > t_0$, the function $\gamma(t)$ is defined as the only root of $q \in \mathbb{C} \mapsto \mathcal{F}(q,t)$ whose real part is positive.

- $\boldsymbol{\nu}^-_{PfPs}$ *is the velocity of the transmitted $PfPs$ wave and satisfies:*

$$\begin{cases} \nu^-_{PfPs,x}(x,y,t) = -\dfrac{\mathcal{P}^-_{12}F^+_{Pf}}{\pi}\Re e\left[\mathrm{i}\,\upsilon(t)\mathcal{T}_{PfPs}(\upsilon(t))\dfrac{d\upsilon}{dt}(t)\right], \\ \\ \nu^-_{PfPs,y}(x,y,t) = \dfrac{\mathcal{P}^-_{12}F^+_{Pf}}{\pi}\Re e\left[\kappa^-_{Ps}(\upsilon(t))\mathcal{T}_{PfPs}(\upsilon(t))\dfrac{d\upsilon}{dt}(t)\right], \end{cases} \quad \textit{if } t_h < t \le t_0 \textit{ and } |\Im m\,(\upsilon(t_0))| > \frac{1}{V_{\max}}$$

$$\begin{cases} \nu^-_{PfPs,x}(x,y,t) = -\dfrac{\mathcal{P}^-_{12}F^+_{Pf}}{\pi}\Re e\left[\mathrm{i}\,\gamma(t)\mathcal{T}_{PfPs}(\gamma(t))\dfrac{d\gamma}{dt}(t)\right], \\ \\ \nu^-_{PfPs,y}(x,y,t) = \dfrac{\mathcal{P}^-_{12}F^+_{Pf}}{\pi}\Re e\left[\kappa^-_{Ps}(\gamma(t))\mathcal{T}_{PfPs}(\gamma(t))\dfrac{d\gamma}{dt}(t)\right], \end{cases} \quad \textit{if } t > t_0$$

$$\boldsymbol{\nu}^-_{PfPs}(x,y,t) = 0. \quad \textit{else}$$

Here t_0 denotes the arrival time of the transmitted $PfPs$ wave and t_h denotes the arrival time of the transmitted $PfPs$ head wave:

$$t_h = -y\sqrt{\frac{1}{V_{Ps}^{-\,2}} - \frac{1}{V_{\max}^2}} + h\sqrt{\frac{1}{V_{Pf}^{+\,2}} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}}.$$

For $t_h < t \le t_0$, the function $\upsilon(t)$ is implicitly defined as the only root of

$$q \in \mathbb{C} \mapsto \mathcal{F}(q,t) = -y\left(\frac{1}{V_{Ps}^{-\,2}} + q^2\right)^{1/2} + h\left(\frac{1}{V_{Pf}^{+\,2}} + q^2\right)^{1/2} + iqx - t,$$

such that $\Im m\left(x\frac{d\upsilon(t)}{dt}\right) \le 0.$
For $t > t_0$, the function $\gamma(t)$ is defined as the only root of $q \in \mathbb{C} \mapsto \mathcal{F}(q,t)$ whose real part is positive.

- $\boldsymbol{\nu}_{PfS}^{-}$ *is the velocity of the transmitted PfS wave and satisfies:*

$$\left\{\begin{array}{l} \nu_{PfS,x}^{-}(x,y,t) = \dfrac{F_{Pf}^{+}}{\pi}\Re e\left[\kappa_S^{-}(\upsilon(t))\mathcal{T}_{PfS}(\upsilon(t))\dfrac{d\upsilon}{dt}(t)\right], \\[2ex] \nu_{PfS,y}^{-}(x,y,t) = \dfrac{F_{Pf}^{+}}{\pi}\Re e\left[\mathrm{i}\,\upsilon(t)\mathcal{T}_{PfS}(\upsilon(t))\dfrac{d\upsilon}{dt}(t)\right], \end{array}\right. \quad \begin{array}{l} \textit{if } t_h < t \le t_0 \\ \textit{and } |\Im m\,(\upsilon(t_0))| > \dfrac{1}{V_{\max}} \end{array}$$

$$\left\{\begin{array}{l} \nu_{PfS,x}^{-}(x,y,t) = \dfrac{F_{Pf}^{+}}{\pi}\Re e\left[\kappa_S^{-}(\gamma(t))\mathcal{T}_{PfS}(\gamma(t))\dfrac{d\gamma}{dt}(t)\right], \\[2ex] \nu_{PfS,y}^{-}(x,y,t) = \dfrac{\mathcal{F}_{Pf}^{+}}{\pi}\Re e\left[\mathrm{i}\,\gamma(t)\mathcal{T}_{PfS}(\gamma(t))\dfrac{d\gamma}{dt}(t)\right], \end{array}\right. \quad \textit{if } t > t_0$$

$$\boldsymbol{\nu}_{PfS}^{-}(x,y,t) = 0. \qquad \textit{else}$$

Here t_0 denotes the arrival time of the transmitted PfS wave and t_h denotes the arrival time of the transmitted PfS head wave:

$$t_h = -y\sqrt{\frac{1}{V_S^{-\,2}} - \frac{1}{V_{\max}^2}} + h\sqrt{\frac{1}{V_{Pf}^{+\,2}} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}}.$$

For $t_h < t \le t_0$, the function $\upsilon(t)$ is implicitly defined as the only root of

$$q \in \mathbb{C} \mapsto \mathcal{F}(q,t) = -y\left(\frac{1}{V_S^{-\,2}} + q^2\right)^{1/2} + h\left(\frac{1}{V_{Pf}^{+\,2}} + q^2\right)^{1/2} + iqx - t,$$

such that $\Im m\left(x\frac{d\upsilon(t)}{dt}\right) \le 0.$
For $t > t_0$, the function $\gamma(t)$ is defined as the only root of $q \in \mathbb{C} \mapsto \mathcal{F}(q,t)$ whose real part is positive.

- $\boldsymbol{\nu}^-_{PsPf}$ *is the velocity of the transmitted* $PsPf$ *wave and satisfies:*

$$\begin{cases} \nu^-_{PsPf,x}(x,y,t) = -\dfrac{\mathcal{P}^-_{11}F^+_{Ps}}{\pi}\Re e\left[\mathrm{i}\,\upsilon(t)\mathcal{T}_{PsPf}(\upsilon(t))\dfrac{d\upsilon}{dt}(t)\right], \\ \nu^-_{PsPf,y}(x,y,t) = \dfrac{\mathcal{P}^-_{11}F^+_{Ps}}{\pi}\Re e\left[\kappa^-_{Pf}(\upsilon(t))\mathcal{T}_{PsPf}(\upsilon(t))\dfrac{d\upsilon}{dt}(t)\right], \end{cases} \quad \textit{if } t_h < t \leq t_0 \textit{ and } |\Im m\,(\upsilon(t_0))| > \frac{1}{V_{\max}}$$

$$\begin{cases} \nu^-_{PsPf,x}(x,y,t) = -\dfrac{\mathcal{P}^-_{11}F^+_{Ps}}{\pi}\Re e\left[\mathrm{i}\,\gamma(t)\mathcal{T}_{PsPf}(\gamma(t))\dfrac{d\gamma}{dt}(t)\right], \\ \nu^-_{PsPf,y}(x,y,t) = \dfrac{\mathcal{P}^-_{11}F^+_{Ps}}{\pi}\Re e\left[\kappa^-_{Pf}(\gamma(t))\mathcal{T}_{PsPf}(\gamma(t))\dfrac{d\gamma}{dt}(t)\right], \end{cases} \quad \textit{if } t > t_0$$

$$\boldsymbol{\nu}^-_{PsPf}(x,y,t) = 0. \qquad \textit{else}$$

Here t_0 *denotes the arrival time of the transmitted* $PsPf$ *wave and* t_h *denotes the arrival time of the transmitted* $PsPf$ *head wave:*

$$t_h = -y\sqrt{\frac{1}{{V^-_{Pf}}^2} - \frac{1}{V^2_{\max}}} + h\sqrt{\frac{1}{{V^+_{Ps}}^2} - \frac{1}{V^2_{\max}}} + \frac{|x|}{V_{\max}}.$$

For $t_h < t \leq t_0$*, the function* $\upsilon(t)$ *is implicitly defined as the only root of*

$$q \in \mathbb{C} \mapsto \mathcal{F}(q,t) = -y\left(\frac{1}{{V^-_{Pf}}^2} + q^2\right)^{1/2} + h\left(\frac{1}{{V^+_{Ps}}^2} + q^2\right)^{1/2} + iqx - t,$$

such that $\Im m\left(x\frac{d\upsilon(t)}{dt}\right) \leq 0$.
For $t > t_0$*, the function* $\gamma(t)$ *is defined as the only root of* $q \in \mathbb{C} \mapsto \mathcal{F}(q,t)$ *whose real part is positive.*

- $\boldsymbol{\nu}^-_{PsPs}$ *is the velocity of the transmitted* $PsPs$ *wave and satisfies:*

$$\begin{cases} \nu^-_{PsPs,x}(x,y,t) = -\dfrac{\mathcal{P}^-_{12}F^+_{Ps}}{\pi}\Re e\left[\mathrm{i}\,\upsilon(t)\mathcal{T}_{PsPs}(\upsilon(t))\dfrac{d\upsilon}{dt}(t)\right], \\ \nu^-_{PsPs,y}(x,y,t) = \dfrac{\mathcal{P}^-_{12}F^+_{Ps}}{\pi}\Re e\left[\kappa^-_{Ps}(\upsilon(t))\mathcal{T}_{PsPs}(\upsilon(t))\dfrac{d\upsilon}{dt}(t)\right], \end{cases} \quad \textit{if } t_h < t \leq t_0 \textit{ and } |\Im m\,(\upsilon(t_0))| > \frac{1}{V_{\max}}$$

$$\begin{cases} \nu^-_{PsPs,x}(x,y,t) = -\dfrac{\mathcal{P}^-_{12}F^+_{Ps}}{\pi}\Re e\left[\mathrm{i}\,\gamma(t)\mathcal{T}_{PsPs}(\gamma(t))\dfrac{d\gamma}{dt}(t)\right], \\ \nu^-_{PsPs,y}(x,y,t) = \dfrac{\mathcal{P}^-_{12}F^+_{Ps}}{\pi}\Re e\left[\kappa^-_{Ps}(\gamma(t))\mathcal{T}_{PsPs}(\gamma(t))\dfrac{d\gamma}{dt}(t)\right], \end{cases} \quad \textit{if } t > t_0$$

$$\boldsymbol{\nu}^-_{PsPs}(x,y,t) = 0. \qquad \textit{else}$$

Here t_0 denotes the arrival time of the transmitted $PsPs$ wave and t_h denotes the arrival time of the transmitted $PsPs$ head wave:

$$t_h = -y\sqrt{\frac{1}{{V_{Ps}^-}^2} - \frac{1}{V_{\max}^2}} + h\sqrt{\frac{1}{{V_{Ps}^+}^2} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}}.$$

For $t_h < t \leq t_0$, the function $\upsilon(t)$ is implicitly defined as the only root of

$$q \in \mathbb{C} \mapsto \mathcal{F}(q,t) = -y\left(\frac{1}{{V_{Ps}^-}^2} + q^2\right)^{1/2} + h\left(\frac{1}{{V_{Ps}^+}^2} + q^2\right)^{1/2} + iqx - t,$$

such that $\Im m\left(x\frac{d\upsilon(t)}{dt}\right) \leq 0.$
For $t > t_0$, the function $\gamma(t)$ is defined as the only root of $q \in \mathbb{C} \mapsto \mathcal{F}(q,t)$ whose real part is positive.

- $\boldsymbol{\nu}_{PsS}^-$ *is the velocity of the transmitted PsS wave and satisfies:*

$$\begin{cases} \nu_{PsS,x}^-(x,y,t) = \dfrac{F_{Ps}^+}{\pi}\Re e\left[\kappa_S^-(\upsilon(t))\mathcal{T}_{PsS}(\upsilon(t))\dfrac{d\upsilon}{dt}(t)\right], \\ \nu_{PsS,y}^-(x,y,t) = \dfrac{F_{Ps}^+}{\pi}\Re e\left[\mathrm{i}\,\upsilon(t)\mathcal{T}_{PsS}(\upsilon(t))\dfrac{d\upsilon}{dt}(t)\right], \end{cases} \quad \textit{if } t_h < t \leq t_0 \textit{ and } |\Im m\,(\upsilon(t_0))| > \frac{1}{V_{\max}}$$

$$\begin{cases} \nu_{PsS,x}^-(x,y,t) = \dfrac{F_{Ps}^+}{\pi}\Re e\left[\kappa_S^-(\gamma(t))\mathcal{T}_{PsS}(\gamma(t))\dfrac{d\gamma}{dt}(t)\right], \\ \nu_{PsS,y}^-(x,y,t) = \dfrac{\mathcal{F}_{Ps}^+}{\pi}\Re e\left[\mathrm{i}\,\gamma(t)\mathcal{T}_{PsS}(\gamma(t))\dfrac{d\gamma}{dt}(t)\right], \end{cases} \quad \textit{if } t > t_0$$

$$\boldsymbol{\nu}_{PsS}^-(x,y,t) = 0. \qquad \textit{else}$$

Here t_0 denotes the arrival time of the transmitted PsS wave and t_h denotes the arrival time of the transmitted PsS head wave:

$$t_h = -y\sqrt{\frac{1}{{V_S^-}^2} - \frac{1}{V_{\max}^2}} + h\sqrt{\frac{1}{{V_{Ps}^+}^2} - \frac{1}{V_{\max}^2}} + \frac{|x|}{V_{\max}}.$$

For $t_h < t \leq t_0$, the function $\upsilon(t)$ is implicitly defined as the only root of

$$q \in \mathbb{C} \mapsto \mathcal{F}(q,t) = -y\left(\frac{1}{{V_S^-}^2} + q^2\right)^{1/2} + h\left(\frac{1}{{V_{Ps}^+}^2} + q^2\right)^{1/2} + iqx - t,$$

such that $\Im m\left(x\frac{d\upsilon(t)}{dt}\right) \leq 0.$
For $t > t_0$, the function $\gamma(t)$ is defined as the only root of $q \in \mathbb{C} \mapsto \mathcal{F}(q,t)$ whose real part is positive.

Remark 2.1. *For the practical computations of the displacement, we won't have to explicitly compute the primitive of the velocities ν, which would be rather tedious, since*

$$\left(\int_0^t \nu(\tau)\,d\tau\right) * f = \nu * \left(\int_0^t f(\tau)\,d\tau\right).$$

Therefore, we'll only have to compute the primitive of the source function f.

The proof of this theorem is similar to the one detailed in [8] for the computation of the analytical solution to the acoustic/poroelastic problem, therefore we won't detail it here.

3 Numerical illustration

To illustrate the use our results, we have compared our analytical solution to a numerical one obtained by C. Morency and J. Tromp [13]. We consider an two-layered poroelastic medium whose characteristic coefficients are

- the solid density: $\rho_s^+ = 2200$ kg/m^3 and $\rho_s^- = 2650$ kg/m^3;
- the fluid density: $\rho_f^+ = 950$ kg/m^3 and $\rho_f^- = 750$ kg/m^3 ;
- the porosity: $\phi^+ = 0.4$ and $\phi^- = 0.2$;
- the tortuosity: $a^+ = 2$ and $a^- = 2$;
- the solid bulk modulus: $K_s^+ = 6.9$ GPa and $K_s^- = 37$ GPa;
- the fluid bulk modulus: $K_f^+ = 2$ GPa and $K_f^- = 1.7$ GPa;
- the frame bulk modulus: $K_b^+ = 6.7$ GPa and $K_b^- = 2.2$ GPa;
- the frame shear modulus $\mu^+ = 3$ GPa and $\mu^- = 4.4$ GPa;

so that the celerity of the waves in the poroelastic medium are:

- for the fast P wave, $V_{Pf}^+ = 2692$ m/s and $V_{Pf}^- = 2535$ m/s;
- for the slow P wave, $V_{Ps}^+ = 1186$ m/s and $V_{Ps}^- = 744$ m/s;
- for the ψ wave, $V_S^+ = 1409$ m/s and $V_S^- = 1415$ m/s.

The source is located in the acoustic layer, at 500 m from the interface. We used two types of sources in space: the first one is a bulk source such that $f_u = f_w = -10^{10}$ and $f_p = 0$; the second one is a pressure source such that $f_u = f_w = 0$ and $f_p = 1$. In each case we used a fifth derivative of a Gaussian of dominant frequency $f_0 = 15$ Hz:

$$f(t) = 4\frac{\pi^2}{f_0^2}\left[9\left(t - \frac{1}{f_0}\right) + 4\frac{\pi^2}{f_0^2}\left(t - \frac{1}{f_0}\right)^3 - 4\frac{\pi^4}{f_0^4}\left(t - \frac{1}{f_0}\right)^5\right] e^{-\frac{\pi^2}{f_0^2}\left(t-\frac{1}{f_0}\right)^2}$$

for the source in time. We compute the solution at two receivers, the first one is in the upper layer, at 533 m from the interface; the second one is in the bottom layer, at 533 m from the interface; both are located on a vertical line at 400 m from the source (see Fig. 3). We represent the y component of the displacement from $t = 0$ to $t = 1$ s in Fig. 4 for the bulk source and in Fig. 5 for the pressure source. The left pictures represents the solution at receiver 1 while the right pictures represents the solution at receiver 2. On all the pictures the blue solid curve is the analytical solution and the red dashed curve is the numerical solution.

All the pictures show a good agreement between the two solutions, which validates the numerical code.

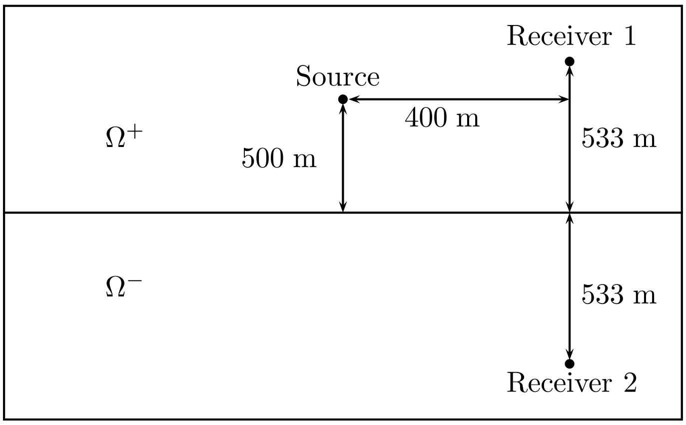

Figure 3: Configuration of the experiment

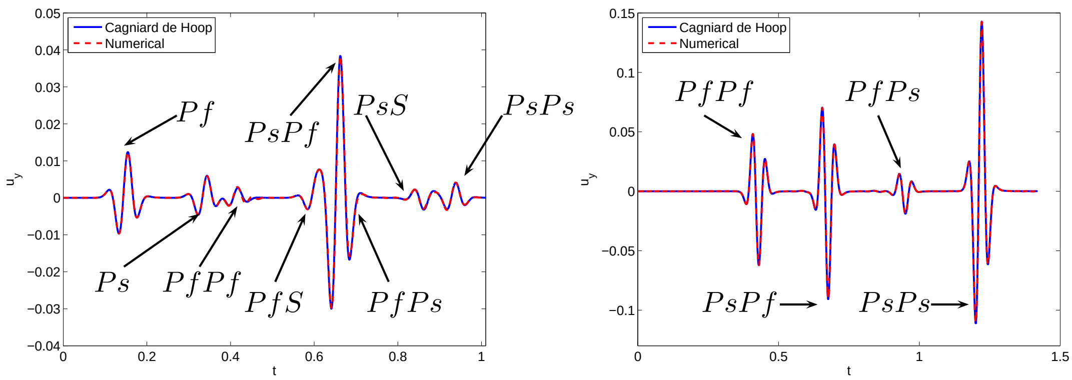

Figure 4: The y component of the displacement at receiver 1 (left picture) and 2 (right picture) in the case of a bulk source. The blue solid curve is the analytical solution computed by the Cagniard-de Hoop method, the red dashed curve is the numerical solution.

4 Conclusion

We provided the complete solution (reflected and transmitted wave) of the propagation of wave in a two-layered 2D poroelastic medium and we used it to validate a numerical code. In a forthcoming paper we will use this solution as a basis to derive the solution in a three dimensional medium.

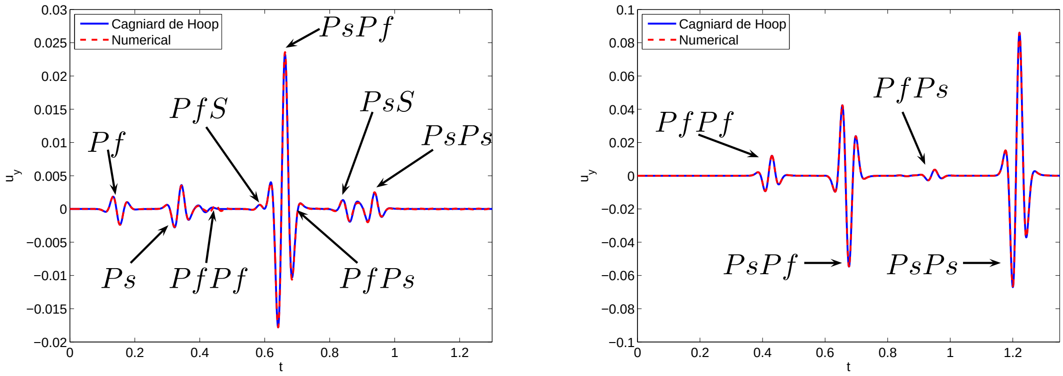

Figure 5: The y component of the displacement at receiver 1 (left picture) and 2 (right picture) in the case of a pressure source. The blue solid curve is the analytical solution computed by the Cagniard-de Hoop method, the red dashed curve is the numerical solution.

Acknowledgments

We thanks Christina Morency who provided us the numerical solutions we have used to validate our analytical solution.

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