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SIAM J. DISCRETE MATH.
Vol. 26, No. 1, pp. 145–168

GRIGGS AND YEH'S CONJECTURE AND $L(p, 1)$-LABELINGS*

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Abstract. An $L(p, 1)$-labeling of a graph is a function f from the vertex set to the positive integers such that $|f(x) - f(y)| \geqslant p$ if $\mathrm{dist}(x, y) = 1$ and $|f(x) - f(y)| \geqslant 1$ if $\mathrm{dist}(x, y) = 2$, where $\mathrm{dist}(x, y)$ is the distance between the two vertices x and y in the graph. The *span* of an $L(p, 1)$-labeling f is the difference between the largest and the smallest labels used by f. In 1992, Griggs and Yeh conjectured that every graph with maximum degree $\Delta \geqslant 2$ has an $L(2, 1)$-labeling with span at most Δ^2. We settle this conjecture for Δ sufficiently large. More generally, we show that for any positive integer p there exists a constant Δ_p such that every graph with maximum degree $\Delta \geqslant \Delta_p$ has an $L(p, 1)$-labeling with span at most Δ^2. This yields that for each positive integer p, there is an integer C_p such that every graph with maximum degree Δ has an $L(p, 1)$-labeling with span at most $\Delta^2 + C_p$.

Key words. channel assignment, graph labeling, graph coloring, generalized coloring

AMS subject classifications. 05C78, 05C15

DOI. 10.1137/090763998

1. Introduction. In the channel assignment problem, transmitters at various nodes within a geographic territory must be assigned channels or frequencies in such a way as to avoid interferences. In a model for the channel assignment problem developed wherein channels or frequencies are represented with integers, "close" transmitters must be assigned different integers and "very close" transmitters must be assigned integers that differ by at least 2. This quantification led to the definition of an $L(p, q)$*-labeling* of a graph $G = (V, E)$ as a function f from the vertex set to the positive integers such that $|f(x) - f(y)| \geqslant p$ if $\mathrm{dist}(x, y) = 1$ and $|f(x) - f(y)| \geqslant q$ if $\mathrm{dist}(x, y) = 2$, where $\mathrm{dist}(x, y)$ is the distance between the two vertices x and y in the graph G. The notion of $L(2, 1)$-labeling first appeared in 1992 [12]. Since then, a large number of articles has been published devoted to the study of $L(p, q)$-labelings. We refer the interested reader to the surveys of Calamoneri [6] and Yeh [25].

Generalizations of $L(p, q)$-labelings in which for each $i \geqslant 1$ a minimum gap of p_i is required for channels assigned to vertices at distance i have also been studied (see, for example, the survey by Griggs and Král' [11], and consult also [3, 15, 16, 18]).

In the context of the channel assignment problem, the main goal is to minimize the number of channels used. Hence, we are interested in the *span* of an $L(p, q)$-labeling f, which is the difference between the largest and the smallest labels of f. The $\lambda_{p,q}$-*number* of G is $\lambda_{p,q}(G)$, the minimum span over all $L(p, q)$-labelings of G. In general, determining the $\lambda_{p,q}$-number of a graph is NP-hard [9]. In their seminal paper, Griggs

*Received by the editors July 6, 2009; accepted for publication (in revised form) November 1, 2011; published electronically February 16, 2012. This work was partially supported by the *Agence nationale de la recherche* under references AGAPE ANR-09-BLAN-0159, GRATEL ANR-09-BLAN-0373-01, and ANR 10 JCJC 204 01, the ÉGLIDE ECO-NET project 16305SB and a PICS CNRS.
http://www.siam.org/journals/sidma/26-1/76399.html

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and Yeh [12] observed that a greedy algorithm yields that $\lambda_{2,1}(G) \leqslant \Delta^2 + 2\Delta$, where Δ is the maximum degree of the graph G. Moreover, they conjectured that this upper bound can be decreased to Δ^2.

CONJECTURE 1 (see [12]). *For every $\Delta \geqslant 2$ and every graph G of maximum degree Δ,*

$$\lambda_{2,1}(G) \leqslant \Delta^2.$$

This upper bound would be tight: there are graphs with degree Δ, diameter 2, and $\Delta^2 + 1$ vertices, namely the 5-cycle, the Petersen graph, and the Hoffman–Singleton graph. Thus, their square is a clique of order $\Delta^2 + 1$, and hence the span of every $L(2,1)$-labeling is at least Δ^2.

However, such graphs exist only for Δ being 2, 3, 7, and possibly 57, as shown by Hoffman and Singleton [13]. So one can ask how large may be the $\lambda_{2,1}$-number of a graph with large maximum degree. As it should be at least as large as the largest clique in its square minus one, one can ask what is the largest clique number $\gamma(\Delta)$ of the square of a graph with maximum degree Δ. If Δ is a prime power plus 1, then $\gamma(\Delta) \geqslant \Delta^2 - \Delta + 1$. Indeed, in the projective plane of order $\Delta - 1$, each point is in Δ lines, each line contains Δ points, each pair of distinct points is in a line, and each pair of distinct lines has a common point. Consider the *incidence graph* of the projective plane: it is the bipartite graph with vertices the set of points and lines of the projective plane, and every line is linked to all the points it contains. The properties of the projective plane imply that the set of points and the set of lines form two cliques in the square of this graph, and there are $\Delta^2 - \Delta + 1$ vertices in each.

Jonas [14] improved slightly on Griggs and Yeh's upper bound by showing that every graph of maximum degree Δ admits an $L(2,1)$-labeling with span at most $\Delta^2+2\Delta-4$. Subsequently, Chang and Kuo [7] provided the upper bound $\Delta^2+\Delta$ which remained the best general upper bound for about a ecade. Král' and Škrekovski [17] brought this upper bound down by 1 as the corollary of a more general result. And, using the algorithm of Chang and Kuo [7], Gonçalves [10] decreased this bound by 1 again, thereby obtaining the upper bound $\Delta^2 + \Delta - 2$. Note that Conjecture 1 is true for planar graphs of maximum degree $\Delta \neq 3$. For $\Delta \geqslant 7$ it follows from a result of van den Heuvel and McGuinness [24], and Bella et al. [4] proved it for the remaining cases.

We prove the following approximate version of the generalization of Conjecture 1 to $L(p,1)$-labeling.

THEOREM 2. *For any fixed integer p, there exists a constant C_p such that for every integer Δ and every graph of maximum degree Δ,*

$$\lambda_{p,1}(G) \leqslant \Delta^2 + C_p.$$

This result is obtained by combining a greedy algorithm (or any of the previously mentioned upper bounds, or their generalization for $L(p,1)$-labelings) with the next theorem, which settles Conjecture 1 for sufficiently large Δ.

THEOREM 3. *For any fixed integer p, there is a Δ_p such that for every graph G of maximum degree $\Delta \geqslant \Delta_p$,*

$$\lambda_{p,1}(G) \leqslant \Delta^2.$$

Actually, we consider a more general setup. We are given a graph G_1 with vertex-set V, along with a spanning subgraph G_2. We want to find a $(p,1)$-*coloring* of (G_1, G_2) that is an assignment of integers from $\{1, 2, \ldots, k\}$ to the elements of V

so that vertices adjacent in G_1 receive different colors and vertices adjacent in G_2 receive colors which differ by at least p. This setup is a particular case of the *constraint matrix* or *weighted graph* model (with unit demands), formalized in the early nineties. Broersma et al. [5] called this particular case the *backbone coloring problem* and Babilon et al. [3] studied its generalization to real weights via the notion of lambda-graphs.

Typically the maximum degree of G_1 is much larger than the maximum degree of G_2. In the case of $L(p,1)$-labeling, G_1 is the square of G_2 and a $(p,1)$-coloring of (G_1,G_2) using k colors is an $L(p,1)$-labeling of G_2 with span $k-1$. We impose the condition that for some integer Δ, the graph G_1 has maximum degree at most Δ^2 and G_2 has maximum degree Δ. We show that under these conditions there exists a $(p,1)$-coloring for $k=\Delta^2+1$ provided that Δ is large enough. The bound is best possible since G_1 may be a clique of size Δ^2+1. Formally, we prove the following result.

THEOREM 4. *Let p be an integer. There is a Δ_p such that for every $\Delta \geqslant \Delta_p$, and $G_2 \subseteq G_1$ with $\Delta(G_1) \leqslant \Delta^2$ and $\Delta(G_2) \leqslant \Delta$, there exists a $(p,1)$-coloring of (G_1,G_2) with $\{1,2,\ldots,\Delta^2+1\}$.*

In the next section we give an outline of the proof. In section 3, we present some needed probabilistic tools. We then turn to the gory details.

In what follows, we use G_1*-neighbor* to mean a neighbor in G_1 and G_2*-neighbor* to indicate a neighbor in G_2. For every vertex v and every subgraph H of G_1, we let $\deg^1_H(v)$ be the number of G_1-neighbors of v in H. We omit the subscript if $H=G_1$.

Moreover, lots of inequalities are correct only when Δ is large enough. In such inequalities, we use the symbols $\leqslant^*$, $\geqslant^*$, $<^*$, and $>^*$ instead of $\leqslant$, $\geqslant$, $<$, and $>$, respectively. We do not explicit the value of the constant Δ_p and make no attempt to minimize it.

We finish this section by pointing out that Theorem 2 can be further generalized as follows: *For every integers $p \geqslant 2$ and q and every real $c \in [0\,,1]$, there exists an integer $C_{p,q,c}$ such that for every graph G of maximum degree Δ^c,*

$$\lambda_{p,q}(G) \leqslant q \cdot \Delta^c + C_{p,q,c}.$$

2. A sketch of the proof.

The general structure of the proof is very close to the one used by Molloy and Reed in [21]. Many parts of it follow proofs from this paper very closely.

We consider a counterexample to Theorem 4 chosen so as to minimize V. Thus, for every proper subset X of the vertices of G_1, there is a $(p,1)$-coloring c of $(G_1[X],G_2[X])$ using at most Δ^2+1 colors. Such a coloring is a *good coloring* of X. In particular, as $G_2 \subseteq G_1$, this implies that every vertex v has more than $\Delta^2-(2p-2)\Delta$ neighbors in G_1, as otherwise we could complete a good coloring of $V-v$ greedily. Indeed, for each vertex, a colored G_2-neighbor forbids $2p-1$ colors, which is $2p-2$ more than being only a G_1-neighbor.

The next lemma follows by setting $d=1000p\Delta$ and applying to G_1 a decomposition result due to Reed [22, Lemma 15.2].

LEMMA 5. *There is a partition of V into disjoint sets $D_1,\ldots,D_\ell,S$ such that*

(a) *Every D_i has between $\Delta^2-8000p\Delta$ and $\Delta^2+4000p\Delta$ vertices;*
(b) *There are at most $8000p\Delta^3$ edges of G_1 leaving any D_i;*
(c) *A vertex has at least $\frac{3}{4}\Delta^2$ G_1-neighbors in D_i if and only if it is in D_i; and*
(d) *For each vertex v of S, the neighborhood of v in G_1 contains at most $\binom{\Delta^2}{2}-1000p\Delta^3$ edges.*

We let H_i be the subgraph of G_1 induced by D_i and $\overline{H_i}$ its complementary graph. An *internal neighbor* of a vertex of D_i is a neighbor in H_i. An *external neighbor* of a vertex of D_i is a neighbor that is not internal.

LEMMA 6. *For every i, the graph $\overline{H_i}$ has no matching of size at least* $10^3 p\Delta$.

Proof. Suppose on the contrary that M is a matching of size $10^3 p\Delta$ in $\overline{H_i}$.

Let R be the unmatched vertices in H_i. Then, $\Delta^2 - 10^4 p\Delta < |R| < \Delta^2 + 10^4 p\Delta$ by Lemma 5(a). For each pair of vertices u and v that are matched in M, the number of internal neighbors of u plus the number of internal neighbors of v is at least $\frac{3}{2}\Delta^2$ by Lemma 5(c). Thus there are at least $\frac{1}{2}\Delta^2 - (|H_i| - \Delta^2) - 2|M| >^* \frac{1}{3}|R|$ vertices in R that are adjacent to both of u and v in G_1. So on average, a vertex of R is adjacent in G_1 to both members of at least $\frac{1}{3}|M|$ pairs. This implies that at least $\frac{1}{5}|R| >^* \frac{1}{10}\Delta^2$ members of R are adjacent in G_1 to both members of at least $\frac{1}{10}|M|$ pairs. Let X be $\frac{1}{10}\Delta^2$ such vertices in R.

Every vertex of $R \backslash X$ that is adjacent in G_1 to less than half of X must have at least $\Delta^2 - (2p-2)\Delta - (|H_i| - \frac{1}{2}|X|) >^* \frac{1}{25}\Delta^2$ G_1-neighbors outside D_i. Thus, Lemma 5(b) implies that there are at least $|R \setminus X| - 200000p\Delta \geqslant \frac{9}{10}\Delta^2 - 10^4 p\Delta - 200000p\Delta \geqslant^* \frac{1}{2}\Delta^2$ vertices in $R \setminus X$ that are adjacent in G_1 to at least half of X. Let Y be a set of $\frac{1}{2}\Delta^2$ such vertices.

We consider a good coloring of $V \setminus D_i$. We obtain a contradiction by extending this good coloring to our desired (Δ^2+1)-coloring of V greedily, as follows:

1. Color the vertices of M, assigning the same color to both members of each matched pair. This is possible because each pair has at most $\frac{1}{2}\Delta^2 + 2|M|$ previously colored G_1-neighbors (by Lemma 5(c)) and 2Δ previously colored G_2-neighbors, so there are at least $\frac{1}{2}\Delta^2 + 1 - 1004p\Delta \geqslant^* 1$ colors available.
2. Color the vertices of $H_i - Y - X - M$. This is possible since each such vertex has at most $\frac{1}{4}\Delta^2$ G_1-neighbors outside of D_1 (by Lemma 5(c)) and at most $|H_i| - |X| - |Y| <^* \frac{1}{2}\Delta^2$ previously colored internal neighbors.
3. Color the vertices of Y. This is possible since each vertex of Y has at least $\frac{1}{2}|X| = \frac{1}{20}\Delta^2$ uncolored G_1-neighbors and hence at least $\frac{1}{20}\Delta^2 + 1 - (2p - 2)\Delta \geqslant^* 1$ colors available.
4. Color the vertices of X. This is possible since each vertex of X has at least $\frac{1}{10}|M| = 100p\Delta$ colors that appear twice in its neighborhood and thus has at least $98p\Delta$ colors available. □

For each $i \in \{1, 2, \ldots, \ell\}$, we let M_i be a maximum matching of $\overline{H_i}$ and K_i be the clique $D_i - V(M_i)$. So, $|K_i| \geqslant \Delta^2 - 10^4 p\Delta$ by Lemmas 5(a) and 6. We let B_i be the set of vertices in K_i that have more than $\Delta^{5/4}$ G_1-neighbors outside D_i, and we set $A_i \coloneqq K_i \setminus B_i$. Considering Lemma 5(b) we can make the following observation.

OBSERVATION 7. *For every index $i \in \{1, 2, \ldots, \ell\}$,*

$$|B_i| \leqslant 8000p\Delta^{7/4} \text{ and so } |A_i| \geqslant \Delta^2 - 9000p\Delta^{7/4}.$$

We are going to color the vertices in three steps. We first color $V_1 \coloneqq V \setminus \cup_{i=1}^{\ell} A_i$ and we then color the vertices of $V_2 \coloneqq \cup_{i=1}^{\ell} A_i$.

In order to extend the coloring of V_1 to V_2, we need some properties. We prove the following.

LEMMA 8. *There is a good coloring c of V_1 such that*

(i) *$c(x) = c(y)$ for each edge xy of every M_i; and*

(ii) *For every color j and clique A_i there are at most $\frac{4}{5}\Delta^2$ vertices of A_i that have either a G_1-neighbor outside D_i colored j or a G_2-neighbor outside D_i with a color in $[j-p+1\,, j+p-1]$.*

We then establish that a coloring that satisfies the conditions of Lemma 8 can be extended to $Y \cup V_2$.

LEMMA 9. *Every good coloring of* V_1 *satisfying conditions* (i)–(ii) *of Lemma* 8 *can be completed to a good coloring of* $V = V_1 \cup V_2$.

Thus to prove our theorem, we need only prove Lemmas 8 and 9. The details follow.

3. Probabilistic preliminaries. In this section, we present a few probabilistic tools that are used in this paper. Each of these tools is presented in the book of Molloy and Reed [22], and most are presented in many other places.

The Lovász local lemma (see [8]). Let $A_1, A_2, \ldots, A_n$ be a set of random events so that for each $i \in \{1, 2, \ldots, n\}$,

(i) $\mathbf{Pr}(A_i) \leqslant p$ and

(ii) A_i is mutually independent of all but at most d other events.

If $pd \leqslant \frac{1}{4}$, then $\mathbf{Pr}(\overline{A_1} \cup \ldots \cup \overline{A_n}) > 0$.

The *binomial random variable* $\mathrm{BIN}(n,p)$ is the sum of n independent zero-one random variables, where each is equal to 1 with probability p.

The Chernoff bound (see [1, 19]). For every $t \in [0\,, np]$,

$$\mathbf{Pr}\left(|\mathrm{BIN}(n,p) - np| > t\right) < 2\exp\left(-\frac{t^2}{3np}\right).$$

Only in the proof of Lemma 20 do we use the following version of the Chernoff bound: for every $t > 0$,

$$\mathbf{Pr}\left(|\mathrm{BIN}(n,p) - np| > t\right) < 2\exp\left(t - \ln\left(1 + \frac{t}{np}\right)(np + t)\right).$$

The following is a simple corollary of Azuma's inequality [2, 22].

The simple concentration bound. Let X be a nonnegative random variable determined by the independent trials $T_1, T_2, \ldots, T_n$. Suppose that for every set of possible outcomes of the trials,

(i) Changing the outcome of any one trial can affect X by at most c.

Then

$$\mathbf{Pr}\left(|X - \mathbf{E}(X)| > t\right) \leqslant 2\exp\left(-\frac{t^2}{2c^2 n}\right).$$

Talagrand's inequality requires another condition but often provides a stronger bound when $\mathbf{E}(X)$ is much smaller than n. Rather than providing Talagrand's original statement [23], we present the following useful corollary [22].

Talagrand's inequality (see [23]). Let X be a nonnegative random variable determined by the independent trials $T_1, , T_2, \ldots, T_n$. Suppose that for every set of possible outcomes of the trials,

(i) Changing the outcome of any one trial can affect X by at most c; and

(ii) For each $s > 0$, if $X \geqslant s$, then there is a set of at most rs trials whose outcomes certify that $X \geqslant s$.

Then for every $t \in [0\,, \mathbf{E}(X)]$,

$$\mathbf{Pr}\left(|X - \mathbf{E}(X)| > t + 60c\sqrt{r\,\mathbf{E}(X)}\right) \leqslant 4\exp\left(-\frac{t^2}{8c^2 r\,\mathbf{E}(X)}\right).$$

McDiarmid extended Talagrand's inequality to the setting where X depends on independent trials and permutations, a setting that arises in this paper. Again, we present a useful corollary [22] rather than the original inequality [20].

McDiarmid's inequality (see [20]). Let X be a nonnegative random variable determined by the independent trials $T_1, \ldots, T_n$ and m independent permutations $\Pi_1, \ldots, \Pi_m$. Suppose that for every set of possible outcomes of the trials,

(i) Changing the outcome of any one trial can affect X by at most c;
(ii) Interchanging two elements in any one permutation can affect x by at most c; and
(iii) For each $s > 0$, if $X \geqslant s$, then there is a set of at most rs trials whose outcomes certify that $X \geqslant s$.

Then for every $t \in [0\,, \mathbf{E}(X)]$,

$$\mathbf{Pr}\left(|X - \mathbf{E}(X)| > t + 60c\sqrt{r\,\mathbf{E}(X)}\right) \leqslant 4\exp\left(-\frac{t^2}{8c^2 r\,\mathbf{E}(X)}\right).$$

In both Talagrand's inequality and McDiarmid's inequality, if $60c\sqrt{r\,\mathbf{E}(X)} \leqslant t \leqslant \mathbf{E}(X)$, then by substituting $t/2$ for t in the above bounds, we obtain the more concise

$$\mathbf{Pr}\left(|X - \mathbf{E}(X)| > t\right) \leqslant 4\exp\left(-\frac{t^2}{32c^2 r\,\mathbf{E}(X)}\right).$$

That is the bound that we usually use.

4. The proof of Lemma 8. In this section, we want to find a good coloring of V_1, which satisfies conditions (i)–(ii) of Lemma 8. We actually construct new graphs G_1^* and G_2^* and consider good colorings of these graphs. This helps us to ensure that the conditions of Lemma 8 hold.

4.1. Forming G_1^* and G_2^*. For $j \in \{1, 2\}$, we obtain G'_j from G_j by contracting each edge of each M_i into a vertex (that is, we consider these vertex pairs one by one, replacing the pair xy with a vertex adjacent to all of the neighbors of both x and y in the graph). We let C_i be the set of vertices obtained by contracting the pairs in M_i. We set $V^* := (V_1 \setminus \bigcup_{i=1}^{\ell} V(M_i)) \cup \bigcup_{i=1}^{\ell} C_i$. For each $i \in \{1, 2, \ldots, \ell\}$, let Big_i be the set of vertices of V^* not in $B_i \cup C_i$ that have more than $\Delta^{9/5}$ neighbors in A_i. We construct G_1^* from G'_1 by removing the vertices of $\cup_{i=1}^{\ell} A_i$ and adding for each i an edge between every pair of vertices in Big_i. The graph G_2^* is obtained from G'_2 by removing the vertices of $\cup_{i=1}^{\ell} A_i$.

Note that $G_2^* \subseteq G_1^*$. Our aim is to color the vertices of V^* except some of S such that vertices adjacent in G_1^* are assigned different colors, and vertices adjacent in G_2^* are assigned colors at distance at least p. Such a coloring is said to be *nice*. To every partial nice coloring of V^* is associated the $(p, 1)$-coloring of V_1 obtained as follows: each colored vertex of $V \cap V^*$ keeps its color, and for each index i, every pair of matched vertices of M_i is assigned the color of the corresponding vertex of C_i. So this partial good coloring verifies condition (ii) of Lemma 8.

Definition 10. *For every vertex u and every subset F of V^*,*

- *The number of G_1^*-neighbors of u in F is $\delta_F^1(u)$;*
- *The number of G_2^*-neighbors of u in F is $\delta_F^2(u)$; and*
- *$\delta_F^*(u) := \delta_F^1(u) + 4p\delta_F^2(u)$.*

For all these notations, we omit the subscript if $F = V^*$.

The next lemma bounds these parameters.

LEMMA 11. *Let v be a vertex of V^*. The following hold:*
(i) $\delta^2(v) \leqslant 2\Delta$, *and if $v \notin \cup_{i=1}^{\ell} C_i$, then* $\delta^2(v) \leqslant \Delta$;
(ii) *If $v \in \mathrm{Big}_i$ for some i, then* $\delta^1(v) \leqslant \Delta^2 - 8p\Delta$; *and*
(iii) $\delta^1(v) \leqslant \Delta^2$, *and if $v \notin S$, then* $\delta^1(v) \leqslant \frac{3}{4}\Delta^2$.

Proof.

(i) To obtain G_2^*, we only removed some vertices and contracted some pairwise disjoint pairs of nonadjacent vertices. Consequently, the degree of each new vertex is at most twice the maximum degree of G_2, i.e., 2Δ, and the degree of the other vertices is at most their degree is G_2, hence at most Δ.

(ii) By Lemma 5(b), we have $|\mathrm{Big}_i| \leqslant 8000p\Delta^{6/5}$ for each index i. Moreover, a vertex v can be in Big_i for at most $\Delta^{1/5}$ values of i. Recall that for each index i such that $v \in \mathrm{Big}_i$, the vertex v has at least $\Delta^{9/5}$ G_1-neighbors in A_i. So, in the process of constructing G_1^*, it loses at least $\Delta^{9/5}$ neighbors and gains at most $8000p\Delta^{7/5}$ neighbors. Consequently, the assertion follows because $\Delta^{9/5} \geqslant^* 8000p\Delta^{7/5} + 8p\Delta$.

(iii) By (ii), if $v \in S$, then $\delta^1(v) \leqslant \deg^1(v) \leqslant \Delta^2$. Assume now that $v \notin S$, hence $v \in B_i \cup C_i$ for some index i. By Lemma 6, each set C_i has at most $1000p\Delta$ vertices and by Observation 7, each set B_i has at most $8000p\Delta^{7/4}$ vertices. Moreover, by Lemma 5(c), each vertex of D_i has at most $\frac{1}{4}\Delta^2$ G_1-neighbors outside of D_i. It follows that each vertex of $B_i \cup C_i$ has at most $\frac{1}{2}\Delta^2 + 1000p\Delta + 8000p\Delta^{7/4} + 8000p\Delta^{7/5} \leqslant^* \frac{3}{4}\Delta^2$ G_1^*-neighbors. ◻

Our construction of G_1' and G_2' is designed to deal with condition (i) of Lemma 8. The edges we add between vertices of Big_i are designed to help with condition (ii). The bound of $\frac{3}{4}\Delta^2$ on the degree of the vertices of $V^* \setminus S$ in the last lemma helps us to ensure that all vertices of $V_1 \setminus S$ will be colored.

In order to color all vertices of S we would like to use the fact that sparse vertices have many nonadjacent pairs of G_1-neighbors. However, in constructing G_1^*, we contracted some pairs of nonadjacent vertices and added edges between some other pairs of nonadjacent vertices. As a result, possibly some vertices in S are no longer sparse. We have to treat such vertices carefully.

We define $\hat{S}$ to be those vertices in S that have at least $90p\Delta$ neighbors outside S. Then $\hat{S}$ contains all the vertices which may no longer be sufficiently sparse, as we note next.

LEMMA 12. *Each vertex of $S \setminus \hat{S}$ has at least $450p\Delta^3$ pairs of G_1-neighbors in S that are not adjacent in G_1^*.*

Proof. Let $s \in S \setminus \hat{S}$. We know that s has at least $\Delta^2 - (2p-2)\Delta$ G_1-neighbors. Hence it has more than $\binom{\Delta^2}{2} - 4p\Delta^3$ pairs of G_1-neighbors. Thus, by Lemma 5(d), the vertex s has more than $996p\Delta^3$ pairs of G_1-neighbors that are not adjacent in G_1. Since $s \notin \hat{S}$, all but at most $90p\Delta^3$ such pairs lie in $N(s) \cap S$. Let Ω be the collection of pairs of G_1-neighbors of s in S that are not adjacent in G_1. Then $|\Omega| \geqslant 906p\Delta^3$. For convenience, we say that a pair of Ω is *suitable* if its vertices are not adjacent in G_1^*.

Let s_1 be a member of a pair of Ω. If s_1 does not belong to $\cup_{i=1}^{\ell} \mathrm{Big}_i$, then every vertex of S that is not adjacent to s_1 in G_1 is also not adjacent to s_1 in G_1^*. Thus every pair of Ω containing s_1 is suitable.

If $s_1 \in \cup_{i=1}^{\ell} \mathrm{Big}_i$, then for each index i such that $s_1 \in \mathrm{Big}_i$, the vertex s_1 has at least $\Delta^{9/5}$ G_1-neighbors in A_i. Hence, there are more than $\Delta^2 - 92p\Delta - (\Delta^2 - \Delta^{9/5}) = \Delta^{9/5} - 92p\Delta$ pairs of Ω containing s_1. Recall from the proof of Lemma 11 that the number of edges added to s_1 by the construction of G_1^* is at most $8000p\Delta^{7/5} <^* \frac{1}{2}\Delta^{9/5} - 46p\Delta$. Consequently, the number of suitable pairs of Ω containing the vertex s_1 is at least half the number of pairs of Ω containing s_1.

Therefore, we conclude that at least $\frac{1}{2}|\Omega| > 450p\Delta^3$ pairs of Ω are suitable. ◻

4.2. High-level overview. Our first step is to color some of S, including all of $\hat{S}$. We do this in two phases. In the first phase, we consider assigning each vertex of S a color at random. We show by analyzing this random procedure that there is a partial nice coloring of S such that for every uncolored vertex in $S \setminus \hat{S}$, at least $6p\Delta$ colors appear on two G_1-neighbors of v. In the second phase, we finish coloring the vertices of $\hat{S}$. We use an iterative quasi-random procedure. In each iteration but the last, each vertex chooses a color, from those which do not yield a conflict with any already colored neighbor, uniformly at random. The last iteration has a similar flavor.

We then turn to coloring the vertices in the sets B_i and C_i and the uncolored vertices in S. Our degree bounds imply that we could do this greedily. However, we will mimic the iterative approach just discussed. We use this complicated coloring process because it allows us to ensure that condition (ii) of Lemma 8 holds for the coloring we obtain. At any point during the coloring process, $\mathrm{Notbig}_{i,j}$ is the set of vertices $v \in A_i$ such that v has either a G_1'-neighbor $u \notin \mathrm{Big}_i \cup D_i$ that has color j or a G_2'-neighbor $u \notin \mathrm{Big}_i \cup D_i$ that has a color in $[j-p+1\,, j+p-1]$. The challenge is to construct a coloring such that $\mathrm{Notbig}_{i,j}$ remains small for every index i and every color j.

4.3. Coloring sparse vertices. As mentioned earlier, we color sparse vertices in two phases. The first phase provides a partial nice coloring of S satisfying the above mentioned condition for uncolored vertices of $S \setminus \hat{S}$. The second phase extends this nice coloring to all the vertices of $\hat{S}$ using an iterative quasi-random procedure.

We need a lemma to bound the size of $\mathrm{Notbig}_{i,j}$. We consider the following setting. We have a collection of at most Δ^2 subsets of vertices. Each set contains at most Q vertices, and no vertex lies in more than $\Delta^{9/5}$ sets. A random experiment is conducted, where each vertex is marked with probability at most $\frac{1}{Q\cdot\Delta^{2/5}}$. We moreover assume that for any set of $s \geqslant 1$ vertices, the probability that all are marked is at most $(\frac{1}{Q\cdot\Delta^{2/5}})^s$. Note that in particular this is the case if the vertices are marked independently.

Applying a lemma of Molloy and Reed [21, Lemma 30] with Δ^2 in place of Δ yields the following.

LEMMA 13. *Under the preceding hypothesis, the probability that at least* $\Delta^{37/20}$ *sets contain a marked vertex is at most* $\exp\left(-\Delta^{1/20}\right)$.

4.3.1. First step.

LEMMA 14. *There exists a nice coloring of a subset H of S with colors in* $\{1, 2, \ldots, \Delta^2 + 1\}$ *such that*

(i) *Every uncolored vertex v of $S \setminus \hat{S}$ has at least $6p\Delta$ colors appearing at least twice in $N_S(v) \coloneqq N_{G_1}(v) \cap S$;*

(ii) *Every vertex of S has at most $\frac{19}{20}\Delta^2$ colored G_1^*-neighbors; and*

(iii) *For every index i and every color j, the size of* $\mathrm{Notbig}_{i,j}$ *is at most* $\Delta^{19/10}$.

Proof. For convenience, let us set $C \coloneqq \Delta^2 + 1$. We use the following coloring procedure:

1. Each vertex of S is activated with probability $\frac{9}{10}$.
2. Each activated vertex is assigned a color of $\{1, 2, \ldots, C\}$, independently and uniformly at random.
3. A vertex that receives a color creating a conflict—i.e., assigned to one of its G_1^*-neighbors, or at distance less than p of a color assigned to one of its G_2^*-neighbors—is uncolored.

We aim at applying the Lovász local lemma to prove that with positive probability, the resulting coloring fulfills the three conditions of the lemma. Let v be a vertex of G. We let $E_1(v)$ be the event that v does not fulfill condition (i) and $E_2(v)$ be the event that v does not fulfill condition (ii). For each i, j, let $E_3(i,j)$ be the event that the size of $\mathrm{Notbig}_{i,j}$ exceeds $\Delta^{19/10}$. It suffices to prove that each of those events occurs with probability less than Δ^{-17}. Indeed, each event is mutually independent of all events involving vertices or dense sets at distance more than 4 in G_1^* or G_1'. Moreover, each vertex of any set A_i has at most $\Delta^{5/4}$ external neighbors in G, and $|A_i| \leqslant \Delta^2 + 1$. Thus, each event is mutually independent of all but at most Δ^{16} other events. Consequently, the Lovász local lemma applies since $\Delta^{-17} \times \Delta^{16} <^* \frac{1}{4}$ and yields the sought result.

Hence, it only remains to prove that the probability of each event is at most Δ^{-17}. We use the results cited in section 3. Let us start with $E_2(v)$. We define W to be the number of activated neighbors of v. Thus, $\mathbf{Pr}(E_2(v)) \leqslant \mathbf{Pr}\left(W > \frac{19}{20}\Delta^2\right)$. We set $m := |N(v) \cap S|$, and we may assume that $m > \frac{19}{20}\Delta^2$. The random variable W is a binomial on m variables with probability $\frac{9}{10}$. In particular, its expected value $\mathbf{E}(W)$ is $\frac{9m}{10}$. Applying the Chernoff bound to W with $t = \frac{m}{20}$, we obtain

$$\begin{aligned}\mathbf{Pr}\left(W > \tfrac{19}{20}\Delta^2\right) &\leqslant \mathbf{Pr}\left(|W - \mathbf{E}(W)| > \tfrac{m}{20}\right)\\ &\leqslant 2\exp\left(-\tfrac{m^2 \cdot 10}{400 \cdot 27m}\right) \leqslant^* \Delta^{-17},\end{aligned}$$

since $\frac{19}{20}\Delta^2 < m \leqslant \Delta^2$.

Let $v \in S \setminus \hat{S}$. We now bound $\mathbf{Pr}(E_1(v))$. By Lemma 12, let Ω be a collection of $450p\Delta^3$ pairs of G_1-neighbors of v in S that are not adjacent in G_1^*. We consider the random variable X defined as the number of pairs of Ω whose members (i) are both assigned the same color j, (ii) both retain that color, and (iii) are the only two vertices in $N(v)$ that are assigned j. Thus, X is at most the number of colors appearing at least twice in $N_S(v)$. The probability that some nonadjacent pair of vertices u, w in $N(v)$ satisfies (i) is $\frac{9}{10} \cdot \frac{9}{10} \cdot \frac{1}{C}$. In total, the number of G_1^*-neighbors of v, u, w in H is at most $3\Delta^2$, and the number of G_2^*-neighbors of u and w is at most 4Δ. Therefore, given that they satisfy (i), the vertices u and w also satisfy (ii) and (iii) with probability at least $\left(1 - \frac{1}{C}\right)^{3\Delta^2} \cdot \left(1 - \frac{6p}{C}\right)^{4\Delta}$. Consequently,

$$\mathbf{E}(X) \geqslant 450p\Delta^3 \cdot \frac{81}{100C} \cdot \exp\left(-\frac{3\Delta^2}{C}\right) \exp\left(-\frac{24p \cdot \Delta}{C}\right) >^* 3p\Delta.$$

Hence, if $E_1(v)$ holds, then X must be smaller than its expected value by at least $p\Delta$. But we assert that

$$\mathbf{Pr}\left(\mathbf{E}(X) - X > p\Delta\right) \leqslant^* \Delta^{-17}, \tag{1}$$

which will yield the desired result.

To establish equation (1), we apply Talagrand's inequality, stated in section 3. We set X_1 to be the number of colors assigned to at least two vertices in $N(v)$, including both members of at least one pair in Ω, and X_2 is the number of colors that (i) are assigned to both members of at least one pair in Ω and (ii) create a conflict with one of their neighbors, or are also assigned to at least one other vertex in $N(v)$. Note that $X = X_1 - X_2$. Therefore, by what precedes, if $E_1(v)$ holds, then either X_1 or X_2 must differ from its expected value by at least $\frac{1}{2}p\Delta$. Notice that

$$\mathbf{E}(X_2) \leqslant \mathbf{E}(X_1) \leqslant C \cdot 450p\Delta^3 \cdot \frac{1}{C^2} \leqslant 450p\Delta.$$

If $X_1 \geqslant t$, then there is a set of at most $4t$ trials whose outcomes certify this, namely, the activation and color assignment for t pairs of variables. Moreover, changing the outcome of any random trial can only affect X_1 by at most 1, since X_1 can decrease by 1 in case the old color is not counted anymore and increase by 1 in case the new color was not counted before and is counted now. Thus Talagrand's inequality applies and, since $\mathbf{E}(X_1) \geqslant \mathbf{E}(X) >^* 3p\Delta$, we obtain

$$\mathbf{Pr}\left(|X_1 - \mathbf{E}(X_1)| > \frac{1}{2}p\Delta\right) \leqslant 4\exp\left(-\frac{p^2\Delta^2}{4\cdot 32\cdot 1\cdot 4\cdot 450p\Delta}\right) \leqslant^* \frac{1}{2}\Delta^{-17}.$$

Similarly, if $X_2 \geqslant t$, then there is a set of at most $6t$ trials whose outcomes certify this fact, namely, the activation and color assignment of t pairs of vertices and, for each of these pairs, the activation and color assignment of a color creating a conflict to a neighbor of a vertex of the pair. As previously, changing the outcome of any random trial can only affect X_2 by at most $2p$. Therefore by Talagrand's inequality, if $\mathbf{E}(X_2) \geqslant \frac{1}{2}p\Delta$, then

$$\mathbf{Pr}\left(|X_2 - \mathbf{E}(X_2)| > \frac{1}{2}p\Delta\right) \leqslant 4\exp\left(-\frac{p^2\Delta^2}{4\cdot 32\cdot 4p^2\cdot 6\cdot 450p\Delta}\right) \leqslant^* \frac{1}{2}\Delta^{-17}.$$

If $\mathbf{E}(X_2) < \frac{1}{2}p\Delta$, then we consider a binomial random variable that counts each vertex of $N_S(v)$ independently with probability $\frac{1}{4|N_S(v)|}p\Delta$. We let X_2' be the sum of this random variable and X_2. Note that $\frac{1}{4}p\Delta \leqslant \mathbf{E}(X_2') \leqslant \frac{3}{4}p\Delta$ by linearity of expectation. Moreover, observe that if $|X_2 - \mathbf{E}(X_2)| > \frac{1}{2}p\Delta$, then $|X_2' - \mathbf{E}(X_2')| > \frac{1}{4}p\Delta$. Therefore, by applying Talagrand's inequality to X_2' with $c = 2p$, $r = 6$, and $t = \frac{1}{4}p\Delta \in [60c\sqrt{r\,\mathbf{E}(X_2')}\,, \mathbf{E}(X_2')]$, we also obtain in this case

$$\begin{aligned}\mathbf{Pr}\left(|X_2 - \mathbf{E}(X_2)| > \frac{1}{2}p\Delta\right) &\leqslant \mathbf{Pr}\left(|X_2' - \mathbf{E}(X_2')| > \frac{1}{4}p\Delta\right)\\ &\leqslant 4\exp\left(-\frac{2\cdot p^2\Delta^2}{16\cdot 32\cdot 4p^2\cdot 6\cdot p\Delta}\right) \leqslant^* \frac{1}{2}\Delta^{-17}.\end{aligned}$$

Consequently, we infer that $\mathbf{Pr}\left(\mathbf{E}(X) - X > \Delta\right) \leqslant^* \Delta^{-17}$, as desired.

It remains now to deal with $E_3(i,j)$. We use Lemma 13. For each i, every vertex of A_i has at most $\Delta^{5/4}$ external neighbors. Moreover, for each color j, each such neighbor is activated and assigned a color in $[j-p+1\,, j+p-1]$ with probability at most $\frac{9}{10}\cdot\frac{(2p-1)}{C} <^* \frac{1}{\Delta^{5/4}\cdot\Delta^{2/5}}$. As these assignments are made independently, the conditions of Lemma 13 are fulfilled, so we deduce that the probability that $E_3(i,j)$ holds is at most $\exp\left(-\Delta^{1/20}\right) \leqslant^* \Delta^{-17}$. Thus, we obtained the desired upper bound on $\mathbf{Pr}(E_3(i,j))$, which concludes the proof. ∎

4.3.2. Second step. In the second step, we extend the partial coloring of S to all the vertices of $\hat{S}$. To do so, we need the following general lemma, which will also be used in the next subsection to color the vertices of the sets $B_i \cup C_i$. Its proof is long and technical, so we postpone it to section 6.

LEMMA 15. *Let F be a subset of V^* with a partial nice coloring and H be a set of uncolored vertices of F. For each vertex u of H, let $L(u)$ be the colors available to color u, that is, that create no conflict with the already colored vertices of $F \cup H$. We assume that for every vertex u, $|L(u)| \geqslant 11p\Delta^{33/20}$ and $|L(u)| \geqslant \delta^1_H(u) + 6p\Delta$.*

Then, the partial nice coloring of F can be extended to a nice coloring of H such that for every index $i \in \{1, 2, \ldots, \ell\}$ and every color j, the size of $\mathrm{Notbig}_{i,j}$ *increases by at most $\Delta^{19/10}$.*

Consider a partial nice coloring of S obtained in the first step. In particular, $\left|\mathrm{Notbig}_{i,j}\right| \leqslant \Delta^{19/10}$. We wish to ensure that every vertex of $\hat{S}$ is colored. This can be done greedily, but to be able to continue the proof we need to have more control on the coloring. We apply Lemma 15 to the set H of uncolored vertices in $\hat{S}$. For each vertex $u \in H$, the list $L(u)$ is initialized as the list of colors that can be assigned to u without creating any conflict. By Lemmas 11–14(ii), $|L(u)| \geqslant \frac{1}{20}\Delta^2 - 4p\Delta \geqslant^* 11p\Delta^{33/20}$.

Suppose that u is in no set Big_i. Then $\delta^1_S(u) \leqslant \deg^1_S(u) \leqslant \Delta^2 - 90p\Delta$, and u has at most Δ G_2^*-neighbors. Hence, we infer that $|L(u)| \geqslant \delta^1_H(u) + 88p\Delta$. Assume now that u belongs to some set Big_i. By Lemma 11(i)–(ii), we have $\delta^1(u) \leqslant \Delta^2 - 8p\Delta$ and $\delta^2(u) \leqslant \Delta$. So, $|L(u)| \geqslant \delta^1_H(u) + 8p\Delta - 2p\Delta = \delta^1_H(u) + 6p\Delta$.

Therefore, by Lemma 15 we can extend the partial nice coloring of S to $\hat{S}$ such that $\left|\mathrm{Notbig}_{i,j}\right| \leqslant 2\Delta^{19/10}$ for every index i and every color j.

4.4. Coloring the sets B_i and C_i. Let H be the set of vertices which are uncolored at this stage. Then H is the union of $\bigcup_{i=1}^{\ell}(B_i \cup C_i)$ and the set of uncolored vertices of $S \setminus \hat{S}$.

We first apply Lemma 15 to extend the partial nice coloring of S to the vertices of H in such a way that $\mathrm{Notbig}_{i,j}$ does not grow too much, for every index i and color j. Next, we show that the good coloring derived from this nice coloring verifies the conditions of Lemma 8.

For each vertex u of H, let $L(u)$ be the list of colors that would not create any conflict with the already colored vertices. If $u \in \bigcup_{i=1}^{\ell}(B_i \cup C_i)$, by Lemma 11(iii), $\delta^1(u) \leqslant \frac{3}{4}\Delta^2$. Hence, $|L(u)| \geqslant \frac{1}{4}\Delta^2 + \delta^1_H(u) - 4p\Delta \geqslant^* \max\left(11p\Delta^{33/20}, \delta^1_H(u) + 6p\Delta\right)$. If $u \in S \setminus \hat{S}$, then by Lemma 14(i), $|L(u)| \geqslant \delta^1_H(u) + 6p\Delta$.

Therefore, by Lemma 15, we extend the partial nice coloring of the vertices of S to the vertices of H. Moreover, for each index i and each color j, the size of each $\mathrm{Notbig}_{i,j}$ is at most $3\Delta^{19/10}$.

Consider now the partial good coloring of V_1 associated to this nice coloring. Let us show that it verifies the conditions of Lemma 8. By the definition, it satisfies condition (i). Hence, it only remains to show that condition (iv) holds.

Fix an index i and a color j. Recall that Big_i is a clique, so there is at most one vertex of Big_i of each color. Consequently, the number of vertices of A_i with a G_1-neighbor in Big_i colored j is at most $\max\left(2 \cdot \frac{1}{4}\Delta^2, \frac{3}{4}\Delta^2\right) = \frac{3}{4}\Delta^2$, by Lemma 5(c). Besides, the number of vertices of A_i with a G_2-neighbor in Big_i with a color in $[j-p+1\,, j+p-1]$ is at most $4p\Delta$. Finally, the number of vertices of A_i with either a G_1-neighbor not in $\mathrm{Big}_i \cup D_i$ colored j or a G_2-neighbor not in $\mathrm{Big}_i \cup D_i$ with a color in $[j-p+1\,, j+p-1]$ is at most $\left|\mathrm{Notbig}_{i,j}\right| \leqslant 3\Delta^{19/10}$. Thus, all together, the number of vertices of A_i with a G_1-neighbor not in $B_i \cup C_i$ colored j or a G_2-neighbor not in $B_i \cup C_i$ with a color in $[j-p+1\,, j+p-1]$ is at most

$$\frac{3}{4}\Delta^2 + 3\Delta^{19/10} + 4p\Delta \leqslant^* \frac{4}{5}\Delta^2,$$

as desired.

This concludes the proof of Lemma 8.

5. The proof of Lemma 9. We consider a good coloring of V satisfying the conditions of Lemma 8. The procedure we apply is composed of two phases. In the first phase, a random permutation of a subset of the colors is assigned to the vertices of A_i. In doing so, we might create two kinds of conflicts: a vertex of A_i colored j might have an external G_1-neighbor colored j or a G_2-neighbor with a color in

$[j-p+1\,,j+p-1]$. We shall deal with these conflicts in a second phase. To be able to do so, we first ensure that the coloring obtained in the first phase fulfills some properties.

PROPOSITION 16.

$$|A_i| + |B_i| + \frac{1}{2}\,|V(M_i)| \leqslant \Delta^2 + 1.$$

Proof. By the maximality of M_i, for every edge $e = xy$ of M_i there is at most one vertex v_e of K_i that is adjacent to both x and y in $\overline{H_i}$. Hence, every edge e of M_i has an endvertex $n(e)$ that is adjacent in H_i to every vertex of K_i except possibly one, called $x(e)$. By Lemmas 5 and 6,

$$|K_i| = |A_i| + |B_i| \geqslant \Delta^2 - 8000p\Delta - 2.10^3 p\Delta \geqslant^* 10^3 p\Delta > |M_i|\,.$$

So there exists a vertex $v \in A_i \cup B_i \setminus \cup_{e\in M_i} x(e)$. The vertex v is adjacent in G_1 to all the vertices of K_i (except itself) and all the vertices $n(e)$ for $e \in M_i$. So

$$|K_i| - 1 + \frac{1}{2}\,|V(M_i)| \leqslant \deg^1(v) \leqslant \Delta^2. \qquad \square$$

Phase 1. For each set A_i, we choose a subset of $a_i \coloneqq |A_i|$ colors as follows. First, we exclude all the colors that appear on the vertices of $B_i \cup C_i$. Moreover, if a color j is assigned to at least $2p-1$ pairs of vertices matched by M_i, we exclude not only the color j but also the colors in $[j-p+1\,,j+p-1]$. By Proposition 16 and because every edge of M_i is monochromatic by Lemma 8(i), we infer that at least a_i colors have not been excluded. Then we assign a random permutation of those colors to the vertices of A_i. We let Temp_i be the subset of vertices of A_i with an external G_1-neighbor of the same color or a G_2-neighbor with a color at distance less than p.

LEMMA 17. *With positive probability, the following hold:*

- (i) *For each i, $|\mathrm{Temp}_i| \leqslant 3\Delta^{5/4}$.*
- (ii) *For each index i and each color j, at most $\Delta^{19/10}$ vertices of A_i have a G_1-neighbor in $\cup_{k\neq i} A_k$ colored j or a G_2-neighbor in $\cup_k A_k$ with a color in $[j-p+1\,,j+p-1]$.*

Proof. We use the Lovász local lemma. For every index i, we let $E_1(i)$ be the event that $|\mathrm{Temp}_i|$ is greater than $3\Delta^{5/4}$. For each index i and each color j, we define $E_2(i,j)$ to be the event that condition (ii) is not fulfilled. Each event is mutually independent of all events involving dense sets at distance greater than 2, so each event is mutually independent of all but at most Δ^9 other events. According to the Lovász local lemma, it is enough to show that each event has probability at most Δ^{-10}, since $\Delta^9 \times \Delta^{-10} <^* \frac{1}{4}$.

Our first goal is to upper bound $\mathbf{Pr}(E_1(i))$. We may assume that both the color assignments for all cliques other than A_i and the choice of the a_i colors to be used on A_i have already been made. Thus it only remains to choose a random permutation of those a_i colors onto the vertices of A_i. Since every vertex $v \in A_i$ has at most $\Delta^{5/4}$ external neighbors and Δ G_2-neighbors, the probability that $v \in \mathrm{Temp}_i$ is at most $(\Delta^{5/4} + 2p\Delta)/a_i$. So we deduce that $\mathbf{E}(|\mathrm{Temp}_i|) \leqslant \Delta^{5/4} + 2p\Delta$. We define a binomial random variable B that counts each vertex of A_i independently with probability $\Delta^{5/4}/(2a_i)$. We set $X \coloneqq |\mathrm{Temp}_i| + B$. By linearity of expectation,

$$\frac{1}{2}\Delta^{5/4} \leqslant \mathbf{E}(X) = \mathbf{E}(|\mathrm{Temp}_i|) + \frac{1}{2}\Delta^{5/4} \leqslant^* 2\Delta^{5/4}.$$

Moreover, if $|\mathrm{Temp}_i| > 3\Delta^{5/4}$, then $|\mathrm{Temp}_i| - \mathbf{E}(|\mathrm{Temp}_i|) > \Delta^{5/4}$, and hence $X - \mathbf{E}(X) > \frac{1}{2}\Delta^{5/4}$. We now apply McDiarmid's inequality to show that X is concentrated. Note that if $|\mathrm{Temp}_i| \geqslant s$, then the colors to $2s$ vertices (that is, s members of Temp_i and one neighbor for each) certify that fact. Moreover, switching the colors of two vertices in A_i may only affect whether those two vertices are in Temp_i and whether at most four vertices with a color at distance less than 2 are in Temp_i. So we may apply McDiarmid's inequality to X with $c = 6, r = 2$ and $t = \frac{1}{2}\Delta^{5/4} \in [60c\sqrt{r\,\mathbf{E}(X)}\,, \mathbf{E}(X)]$. We deduce that the probability that the event $E_1(i)$ holds is at most

$$\begin{aligned}\mathbf{Pr}\left(|\mathrm{Temp}_i| - \mathbf{E}\left(|\mathrm{Temp}_i|\right) > \Delta^{5/4}\right) &\leqslant \mathbf{Pr}\left(|X - \mathbf{E}(X)| > \frac{1}{2}\Delta^{5/4}\right)\\ &< 4\exp\left(-\frac{\Delta^{5/2}}{4\times 32\times 36\times 2\Delta^{5/4}}\right)\\ &<^* \Delta^{-10}.\end{aligned}$$

We now upper bound $\mathbf{Pr}(E_2(i,j))$. To this end, we use Lemma 13. Recall that the vertices of A_i get different colors. Every vertex $v \in A_i$ has at most $\Delta^{5/4}$ external neighbors and Δ G_2-neighbors. We set $Q \coloneqq \Delta^{5/4} + \Delta$. We let $S(v)$ be the set of all vertices that are either external G_1-neighbors of v or G_2-neighbors of v. Hence, $|S(v)| \leqslant Q$. Note that each vertex is in at most $\Delta^{5/4}$ sets $S(v)$ for $v \in A_i$. Each vertex of a set $S(v)$ is assigned a color in $[j-p+1\,, j+p-1]$ with probability at most

$$\max_k \frac{2p-1}{a_k} <^* \frac{1}{(2p-1)Q\times\Delta^{2/5}},$$

because $\min a_k \geqslant \Delta^2 - 9000p\Delta^{7/4}$ by Observation 7. Moreover, at most $2p-1$ vertices in each set A_k are assigned a color in $[j-p+1\,, j+p-1]$. As the random permutations for different cliques are independent, Lemma 13 implies that the probability that more than $\Delta^{37/20}$ vertices of A_i have an external G_1-neighbor in some A_k colored j or a G_2-neighbor in some A_k colored in $[j-p+1\,, j+p-1]$ is at most $\exp\left(-\Delta^{1/20}\right) <^* \Delta^{-10}$. This concludes the proof. ∎

Phase 2. We consider a coloring γ satisfying the conditions of Lemma 17. For each set A_i and each vertex $v \in \mathrm{Temp}_i$ we let $\mathrm{Swappable}_v$ be the set of vertices u such that

(a) $u \in A_i \setminus \mathrm{Temp}_i$;
(b) $\gamma(u)$ does not appear on an external G_1-neighbor of v;
(c) $\gamma(v)$ does not appear on an external G_1-neighbor of u;
(d) No color of $[\gamma(u)-p+1\,, \gamma(u)+p-1]$ appears on a G_2-neighbor of v; and
(e) No color of $[\gamma(v)-p+1\,, \gamma(v)+p-1]$ appears on a G_2-neighbor of u.

LEMMA 18. *For every $v \in \mathrm{Temp}_i$, the set* $\mathrm{Swappable}_v$ *contains at least* $\frac{1}{10}\Delta^2$ *vertices.*

Proof. Let us upper bound the number of vertices that are not in $\mathrm{Swappable}_v$. By Lemma 17(i), at most $3\Delta^{5/4}$ vertices of A_i violate condition (a) and at most $\Delta^{5/4}$ vertices violate condition (b) by the definition of A_i. As v has at most Δ neighbors in G_2, the number of vertices violating condition (d) is at most $2p\Delta$. According to Lemma 8(ii), the number of vertices of A_i violating condition (c) or (e) because of a neighbor not in $\left(\cup_{k=1}^{\ell} A_k\right) \cup (B_i \cup C_i)$ is at most $\frac{4}{5}\Delta^2$. Moreover, by the way we chose the a_i colors for A_i, for any color $\alpha \in [\gamma(v)-p+1\,, \gamma(v)+p-1] \setminus \{\gamma(v)\}$, at most $2\cdot(2p-2)$ vertices of M_i and one vertex of B_i are colored α. Each of these vertices has at most Δ neighbors in G_2. Hence, as there are $2p-2$ choices for the color α, the number of vertices violating condition (e) because of a neighbor in $B_i \cup C_i$ is at most

$$(2p-2)\cdot(2\cdot(2p-2)+1)\cdot\Delta = (8p^2-14p+6)\cdot\Delta.$$

Finally, the number of vertices violating condition (c) or (e) because of a color assigned during Phase 1 is at most $\Delta^{19/10}$ thanks to Lemma 17(ii). Therefore, we deduce that

$$|\mathrm{Swappable}_v| \geqslant |A_i| - \frac{4}{5}\Delta^2 - \Delta^{19/10} - 4\Delta^{5/4} - (8p^2-14p+2p+6)\cdot\Delta - 1 \geqslant^* \frac{1}{10}\Delta^2,$$

as $|A_i| \geqslant \Delta^2 - 9000p\Delta^{\frac{7}{4}}$ by Observation 7. □

For each index i and each vertex $v \in \mathrm{Temp}_i$, we choose 100 uniformly random members of $\mathrm{Swappable}_v$. These vertices are called *candidates* of v.

DEFINITION 19. *A candidate u of v is* unkind *if either*

(a) *u is a candidate for some other vertex;*

(b) *v has an external neighbor w that has a candidate w' with the same color as u;*

(c) *v has a G_2-neighbor w that has a candidate w' with a color in $[\gamma(u)-p+1\,,\gamma(u)+p-1]$;*

(d) *v has an external neighbor w that is a candidate for exactly one vertex w', with $\gamma(w')=\gamma(u)$;*

(e) *v has a G_2-neighbor w that is a candidate for exactly one vertex w', which has a color in $[\gamma(u)-p+1\,,\gamma(u)+p-1]$;*

(f) *u has an external neighbor w that has a candidate w' with the same color as v;*

(g) *u has a G_2-neighbor w that has a candidate w' with a color in $[\gamma(v)-p+1\,,\gamma(v)+p-1]$;*

(h) *u has an external neighbor w that is a candidate for exactly one vertex w' with the same color as v; or*

(i) *u has a G_2-neighbor w that is a candidate for exactly one vertex w' with a color in $[\gamma(v)-p+1\,,\gamma(v)+p-1]$.*

A candidate of v is kind *if it is not unkind.*

LEMMA 20. *With positive probability, for each index i, every vertex of* Temp_i *has a kind candidate.*

We choose candidates satisfying the preceding lemma. For each vertex $v \in \mathrm{Temp}_i$ we swap the color of v and one of its kind candidates. The obtained coloring is the desired one. So to conclude the proof of Lemma 9, it only remains to prove Lemma 20.

Proof of Lemma 20. For every vertex v in some Temp_i, let $E_1(v)$ be the event that v does not have a kind candidate. Each event is mutually independent of all events involving dense sets at distance greater than 2. So each event is mutually independent of all but at most Δ^9 other events. Hence, if we prove that the probability of each event is at most Δ^{-10}, then the conclusion would follow from the Lovász local lemma since $\Delta^{-10}\cdot\Delta^9 <^* \frac{1}{4}$.

Observe that the probability that a particular vertex of $\mathrm{Swappable}_v$ is chosen is $100/|\mathrm{Swappable}_v|$, which is at most $1000\Delta^{-2}$.

We wish to upper bound $\mathbf{Pr}(E_1(v))$ for an arbitrary vertex $v \in \mathrm{Temp}_i$, so we can assume that all vertices but v have already chosen candidates. Hence, we can consider that the candidates are chosen in two rounds: in the first round, we choose the candidates for all vertices but v; in the second round we choose the candidates for v.

Let Y be the number of vertices $u \in \mathrm{Swappable}_v$ that meets conditions (f)–(i) of the definition of unkind; note that Y is determined by the candidates selected in the first round. We shall use Lemma 13 to show that with high probability, Y is not too large. For each vertex $u \in \text{Swappable}_v$, we define N_u to be the set of external and G_2-neighbors of u. Since every vertex in V_2 has at most $\Delta^{5/4}$ external neighbors and at most Δ G_2-neighbors, each set N_u has size at most $\Delta^{5/4} + \Delta \leqslant 2\Delta^{5/4} = Q$ and no vertex lies in more than $\Delta^{5/4} + \Delta \leqslant \Delta^{9/5}$ of these sets. We consider a vertex in $\bigcup_{u \in \text{Swappable}_v} N_u$ to be *marked* if it chooses a candidate with color in $[\gamma(v) - p + 1\,, \gamma(v) + p - 1]$ or it is chosen as a candidate for a vertex with color in $[\gamma(v) - p + 1\,, \gamma(v) + p - 1]$. Each of these vertices has at most $2p - 1$ potential candidates with color in $[\gamma(v) - p + 1\,, \gamma(v) + p - 1]$ and can be chosen for at most $2p - 1$ vertices with color in $[\gamma(v) - p + 1\,, \gamma(v) + p - 1]$. So the probability that a vertex is marked is at most $(4p-2)10^3/\Delta^2 <^* 1/(Q \times \Delta^{2/5})$. Furthermore, it is easy to check that for any set of $s \geqslant 1$ vertices, the probability that all are marked is at most $(\frac{1}{Q \cdot \Delta^{2/5}})^s$. Therefore, by Lemma 13

$$\mathbf{Pr}(Y > \Delta^{\frac{37}{20}}) \leqslant \exp\left(-\Delta^{1/20}\right) \leqslant \frac{1}{2}\Delta^{-10}.$$

We now analyze the second round. By Lemma 17(i), the number of vertices that satisfy condition (a) of Definition 19 is at most $300\Delta^{5/4}$. Note that the vertex v has at most $\Delta^{5/4}$ external neighbors, each having at most 100 candidates. Since each color appears on at most one member of Swappable_v, we deduce that the number of vertices satisfying one of the conditions (b) and (d) is at most $101\Delta^{5/4}$. Similarly, the number of vertices satisfying one of the conditions (c) and (e) is at most $202p\Delta$. If $Y \leqslant \Delta^{37/20}$, then the number of unkind members of Swappable_v is at most $\Delta^{37/20} + 300\Delta^{5/4} + 101\Delta^{5/4} + 202p\Delta <^* 2\Delta^{37/20}$. So the probability that v chooses an unkind candidate during the second round is at most

$$\left(\frac{2\Delta^{37/20}}{\Delta^2/1000}\right)^{100} \leqslant^* \frac{1}{2}\Delta^{-10}.$$

Consequently, the probability that $E_1(v)$ holds is at most $\frac{1}{2}\Delta^{-10} + \frac{1}{2}\Delta^{-10} = \Delta^{-10}$, as desired. ☐

6. The proof of Lemma 15. In this subsection we prove Lemma 15. The proof is similar to that of a lemma of Molloy and Reed [21, Lemma 31]. However the existence of the G_2-edges introduce many small technical changes. Therefore, for clarity and to make this paper self-contained, we include a complete proof rather than simply an explanation of the changes that would be necessary to that proof in order to prove our result.

We color H using a two-phase quasi-random procedure.

Phase 1. We fix a small real number $\varepsilon \in (0\,, \frac{1}{10000}]$ and carry out $K := 2\Delta^{\varepsilon}\log\Delta$ iterations. In each iteration, we analyze the following random procedure, which produces a partial coloring. Note that at every time of the procedure, $|L(v)| \geqslant \delta^1_U(v) + 2p\Delta$ for every vertex v of H, where U is the subgraph of H induced by the uncolored vertices.

1. Each uncolored vertex of H is activated with probability $\alpha := \Delta^{-\varepsilon}$.
2. Each activated vertex v chooses a uniformly random color $\lambda(v) \in L(v)$.
3. If two activated neighbors create a conflict, both are uncolored.
4. Each activated vertex u that is still colored is uncolored with probability $q(v)$, where $q(v)$ is defined so that v has probability exactly $\frac{1}{2}\alpha$ of being activated and retaining its color.

5. For each vertex v that retains a color, we remove from the lists of each yet uncolored vertex every color whose assignment to this vertex would create a conflict.

First, we have to show that the parameter $q(v)$ is well defined. Let $N_1(v)$ be the set of all uncolored G_1^*-neighbors of v. Given that v is activated, the probability that it is uncolored in the third step of the procedure is at most

$$\begin{aligned}\sum_{j\in L(v)} \mathbf{Pr}(\lambda(v)=j)\times \sum_{u\in N_1(v)} \alpha\,\mathbf{Pr}(\lambda(u)\in[j-p+1\,,j+p-1])\\ = \frac{\alpha}{|L(v)|}\sum_{u\in N_1(v)}\sum_{j\in L(v)} \mathbf{Pr}(\lambda(u)\in[j-p+1\,,j+p-1])\\ \leqslant \frac{\alpha}{|L(v)|}\sum_{u\in N_1(v)}\sum_{k\in L(u)}(2p-1)\cdot\mathbf{Pr}(\lambda(v)=k)\\ \leqslant \frac{\alpha}{|L(v)|}\sum_{u\in N_1(v)}(2p-1)\\ = (2p-1)\alpha\frac{|N_1(v)|}{|L(v)|}\leqslant(2p-1)\alpha <^* \frac{1}{2},\end{aligned}$$

since $|L(v)| > |N_1(v)|$. Thus, the probability of being activated and not being uncolored after the third step of the procedure is more than $\frac{1}{2}\alpha$. So $q(v)$ is well defined.

LEMMA 21. *After K iterations, with positive probability,*

(i) *Each vertex of $\cup_{i=1}^{\ell} A_i$ has at most $\Delta^{200\varepsilon}$ uncolored external neighbors in H;*

(ii) *Each vertex of H has at most $\Delta^{200\varepsilon}$ uncolored neighbors in H; and*

(iii) *For every i and every color j, the size of* $\mathrm{Notbig}_{i,j}$ *grows by at most $\frac{1}{2}\Delta^{19/10}$.*

We postpone the proof of this lemma to the end of this section. We choose a partial coloring of H that verifies the conditions of the preceding lemma and proceed with Phase 2.

Phase 2. For every uncolored vertex of H, let $L_1(v)$ be the list of available colors after Phase 1. At most $\delta_H^*(v) \leqslant \delta_H^1(v) + 4p\Delta$ colors have been removed from $L(v)$. Hence, $|L_1(v)| \geqslant 2p\Delta$. We apply the following procedure:

1. For each uncolored vertex v of H, we choose a uniformly random subset $L'(v) \subset L_1(v)$ of size $2p\Delta^{200\varepsilon}$.
2. We color all such vertices v from their sublist $L'(v)$, greedily one at a time.

Observe that the second step is possible thanks to Lemma 21(ii). Thus, we obtain a good coloring of H. It only remains to prove that it fulfills the condition of Lemma 15. To this end, we first establish the following result about the coloring constructed in Phase 2.

LEMMA 22. *With positive probability, for every i and every color j, the size of* $\mathrm{Notbig}_{i,j}$ *grows by at most $\frac{1}{2}\Delta^{19/10}$ during Phase* 2.

Proof. We want to apply the Lovász local lemma. For each set A_i and each color j, let $E(i,j)$ be the event that more than $\frac{1}{2}\Delta^{19/10}$ vertices of A_i have neighbors outside of $\mathrm{Big}_i \cup D_i$ with a color in $[j-p+1\,, j+p-1]$ in their sublist. We bound $\mathbf{Pr}\,(E(i,j))$ using Lemma 13. By Lemma 21(i), every vertex of A_i has at most $Q \coloneqq \Delta^{200\varepsilon}$ uncolored external neighbors in H. Each such neighbor u chooses a color in $[j-p+1\,, j+p-1]$ in its sublist with probability at most $4p^2\Delta^{200\varepsilon}/\,|L_1(u)| <^* \frac{1}{Q\Delta^{2/5}}$, because $|L_1(u)| \geqslant 2p\Delta$. Besides, these assignments are made independently. So, as $\frac{1}{2}\Delta^{19/10} >^* \Delta^{37/20}$, Lemma 13 yields that $\mathbf{Pr}\,(E(i,j)) < \exp(-\Delta^{1/20}) <^* \Delta^{-10}$.

Observe that each event is mutually independent of all events involving dense sets at distance more than 2, and each dense set is adjacent to at most $8000p\Delta^3$ other dense sets. As a result, each event is mutually independent of all but at most Δ^9 other events. Consequently, the Lovász local lemma applies and yields the conclusion. $\square$

Using the last two lemmas, we can prove Lemma 15.

Proof of Lemma 15. We consider a coloring obtained after Phases 1 and 2. By Lemmas 21(iii) and 22, $\mathrm{Notbig}_{i,j}$ grows by at most $\frac{1}{2}\Delta^{19/10}$ during each phase for every index i and every color j. $\square$

Thus, to complete the proof, it only remains to prove Lemma 21. To this end, we inductively obtain an upper bound U_k on the number of uncolored external G'_1-neighbors of a vertex of $\cup_{i=1}^{l} A_i$ after the kth iteration and lower and upper bounds $m_k^-(v)$ and $m_k^+(v)$ on the number of neighbors in U of a vertex v after the kth iteration. Let $\theta \coloneqq (1-\frac{1}{2}\Delta^{-\varepsilon})$. Note that $\theta > \frac{1}{2}$ since $\Delta^{\varepsilon} >^* 1$. We set

$$U_0 \coloneqq \Delta^{5/4} \quad \text{and} \quad \forall k>0,\ U_k \coloneqq \theta U_{k-1} + U_{k-1}^{49/50}, \tag{2}$$

and for every vertex v,

$$m_0^+(v) \coloneqq \delta_H^1(v) \quad \text{and} \quad \forall k>0,\ m_k^+(v) \coloneqq \theta m_{k-1}^+(v) + m_{k-1}^+(v)^{49/50}, \tag{3}$$

$$m_0^-(v) \coloneqq \delta_H^1(v) \quad \text{and} \quad \forall k>0,\ m_k^-(v) \coloneqq \theta m_{k-1}^-(v) - m_{k-1}^-(v)^{49/50}. \tag{4}$$

As shown by Lemma 34 of [21], parameters satisfying the above inequalities fulfill some useful properties.

LEMMA 23. *The following hold:*

(i) *If $U_k \geqslant \Delta^{150\varepsilon}$, then $U_k \leqslant^* 2\theta^k U_0$.*

(ii) *If $m_k^-(v) \geqslant \Delta^{150\varepsilon}$, then*

$$\frac{1}{2}\theta^k \delta_H^1(v) \leqslant^* m_k^-(v) \leqslant m_k^+(v) \leqslant^* 2\theta^k \delta_H^1(v).$$

Proof of Lemma 21. We apply the Lovász local lemma to each iteration of the procedure to prove inductively that with positive probability, after $k \leqslant K$ iterations the following hold:

(a) If $U_k \geqslant \frac{1}{2}\Delta^{200\varepsilon}$, then every vertex in $\cup_{i=1}^{l} A_i$ has at most U_k uncolored external G'_1-neighbors in H.

(b) For every vertex v of H, if $m_k^-(v) \geqslant \frac{1}{8}\Delta^{200\varepsilon}$, then $m_{k-1}^-(v) \leqslant \delta_U^1(v) \leqslant m_{k-1}^+(v)$.

(c) For every index i and every color j, the size of $\mathrm{Notbig}_{i,j}$ increases by at most $\frac{1}{4\log\Delta}\Delta^{19/10-\varepsilon}$ during iteration k.

Assuming this, we can finish the proof as follows. Note that

$$2\theta^K \Delta^2 <^* 1 <^* \Delta^{150\varepsilon}.$$

Since $U_0 = \Delta^{5/4}$ and $\delta_H^1(v) \leqslant \Delta^2$ for every vertex v, the contrapositive of Lemma 23 implies that both U_K and $m_K^-(v)$ are less than $\Delta^{150\varepsilon} \leqslant^* \frac{1}{8}\Delta^{200\varepsilon}$. Furthermore, all these parameters decrease with k. Note that U_k and $m_k^+(v)$ decrease by less than half at each iteration, and this is also true for $m_k^-(v)$ provided it is large enough, e.g., if $m_k^-(v) \geqslant \Delta^{150\varepsilon}$. Therefore, as $\Delta^{200\varepsilon} < \Delta^{5/4}$, there exist two integers k_1 and $k_2(v)$, both at most K, such that

$$\frac{1}{2}\Delta^{200\varepsilon} \leqslant U_{k_1} < \Delta^{200\varepsilon}$$

and, if $\delta^1_H(v) > \Delta^{200\varepsilon}$, then

$$\frac{1}{8}\Delta^{200\varepsilon} \leqslant m^-_{k_2(v)}(v) < \frac{1}{4}\Delta^{200\varepsilon}.$$

Note that the number of uncolored vertices cannot increase; therefore applying (a) at iteration k_1 yields (i). Similarly, applying (b) at iteration $k_2(v)$ yields (ii), since $m^+_{k_2(v)}(v) \leqslant 4m^-_{k_2(v)}(v) < \Delta^{200\varepsilon}$, using Lemma 23(ii). Finally, (iii) follows from (c) because the number of iterations is $K = 2\Delta^\varepsilon \log \Delta$.

It only remains to prove (a), (b), and (c). We proceed by induction on k, the three assertions holding trivially when $k = 0$. Let k be a positive integer such that the assertions hold for all smaller integers.

For every uncolored vertex v of $\cup_{i=1}^l A_i$, we define $E_1(v)$ to be the event that v violates (a). For every vertex u of H, we define $E_2(u)$ to be the event that u violates (b). For every index i and each color j, we define $E_3(i, j)$ to be the event that $\text{Notbig}_{i,j}$ violates (c). Each event is mutually independent of all other events involving vertices or dense sets at distance more than 4 in G^*_1 and hence is mutually independent of all but at most Δ^{16} other events. We prove that each event $E_1(v), E_2(v)$ and $E_3(i, j)$ occurs with probability at most Δ^{-17}. Consequently, the Lovász local lemma applies since $3\Delta^{-17} \cdot \Delta^{16} <^* \frac{1}{4}$, and therefore with positive probability none of these events occurs.

Bounding $\mathbf{Pr}(E_3(i, j))$. Fix an index i and a color j. We apply Lemma 13 with $Q \coloneqq \max(U_{k-1}, \Delta^{200\varepsilon})$. By induction, we know that every vertex in A_i has at most Q uncolored external G'_1-neighbors at the beginning of iteration k. Moreover, the probability that a vertex v of H is assigned a color in $[j - p + 1 , j + p - 1]$ is at most $\frac{2p}{|L(v)|}$. Note that these color assignments are independent. Consequently, provided that $|L(v)| \geqslant 2pQ\Delta^{2/5}$, Lemma 13 implies that $\mathbf{Pr}(E_3(i, j)) < \exp(-\Delta^{1/20}) \leqslant^* \Delta^{-17}$, since $\frac{\Delta^{19/10-\varepsilon}}{4 \log \Delta} \geqslant^* \Delta^{37/20}$.

Now, let us show that $|L(v)| \geqslant 2pQ\Delta^{2/5}$. Note that at most $\delta^*_H(v)$ colors can be removed from $L(v)$, so by hypothesis $|L(v)| \geqslant 2p\Delta$. This remark establishes the result if $Q \leqslant \Delta^{3/5}$. Notice that $\Delta^{200\varepsilon} < \Delta^{3/5}$, since $\varepsilon < \frac{3}{1000}$. So we may assume now that $U_{k-1} > \Delta^{3/5}$, and hence $Q = U_{k-1}$. Recall that at the beginning $|L(v)| \geqslant 11p\Delta^{33/20}$ by hypothesis. Thus, if $\delta^1_H(v) \leqslant 8p\Delta^{33/20}$, then $|L(v)| \geqslant 3p\Delta^{33/20} - 4p\Delta \geqslant^* 2p\Delta^{33/20} \geqslant 2pQ\Delta^{2/5}$ since $Q = U_{k-1} \leqslant U_0 = \Delta^{5/4}$. If $\delta^1_H(v) > 8p\Delta^{33/20}$, then as $U_{k-1} > \Delta^{3/5}$ observe that $m^-_k(v) \geqslant \Delta^{150\varepsilon}$. Indeed, $m^+_{k-1}(v) > \Delta^{3/5}$ since $m^+_0(v) = \delta^1_H(v) > U_0$. Hence $m^+_k(v) >^* \frac{1}{2} \cdot \Delta^{3/5}$. Consequently, $m^-_k(v) > \frac{1}{8} \cdot \Delta^{3/5} \geqslant \Delta^{150\varepsilon}$ by Lemma 23(ii). So by Lemma 23(i)–(ii), we deduce that

$$\begin{aligned}|L(v)| \geqslant \delta^1_U(v) \geqslant m^-_{k-1}(v) &\geqslant \frac{1}{2}\theta^{k-1}\delta^1_H(v)\\ &> 4p\theta^{k-1}U_0\Delta^{2/5}\\ &\geqslant^* 2pU_{k-1}\Delta^{2/5} = 2pQ\Delta^{2/5}.\end{aligned}$$

Bounding $\mathbf{Pr}(E_1(v))$. Fix a vertex v of $\cup_{i=1}^l A_i$. We assume that $U_k \geqslant \frac{1}{2}\Delta^{200\varepsilon}$. Let m be the number of uncolored external neighbors of v in H at the beginning of iteration k. By induction, $m \leqslant U_{k-1}$. We define Y to be the number of those vertices that are colored during iteration k. The probability of an uncolored vertex becoming colored during iteration k is exactly $\frac{1}{2}\Delta^{-\varepsilon}$. Hence, $\mathbf{E}(Y) = \frac{1}{2}\Delta^{-\varepsilon}m$. Consequently, if $E_1(v)$ holds, then Y must differ from its expected value by more than $U^{49/50}_{k-1}$.

As in the proof of Lemma 14, we express Y as the difference of two random variables. Let Y_1 be the number of uncolored external G_1'-neighbors of v that are activated during iteration k. Let Y_2 be the number of uncolored external G_1'-neighbors of v that are activated and uncolored during iteration k. Thus, $Y = Y_1 - Y_2$ and hence if $E_1(v)$ holds, then either Y_1 or Y_2 differs from its expected value by more than $\frac{1}{2}U_{k-1}^{49/50}$.

Note that $Y_1 \leqslant U_{k-1}$; hence $\mathbf{E}(Y_1) \leqslant U_{k-1}$. Moreover, Y_1 is a binomial random variable, so Chernoff's bound implies that

$$\mathbf{Pr}\left(|Y_1 - \mathbf{E}(Y_1)| > \frac{1}{2}U_{k-1}^{49/50}\right) \leqslant 2\exp\left(-\frac{U_{k-1}^{49/25}}{12U_{k-1}}\right) \leqslant^* \frac{1}{2}\Delta^{-10},$$

since $U_{k-1} \geqslant U_k > \frac{1}{2}\Delta^{200\varepsilon}$.

The random variable Y_2 is upper-bounded by the random variable Y_2', defined as the number of uncolored external G_1'-neighbors of v that are activated and (i) uncolored or (ii) assigned a color that is assigned to at least $\log\Delta$ G_1'-neighbors of v. Furthermore, we assert that $Y_2 = Y_2'$ with high probability. Indeed, if $Y_2 \neq Y_2'$, then there exists a color assigned to at least $\log\Delta$ G_1'-neighbors of v. By Lemma 23(i), the number of uncolored G_1'-neighbors of v in H is at most $d \coloneqq 2\theta^{k-1}U_0$. Moreover, by the induction hypothesis, $\delta_U^1(u) \geqslant m_{k-1}^-(u) \geqslant^* \frac{1}{2}\theta^{k-1}\delta_H^1(u)$ for every G_1'-neighbor u of v in H. Therefore, the number of colors available for u is at least

$$\begin{aligned}
&\max\left(\delta_H^1(u) + 6p\Delta, 11p\Delta^{33/20}\right) - \delta_H^*(u) + m_{k-1}^-(u)\\
&\quad\geqslant \max\left(\delta_H^1(u) + 6p\Delta, 11p\Delta^{33/20}\right) - 4p\Delta - \left(1 - \frac{1}{2}\theta^{k-1}\right)\delta_H^1(u)\\
&\quad\geqslant \max\left(\delta_H^1(u), 8p\Delta^{33/20}\right)\cdot\left(1 - \left(1 - \frac{1}{2}\theta^{k-1}\right)\right)\\
&\quad\geqslant \frac{1}{2}\theta^{k-1}\times 8p\Delta^{33/20}\\
&\quad\geqslant 2p\Delta^{2/5}d.
\end{aligned}$$

Consequently,

$$\mathbf{Pr}(Y_2 \neq Y_2') \leqslant \Delta^2 \times \binom{d}{\log\Delta}\left(2p\Delta^{2/5}d\right)^{-\log\Delta} \leqslant \Delta^2 \times \left(\frac{e}{2p\Delta^{2/5}}\right)^{\log\Delta} <^* \frac{1}{4}\Delta^{-17},$$

which proves the assertion.

Since $|Y_2 - Y_2'| \leqslant \Delta^2$, this implies that $|\mathbf{E}(Y_2) - \mathbf{E}(Y_2')| = o(1)$. As a result, it is enough to establish that $\mathbf{Pr}(|Y_2' - \mathbf{E}(Y_2')| > \frac{1}{4}U_{k-1}^{49/50}) <^* \frac{1}{4}\Delta^{-17}$ to deduce that $\mathbf{Pr}(|Y_2 - \mathbf{E}(Y_2)| > \frac{1}{2}U_{k-1}^{49/50}) \leqslant^* \frac{1}{2}\Delta^{-17}$.

We apply Talagrand's inequality. For convenience, we consider that each vertex v of H is involved in two random trials. The first, which combines steps 1 and 2 of our procedure, is to be assigned the label "unactivated" or "activated with color j" for some color j in $L(v)$. The former label is assigned with probability $1 - \Delta^{-\varepsilon}$ and the latter with probability $\frac{\Delta^{-\varepsilon}}{|L(v)|}$. The second random trial assigns to v the label "uncolored" with probability $q(v)$, whatever the result of the first trial is. The technical benefit of this approach is to obtain independent random trials.

If $Y_2' \geqslant s$, then there is a set of at most $s\log\Delta$ random trials that certify this fact, i.e., for each of the s vertices counted by Y_2', the activation and color assignment of the vertex and either the choice to uncolor it in step 4, or the activation and assignment of a conflicting color to a neighbor of that vertex, or the activation and assignment of the same color to $\log\Delta - 1$ other G'_1-neighbors of v. Furthermore, changing the outcome of one of the random trials can affect Y'_2 by at most $\log\Delta$. Recalling that $\mathbf{E}(Y_2) \leqslant U_{k-1}$ and $U_{k-1} \geqslant \Delta^{200\varepsilon}$, Talagrand's inequality yields that

$$\mathbf{Pr}\left(|Y'_2 - \mathbf{E}(Y'_2)| > \frac{1}{4}U_{k-1}^{49/50}\right) < 4\exp\left(-\frac{U_{k-1}^{49/25}}{16\times 32\log^3\Delta U_{k-1}}\right) <^* \frac{1}{4}\Delta^{-17},$$

provided that $\mathbf{E}(Y_2) \geqslant \frac{1}{4}U_{k-1}^{49/50}$. If $\mathbf{E}(Y_2) < \frac{1}{4}U_{k-1}^{49/50}$, then we consider the random variable Y'_2 defined to be the sum of Y_2 and a binomial random variable that counts each of the m uncolored external neighbors of v in H independently at random with probability $\max(1, \frac{1}{8m}U_{k-1}^{49/50})$. Notice that $\frac{1}{8}U_{k-1}^{49/50} \leqslant \mathbf{E}(Y'_2) \leqslant \frac{3}{8}U_{k-1}^{49/50}$. Moreover, if $|Y_2 - \mathbf{E}(Y_2)| > \frac{1}{4}U_{k-1}^{49/50}$, then $|Y'_2 - \mathbf{E}(Y'_2)| > \frac{1}{8}U_{k-1}^{49/50}$. Applying Talagrand's inequality to Y'_2, we infer that the last inequality occurs with probability less than $\frac{1}{4}\Delta^{-17}$, as wanted. Therefore, we obtain $\mathbf{Pr}(E_1(v)) \leqslant \Delta^{-17}$, as desired.

Bounding $\mathbf{Pr}(E_2(v))$. We fix a vertex v of H, and we assume that $m_k^-(v) \geqslant \frac{1}{8}\Delta^{200\varepsilon}$. Our aim is to prove that $\mathbf{Pr}(E_2(v)) \leqslant^* \Delta^{-17}$. To this end, we wish to use a similar approach to that for $E_1(v)$. However, for every G_1^*-neighbor u of v in H, the degree of u in H may be a lot bigger than the degree of v in H, which makes it more difficult to bound the analogue of $\mathbf{Pr}(Y_2 \neq Y'_2)$.

For every vertex u, let $\tilde{\delta}(u) \coloneqq \delta_U^1(u)$. Let L_u be the set of colors available to color u. Recall that $|L_u| \geqslant \delta_H^1(u) + 2p\Delta > \tilde{\delta}(u)$.

We define Z_1 and Z_2 analogously to Y_1 and Y_2, that is, we let Z_1 be the number of uncolored G_1^*-neighbors of v that get activated, and we let Z_2 be the number of those activated neighbors of v that get uncolored. Similarly as before, it suffices to prove that with high probability, neither Z_1 nor Z_2 differs from its expected value by more than $\frac{1}{2}\left(m_{k-1}^-(v)\right)^{49/50}$. Observe that $Z_1 \leqslant m_{k-1}^+(v) < 4m_{k-1}^-(v)$, and so $\mathbf{E}(Z_1) \leqslant 4m_{k-1}^-(v)$. Therefore, Chernoff's bound implies that

$$\mathbf{Pr}\left(|Z_1 - \mathbf{E}(Z_1)| > \frac{1}{2}\left(m_{k-1}^-(v)\right)^{49/50}\right) \leqslant 2\exp\left(-\frac{m_{k-1}^-(v)^{49/25}}{48\cdot m_{k-1}^-(v)}\right) <^* \frac{1}{2}\Delta^{-17},$$

since $m_{k-1}^-(v) \geqslant \frac{1}{8}\Delta^{200\varepsilon}$.

We partition the neighbors of v in H into two parts N_A and N_B, where N_A contains those vertices u with $\tilde{\delta}(u) \geqslant \tilde{\delta}(v)^{3/4}$ and N_B those with $\tilde{\delta}(u) < \tilde{\delta}(v)^{3/4}$. We define Z_A and Z_B to be the number of vertices that get activated and uncolored during this iteration in N_A and N_B, respectively. Thus, $Z_2 = Z_A + Z_B$.

We use a similar argument as the one for Y_2 to show that Z_A is concentrated. Let $Z_{A'}$ be the number of vertices in N_A that get activated and are (i) uncolored or (ii) assigned a color that is assigned to at least $\tilde{\delta}(v)^{3/10}$ members of N_A. As $|N_A| \leqslant \tilde{\delta}(v)$, and $|L_u| \geqslant \tilde{\delta}(u) \geqslant \tilde{\delta}(v)^{3/4}$ for every vertex of $u \in N_A$, the probability that Z_A and Z'_A are different is at most

$$(\Delta^2+1)\binom{\tilde{\delta}(v)}{\tilde{\delta}(v)^{3/10}}\left(\tilde{\delta}(v)^{3/4}\right)^{-\tilde{\delta}(v)^{3/10}} < (\Delta^2+1)\left(\frac{e\tilde{\delta}(v)}{\tilde{\delta}(v)^{3/10}\tilde{\delta}(v)^{3/4}}\right)^{\tilde{\delta}(v)^{3/10}}$$
$$<^* \frac{1}{8}\Delta^{-17},$$

since $\tilde{\delta}(v) \geqslant m^-_{k-1}(v) > \frac{1}{8}\Delta^{200\varepsilon}$. As $|Z_A - Z'_A| \leqslant \Delta^2$, we infer that $|\mathbf{E}(Z_A) - \mathbf{E}(Z'_A)| = o(1)$.

By the same argument as for Y'_2, we deduce that if $Z'_A \geqslant s$, then there are at most $\tilde{\delta}(v)^{3/10} \cdot s$ trials whose outcomes certify this fact. Furthermore, each trial can affect Z'_A by at most $\tilde{\delta}(v)^{3/10}$. Therefore, if $\mathbf{E}(Z'_A) \geqslant \frac{1}{4} m^-_{k-1}(v)^{49/50}$, then Talagrand's inequality yields that

$$\begin{aligned}
&\mathbf{Pr}\left(|Z'_A - \mathbf{E}(Z'_A)| > \frac{1}{4}\left(m^-_{k-1}(v)\right)^{49/50}\right) \\
&\quad \leqslant 4\exp\left(-\frac{m^-_{k-1}(v)^{49/25}}{32 \cdot m^+_{k-1}(v)^{6/10} \cdot m^+_{k-1}(v)^{3/10} \cdot 4 \cdot m^-_{k-1}(v)}\right) \\
&\quad \leqslant 4\exp\left(-\frac{m^-_{k-1}(v)^{49/25}}{128 \cdot 4^{9/10} \cdot m^-_{k-1}(v)^{6/10} \cdot m^-_{k-1}(v)^{3/10} \cdot m^-_{k-1}(v)}\right) \\
&\quad <^* \frac{1}{8}\Delta^{-17}.
\end{aligned}$$

If $\mathbf{E}(Z'_A) < \frac{1}{4} m^-_{k-1}(v)^{49/50}$, then we define Z''_A to be the sum of Z'_A and a binomial random variable that counts each vertex of N_A independently with probability $\max(1, \frac{1}{8|N_A|} m^-_{k-1}(v)^{49/50})$. By linearity of expectation, $\frac{1}{8} m^-_{k-1}(v)^{49/50} \leqslant \mathbf{E}(Z''_A) \leqslant \frac{3}{8} m^-_{k-1}(v)^{49/50}$. Furthermore, if $|Z'_A - \mathbf{E}(Z'_A)| > \frac{1}{4} m^-_{k-1}(v)^{49/50}$, then $|Z''_A - \mathbf{E}(Z''_A)| > \frac{1}{8} m^-_{k-1}(v)^{49/50}$. Applying Talagrand's inequality to Z''_A yields that this latter inequality occurs with probability less than $\frac{1}{8}\Delta^{-17}$, as desired. Consequently,

$$\mathbf{Pr}\left(|Z_A - \mathbf{E}(Z_A)| > \frac{1}{2} m^-_{k-1}(v)^{49/50}\right) \leqslant^* \frac{1}{4}\Delta^{-17}. \tag{5}$$

We finish by considering Z_B. We first expose the assignments to all vertices other than N_B. Let $\mathcal{H}$ be this assignment. We now condition on $\mathcal{H}$. First, we consider the conditional expected value of Z_B regarding $\mathcal{H}$. We assert that

$$\mathbf{Pr}\left(|\mathbf{E}(Z_B|\mathcal{H}) - \mathbf{E}(Z_B)| > \frac{1}{2} m^-_{k-1}(v)^{49/50}\right) <^* \frac{1}{8}\Delta^{-17}. \tag{6}$$

To see this, let $\mu_{\mathcal{H}}$ be the conditional expectation $\mathbf{E}(Z_B|\mathcal{H})$. Note that the expected value of $\mu_{\mathcal{H}}$ over the space of random colorings of $H - N_B$ is equal to the expected value of Z_B over the space of random colorings of H. So our assertion is that $\mu_{\mathcal{H}}$ is indeed concentrated.

For each vertex u of N_B, let $F_u = F_u(\mathcal{H}) \subseteq L_u$ be the set of colors of L_u that conflict with the assignments made by $\mathcal{H}$ to the neighbors of u in H that are not in N_B. First, we use Talagrand's inequality to prove that $|F_u|$ is concentrated.

The random variable $|F_u|$ is determined by the independent color assignments to the vertices of $H - N_B$. If $|F_u| \geqslant s$, then there is a set of at most s assignments that certify this fact, namely, the assignments of colors to s vertices. Observe that the assignments to one vertex can affect $|F_u|$ by at most $2p$. Moreover, $|F_u| \leqslant |L_u|$ so $\mathbf{E}(F_u) \leqslant |L_u|$. Therefore, Talagrand's inequality implies that

$$\mathbf{Pr}\left(||F_u| - \mathbf{E}(|F_u|)| > \Delta^{-2/5}|L_u|\right) < 4\exp\left(-\frac{\Delta^{-4/5}|L_u|^2}{128p^2|L_u|}\right) <^* \frac{1}{8}\Delta^{-19},$$

since $|L_u| \geqslant 2\Delta$, in the case where $\mathbf{E}(|F_u|) \geqslant \Delta^{-2/5}|L_u|$. In the opposite case, we define F to be the sum of the random variable F_u and a binomial random variable that counts each color of L_u independently at random with probability $\frac{1}{2}\Delta^{-2/5}$. Note that $\frac{1}{2}\Delta^{-2/5}|L_u| \leqslant \mathbf{E}(F) \leqslant \frac{3}{2}\Delta^{-2/5}|L_u|$. Moreover, if $||F_u| - \mathbf{E}(|F_u|)| > \Delta^{-2/5}|L_u|$, then $|F - \mathbf{E}(F)| > \frac{1}{2}\Delta^{-2/5}|L_u|$. Applying Talagrand's inequality to F shows that this latter inequality occurs with probability less than $\frac{1}{8}\Delta^{-19}$, as desired.

Consequently, the probability that there is at least one vertex u of N_B for which $|F_u|$ differs from its expected value by more than $\Delta^{-2/5}|L_u|$ is at most $|N_B|\frac{1}{8}\Delta^{-19} \leqslant \frac{1}{8}\Delta^{-17}$. Hence, we assume that there is no such vertex u, and we prove that this implies that $|\mu_{\mathcal{H}} - \mathbf{E}(\mu_{\mathcal{H}})| < \frac{1}{2}m^-_{k-1}(v)^{49/50}$.

Given a particular assignment $\mathcal{H}$ to $H \setminus \cup_{i=1}^{\ell} A_i$ and a color $j \in L_u$, the probability that u keeps the color j is 0 if $j \in F_u$ and at most

$$(1-q(u)) \prod_{\substack{w \in N(u) \cap N_B \\ j \in L_w}} \left(1 - \frac{2p-1}{|L_w|}\right)$$

otherwise. Note that the product is at most 1 and does not depend on F_u. Hence, changing whether j belongs to F_u affects the probability that u retains its color by at most $\frac{2p-1}{|L(u)|}$. So, as $|F_u|$ differs from its expected value by at most $\Delta^{-2/5}|L_u|$, the conditional probability that u is uncolored differs from its expected value by at most $(2p-1)\Delta^{-2/5}$. Since $\mu_{\mathcal{H}}$ is the sum of these probabilities over all the vertices u of N_B, we deduce that

$$|\mu_{\mathcal{H}} - \mathbf{E}(\mu_{\mathcal{H}})| \leqslant (2p-1)\Delta^{-2/5}|N_B| < \frac{1}{2}m^-_{k-1}(v)^{49/50},$$

because $|N_B| < 2m^-_{k-1}(v)$ and $m^-_{k-1}(v) \leqslant \Delta^2$. This concludes the proof of our assertion.

We define Z'_B to be the number of vertices of N_B that are activated and uncolored because (i) they are assigned a color conflicting with a neighbor outside of N_B or (ii) they are assigned a color conflicting with a neighbor $w \in N_B$ and this color is assigned to at least $\tilde{\delta}(v)^{3/10}$ vertices of $N_{G_1^*}(w) \cap N_B$.

If $Z_B \neq Z'_B$, then some vertex u of N_B receives the same color as at least $\tilde{\delta}(v)^{3/10}$ of its neighbors. Since each vertex u of N_B has at most $\tilde{\delta}(v)^{3/4}$ neighbors, and $|L_w| \geqslant 2\Delta$ for every vertex w, we deduce that

$$\begin{aligned}
\mathbf{Pr}\left(Z_B \neq Z'_B\right) &\leqslant |N_B| \times \binom{\tilde{\delta}(v)^{3/4}}{\tilde{\delta}(v)^{3/10}} \cdot (2\Delta)^{-\tilde{\delta}(v)^{3/10}} \\
&\leqslant \Delta^2 \left(\frac{e\tilde{\delta}(v)^{3/4}}{2\tilde{\delta}(v)^{3/10}\Delta}\right)^{\tilde{\delta}(v)^{3/10}} = \Delta^2 \left(\frac{\tilde{\delta}(v)^{9/20}}{2\Delta}\right)^{\tilde{\delta}(v)^{3/10}} \\
&\leqslant \Delta^2 \left(\frac{\Delta^{18/20}}{2\Delta}\right)^{\tilde{\delta}(v)^{3/10}} <^* \frac{1}{8}\Delta^{-17}, \qquad (7)
\end{aligned}$$

as $\frac{1}{8}\Delta^{200\varepsilon} \leqslant \tilde{\delta}(v) \leqslant \Delta^2$. Since $|Z_B - Z'_B| \leqslant \Delta^2$ for every choice of $\mathcal{H}$, we infer that $|\mathbf{E}(Z_B|\,\mathcal{H}) - \mathbf{E}(Z'_B\,|\mathcal{H})| = o(1)$.

After conditioning on $\mathcal{H}$, the random variable Z'_B is determined by at most $\delta^1_H(v)$ assignments and each assignment can affect Z'_B by at most $\delta^1_H(v)^{1/3}$. Note that

$\delta^1_H(v) \leqslant m^+_{k-1}(v) \leqslant 2m^-_{k-1}(v)$. So, for every choice of $\mathcal{H}$, the simple concentration bound yields that

$$\begin{aligned}
&\mathbf{Pr}\left(|Z'_B - \mathbf{E}(Z'_B|\,\mathcal{H})\,\middle| > \frac{1}{4} m^-_{k-1}(v)^{49/50} \,\middle|\, \mathcal{H}\right)\\
&< 2\exp\left(-\frac{m^-_{k-1}(v)^{49/25}}{32 \times 2m^-_{k-1}(v)^{2/3} \times 2m^-_{k-1}(v)}\right)\\
&<^* \frac{1}{8}\Delta^{-17}. \qquad (8)
\end{aligned}$$

Therefore, by (6)–(8), we infer that $\mathbf{Pr}\left(|Z_B - \mathbf{E}(Z_B)| > \frac{1}{2} m^-_{k-1}(v)^{49/50}\right) <^* \frac{1}{2}\Delta^{-17}$. Thus, along with (5), we deduce that

$$\mathbf{Pr}(E_2(v)) <^* \Delta^{-17},$$

which concludes the proof. ☐

Acknowledgments. The authors thank one of the referees for shortening some of the proofs. The first and second authors thank the Department of Applied Mathematics KAM-DIMATIA and the Institute for Theoretical Computer Science (ITI) of Charles University for their hospitality during their stay in Prague, where part of this research was conducted.

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