Exact simulation of prices and greeks: application to CIR

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Abstract

We generalize the exact simulation algorithm of one dimensional solution of SDE proposed by Beskos et al. [6]. We apply Malliavin Calculus to simulate exactly the greeks, that is the derivatives with respect to the initial condition. We obtain estimations of these derivatives without time discretization or space discretization. We detail the method for the CIR process and give numerical results.

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1 Introduction

The simulation of diffusion is neither accurate nor efficient for a general form of Stochastic Differential Equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t.$$

Classical methods of approximation of the law of X requires some kind of discrete approximation. An Euler approximation of the SDE might be used

$$\bar{X}_{(k+1)\delta} \sim \bar{X}_{k\delta} + b(\bar{X}_{k\delta})\delta + \sqrt{\delta}\sigma(\bar{X}_{k\delta})\mathcal{N}(0,1),$$

where δ is the time step. In [3, 4], Bally and Talay prove the one order rate of convergence of the Euler approximation for $\mathbb{E}(f(X_T))$, where the coefficient b and σ are smooth and the function f is only assumed to be bounded and measurable. For particular equations, better schemes are available [14].

Beskos, Papaspiliopoulos and Roberts in [8] have proposed an exact simulation method to price options in a model with a factor following a one dimensional SDE with constant volatility.

$$dX_t = \alpha(X_t)dt + dB_t.$$

This supposes some hypothesis: $\alpha^2(x) + \alpha'(x)$ has to be bounded from below and does not go to $+\infty$ when x goes to $-\infty$ and x goes to $+\infty$. Their method is a rejection method. In this case, we have to wait a random time for the simulation of a fixed number of sample paths.

In Section 2, we first propose a small modification of their algorithm. It permits to reject faster the wrong trajectories. We also generalize their algorithm to an almost exact one not requiring all their assumptions. In Section 3, we give a control of the error obtained with this generalization.

In Section 4, we apply this method to positive Cox-ingersoll-Ross processes. The coefficient of the CIR are not Lipschtiz and classical results cannot be applied. Recently, many papers are published on the simulation of the CIR (see for instance [1, 5]).

In Section 5, we propose an unbiased scheme for the derivatives with respect to the initial value of the expectation (known as Deltas and Gammas in finance). The main tools are Malliavin Calculus and Beskos et al. method.

In Section 6, we give some numerical results for the CIR process.

2 Exact Simulation

2.1 Beskos-Papaspiliopoulos-Roberts Method

Beskos et al. [7, 6] proposed an exact simulation algorithm for one dimensional SDE with constant diffusion coefficient. Let us briefly recall here the main idea of their method. They can simulate the process X solution to the SDE:

$$dX_t = \alpha(X_t)dt + dB_t. \tag{1}$$

Thanks to Girsanov Theorem, they obtain

$$\mathbb{E}\left[f(X_t, t \in [0,T])\right] = \mathbb{E}\left[f(B_t, t \in [0,T]) \exp\left(\int_0^T \alpha(B_s)dB_s - \int_0^T \frac{\alpha^2(B_s)}{2}ds\right)\right]$$

where B_t is a one dimensional Brownian motion. Because of the dimension, they remove the stochastic integral:

$$\int_0^T \alpha(B_s)dB_s = A(B_T) - A(B_0) - \int_0^T \frac{\alpha'(B_s)}{2}ds,$$

where $A(x) = \int_0^x \alpha(y)dy$. Finally,

$$\mathbb{E}\left[f(X_t, t \in [0,T])\right] = \\ \mathbb{E}\left[f(B_T, t \in [0,T]) \exp\left(A(B_t) - A(B_0) - \int_0^T \frac{\alpha^2(B_t) + \alpha'(B_t)}{2} dt\right)\right].$$

In [8], the authors replace the Brownian motion B by a Brownian bridge $\tilde{B}$ with

$$\tilde{B}_T \sim C \exp\left(-\frac{(x-x_0)^2}{2T} + \int_0^x \alpha(y)dy\right) dx$$

(they assume that the function $\exp\left(-\frac{(x-x_0)^2}{2T} + \int_0^x \alpha(y)dy\right)$, $x \in \mathbb{R}$ is integrable).

They use the notation $\varphi(x) = \frac{1}{2}\left(\alpha^2(x) + \alpha'(x)\right)$. Assume first that φ is non-negative and $t \mapsto \varphi(\tilde{B}_t)$ is almost surely bounded.

$$\mathbb{E}\left[f(X_t, t \in [0,T])\right] = K\mathbb{E}\left[f(\tilde{B}_t, 0 \le t \le T) \exp\left(-\int_0^T \varphi(\tilde{B}_t)dt\right)\right] \quad (2)$$

This identity permits to simulate exactly the diffusion X with a rejection procedure. They simulate the Brownian bridge and accept the trajectory with probability $\exp\left(-\int_0^T \varphi(\tilde{B}_t)dt\right)$. Let us denote by $K(\omega)$ the upper bound, we can simulate a Poisson Point Process on $[0,T] \times [0, K(\omega)]$. The probability that there is no point of the Poisson Point Process in the domain

$$D(\omega) = \left\{(t,y) \in [0,T] \times [0, K(\omega)], y \le \varphi(\tilde{B}_t)\right\}$$

is exactly $\exp\left(-\int_0^T \varphi(\tilde{B}_t)dt\right)$. So, in order to accept a trajectory, we only need to know the value of the Brownian bridge at the times of the realization of the Poisson Point Process, that is at a finite number of times. A trajectory of Brownian bridge is accepted with probability

$$a = \mathbb{E}\left(\exp\left(-\int_0^T \varphi(\tilde{B}_t)dt\right)\right). \quad (3)$$

In their paper, Beskos et al. give the following algorithm:

Step 1 Simulate a random variable Y with density

$$h(u) = C_1 \exp\left(\int_0^u \alpha(y)dy - \frac{(u-x_0)^2}{2T}\right), \quad (4)$$

where C_1 is a normalization

Step 2 Simulate a Poisson Point Process with unit intensity on $[0,T] \times [0,K]$. The result is a random number n of points: $((t_1, z_1), \cdots, (t_n, z_n))$

Step 3 Simulate the Brownian Bridge at times $t_1, \cdots, t_n$

Step 4 If $\forall i \in 1, \cdots, n$, $\varphi(\tilde{B}_{t_i}) \leq z_i$, accept the trajectory. Else, return to Step 1

Remark 1. 1. If φ is no more supposed to be non-negative but only bounded bellow, we replace φ by $\varphi(x) - \inf_{y \in \mathbb{R}} \varphi(y)$.

2. In [6], the authors generalize this method to drift α such that at least one of the following condition is satisfied:

   (a) $\limsup_{x \to +\infty} \alpha^2(x) + \alpha'(x) < +\infty$

   (b) $\limsup_{x \to -\infty} \alpha^2(x) + \alpha'(x) < +\infty$

   Suppose for instance that the first condition is satisfied: they first simulate the minimum m of the Brownian Bridge $\tilde{B}$ and the instant θ where this minimum is realized. The law of $\tilde{B}$ conditioned by (m, θ) is known and the exact simulation of this process is available.

3. We can then observe that every Brownian trajectory has a positive probability of beeing accepted and therefore the method of simulation is efficient, it won't reject every trajectory.

2.2 Another way to simulate the Poisson Process

In order to decide if a Brownian bridge is accepted, we need only to know if (at most) one point of the Poisson Process is below the curve $((t, \varphi(\tilde{B}_t)),\ 0 \leq t \leq T)$. So we do not need to simulate the whole Poisson Process. We stop the simulation as soon as one point of the PP reject the trajectory. Sooner the rejection occurs, faster is the procedure. So, we propose to simulate the PP $(t_1, z_1), \cdots (t_n, z_n)$ with $z_1 \leq z_2 \leq \cdots \leq z_n$.

In practice, we generate a random variable z_1 with exponential law of parameter T ($\mathcal{E}(T)$) and t_1 has a uniform law on $[0, T]$. We simulate the Brownian bridge $\tilde{B}_{t_1}$ at time t_1.

- If $z_1 < \varphi(\tilde{B}_{t_1})$, we reject the simulation
- Else, we generate $z_2 - z_1 \sim \mathcal{E}(T)$, $t_2 \sim U(0, T)$, we simulate the Brownian bridge $\tilde{B}_{t_2}$ at time t_2, conditionally on $(\tilde{B}_0, \tilde{B}_{t_1}, \tilde{B}_T)$, and compare z_2 and $\varphi(\tilde{B}_{t_2})$, etc.

2.3 Extension

This method permits to start the algorithm even if $K(\omega) = \sup_{t \in [0,T]} \varphi(\tilde{B}_t)$ is not easy to estimate. Indeed, with our method, we only need $K(\omega)$ in order to decide to stop the simulation of the Poisson Point Process. Suppose that we have generated the n first points of the PPP, that is $(t_1, z_1), \cdots (t_n, z_n)$ with $z_1 \leq z_2 \leq \cdots \leq z_n$. If $z_n \geq K(\omega)$, we accept the trajectory and we obtain an exact realization of X.

If we don't know exactly how to compute $K(\omega)$ (or an upper bound), we can generalize our method. We generate the n first points $(t_1, z_1), \cdots (t_n, z_n)$ of the

Poisson Point Process. We suppose that we have not rejected the trajectory, that is the result is such that $\forall i = 1, \cdots, n$, $\varphi(\tilde{B}_{t_i}) < z_i$. In the next Section, we estimate the probability of accepting a wrong trajectory.

3 Control of the error

We suppose in this Section that we are not able to obtain exactly an upper bound of $K(\omega) = \sup_{0 \le t \le T} \varphi(\tilde{B}_t)$ and we simulate the Poisson Point Process with increasing ordinate z. If we decide to stop the procedure and to accept the trajectory, we need to estimate the probability of accepting a wrong trajectory (we say that we have missed a rejection). An upper bound of this probability is given by the probability that the maximum of the trajectory is greater than z_n:

$$\mathbb{P}\left(\sup_{t \in [0,T]} \varphi(\tilde{B}_t) > z_n \middle| \tilde{B}_{t_1}, \cdots, \tilde{B}_{t_n} \right)$$

In Section 3.1, we estimate the probability of missed rejection for a given path of Brownian Bridge. In Section 3.2, we decide to simulate the Poisson Point Process on $[0, T] \times [0, K]$ for a given K instead of the complete Poisson Point Process on $[0, T] \times \mathbb{R}_+$.

3.1 Estimation for a given trajectory

In order to give the result, we introduce some notations.

- the ordered times $0 = t_{(0)} < t_{(1)} < \cdots < t_{(n)} < t_{(n+1)} = T$
- the set $S(z_n)$

$$S(z_n) = \varphi^{-1}((z_n, +\infty)) = \{x \in \mathbb{R} \text{ s.t. } \varphi(x) > z_n\} \tag{5}$$

- the distance d between a point x and a set S: $d(x, S) = \inf_{y \in S} |x - y|$.

Theorem 1. *Let us assume that*

- *The trajectory has not been rejected by the n first points of the Poisson Point Process*
- *The points $\tilde{B}_{(t_i)}$, $i = 0, \cdots, n+1$ are in an unique connected set of the complement of $S(z_n)$:*

$$\left[\min_{i=0,\cdots,n+1} \tilde{B}_{(t_i)}, \max_{i=0,\cdots,n+1} \tilde{B}_{(t_i)} \right] \cap S(z_n) = \emptyset$$

.

The probability of missed rejection, that is the probability of accepting the trajectory that we have to reject is bounded by

$$2 \sum_{i=1}^{n+1} \exp\left(-2 \frac{d(\tilde{B}_{t_{(i)}}, S(z_n)) d(\tilde{B}_{t_{(i-1)}}, S(z_n))}{t_{(i)} - t_{(i-1)}} \right)$$

Proof. The realization of the Poisson Point Process on $[0,T]\times\mathbb{R}$ is a set of points $(t_i, z_i), i \geq 1$. The simulation has to be accepted if it satisfies the condition: $\forall i \in \mathbb{N}$, $\varphi\left(\tilde{B}_{t_i}\right) < z_i$.

We have chosen to simulate the PPP such that $\forall k > n$, $z_k > z_n$. So, we have the inclusion

$$\bigcap_{k>n} \left\{\varphi(\tilde{B}_{t_k}) > z_k\right\} \subset \left\{\sup_{t\in[0,T]} \varphi(\tilde{B}_t) > z_n\right\}$$

We introduce two particular extreme points of $S(z_n)$:

$$m_n = \sup\left\{x \in S(z_n) \text{ s.t. } x < \varphi(\tilde{B}_{(t_i)}) \text{ for } i = 0, \cdots, n+1\right\}$$
$$M_n = \inf\left\{x \in S(z_n) \text{ s.t. } x > \varphi(\tilde{B}_{(t_i)}) \text{ for } i = 0, \cdots, n+1\right\}$$

So, we obtain:

$$\mathbb{P}\left(\sup_{t\in[0,T]} \varphi(\tilde{B}_t) > z_n\right) \leq \sum_i \mathbb{P}\left(\sup_{t\in[t_{(i-1)},t_{(i)}]} \tilde{B}_t > M_n\right) + \mathbb{P}\left(\inf_{t\in[t_{(i-1)},t_{(i)}]} \tilde{B}_t < m_n\right)$$

We recall the law of the maximum of a Brownian bridge

$$\mathbb{P}\left(\sup_{0\leq s\leq t} B_s > y \middle| B_0 = x, B_t = z\right) = \exp\left(-\frac{2(y-x)(y-z)}{t}\right). \tag{6}$$

This ends the proof. □

Remark 2. a- The relevant quantity is $\beta_i = \dfrac{d(\tilde{B}_{t_{(i)}}, S(z_n))d(\tilde{B}_{t_{(i-1)}}, S(z_n))}{t_{(i)} - t_{(i-1)}}$. We introduce $\beta = \min_i \beta_i$. So, the upper bound in Theorem 1 can be replaced by $2(n+1)\exp(-2\beta)$. If β is not big enough (that is $\exp(-2\beta)$ is not small enough), we have to generate more points of the Poisson Point Process. In order to have a fast algorithm, we can decide to generate the Poisson Point Process on $(t_{(i-1)}, t_{(i)}) \times (z_n, +\infty)$ for i such that β_i is small.

b- Obviously, we also need to compute the probability conditioned by θ and $\tilde{B}_\theta = m(= \inf_{t\in[0,T]} \tilde{B}_t)$ if we have generated these random variables. In this case, we know that $(\tilde{B}_{\theta+t} - m, t \in [0, T-\theta])$ and $(\tilde{B}_{\theta-t} - m, t \in [0,\theta])$ are two independent Bessel Bridges starting from 0, that is the norm of a standard Brownian bridge of dimension 3. To simulate the Bessel Bridge $\bar{B}$ starting from 0 paths, we generate three independent Brownian bridges W^1, W^2 and W^3 and we use the identity $\bar{B}_t = \sqrt{(W_t^1)^2 + (W_t^2)^2 + (W_t^3)^2}$. So, we can deduce an upper bound of the probability of missed reject in this case.

3.2 Restriction of the domain for the Poisson Point Process

In this Section, we decide to simulate a Poisson Point Process with unit intensity on $[0,T] \times [0,K]$ instead of $[0,T] \times \mathbb{R}_+$. So, the probability of accepting a trajectory is $\mathbb{E}\left(\exp\left(-\int_0^T \varphi(\tilde{B}(t)) \wedge K dt\right)\right)$ instead of $\mathbb{E}\left(\exp\left(-\int_0^T \varphi(\tilde{B}(t))dt\right)\right)$.

As in Section 3.1, we introduce $S(K)$ (5). We assume that K is large enough such that $S(K) = (-\infty, m_K) \cup (M_K, +\infty)$ $(-\infty \leq m_K < x < M_K \leq +\infty)$.

Theorem 2. *We keep the notation, the realization of the Poisson Point Process on $[0,T] \times \mathbb{R}_+$ is a countable set of points (t_i, z_i) where $(z_i)_{i\in\mathbb{N}}$ is an increasing sequence. We denote by $A(K)$ the event of not rejecting the trajectory with the points such that $z_i \leq K$, that is*

$$A(K) = \bigcap_{z_i \leq K} \left\{ \varphi\left(\tilde{B}_{t_i}\right) < z_i \right\}$$

An upper bound of the probability of missed rejection is given by

$$\mathbb{P}\left(\bigcup_{x>K} A(x)^C | A(K) \right) \leq \mathbb{P}\left(Y > \frac{M_K}{2} \right) + \exp\left(-\frac{M_K(M_K - x)}{T} \right) + \mathbb{P}\left(Y < \frac{m_K}{2} \right) + \exp\left(-\frac{m_K(m_K - x)}{T} \right)$$

*where Y is the random variable generated at **Step 1** of the algorithm, with density given by (4).*

Proof. We again control the probability of rejection of a trajectory in $A(K)$ by the probability that it reaches m_K or M_K. Given $Y = u$, we use the law of the maximum of a Brownian bridge (6)

$$\mathbb{P}\left(\bigcup_{x>K} A(x)^C \middle| A(K) \right) = \frac{\mathbb{P}\left(\bigcup_{x>K} A(x)^C \cap A(K)\right)}{\mathbb{P}(A(K))} \leq \frac{\mathbb{P}\left(\bigcup_{x>K} A(x)^C\right)}{\mathbb{P}(A(\infty))}$$
$$\leq \frac{\int \mathbb{P}\left(\sup_{0\leq t\leq T} \tilde{B}_t > M_K \middle| Y = u \right) h(u)du}{\mathbb{P}(A(\infty))} + \frac{\int \mathbb{P}\left(\inf_{0\leq t\leq T} \tilde{B}_t < m_K \middle| Y = u \right) h(u)du}{\mathbb{P}(A(\infty))}$$

As noted in Remark 1 the denominator is strictly positive, $\mathbb{P}(A(\infty))$ is the global probability of acceptation of a Brownian trajectory.

Let us detail the inequality for the first term

$$\begin{aligned} I_1 &= \int \mathbb{P}\left(\sup_{0\leq t\leq T} \tilde{B}_t > M_K \middle| Y = u \right) h(u)du \\ &\leq \int_{\frac{M_K}{2}}^{\infty} h(u)du \\ &\quad + \int_{-\infty}^{\frac{M_K}{2}} \exp\left(-\frac{2(M_K - x)(M_K - u)}{T} \right) h(u)du \\ &\leq \mathbb{P}\left(Y > \frac{M_K}{2} \right) + \exp\left(-\frac{M_K(M_K - x)}{T} \right) \end{aligned}$$

We use a similar inequality for the minimum of a Brownian bridge to end the proof $\square$

Remark 3. 1. The rate of convergence of this method depends on the drift α of the diffusion. If $\lim_{x \pm \infty} \frac{\alpha(x)}{x} = 0$, we obtain a gaussian rate of convergence.

2. Another algorithm of simulation consists in keeping fixed the number n of generated points of the Poisson Point Process. This method permits to control the maximum time of simulation for a trajectory. We also are able to give a control of the error. (we integrate the last result in K against the law of the sum of z_n).

4 CIR Model: Details of the Algorithm

In this part, we apply our algorithm to the simulation of the Cox Ingersoll Ross Process (CIR), usually used to model short rate or volatility for stochastic volatility model on asset. This process is governed by

$$V_T = V_0 + \int_0^T \kappa \left(V_\infty - V_t\right) dt + \varepsilon \int_0^T \sqrt{V_t} dB_t \tag{7}$$

where κ, V_∞ and ε are constant. In dimension 1, a classical transform (Lamperti) permits to replace the volatility by a constant.

$$X_t = \frac{2\sqrt{V_t}}{\varepsilon} =: \eta(V_t)$$

The SDE satisfied by X is:

$$\begin{aligned} dX_t &= \eta'(V_t)dV_t + \frac{1}{2}\eta''(V_t)d\langle V\rangle_t \\ &= \frac{1}{\varepsilon\sqrt{V_t}}\left(\kappa(V_\infty - V_t)dt + \varepsilon\sqrt{V_t}dB_t\right) - \frac{\varepsilon^2 V_t}{4\varepsilon V_t^{3/2}}dt \\ &= \left(\frac{1}{X_t}\left(\frac{2\kappa V_\infty}{\varepsilon^2} - \frac{1}{2}\right) - \frac{\kappa X_t}{2}\right) dt + dB_t \end{aligned}$$

Thus X satisfies an SDE of type (1) with

$$\alpha(x) = \frac{1}{x}\left(\frac{2\kappa V_\infty}{\varepsilon^2} - \frac{1}{2}\right) - \frac{\kappa x}{2} \text{ for } x > 0.$$

Following Beskos et al. algorithm, we have first to be sure that $\varphi_1(x) = \frac{1}{2}(\alpha^2(x) + \alpha'(x))$ is bounded below.

$$\begin{aligned} \varphi_1(x) =& \frac{\alpha^2(x) + \alpha'(x)}{2} \\ =& \left(\left(\frac{2\kappa V_\infty}{\varepsilon^2} - 1\right)^2 - \frac{1}{4}\right)\frac{1}{2x^2} + \frac{\kappa^2}{8}x^2 - \frac{\kappa^2 V_\infty}{\varepsilon^2} \end{aligned}$$

This function is bounded below on $(0, +\infty)$ if and only if

$$\left(\frac{2\kappa V_\infty}{\varepsilon^2} - 1\right)^2 \geq \frac{1}{4}$$

This means that the degree d (defined as $d = \frac{4\kappa V_\infty}{\varepsilon^2}$) of the CIR has to satisfy:

$$d \in (0, 1] \cup [3, \infty).$$

In this paper, we limit our work to $d \geq 3$.

We follow the general framework to apply the Exact Simulation Algorithm

4.1 Final Value

In the first step, we have to generate a random variable Y with density:

$$h(x) = Rx^c \exp\left(-\frac{-(x - \hat{x})^2}{2\sigma^2}\right) \mathbb{1}_{x \geq 0}$$

where

$$c = \frac{2\kappa V_\infty}{\epsilon^2} - \frac{1}{2} \qquad \hat{x} = 2\sigma^2 \frac{x_0}{2T} \qquad x_0 = \frac{2}{\epsilon}\sqrt{V_0} \qquad \sigma^2 = \frac{1}{\frac{\kappa}{2} + \frac{1}{T}}$$

We use a classical rejection procedure to simulate this random variable.

$$h(x) \leq K \exp\left(-\frac{(x - \bar{x})^2}{2\bar{\sigma}^2}\right)$$

We have to choose $\bar{x}$ and $\bar{\sigma}$ such that the constant K is minimal. Our choice is

$$\bar{\sigma} = \sigma \qquad \bar{x} = \frac{\hat{x} + \sqrt{\hat{x}^2 + 4c\sigma^2}}{2}$$

4.2 Simulation of the minimum

The second step consists in generating the random variables (m, θ) where

$$m = \inf_{0 \leq t \leq T} \left\{ \tilde{B}_t \middle| \tilde{B}_0 = x_0, \tilde{B}_T = Y \right\} \qquad \tilde{B}_\theta = m$$

This law is known (see for instance Karatzas-Shreve[12, p. 102])

$$\mathbb{P}\left[m \in d\alpha, \theta \in ds \middle| \tilde{B}_T = Y\right] = \frac{\alpha(\alpha - Y)}{\sqrt{t^3(T - s)^3}} \exp\left(-\frac{\alpha^2}{2s} - \frac{(\alpha - Y)^2}{2(T - s)}\right) d\alpha ds$$

In Beskos et al. [6, Prop. 2], the authors give the detail of the random variables used to simulate (m, θ), that is with uniform law, exponential law and Inverse Gaussian distribution (See Devroye [9, p.149] for Inverse Gaussain distribution).

4.3 Simulation of the Poisson Point Process

We apply the method detailed in Section 2.2. We generate $z_1 \sim \mathcal{E}(T)$, $t_1 \sim \mathcal{U}(0, T)$, $\tilde{B}_{t_1}$ conditioned by $\tilde{B}_0, \tilde{B}_T, m, \theta$. If $\varphi(\tilde{B}_{t_1}) > z_1$, we reject the trajectory. Else, we generate $z_2 - z_1 \sim \mathcal{E}(T)$, $t_2 \sim \mathcal{U}(0, T)$, $\tilde{B}_{t_2}$ conditioned by $\tilde{B}_0, \tilde{B}_{t_1}, \tilde{B}_T, m, \theta$. If $\varphi(\tilde{B}_{t_2}) > z_2$, we reject the trajectory, etc.

4.4 Stopping condition

In this example, we are not able to estimate exactly $K(\omega)$. So, we have to choose a stopping condition for the algorithm. We stop (and accept finally the trajectory) when both following conditions are satisfied

$$z_n \geq \varphi(m)$$
$$n \geq N_0$$

where N_0 is a fixed number of points.

5 Simulation of the greeks (Delta and Gamma)

In the previous sections, we have detailed our method to computing prices of options without discretization error. The sensitivity of the prices with respect to the initial condition and the parameters of the equations are also very important in practice.

For instance, in order to hedge the option, we need to estimate the Delta, that is the derivatives of the prices with respect to the initial condition.

$$\Delta(x) = \frac{d}{dx}\mathbb{E}_x\left[\varphi(X_t, 0 \leq t \leq T)\right]$$

We can use

$$\Delta(x) = \lim_{\delta x \to 0} \frac{\mathbb{E}_{x+\delta x}\left[\varphi(X_t, 0 \leq t \leq T)\right] - \mathbb{E}_x\left[\varphi(X_t, 0 \leq t \leq T)\right]}{\delta x}.$$

But we really need a very good accuracy for the estimation of the both expectations in order to obtain a sufficient accuracy for $\Delta(x)$.

5.1 Preliminaries

In this part, we apply Malliavin calculus to estimate the greeks (see Fournié et al. [10]).

Let us recall the model: we have a one dimensional diffusion with constant diffusion coefficient (possibly after a Lamperti transform)

$$dX_t = \alpha(X_t)dt + dW_t.$$

where the drift coefficient α satisfied $\inf_{x\in\mathbb{R}} \alpha^2(x) + \alpha'(x) > -\infty$.

In Section 2, we have explained the algorithm used to estimate the prices of options $V(T, x) = \mathbb{E}_x\left[\varphi(X_T)\right]$.

We denote by Y^x the first variation process associated to X^x (for convenience, we use in this section the notation X^x for the process X such that $X_0 = x$). We give the definition and the equation satisfied by Y^x.

$$\begin{cases} Y_t^x = \dfrac{d}{dx} X_t^x \\ dY_t^x = Y_t^x \alpha'(X_t^x)dt \\ Y_0^x = 1 \end{cases}$$

The solution of this linear equation is

$$Y_t^x = \exp\left(\int_0^t \alpha'(X_s^x)ds\right). \tag{8}$$

We denote by $D_t X_T^x$ the Malliavin derivative of the process X^x; this process satisfies

$$\begin{cases} dD_t X_s^x = D_t X_s^x \alpha'(X_s^x)ds \\ D_t X_t^x = 1 \end{cases} \tag{9}$$

The processes Y^x and $D_t X^x$ satisfy the identity:

$$\boxed{D_t X_T^x = \frac{Y_T^x}{Y_t^x}} \tag{10}$$

So, we obtain

$$Y_T^x = Y_t^x D_t X_T^x = \int_0^T \frac{1}{T} Y_t^x D_t X_T^x dt.$$

In this relation, we can replace $\frac{1}{T}$ by any L^2 function a such that $\int_0^T a(t)dt = 1$.

5.2 Delta

Following Fournié et al. [10], and using classical results on Malliavin calculus (integration by parts formula, see [15]), we obtain

$$\begin{aligned} \Delta(x) &= \mathbb{E}_x\left[\varphi'(X_T^x)Y_T^x\right] \\ &= \frac{1}{T}\mathbb{E}_x\left[\int_0^T \varphi'(X_T^x)D_t(X_T^x)Y_t^x dt\right] = \frac{1}{T}\mathbb{E}_x\left[\int_0^T D_t(\varphi(X_T^x))Y_t^x dt\right] \\ &= \frac{1}{T}\mathbb{E}_x\left[\varphi(X_T^x)\delta(Y_t^x)\right] = \frac{1}{T}\mathbb{E}_x\left[\varphi(X_T^x)\int_0^T Y_t^x dW_t\right] \end{aligned}$$

Remark 4. This relation is still available even if φ is not a smooth function.

We again use the dimension of our problem to remove the stochastic integral.

$$\int_0^T Y_t^x dW_t = W_T Y_T^x - W_0 Y_0^x - \int_0^T W_t dY_t^x$$
$$\int_0^T Y_t^x dW_t = W_T Y_T^x - W_0 Y_0^x - \int_0^T W_t Y_t^x \alpha'(X_t^x)dt \tag{11}$$

In order to use an exact simulation scheme, we don't want to estimate the integral (for instance, we could use a discretisation of this integral but it would introduce a bias). So, we now use the classical property: consider a stochastic process γ,

$$\int_0^T \gamma_t dt = T\bar{\mathbb{E}}(\gamma_{UT}) \tag{12}$$

where U is random variable with uniform distribution on $[0,1]$, independent of γ and $\bar{\mathbb{E}}$ is the expectation with respect to U.

Remark 5. The drawback of this method is the increase of the variance. See [13] for a discussion on this topic.

Using this property and the expression of Y (8), we obtain

$$\Delta(x) = \mathbb{E}\left[\frac{\varphi(X_T^x)}{T}\left(W_T \exp\int_0^T \alpha'(X_s^x)ds - \frac{W_0}{T}\right.\right.$$
$$\left.\left. -W_{U_1T}\alpha'(X_{U_1T}^x)\exp\int_0^{U_1T}\alpha'(X_s^x)ds\right)\right]$$

We now apply Girsanov Theorem and we use the same notation as in Section 2

$$\Delta(x) = \mathbb{E}\left[\frac{\varphi(\tilde{B}_T)}{T}\exp\left(-\frac{1}{2}\int_0^T \alpha^2(\tilde{B}_s) + \alpha'(\tilde{B}_s)ds\right)\right.$$
$$\times\left((\tilde{B}_T - \int_0^T \alpha(\tilde{B}_t)dt)\exp\int_0^T \alpha'(\tilde{B}_s^x)ds - \frac{x}{T}\right.$$
$$\left.\left. - \left(\tilde{B}_{U_1T} - \int_0^{U_1T}\alpha(\tilde{B}_s)ds\right)\alpha'(\tilde{B}_{U_1T})\exp\int_0^{U_1T}\alpha'(\tilde{B}_s^x)ds\right)\right]$$

We assume that $\alpha^2 + \alpha'$ and $\alpha^2 - \alpha'$ are both bounded bellow functions and we apply the same rejection procedure as in Section 2

$$\Delta(x) = -\mathbb{E}\left[\frac{x\varphi(\tilde{B}_T)}{T^2}\exp\left(-\frac{1}{2}\int_0^T \alpha^2(\tilde{B}_s) + \alpha'(\tilde{B}_s)ds\right)\right]$$
$$+\,\mathbb{E}\left[\frac{\varphi(\tilde{B}_T)}{T}\left(\tilde{B}_T - T\alpha(\tilde{B}_{U_2T})\right)\exp\left(-\frac{1}{2}\int_0^T \alpha^2(\tilde{B}_s) - \alpha'(\tilde{B}_s)ds\right)\right]$$
$$-\,\mathbb{E}\left[\frac{\varphi(\tilde{B}_T)}{T}\left(\tilde{B}_{U_1T} - U_1T\alpha(\tilde{B}_{U_1U_2T})\right)\alpha'(\tilde{B}_{U_1T})\right.$$
$$\left.\exp\left(-\frac{1}{2}\int_0^T \alpha^2(\tilde{B}_s) + (1 - 2\mathbb{1}_{U_1T\le s\le T})\alpha'(\tilde{B}_s)ds\right)\right]$$

This expression permits to simulate exactly $\Delta(x)$ with the same procedure as the computation of the prices.

5.3 Auxiliary Computations

In this Section, we give some technical results useful to apply a similar algorithm for the computation of the Gamma. Due to the quantity of calculus, we choose to give only the main ideas and leave the details of the computation to the reader.

We denote by Z_t^x the second variation process associated to X_t^x

$$\begin{cases} Z_t^x = \dfrac{d^2}{dx^2}X_t^x \\ Z_T^x = \displaystyle\int_0^T \alpha''(X_t^x)\,(Y_t^x)^2 + \alpha'(X_t^x)Z_t^x dt \end{cases}$$

We solve this linear equation and obtain

$$\boxed{Z_T^x = Y_T^x \int_0^T \alpha''(X_s^x) Y_s^x ds.} \tag{13}$$

We need also the Malliavin derivative of the first variation process Y^x which is solution of

$$D_t Y_T^x = \int_0^T \alpha''(X_s^x) Y_s^x D_t X_s^x + \alpha'(X_s^x) D_t Y_s^x ds$$

We solve this linear equation and obtain

$$\boxed{D_t Y_T^x = \frac{Y_T^x}{Y_t^x} \int_t^T \alpha''(X_s^x) Y_s^x ds} \tag{14}$$

5.4 Gamma

Following the idea of previous section, we first suppose that φ is smooth. Let us recall an expression of $\Delta(x)$ and give $\Gamma(x)$

$$\Delta(x) = \mathbb{E}\left[\frac{\varphi(X_T^x)}{T} \int_0^T Y_t^x dW_t\right]$$

$$\Gamma(x) = \underbrace{\mathbb{E}\left[\frac{\Psi'(X_T^x)}{T} Y_T^x \int_0^T Y_t^x dW_t\right]}_{\Gamma_1(x)} + \underbrace{\mathbb{E}\left[\frac{\varphi(X_T^x)}{T} \int_0^T Z_t^x dW_t\right]}_{\Gamma_2(x)}$$

The computation of Γ_2 does not present difficulties

$$\begin{aligned}\Gamma_2(x) = \mathbb{E}\Bigg[\frac{\varphi(X_T)}{T} \Bigg(& W_T Y_T^x \int_0^T \alpha''(X_t^x) Y_t^x dt - \int_0^T W_t \left(Y_t^x\right)^2 \alpha''(X_t^x) dt \\ & - \int_0^T \int_0^t W_t \alpha'(X_t^x) \alpha''(X_u^x) Y_u^x Y_t^x du dt \Bigg)\Bigg]\end{aligned} \tag{15}$$

We will use the Malliavin integration by part formula to obtain a tractable formulation for Γ_1. Thanks to (10),

$$\begin{aligned}\Gamma_1(x) &= \frac{1}{T^2} \mathbb{E}\left[\int_0^T D_t \varphi(X_T^x) Y_t^x \int_0^T Y_s^x dW_s dt\right] \\ &= \frac{1}{T^2} \mathbb{E}\left[\varphi(X_T^x) \delta\left(Y_t^x \int_0^T Y_s^x dW_s\right)\right]\end{aligned}$$

We finally have to make explicit the divergence operator in the last equation. We apply [15, Prop. 1.3.3] to obtain

$$\delta\left(Y_t^x \int_0^T Y_s^x dW_s\right) = \int_0^T Y_s^x dW_s \delta\left(Y_t^x\right) - \int_0^T D_t \left(\int_0^T Y_s^x dW_s\right) Y_t^x dt$$

and $D_t \left(\int_0^T Y_s^x dW_s\right) = Y_t^x + \int_t^T D_t Y_s^x dW_s$

Due to the dimension, we again can remove the stochastic integrals:

$$\int_t^T D_t Y_s^x dW_s = (D_t Y_T^x)\, W_T - \int_t^T W_s \alpha''(X_s^x) Y_s^x \frac{Y_s^x}{Y_t^x} ds$$
$$- \int_t^T W_s \frac{\alpha'(X_s^x) Y_s^x}{Y_t^x} \int_t^s \alpha''(X_u^x) Y_u^x du ds$$

We summarize the expression of Γ and use again (12). We denote by U_1, U_2 and U_3 three uniform independent random variable, independent of W.

$$\begin{aligned}\Gamma(x) = \mathbb{E}\Bigg[\Psi(X_T^x)\Bigg(&\frac{x^2}{T^2} - \frac{2x}{T^2} W_T Y_T^x + \frac{1}{T^2}(W_T)^2 (Y_T^x)^2 \\ &+ \frac{2x}{T} W_{U_1 T} \alpha'(X_{U_1 T}^x) Y_{U_1 T}^x \\ &- \frac{1}{T}\left(Y_{U_1 T}^x\right)^2 + (U_1 - 1) W_{U_1 T} \alpha''(X_{U_1 T}^x) \left(Y_{U_1 T}^x\right)^2 \\ &+ W_{U_1 T} \alpha'(X_{U_1 T}^x) W_{U_2 T} \alpha'(X_{U_2 T}^x) Y_{U_1 T}^x Y_{U_2 T}^x \\ &- \frac{2W_T}{T} W_{U_1 T} \alpha'(X_{U_1 T}^x) Y_T^x Y_{U_1 T}^x + W^x(T)(1 - U_1)\alpha''(X_{U_1 T}^x) Y_T^x Y_{U_1 T}^x \\ &+ U_1 T (U_1 U_2 - 1) W_{U_1 T} \alpha'(X_{U_1 T}^x) \alpha''(X_{U_1 U_2 T}^x) Y_{U_1 U_2 T}^x Y_{U_1 T}^x\Bigg)\Bigg]\end{aligned}$$

Finally, as in Section 2, we use Girsanov Theorem to replace the law of X by the law of a Brownian bridge; we replace W_t by $\tilde{B}_t - t\alpha(\tilde{B}_{Ut})$

We need now to assume that $\alpha^2 + \alpha'$, $\alpha^2 - \alpha'$ and $\alpha^2 - 3\alpha'$ are three functions bounded bellow. Then, we can simulate exactly the quantity in the last expectation.

Remark 6. In [2], the authors give explicit formula of the Malliavin derivative in the particular case of the CIR,

6 Numerical Results

In this Section, we present the numerical results obtained with our method. We have first studied an academic example in order to improve the advantage of our way to generate the Poisson Point Process. In a second section, we studied the CIR.

6.1 An academic example: a modified Ornstein Uhlenbeck

We construct an example with a regular drift α such that $\lim_{-\infty} \alpha^2 + \alpha' = +\infty$ and $\lim_{+\infty} \alpha^2 + \alpha' < \infty$

$$\begin{aligned} X_0 &= x_0 \\ dX_t &= \alpha(X_t)dt + dB_t \\ \alpha(x) &= \left(-Mx + \frac{x}{2}\right) \mathbb{1}_{x \le -1} + \frac{M}{2} x^2 \mathbb{1}_{-1 \le x \le 0}. \end{aligned}$$

We generate exact realisations of the process X, and we compare the times of simulation as a function of the maturity T. Fig. 1 present the results. We

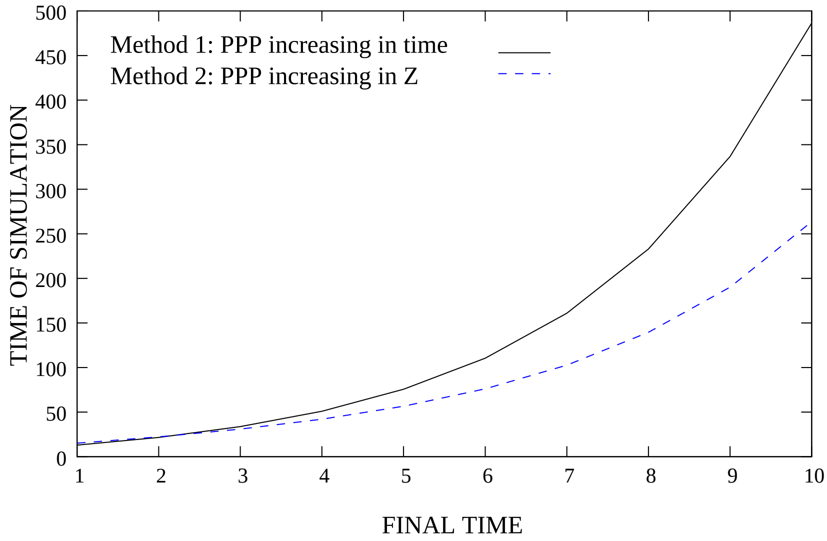

Figure 1: Comparison of the time of simulation for two methods of simulation of the Poisson Point Process. The result are given as function of the final time T

observe that the times of simulation seem to be very close for small value of T. They increase both with an exponential rate but the constant is smaller with our method.

6.2 CIR

In this Section, we present the numerical results obtained for the simulation of the CIR (see Section 4). We recall the SDE (7)

$$V_T = V_0 + \int_0^T \kappa\left(V_\infty - V_t\right) dt + \varepsilon \int_0^T \sqrt{V_t} dB_t.$$

The parameters are fixed to the values

$$\kappa = 0.5 \qquad V_\infty = 0.04 \qquad \varepsilon = 0.1 \qquad T = 1.$$

We apply our method to the estimation of $\mathbb{E}_{V_0} f(V_T)$ (Prices), $\frac{\partial}{\partial V_0}\mathbb{E}_{V_0} f(V_T)$ (Delta) and $\frac{\partial^2}{\partial V_0^2}\mathbb{E}_{V_0} f(V_T)$ (Gamma) where f is a smooth function ($f(x) = x^2$) or f is not smooth ($f(x) = \mathbb{1}_{x>0.06}$).

A Smooth function: $f(z) = z^2$ The estimation of the second moment and the derivatives are

$$V_0 = 0.04 \qquad \frac{\partial}{\partial V_0}\mathbb{E}_{V_0}(V_T^2) = 0.0533178 \pm 1.6 \times 10^{-5}$$

$$\mathbb{E}_{V_0}(V_T^2) = 0.00185282 \pm 5.8 \times 10^{-8} \qquad \frac{\partial^2}{\partial V_0^2}\mathbb{E}_{V_0}(V_T^2) = 0.736373 \pm 4.6 \times 10^{-3},$$

where the value after $\pm$ is the standard deviation. In this test case, we can compute the exact value of the second moment. The Fig. 2 gives the error when we approximate the price with a linear function or a quadratic function.

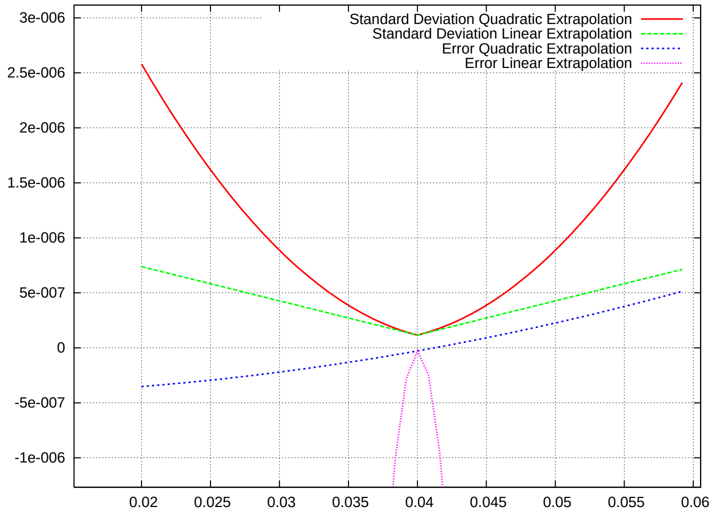

Figure 2: Error for $\mathbb{E}(f(V_T)$ for linear and quadratic approximations

A non smooth function: $f(z) = \mathbb{1}_{z>0.06}$ In Table 1 and Fig. 3, we give the approximation of $V_0 \mapsto \mathbb{P}_{V_0}(V_1 > 0.06)$ obtained by Monte-Carlo simulations and the approximation obtained by an extrapolation of order two of this function arround $V_0 = 0.04$ with the estimation of the Delta and Gamma.

Remark 7. For the simulation of the CIR, we have first generated the infimum m of the Brownian bridge and we have only accepted the trajectories with a non negative infimum. We have also imposed the condition $K \geq \varphi(m)$ in order to accept a trajectory.

In Fig. 4, we illustrate Theorem 1. We denote by F the ordinate of the point of the Poisson Point Process used to reject a trajectory.

7 Conclusion

In this paper, we have relaxed the assumption to apply the exact simulation algorithm proposed by Beskos et al. [6]. We have also used Malliavin calculus to give exact expression for the Delta and the Gamma. This methodology can be obviously apply to estimate the sensitivity of the price to the parameters of the model. We have removed the error due to the time and space discretizations and we have only the Monte Carlo error.

However, we have a strong restriction: this method cannot be generalized easily to dimension greater than one. The second restriction is due to a big variance (especially for the computation of the Gamma) and it should probably be a big advance if we could propose more efficient variance reduction techniques.

V_0	Price	stddev	Delta	stddev	Gamma	stddev
0.01	0.996599	1.0629e-05	-0.718488	0.0334322	-107.652	11.3333
0.02	0.982925	2.36527e-05	-2.17646	0.0276732	-215.864	8.24132
0.03	0.949189	4.00954e-05	-4.63846	0.024441	-280.823	6.9928
0.036	0.916405	5.05328e-05	-6.34233	0.0228411	-287.007	6.45608
0.038	0.903051	5.40215e-05	-6.91021	0.0223331	-276.066	6.29249
0.04	0.8886765	5.74232e-05	-7.47226	0.0218326	-276,908	6.13585
0.042	0.873246	6.0742e-05	-8.01353	0.021337	-272.994	5.98258
0.044	0.856625	6.3984e-05	-8.54554	0.0208508	-253.826	5.83276
0.04	0.888686	5.74232e-05	-7.48546	0.0218326	-279.677	6.13585
0.05	0.801064	7.28836e-05	-9.99319	0.0194123	-213.473	5.40375
0.06	0.69216	8.42763e-05	-11.5853	0.0170785	-111.631	4.72251
0.07	0.572886	9.0312e-05	-12.0874	0.0148461	13.2656	4.07576
0.08	0.454207	9.09034e-05	-11.4768	0.0127356	106.396	3.46544
0.09	0.345545	8.68223e-05	-10.1217	0.0107925	160.336	2.90437

Table 1: Estimation of the Prices ($A_0(f)$), Delta ($A_1(f)$), Gamma ($A_2(f)$) and the corresponding standard deviation for the CIR with a digital option, that is $f(z) = \mathbb{1}_{z>0.06}$. The number of Monte Carlo simulations is 30×10^6.

Another direction should consist in using weights instead of exact trajectory simulation (see [11]). It should be possible to estimate the greeks with such a method.

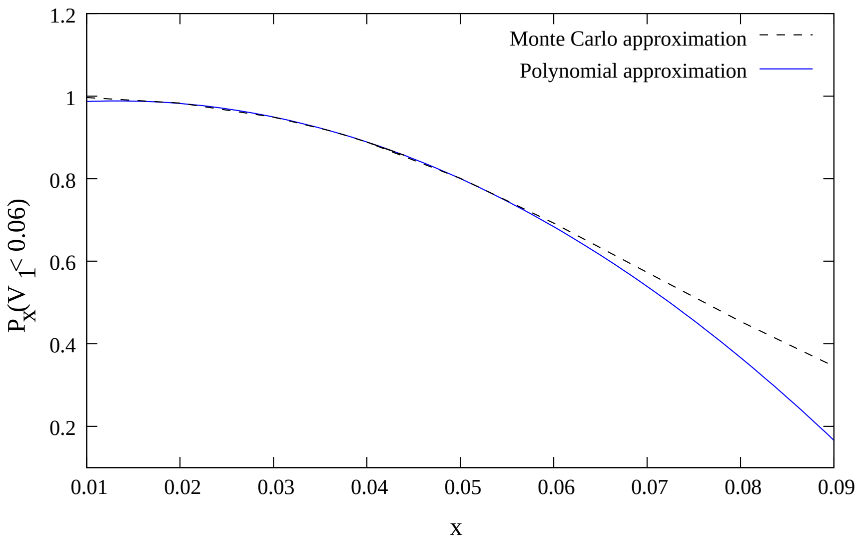

Figure 3: Estimation of the Prices

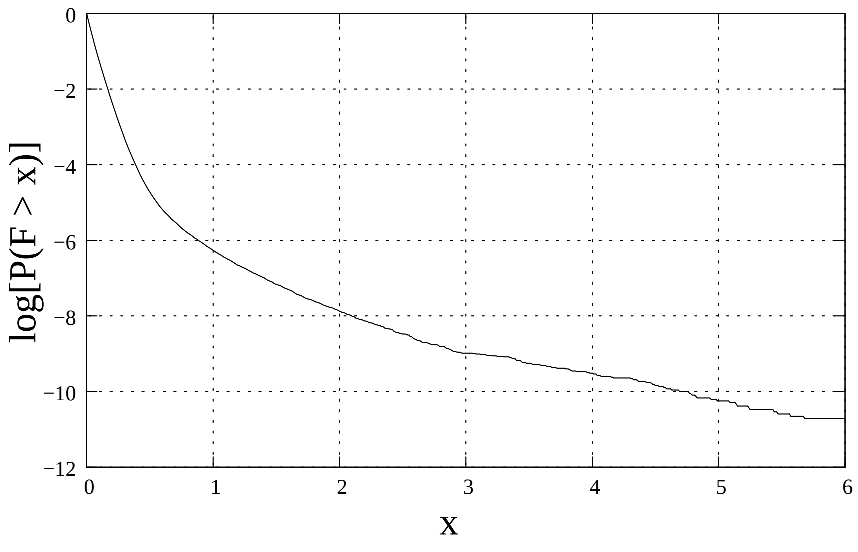

Figure 4: Cumulative distribution function of the ordinate of the point used for the reject

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