

# Structural abstract interpretation, A formal study using Coq

Yves Bertot

► **To cite this version:**

Yves Bertot. Structural abstract interpretation, A formal study using Coq. LERNET Summer School, Ana Bove and Jorge Sousa Pinto, Feb 2008, Piriapolis, Uruguay. inria-00329572v1

**HAL Id: inria-00329572**

**<https://hal.inria.fr/inria-00329572v1>**

Submitted on 13 Oct 2008 (v1), last revised 20 Oct 2008 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Structural abstract interpretation

## A formal study using Coq

Yves Bertot\*

INRIA Sophia-Méditerranée

**Abstract.** Abstract interpreters are tools to compute approximations for behaviors of a program. These approximations can then be used for optimisation or for error detection. In this paper, we show how to describe an abstract interpreter using the type-theory based theorem prover Coq, using inductive types for syntax and structural recursive programming for the abstract interpreter's kernel. The abstract interpreter can then be proved correct with respect to a Hoare logic for the programming language.

### 1 Introduction

Higher-order logic theorem provers provide a description language that is powerful enough to describe programming languages. Inductive types can be used to describe the language's main data structure (the syntax) and recursive functions can be used to describe the behavior of instructions (the semantics). Recursive functions can also be used to describe tools to analyse or modify programs. These tools are themselves programs and we can reason about their correctness. The purpose of this paper is to illustrate all of this with a case study concentrating on abstract interpretation [7] using the theorem prover Coq [13,3]. The development described in this paper is available on the net at the following address (there are two versions, compatible with latest stable release of Coq and with upcoming version).

<http://hal.inria.fr/inria-00329572>

This paper has 7 more sections. Section 2 attempts to give a motivation to the notion of abstract interpretation. Section 3 describes the programming language that is used as our playground. The semantics of this programming language is described using a weakest pre-condition calculus. This weakest pre-condition calculus is later used to argue on the correctness of abstract interpreters. In particular, abstract interpretation returns an annotated instruction and an abstract state, where the abstract state is used as a post-condition and the annotations in the instruction describe the abstract state at the corresponding point in the program. Section 4 describes a first simple abstract interpreter, where the main ideas around abstractly interpreting assignments and sequences are covered, but

---

\* This work was partially supported by ANR contract Compcert, ANR-05-SSIA-0019.

while loops are left basically uncovered. In Section 4, we also show that the abstract interpreter can be formally proved correct. In Section 5, we address while loops in more detail and in particular we show how tests can be handled in abstract interpretation, with applications to dead-code elimination. In Section 6, we observe that abstract interpretation is a general method that can be applied to a variety of abstract domains and we recapitulate the types, functions, and properties that are expected from each abstract domain. In Section 7, we show how the main abstract interpreter can be instantiated for a domain of intervals and we show a little example. In Section 8, we give a few concluding remarks.

## 2 An intuitive view of abstract interpretation

Abstract interpretation is a technique for the static analysis of programs. The objective is to obtain a tool that will take programs as data, perform some symbolic computation, and return information about all executions of the input programs. One important aspect is that this tool should always terminate (hence the adjective *static*). The tool can then be used either directly to provide information about properties of variables in the program (as in the Astree tool [8]), or as part of a compiler, where it can be used to guide optimization. For instance, the kind of interval-based analysis that we describe in this paper can be used to avoid runtime array-bound checking in languages that impose this kind of discipline like Java.

The central idea of abstract interpretation is to replace the values normally manipulated in a program by sets of values, in such a way that all operations still make sense.

For instance, if a program manipulates integer values and performs additions, we can decide to take an abstract point of view and only consider whether values are odd or even. With respect to addition, we can still obtain meaningful results, because we know, for instance, that adding an even and an odd value returns an odd value. Thus, we can decide to run programs with values taken in a new type that contains values `even` and `odd`, with an addition that respects the following table:

$$\begin{aligned} \text{odd} + \text{even} &= \text{odd} \\ \text{even} + \text{odd} &= \text{odd} \\ \text{odd} + \text{odd} &= \text{even} \\ \text{even} + \text{even} &= \text{even}. \end{aligned}$$

When defining abstract interpretation for a given abstract domain, all operations must be updated accordingly. The behavior of control instructions is also modified, because abstract values may not be precise enough to decide how a given decision should be taken.

For instance, if we know that the abstract value for a variable `x` is `odd`, then we cannot tell which branch of a conditional statement of the following form will be taken:

`if x < 10 then x := 0 else x := 1.`

After the execution of this conditional statement, the abstract value for  $x$  cannot be `odd` or `even`. This example also shows that the domain of abstract values must contain an abstract value that represents the whole set of values, or said differently, an abstract value that represents the absence of knowledge. This value will be called `top` later in the paper.

There must exist a connection between abstract values and concrete values for abstract interpretation to work well. This connection has been studied since [7] and is known as a Galois connection. For instance, if the abstract values are `even`, `or`, and `top`, and if we can infer that a value is in  $\{1,2\}$ , then correct choices for the abstract value are `top` or `even`, but obviously the abstract interpreter will work better if the more precise `even` is chosen.

Formal proofs of correctness for abstract interpretation were already studied before, in particular in [11]. The approach taken in this paper is different, in that it follows directly the syntax of a simple structured programming language, while traditional descriptions are tuned to studying a control-flow graph language.

### 3 The programming language

In this case study, we work with a very small language containing only assignments, sequences, and while loops. The right-hand sides for assignments are expressions made of numerals, variables, and addition. The syntax of the programming language is as follows:

- variable names are noted  $x, y, x_1, x'$ , etc.
- integers are noted  $n, n_1, n'$ , etc.
- Arithmetic expressions are noted  $e, e', e_1$ , etc. For our case study, these expressions can only take three forms:

$$e ::= n \mid x \mid e_1 + e_2$$

- boolean expressions are noted  $b, b_1, b'$ , etc. For our case study, these expressions can only take one form:

$$b ::= e_1 < e_2$$

- instructions are noted  $i, i_1, i'$ , etc. For our case study, these instructions can only take three forms:

$$i ::= x := e \mid i_1; i_2 \mid \text{while } b \text{ do } i \text{ done}$$

For the Coq encoding, we use pre-defined strings for variable names and integers for the numeric values. Thus, we use unbounded integers, which is contrary to usual programming languages, but the question of using bounded integers or not is irrelevant for the purpose of this example.

### 3.1 Encoding the language

In our Coq encoding, the description of the various kinds of syntactic components is given by inductive declarations.

```
Require Import String ZArith List.  
Open Scope Z_scope.
```

```
Inductive aexpr : Type :=  
  anum (x:Z) | avar (s:string) | aplus (e1 e2:aexpr).
```

```
Inductive bexpr : Type := blt (e1 e2 : aexpr).
```

```
Inductive instr : Type :=  
  assign (x:string)(e:expr)  
| seq (i1 i2:instr)  
| while (b:bexpr)(i:instr).
```

The first two lines instruct Coq to load pre-defined libraries and to tune the parsing mechanism so that arithmetic formulas will be understood as formulas concerning integers by default.

The definition for `aexpr` states that expressions can only have the three forms `anum`, `avar`, and `aplus`, it also expresses that the names `anum`, `avar`, and `aplus` can be used as function of type, `Z -> aexpr`, `string -> aexpr`, and `aexpr -> aexpr -> aexpr`, respectively. The definition of `aexpr` as an inductive type also implies that we can write recursive functions on this type. For instance, we will use the following function to evaluate an arithmetic expression, given a *valuation* function `g`, which maps every variable name to an integer value.

```
Fixpoint af (g:string->Z)(e:aexpr) : Z :=  
  match e with  
  | anum n => n  
  | avar x => g x  
  | aplus e1 e2 => af g e1 + af g e2  
  end.
```

This function is defined by pattern-matching. There is one pattern for each possible form of arithmetic expression. The third line indicates that when the input `e` has the form `anum n`, then the value `n` is the result. The fourth line indicates that when the input has the form `avar x`, then the value is obtained by applying the function `g` to `x`. The fifth line describes the computation that is done when the expression is an addition. There are two recursive calls to the function `af` in the expression returned for the addition pattern. The recursive calls are made on direct subterms of the initial instruction, this is known as *structural recursion* and guarantees that the recursive function will terminate on all inputs.

A similar function `bf` is defined to describe the boolean value of a boolean expression.

### 3.2 The semantics of the programming language

To describe the semantics of the programming language, we simply give a *weakest pre-condition calculus* [9]. We describe the conditions that are necessary to ensure that a given logical property is satisfied at the end of the execution of an instruction, when this execution terminates. This weakest pre-condition calculus is defined as a pair of functions whose input is an instruction annotated with logical information at various points in the instruction. The output of the first function call `pc` is a condition that should be satisfied by the variables at the beginning of the execution (this is the *pre-condition* and it should be as easy to satisfy as possible, hence the adjective *weakest*); the output of the second function, called `vc`, is a collection of logical statements. When these statements are valid, we know that every execution starting from a state that satisfies the pre-condition will make the logical annotation satisfied at every point in the program and make the post-condition satisfied if the execution terminates.

**annotating programs** We need to define a new data-type for instructions annotated with assertions at various locations. Each assertion is a quantifier-free logical formula where the variables of the program can occur. The intended meaning is that the formula is guaranteed to hold for every execution of the program that is consistent with the initial assertion.

The syntax for assertions is described as follows:

```
Inductive assert : Type :=
  pred (p:string)(l:list aexpr)
| a_b (b:bexpr)
| a_conj (a1 a2:assert)
| a_not (a: assert)
| a_true
| a_false.
```

This definition states that assertions can have six forms: the first form represents the application of a predicate to an arbitrary list of arithmetic expressions, the second represents a boolean test: this assertion holds when the boolean test evaluates to `true`, the third form is the conjunction of two assertions, the fourth form is the negation of an assertion, the fifth and sixth forms give two constant assertions, which are always and never satisfied, respectively. In a minimal description of a weakest pre-condition calculus, as in [2], the last two constants are not necessary, but they will be useful in our description of the abstract interpreter.

To consider whether an assertion holds, we need to know what meaning is attached to each predicate name and what value is attached to each variable name. We suppose the meaning of predicates is given by a function `m` that maps predicate names and list of integers to propositional values and the value of variables is given by a valuation as in the function `af` given above. Given such a meaning for predicates and such a valuation function for variables, we describe the computation of the property associated to an assertion as follows:

```

Fixpoint ia (m:string->list Z->Prop)(g:string->Z)
  (a:assert) : Prop :=
  match a with
  | pred s l => m s (map (af g) l)
  | a_b b => bf g b = true
  | a_conj a1 a2 => (ia m g a1) /\ (ia m g a2)
  | a_not a => not (ia m g a)
  | a_true => True
  | a_false => False
  end.

```

The type of this function exhibits a specificity of type theory-based theorem proving: propositions are described by *types*. The Coq system also provides a type of types, named `Prop`, whose elements are the types that are intended to be used as propositions. Each of these types contains the proofs of the proposition they represent, and for instance the propositions that are unprovable are represented by types that contain no elements. Here, assertions are data, their interpretation as propositions are types, which belongs to the `Prop` type. More details about this description of propositions as types is given in another article on type theory in the same volume.

Annotated instructions are represented using a new data-type, named `a_instr`, which is very close to the `instr` data-type. The two modifications are as follows: first an extra operator `pre` is added to make it possible to attach assertions to any instruction, second `while` loops are mandatorily annotated with an *invariant* assertion. In concrete syntax, we will write `{ a } i` for the instruction `i` carrying the assertion `a` (noted `pre a i` in the Coq encoding).

```

Inductive a_instr : Type :=
  pre (a:assert)(i:a_instr)
| a_assign (x:string)(e:aexpr)
| a_seq (i1 i2:a_instr)
| a_while (b:bexpr)(a:assert)(i:a_instr).

```

**Reasoning on assertions** We can reason on annotated programs, because there are logical reasons for programs to be consistent with assertions. The idea is to compute a collection of logical formulas which must be true for an annotated program to be consistent with a post-condition. Annotations added to an instruction (with the help of the `pre` construct) must be understood as formulas that hold just before executing the annotated instruction. Assertions added to `while` loops must be understood as *invariants*, they are meant to hold at the beginning and the end every time the inner part of the while loop is executed.

The computation works in two phases: the first phase takes as input an annotated program and an assertion, known as the *post-condition*, that should hold at the end and computes a condition that should be strong enough to ensure that the post-condition will be satisfied at the end of execution.

When assertions are present in the annotated instruction, they are taken for granted. For instance, when the instruction is  $\{x = 3\} x := x + 1$ , the computed pre-condition is  $x = 3$ , whatever the post-condition is.

When the instruction is a plain assignment, one can find the pre-condition by substituting the assigned variable with the assigned expression in the post-condition. For instance, when the post condition is  $x = 4$  and the instruction is the assignment  $x := x + 1$ , it suffices that the pre-condition  $x + 1 = 4$  is satisfied before executing the assignment to ensure that the post-condition is satisfied after executing it.

When the annotated instruction is a while loop, the pre-condition simply is the invariant for this while loop. When the annotated instruction is a sequence of two instructions, the pre-condition is the pre-condition computed for the first of the two instructions, but using the pre-condition of the second instruction as the post-condition for the first instruction.

**Coq encoding for pre-condition computation** To encode this pre-condition function in Coq, we need to describe functions that perform the substitution of a variable with an arithmetic expression in arithmetic expressions, boolean expressions, and assertions. These substitution functions are given as follows:

```
Fixpoint asubst (x:string) (s:aexpr) (e:aexpr) : aexpr :=
  match e with
  | anum n => anum n
  | avar x1 => if string_dec x x1 then s else e
  | aplus e1 e2 => aplus (asubst x s e1) (asubst x s e2)
  end.
```

```
Definition bsubst (x:string) (s:aexpr) (b:bexpr) : bexpr :=
  match b with
  | blt e1 e2 => blt (asubst x s e1) (asubst x s e2)
  end.
```

```
Fixpoint subst (x:string) (s:aexpr) (a:assert) : assert :=
  match a with
  | pred p l => pred p (map (asubst x s) l)
  | a_b b => a_b (bsubst x s b)
  | a_conj a1 a2 => a_conj (subst x s a1) (subst x s a2)
  | a_not a => a_not (subst x s a)
  | any => any
  end.
```

In the definition of `asubst`, the function `string_dec` compares two strings for equality. The value returned by this function can be used in an `if-then-else` construct, but it is not a boolean value (more detail can be found in [3]). The rest of the code is just a plain traversal of the structure of expressions and assertions. Note also that the last pattern-matching rule in `subst` is used for both `a_true` and `a_false`.



Once we know how to substitute a variable with an expression, we can easily describe the computation of the pre-condition for an annotated instruction and a post-condition. This is given by the following simple recursive procedure:

```

Fixpoint pc (i:a_instr) (post : assert) : assert :=
  match i with
  | pre a i => a
  | a_assign x e => subst x e post
  | a_seq i1 i2 => pc i1 (pc i2 post)
  | a_while b a i => a
  end.

```

**A verification condition generator** When it receives an instruction carrying an annotation, the function `pc` simply returns the annotation. In this sense, the pre-condition function takes the annotation for granted. To make sure that an instruction is consistent with its pre-condition, we need to check that the assertion really is strong enough to ensure the post-condition.

For instance, when the post-condition is  $x < 10$  and the instruction is the annotated assignment  $\{ x = 2 \} x := x + 1$ , satisfying  $x = 2$  before the assignment is enough to ensure that the post-condition is satisfied. On the other hand, if the annotated instruction was  $\{ x < 10 \} x := x + 1$ , there would be a problem because there are cases where  $x < 10$  holds before executing the assignment and  $x < 10$  does not hold after.

In fact, for assignments that are not annotated with assertions, the function `pc` computes the best formula, the *weakest pre-condition*. Thus, in presence of an annotation, it suffices to verify that the annotation does imply the weakest pre-condition. We are now going to describe a function that collects all the verifications that need to be done. More precisely, the new function will compute conditions that are sufficient to ensure that the pre-condition from the previous section is strong enough to guarantee that the post-condition holds after executing the program, when the program terminates.

For instance, when the post-condition is  $x=3$  and the instruction is the assignment  $x := x+1$ , the pre-condition computed by `pc` is  $x + 1 = 3$  and this is obviously good enough for the post-condition to be satisfied. On the other hand, when the instruction is an annotated instruction,  $\{P\} x := x+1$ , we need to verify that  $P \Rightarrow x + 1 = 3$  holds.

The verification that an annotated instruction is consistent with a post-condition thus returns a sequence of implications between assertions. When all these implications are logically valid, there is a guarantee that satisfying the pre-condition before executing the instruction is enough to ensure that the post-condition will also be satisfied after executing the instruction.

When the instruction is a plain assignment without annotation, there is no need to verify any implication because the computed pre-condition is already good enough. When the instruction is an annotated instruction  $\{ A \} i$  and the post-condition is  $P$ , we can first compute the pre-condition  $P'$  and a list of

implications  $l$  for the instruction  $i$  and the post-condition  $P$ . We then only need to add  $A \Rightarrow P'$  to  $l$  to get the list of conditions for the whole instruction.

When the instruction is a sequence of two instructions  $i_1; i_2$  and the post-condition is  $P$ , we need to compute lists of conditions for both sub-components  $i_1$  and  $i_2$ . The list of conditions for  $i_2$  is computed for the post-condition for the whole construct  $P$ , while the list of conditions of  $i_1$  is computed taking as post-condition the pre-condition of  $i_2$  for  $P$ . This is consistent with the intuitive explanation that it suffices that the pre-condition for an instruction holds to ensure that the post-condition will hold after executing that instruction. If we want  $P$  to hold after executing  $i_2$ , we need the pre-condition of  $i_2$  for  $P$  to be satisfied and it is the responsibility of the instruction  $i_1$  to guarantee this. Thus, the conditions for  $i_1$  can be computed with this assertion as a post-condition.

When the instruction is a while loop, of the form `while b do { A } i done` we must remember that the assertion  $A$  should be an invariant during the loop execution. This is expressed by requiring that  $A$  is satisfied before executing  $i$  should be enough to guarantee that  $A$  is also satisfied after executing  $i$ . However, this is needed only in the cases where the loop test  $b$  is also satisfied, because when  $b$  is not satisfied the inner instruction of the while loop is not executed. At the end of the execution, we can use the information that the invariant  $A$  is satisfied and the information that we know the loop has been executed because the test eventually failed. The program is consistent when these two logical properties are enough to imply the initial post-condition  $P$ . Thus, we must first compute the pre-condition  $A'$  for the inner instruction  $i$  and the post-condition  $A$ , compute the list of conditions for  $i$  with  $A$  as post-condition, add the condition  $A \wedge b \Rightarrow A'$ , and add the condition  $A \wedge \neg b \Rightarrow P$ .

**Coq encoding of the verification condition generator** The verification conditions always are implications. We provide a new data-type for these implications:

```
Inductive cond : Type := imp (a1 a2:assert).
```

The computation of verification conditions is then simply described as a plain recursive function, which follows the structure of annotated instructions.

```
Fixpoint vc (i:a_instr)(post : assert) : list cond :=
  match i with
  | pre a i => (imp a (pc i post))::vc i post
  | a_assign _ _ => nil
  | a_seq i1 i2 => vc i1 (pc i2 post)::vc i2 post
  | a_while b a i =>
    (imp (a_conj a (a_b b)) (pc i a))::
    (imp (a_conj a (a_not (a_b b))) post)::
    vc i a
  end.
```

Describing the semantics of programming language using a verification condition generator is not the only approach that can be used to describe the language. In fact, this approach is partial, because it describes properties of inputs and outputs when instruction execution terminates, but it gives no information about termination. More precise descriptions can be given using operational or denotational semantics and the consistency of this verification condition generator with such a complete semantics can also be verified formally. This is done in [2], but it is not the purpose of this article.

When reasoning about the correctness of a given annotated instruction, we can use the function `vc` to obtain a list of conditions. It is then necessary to reason on the validity of this list of conditions. What we want to verify is that the implications hold for every possible instantiation of the program variables. This is described by the following function.

```

Fixpoint valid (m:string->list Z ->Prop) (l : list cond) :
  Prop :=
  match l with
  | nil => True
  | c::tl =>
    (let (a1, a2) := c in forall g, ia m g a1 -> ia m g a2)
    /\ valid m tl
  end.

```

An annotated program  $i$  is consistent with a given post-condition  $p$  when the property `valid (vc i p)` holds. This means that the post-condition is guaranteed to hold after executing the instruction if the computed pre-condition was satisfied before the execution and the execution of the instruction terminates.

### 3.3 A monotonicity property

In our study of an abstract interpreter, we will use a property of the condition generator.

**Theorem 1.** *For every annotated instruction  $i$ , if  $p_1$  and  $p_2$  are two post-conditions such that  $p_1$  is stronger than  $p_2$ , if the pre-condition for  $i$  and  $p_1$  is satisfied and all the verification conditions for  $i$  and the post-condition  $p_1$  are valid, then the pre-condition for  $i$  and  $p_2$  is also satisfied and the verification conditions for  $i$  and  $p_2$  are also valid.*

*Proof.* This proof is done in the context of a given mapping from predicate names to actual predicates,  $m$ . The property is proved by induction on the structure of the instruction  $i$ . The statement  $p_1$  is stronger than  $p_2$  when the implication  $p_1 \Rightarrow p_2$  is valid. In other words, for every assignment of variables  $g$ , the logical value of  $p_1$  implies the logical value of  $p_2$ .

If the instruction is an assignment, we can rely on a lemma: the value of any assertion `subst x e p` in any valuation  $g$  is equal to the value of the assertion  $p$  in the valuation  $g'$  that is equal to  $g$  on every variable but  $x$ , for which it returns

the value of  $e$  in the valuation  $g$ . Thus, the precondition for the assignment  $x := e$  for  $p_i$  is  $\text{subst } x \ e \ p_i$  and the validity of  $\text{subst } x \ e \ p_1 \Rightarrow \text{subst } x \ e \ p_2$  simply is an instance of the validity of  $p_1 \Rightarrow p_2$ , which is given by hypothesis. Also, when the instruction is an assignment, there is no generated verification condition and the second part of the statement holds.

If the instruction is a sequence  $i_1; i_2$ , then we know by induction hypothesis that the pre-condition  $p'_1$  for  $i_2$  and  $p_1$  is stronger than the pre-condition  $p'_2$  for  $i_2$  and  $p_2$  and all the verification conditions for that part are valid; we can use an induction hypothesis again to obtain that the pre-condition for  $i_1$  and  $p'_1$  is stronger than the pre-condition for  $i_1$  and  $p'_2$ , and the corresponding verification conditions are all valid. The last two pre-conditions are the ones we need to compare, and the whole set of verification conditions is the union of the sets which we know are valid.

If the instruction is an annotated instruction  $\{a\}i$ , the two pre-conditions for  $p_2$  and  $p_1$  are always  $a$ , so the first part of the statement trivially holds. Moreover, we know by induction hypothesis that the pre-condition  $p'_1$  for  $i$  and  $p_1$  is stronger than the pre-condition  $p'_2$  for  $i$  and  $p_2$ . The verification conditions for the whole instruction and  $p_1$  (resp.  $p_2$ ) are the same as for the sub-instruction, with the condition  $a \Rightarrow p'_1$  (resp.  $a \Rightarrow p'_2$ ) added. By hypothesis,  $a \Rightarrow p'_1$  holds, by induction hypothesis  $p'_1 \Rightarrow p'_2$ , we can thus deduce that  $a \Rightarrow p'_2$  holds.

If the instruction is a loop **while**  $b$  **do**  $\{a\}$   $i$  **done**, most verification conditions and generated pre-conditions only depend on the loop invariant. The only thing that we need to check is the verification condition containing the invariant, the negation of the test and the post-condition. By hypothesis,  $a \wedge \neg b \Rightarrow p_1$  and  $p_1 \Rightarrow p_2$  are valid. By transitivity of implication we obtain  $a \wedge \neg b \Rightarrow p_2$  easily.

In Coq, we first prove a lemma that expresses that the satisfiability of an assertion  $a$  where a variable  $x$  is substituted with an arithmetic expression  $e'$  for a valuation  $g$  is the same as the satisfiability of the assertion  $a$  without substitution, but for a valuation that maps  $x$  to the value of  $e'$  in  $g$  and coincides with  $g$  for all other variables.

```

Lemma subst_sound :
  forall m g a x e',
    ia m g (subst x e' a) =
      ia m (fun y => if string_dec x y then af g e' else g y) a.

```

This lemma requires similar lemmas for arithmetic expressions, boolean expressions, and lists of expressions. All are proved by induction on the structure of expressions.

**An example proof for substitution** For instance, the statement for the substitution in arithmetic expressions is as follows:

```

Lemma subst_sound_a :
  forall g e x e',
    af g (asubst x e' e) =
      af (fun y => if string_dec x y then af g e' else g y) e.

```

The proof can be done in Coq by an induction on the expression `e`. This leads the system to generate three cases, corresponding to the three constructors of the `aexpr` type. The combined tactic we use is as follows:

```
intros g e x e'; induction e; simpl; auto.
```

The tactic `induction e` generates three goals and the tactics `simpl` and `auto` are applied to all of them. One of the cases is the case for the `anum` constructor, where both instances of the `af` function compute to the value carried by the constructor, thus `simpl` forces the computation and leads to an equality where both sides are equal. In this case, `auto` solves the goal. Only the other two goals remain.

The first other goal is concerned with the `avar` construct. In this case the expression has the form `avar s` and the expression `subst x e' (avar s)` is transformed into the following expression by the `simpl` tactic.

```
if string_dec x s then e' else (avar s)
```

For this case, the system displays a goal that has the following shape:

```
g : string -> Z
s : string
x : string
e' : aexpr
=====
af g (if string_dec x s then e' else avar s) =
(if string_dec x s then af g e' else g s)
```

In Coq goals, the information that appears above the horizontal bar is data that is known to exist, the information below the horizontal bar is the expression that we need to prove. Here the information that is known only corresponds to typing information.

We need to reason by cases on the values of the expression `string_dec x s`. The tactic `case ...` is used for this purposes. It generate two goals, one corresponding to the case where `string_dec x s` has an affirmative value and one corresponding to the case where `string_dec x s` has a negative value. In each the goal, the `if-then-else` constructs are reduced accordingly. In the goal where `string_dec x s` is affirmative, both sides of the equality reduce to `af g e'`; in the other goal, both sides of the equality reduce to `g x`. Thus in both cases, the proof becomes easy. This reasoning step is easily expressed with the following combined tactic:

```
case (string_dec x s); auto.
```

There only remains a goal for the last possible form of arithmetic expression, `aplus e1 e2`. The induction tactic provides *induction hypotheses* stating that the property we want to prove already holds for `e1` and `e2`. After symbolic computation of the functions `af` and `asubst`, as performed by the `simpl` tactic, the goal has the following shape:

```

...
IHe1 : af g (asubst x e' e1) =
      af (fun y : string =>
          if string_dec x y then af g e' else g y) e1
IHe2 : af g (asubst x e' e2) =
      af (fun y : string =>
          if string_dec x y then af g e' else g y) e2
=====
af g (asubst x e' e1) + af g (asubst x e' e2) =
af (fun y : string =>
    if string_dec x y then af g e' else g y) e1 +
af (fun y : string =>
    if string_dec x y then af g e' else g y) e2

```

This proof can be finished by rewriting with the two equalities named `IHe1` and `IHe2` and then recognizing that both sides of the equality are the same, as required by the following tactics.

```

rewrite IHe1, IHe2; auto.
Qed.

```

We can now turn our attention to the main result, which is then expressed as the following statement:

```

Lemma vc_monotonic :
  forall m i p1 p2, (forall g, ia m g p1 -> ia m g p2) ->
    valid m (vc i p1) ->
    valid m (vc i p2) /\
    (forall g, ia m g (pc i p1) -> ia m g (pc i p2)).

```

To express that this proof is done by induction on the structure of instructions, the first tactic sent to the proof system has the form:

```

intros m; induction i; intros p1 p2 p1p2 vc1.

```

The proof then has four cases, which are solved in about 10 lines of proof script.

## 4 A first simple abstract interpreter

We shall now define an abstract interpreter that runs instructions symbolically, updating a state where each variable is associated to an abstract value. The result of the interpreter contains two parts. The first part is a state. The second part is an annotated version of the instruction, where annotations describe the abstract value for each variable at each location in the instruction.

For example, if we give our abstract interpreter an input state stating that `x` is even and `y` is odd and the instruction `x:= x+y; y:=y+1`, the resulting value will be:

```
((x, odd)::(y, even)::nil,
  {even x /\ odd y} x:=x+y; {odd x /\ odd y} y:= y+1)
```

We suppose there exists a data-type  $A$  whose elements will represent abstract values on which instructions are supposed to compute. For instance, the data-type  $A$  could be the type containing three values `even`, `odd`, and `top`. Another traditional example of abstract data-type is the type of intervals, that are either of the form  $[m, n]$ , with  $m \leq n$ ,  $[-\infty, n]$ ,  $[m, +\infty]$ , or  $[-\infty, +\infty]$ .

The data-type of abstract values should come with a few elements and functions, which we will describe progressively.

#### 4.1 Using Galois connections

Abstract values represent specific sets of concrete values. There is a natural order on sets : set inclusion. Similarly, we can consider an order on abstract values, which mimics the order between the sets they represent. The traditional approach to describe this correspondance between the order on sets of values and the order on abstract values is to consider that the type of abstract values is given with a pair of functions  $\alpha$  and  $\gamma$ , where  $\alpha : \mathcal{P}(\mathbb{Z}) \rightarrow A$  and  $\gamma : A \rightarrow \mathcal{P}(\mathbb{Z})$ . The function  $\gamma$  maps any abstract value to the set of concrete values it represents. The function  $\alpha$  maps any set of concrete values to the smallest abstract value whose interpretation as a set contains the input. Written in a mathematical formula where  $\sqsubseteq$  denotes the order on abstract values, the two functions and the orders on sets of concrete values and on abstract values are related by the following statement:

$$\forall a \in A, \forall b \in \mathcal{P}(\mathbb{Z}). b \subset \gamma(a) \Leftrightarrow \alpha(b) \sqsubseteq a.$$

When the functions  $\alpha$  and  $\gamma$  are given with this property, one says that there is a *Galois connection*.

In our study of abstract interpretation, the functions  $\alpha$  and  $\gamma$  do not appear explicitly. In a sense,  $\gamma$  will be represented by a function `to_pred` mapping abstract values to assertions depending on arithmetic expressions. However, it is useful to keep these functions in mind when trying to figure out what properties are expected for the various components of our abstract interpreters, as we will see in the next section.

#### 4.2 Abstract evaluation of arithmetic expressions

Arithmetic expressions contain integer constants and additions, neither of which are concerned with the data-type of abstract values. To be able to associate an abstract value to an arithmetic expression, we need to find ways to establish a correspondance between concrete values and abstract values. This is done by supposing the existence of two functions and a constant, which are the first three values axiomatized for the data-type of abstract values (but there will be more later):

- `from_Z` :  $\mathbb{Z} \rightarrow A$ , this is used to associate a relevant abstract value to any concrete value,
- `a_add` :  $A \rightarrow A \rightarrow A$ , this is used to add two abstract values,
- `top` :  $A$ , this is used to represent the abstract value that carries no information.

In terms of Galois connections, the function `from_Z` corresponds to the function  $\alpha$ , when applied to singletons. The function `a_add` must be designed in such a way that the following property is satisfied:

$$\forall v_1 v_2, \{x + y \mid x \in (\gamma(v_1), y \in (\gamma(v_2)))\} \subset \gamma(\mathbf{a\_add} \ v_1 \ v_2).$$

With this constraint, a function that maps any pairs of abstract values to `top` would be acceptable, however it would be useless. It is better if `a_add v1 v2` is the least satisfactory abstract value such that the above property is satisfied.

The value `top` is the maximal element of  $A$ , the image of the whole  $\mathbb{Z}$  by the function  $\alpha$ .

### 4.3 Handling abstract states

When computing the value of a variable, we suppose that this value is given by looking up in a state, which actually is a list of pairs of variables and abstract values.

Definition `state := list(string*A)`.

```
Fixpoint lookup (s:state) (x:string) : A :=
  match s with
  | nil => top
  | (y,v)::tl => if string_dec x y then v else lookup tl x
  end.
```

As we see in the definition of `lookup`, when a value is not defined in a state, the function behaves as if it was defined with `top` as abstract value. The computation of abstract values for arithmetic expressions is then described by the following function.

```
Fixpoint a_af (s:state)(e:aexpr) : A :=
  match e with
  | avar x => lookup s x
  | anum n => from_Z n
  | aplus e1 e2 => a_add (a_af s e1) (a_af s e2)
  end.
```

When executing assignments abstractly, we are also supposed to modify the state. If the state contained no previous information about the assigned variable, a new pair is created. Otherwise, the first existing pair must be updated. This is done with the following function.



```

Fixpoint a_upd(x:string)(v:A)(l:state) : state :=
  match l with
  | nil => (x,v)::nil
  | (y,v')::tl =>
    if string_dec x y then (y, v)::tl else (y,v')::a_upd x v tl
  end.

```

Later in this paper, we define a function that generates assertions from states. For this purpose, it is better to update by modifying existing pairs of a variable and a value rather than just inserting the new pair in front.

#### 4.4 The interpreter's main function

When computing abstract interpretation on instructions we want to produce a final abstract state and an annotated instruction. We will need a way to transform an abstract value into an assertion. This is given by a function with the following type:

- `to_pred` : `A -> aexpr -> assert` this is used to express that the value of the arithmetic expression in a given valuation will belong to the set of concrete values represented by the given abstract value. So `to_pred` is axiomatized in the same sense as `from_Z`, `a_add`, `top`.

Relying on the existence of `to_pred`, we can define a function that maps states to assertions:

```

Fixpoint s_to_a (s:state) : assert :=
  match s with
  | nil => a_true
  | (x,a)::tl => a_conj (to_pred a (avar x)) (s_to_a tl)
  end.

```

This function is implemented in a manner that all pairs present in the state are transformed into assertions. For this reason, it is important that `a_upd` works by modifying existing pairs rather than hiding them.

Our first simple abstract interpreter only implements a trivial behavior for while loops. Basically, this says that no information can be gathered for while loops (the result is `nil`, and the while loop's invariant is also `nil`).

```

Fixpoint ab1 (s:state)(i:instr) : state*a_instr :=
  match i with
  | assign x e =>
    (a_upd x (a_af s e) s, pre (s_to_a s) (a_assign x e))
  | seq i1 i2 =>
    let (s1, a_i1) := ab1 s i1 in
    let (s2, a_i2) := ab1 s1 i2 in
    (s2, a_seq a_i1 a_i2)
  | while b i =>

```

```

    let (_, a_i) := ab1 nil i in
      (nil, a_while b (s_to_a nil) a_i)
    end.

```

In this function, we see that the abstract interpretation of sequences is simply described as composing the effect on states and recombining the instruction obtained from each component of the sequence.

#### 4.5 Expected properties for abstract values

To prove the correctness of the abstract interpreter, we only need to know that the various functions and values provided around the type `A` satisfy a collection of properties. These are gathered as a set of hypotheses.

One value that we have not talked about yet is the mapping from predicate names to actual predicates on integers, which is necessary to interpret the assertions generated by `to_pred`. This is given axiomatically, like `top` and the others:

- `m : string -> list Z -> Prop`, maps all predicate names used in `to_pred` to actual predicates on integers.

The first hypothesis expresses that `top` brings no information.

Hypothesis `top_sem` : forall e, (to\_pred top e) = a\_true.

The next two hypotheses express that the predicates associated to each abstract value are *parametric* with respect to the arithmetic expression they receive. Their truth does not depend on the exact shape of the expressions, but only on the concrete value such an arithmetic expression may take in the current valuation. Similarly, substitution basically affects the arithmetic expression part of the predicate, not the part that depends on the abstract value.

```

Hypothesis to_pred_sem :
  forall g v e, ia m g (to_pred v e) =
    ia m g (to_pred v (anum (af g e))).
Hypothesis subst_to_pred :
  forall v x e e', subst x e' (to_pred v e) =
    to_pred v (asubst x e' e).

```

For instance, if the abstract values are intervals, it is natural that the `to_pred` function will map an abstract value `[3,10]` and an arithmetic expression `e` to an assertion `between(3, e, 10)`. When evaluating this assertion with respect to a given valuation `g`, the integers 3 and 10 will not be affected by `g`. Similarly, substitution will not affect these integers.

The last two hypotheses express that the interpretation of the associated predicates for abstract values obtained through `from_Z` and `a_add` are consistent with the concrete values computed for immediate integers and additions. The hypothesis `from_Z_sem` actually establishes the correspondence between `from_Z`

and the abstraction function  $\alpha$  of a Galois connection. The hypothesis `a_add_sem` expresses the condition which we described informally when introducing the function `a_add_sem`.

```
Hypothesis from_Z_sem :
  forall g x, ia m g (to_pred (from_Z x) (anum x)).
Hypothesis a_add_sem : forall g v1 v2 x1 x2,
  ia m g (to_pred v1 (anum x1)) ->
  ia m g (to_pred v2 (anum x2)) ->
  ia m g (to_pred (a_add v1 v2) (anum (x1+x2))).
```

#### 4.6 Avoiding duplicates in states

The way `s_to_a` and `a_upd` are defined is not consistent: `s_to_a` maps every pair occurring in a state to an assertion fragment, while `a_upd` only modifies the first pair occurring in the state.

For instance, when the abstract interpretation computes with intervals, `s` is `("x", [1,1])::("x", [1,1])::nil`, and the instruction is `x := x + 1`, the resulting state is `("x", [2,2])::("x", [1,1])::nil` and the resulting annotated instruction is `{ 1 ≤ x ≤ 1 ∧ 1 ≤ x ≤ 1 } x:= x+1`. The post-condition corresponding to the resulting state is `2 ≤ x ≤ 2 ∧ 1 ≤ x ≤ 1`. It is contradictory and cannot be satisfied when executing from valuations satisfying the pre-condition, which is not contradictory.

To cope with this difficulty, we need to express that the abstract interpreter works correctly only with states that contain no duplicates. We formalize this with a predicate `consistent`, which is defined as follows:

```
Fixpoint mem (s:string)(l:list string): bool :=
  match l with
  | nil => false
  | x::l => if string_dec x s then true else mem s l
  end.

Fixpoint no_dups (s:state)(l:list string) :bool :=
  match s with
  | nil => true
  | (s,_)::tl => andb (no_dups tl (s::l)) (negb (mem s l))
  end.
```

**Definition** `consistent (s:state) := no_dups s nil = true`.

The function `no_dups` actually returns `true` when the state `s` contains no duplicates and no element from the exclusion list `l`. We prove, by induction on the structure of `s`, that updating a state that satisfies `no_dups` for an exclusion list `l`, using `a_upd` for a variable `x` outside the exclusion list returns a new state that still satisfies `no_dups` for `l`. The statement is as follows:

```

Lemma no_dups_update :
  forall s l x v, mem x l = false ->
    no_dups s l = true -> no_dups (a_upd x v s) l = true.

```

The proof of this lemma is done by induction on  $s$ , making sure that the property that is established for every  $s$  is universally quantified over  $l$ : the induction hypothesis is actually used for a different value of the the exclusion list.

The corollary from this lemma corresponding to the case where  $l$  is instantiated with the empty list expresses that `a_upd` preserves the `consistent` property.

```

Lemma consistent_update :
  forall s x v, consistent s -> consistent (a_upd x v s).

```

#### 4.7 Proving the correctness of this interpreter

When the interpreter runs on an instruction  $i$  and a state  $s$  and returns a new state  $s'$  and an annotated instruction  $i'$ , the correctness of the run is expressed with three properties:

- The assertion `s_to_a s` is stronger than the pre-condition `pc i' (s_to_a s')`,
- All the verification conditions in `vc i' (s_to_a s')` are valid,
- The annotated instruction  $i'$  is an annotated version of the input  $i$ .

In the next few sections, we will prove that all runs of the abstract interpreter are correct.

#### 4.8 Soundness of abstract evaluation for expressions

When an expression  $e$  evaluates abstractly to an abstract value  $a$  and concretely to an integer  $z$ ,  $z$  should satisfy the predicate associated to the value  $a$ . Of course, the evaluation of  $e$  can only be done using a valuation that takes care of providing values for all variables occurring in  $e$ . This valuation must be consistent with the abstract state that is used for the abstract evaluation leading to  $a$ . The fact that a valuation is consistent with an abstract state is simply expressed by saying that the interpretation of the corresponding assertion for this valuation has to hold. Thus, the soundness of abstract evaluation is expressed with a lemma that has the following shape:

```

Lemma a_af_sound :
  forall s g e, ia m g (s_to_a s) ->
    ia m g (to_pred (a_af s e) (anum (af g e))).

```

This lemma is proved by induction on the expression  $e$ . The case where  $e$  is a number is a direct application of the hypothesis `from_Z_sem`, the case where  $e$  is an addition is a consequence of `a_add_sem`, combined with induction hypotheses. The case where  $e$  is a variable relies on another lemma:

```

Lemma lookup_sem : forall s g, ia m g (s_to_a s) ->
  forall x, ia m g (to_pred (lookup s x) (anum (g x))).

```

This other lemma is proved by induction on  $s$ . In the base case,  $s$  is empty,  $\text{lookup } s \ x$  is  $\text{top}$ , and the hypothesis  $\text{top\_sem}$  makes it possible to conclude; in the step case, if  $s$  is  $(y,v)::s'$  then the hypothesis

```

ia m g (s_to_a s)

```

reduces to

```

to_pred v (avar y) /\ ia m g (s_to_a s')

```

We reason by cases on whether  $x$  is  $y$  or not. If  $x$  is equal to  $y$  then  $\text{to\_pred } v \ (\text{avar } y)$  is the same as  $\text{to\_pred } v \ (\text{anum } (g \ x))$  according to  $\text{to\_pred\_sem}$  and  $\text{lookup } s \ x$  is the same as  $v$  by definition of  $\text{lookup}$ , this is enough to conclude this case. If  $x$  and  $y$  are different, we use the induction hypothesis on  $s'$ .

#### 4.9 Soundness of update

In the weakest pre-condition calculus, assignments of the form  $x := e$  are taken care of by substituting all occurrences of the assigned variable  $x$  with the arithmetic expression  $e$  in the post-condition to obtain the weakest pre-condition. In the abstract interpreter, assignment is taken care of by updating the first instance of the variable in the state. There is a discrepancy between the two approaches, where the first approach acts on all instances of the variable and the second approach acts only on the first one. This discrepancy is resolved in the conditions of our experiment, where we work with abstract states that contain only one binding for each variable: in this case, updating the first variable is the same as updating all variables. We express this with the following lemmas:

```

Lemma subst_no_occur :
  forall s x l e,
    no_dups s (x::l) = true -> subst x e (s_to_a s) = (s_to_a s).

```

```

Lemma subst_consistent :
  forall s g v x e, consistent s -> ia m g (s_to_a s) ->
    ia m g (to_pred v (anum (af g e))) ->
    ia m g (subst x e (s_to_a (a_upd x v s))).

```

Both lemmas are proved by induction on  $s$  and the second one uses the first in the case where the substituted variable  $x$  is the first variable occurring in  $s$ . This proof also relies on the hypothesis  $\text{subst\_to\_pred}$ .

#### 4.10 Relating input abstract states and pre-conditions

For the correctness proof we consider runs starting from an initial abstract state  $s$  and an instruction  $i$  and obtaining a final abstract state  $s'$  and an annotated instruction  $i'$ . We are then concerned with the verification conditions and the pre-condition generated for the post-condition corresponding to  $s'$  and the annotated instruction  $i'$ . The pre-condition we obtain is either the assertion corresponding to  $s$  or the assertion `a_true`, when the first sub-instruction in  $i$  is a while loop. In all cases, the assertion corresponding to  $s$  is stronger than the pre-condition. This is expressed with the following lemma, which is easily proved by induction on  $i$ .

Lemma `ab1_pc` :

```
forall i i' s s', ab1 i s = (i', s') ->
  forall g a, ia m g (s_to_a s) -> ia m g (pc i' a).
```

This lemma is actually stronger than needed, because the post-condition used for computing the pre-condition does not matter, since the resulting annotated instruction is heavily annotated with assertions and the pre-condition always comes from one of the annotations.

#### 4.11 Validity of generated conditions

The main correctness statement only concerns states that satisfy the `consistent` predicate, that is, states that contain at most one entry for each variable. The statement is proved by induction on instructions. As is often the case, what we prove by induction is a stronger statement; Such a stronger statement also means stronger induction hypotheses. Here we add the information that the resulting state is also consistent.

**Theorem 2.** *If  $s$  is a consistent state and running the abstract interpreter `ab1` on  $i$  from  $s$  returns a new annotated instruction  $i'$  and a final state  $s'$ , then all the verification conditions generated for  $i'$  and the post-condition associated to  $s'$  are valid. Moreover, the state  $s'$  is consistent.*

The Coq encoding of this theorem is as follows:

```
Theorem ab1_correct : forall i i' s s',
  consistent s ->
  (i', s') = ab1 i s ->
  valid m (vc i' (s_to_a s')) /\ consistent s'.
```

This statement is proved by induction on  $i$ . Three cases arise, corresponding to the three instructions in the language.

1. When  $i$  is an assignment  $x := e$ , this is the base case. `ab1 i s` computes to

```
(pre (s_to_a s) (a_assign x e), a_upd x (a_af s e) s)
```

From the lemma `a_af_sound` we obtain that the concrete value of `e` in any valuation `g` that satisfies `ia m g (s_to_a s)` satisfies the following property:  
`ia m g (to_pred (a_af s e) (anum (af g e)))`

The lemma `subst_consistent` can then be used to obtain the validity of the following condition.

```
imp (s_to_a s) (subst x e (s_to_a (a_upd x (a_af s e) s)))
```

This is the single verification condition generated for this instruction. The second part is taken care of by `consistent_update`.

2. When the instruction `i` is a sequence `seq i1 i2`, the abstract interpreter first processes `i1` with the state `s` as input to obtain an annotated instruction `a_i1` and an output state `s1`, it then processes `i2` with `s1` as input to obtain an annotated instruction `a_i2` and a state `s2`. The state `s2` is used as the output state for the whole instruction. We then need to verify that the conditions generated for `a_seq a_i1 a_i2` using `s_to_a a2` as post-condition are valid and `s2` satisfies the `consistent` property. The conditions can be split in two parts. The second part is `vc a_i2 (s_to_a a2)`. the validity of these conditions is a direct consequence of the induction hypotheses. The first part is `vc a_i1 (pc a_i2 (s_to_a s2))`. This is not a direct consequence of the induction hypothesis, which only states `vc a_i1 (s_to_a s1)`. However, the lemma `ab1_pc` applied on `a_i2` states that `s_to_a s1` is stronger than `pc (s_to_a s2)` and the lemma `vc_monotonic` makes it possible to conclude. With respect to the `consistent` property, it is recursively transmitted from `s` to `s1` and from `s1` to `s2`.
3. When the instruction is a while loop, the body of the loop is recursively processed with the `nil` state, which is always satisfied. Thus, the verification conditions all conclude to `a_true` which is trivially true. Also, the `nil` state also trivially satisfies the `consistent` property.

#### 4.12 The annotated instruction

We also need to prove that the produced annotated instruction really is an annotated version of the initial instruction. To state this new lemma, we first define a simple function that forgets the annotations in an annotated instruction:

```
Fixpoint cleanup (i: a_instr) : instr :=
  match i with
  | pre a i => cleanup i
  | a_assign x e => assign x e
  | a_seq i1 i2 => seq (cleanup i1) (cleanup i2)
  | a_while b a i => while b (cleanup i)
  end.
```

We then prove a simple lemma about the abstract interpreter and this function.

```
Theorem ab1_clean : forall i i' s s',
  ab1 i s = (i', s') -> cleanup i' = i.
```

The proof of this lemma is done by induction on the structure of `i`.

### 4.13 Instantiating the simple abstract interpreter

We can instantiate this simple abstract interpreter on a data-type of odd-even values, using the following inductive type and functions:

```
Inductive oe : Type := even | odd | oe_top.
```

```
Definition oe_from_Z (n:Z) : oe :=  
  if Z_eq_dec (Zmod n 2) 0 then even else odd.
```

```
Definition oe_add (v1 v2:oe) : oe :=  
  match v1,v2 with  
  | odd, odd => even  
  | even, even => even  
  | odd, even => odd  
  | even, odd => odd  
  | _, _ => oe_top  
  end.
```

The abstract values can then be mapped into assertions in the obvious way using a function `oe_pred` which we do not describe here for the sake of conciseness. Running this simple interpreter on a small example, representing the program

```
x := x + y; y := Y + 1
```

for the state `("x", odd)::("y", even)::nil` is represented by the following dialog:

```
Definition ab1oe := ab1 oe oe_from_Z oe_top oe_add oe_to_pred.
```

```
Eval vm_compute in  
ab1oe (seq (assign "x" (applus (avar "x") (avar "y")))  
        (assign "y" (applus (avar "y") (anum 1))))  
(("x",even)::("y",odd)::nil).  
= (a_seq  
  (pre  
    (a_conj (pred "even" (avar "x" :: nil))  
            (a_conj (pred "odd" (avar "y" :: nil)) a_true))  
    (a_assign "x" (applus (avar "x") (avar "y"))))  
  (pre  
    (a_conj (pred "odd" (avar "x" :: nil))  
            (a_conj (pred "odd" (avar "y" :: nil)) a_true))  
    (a_assign "y" (applus (avar "y") (anum 1))),  
  ("x", odd) :: ("y", even) :: nil)  
: a_instr * state oe
```



## 5 A stronger interpreter

More precise results can be obtained for while loops. For each loop we need to find a state whose interpretation as an assertion will be an acceptable invariant for the loop. We want this invariant to take into account any information that can be extracted from the boolean test in the loop: when entering inside the loop, we know that the test succeeded; when exiting the loop we know that the test failed. It turns out that this information can help us detect cases where the body of a loop is never executed and cases where a loop can never terminate. To describe non-termination, we change the type of values returned by the abstract interpreter: instead of returning an annotated instruction and a state, our new abstract interpreter returns an annotated instruction and an optional state: the optional value is `None` when we have detected that execution cannot terminate. This detection of guaranteed non-termination is conservative: when the analyser cannot guarantee that an instruction loops, it returns a state as usual. The presence of optional states will slightly complexify the structure of our static analysis.

We assume the existence of two new functions for this purpose.

- `learn_from_success : state -> bexpr -> option state`, this is used to encode the information learned when the test succeeded. For instance if the environment initially contains an interval `[0,10]` for the variable `x` and the test is `x < 6`, then we can return the environment so that the value for `x` becomes `[0, 5]`. Sometimes, the initial environment is so that the test can never be satisfied, in this case a value `None` is returned instead of an environment.
- `learn_from_failure : state -> bexpr -> option state`, this is used to compute information about a state knowing that a test failed.

The body of a while loop is often meant to be run several times. In abstract interpretation, this is also true. At every run, the information about each variable at each location of the instruction needs to be updated to take into account more and more concrete values that may be reached at this location. In traditional approaches to abstract interpretation, a binary operation is applied at each location, to combine the information previously known at this location and the new values discovered in the current run. This is modeled by a binary operation.

- `join : A -> A -> A`, this function takes two abstract values and returns a new abstract value whose interpretation as a set is larger than the two inputs.

The theoretical description of abstract interpretation insists that the set `A`, together with the values `join` and `top` should constitute an upper semi-lattice. In fact, We will use only part of the properties of such a structure in our proofs about the abstract interpreter.

When the functions `learn_from_success` and `learn_from_failure` return a `None` value, we actually detect that some code will never be executed. For

instance, if `learn_from_success` returns `None`, we can know that the test at the entry of a loop will never be satisfied and we can conclude that the body of the loop is not executed. In this condition, we can mark this loop body with a false assertion. We provide a function for this purpose:

```
Fixpoint mark (i:instr) : a_instr :=
  match i with
  | assign x e => pre a_false (a_assign x e)
  | seq i1 i2 => a_seq (mark i1) (mark i2)
  | while b i => a_while b a_false (mark i)
  end.
```

Because it marks almost every instruction, this function makes it easy to recognize at first glance the fragments of code that are dead code. A more lightweight approach could be to mark only the sub-instructions for which an annotation is mandatory: while loops.

## 5.1 Main structure of invariant search

In general, finding the most precise invariant for a while loop is an undecidable problem. Here we are describing a static analysis tool. We will trade preciseness for guaranteed termination. The approach we will describe will be as follows:

1. Run the body of the loop abstractly for a few times, progressively widening the sets of values for each variable at each run. If this process stabilizes, we have reached an invariant,
2. If no invariant was reached, try taking over-approximations of the values for some variables and run again the loop for a few times. This process may also reach an invariant,
3. If no invariant was reached by progressive widening, pick an abstract state that is guaranteed to be an invariant (as we did for the first simple interpreter: take the *top* state that gives no information about any variable),
4. Invariants that were obtained by over-approximation can then be improved by a *narrowing* process: when run through the loop again, even if no information about the state is given at the beginning of the loop, we may still be able to gather some information at the end of executing the loop. The state that gathers the information at the end of the loop and the information before entering the loop is most likely to be an invariant, which is more precise (narrower) than the top state. Again this process may be run several times.

We shall now review the operations involved in each of these steps.

## 5.2 Joining states together

Abstract states are finite list of pairs of variable names and abstract values. When a variable does not occur in a state, the associated abstract value is `top`. When joining two states together every variable that does not occur in one of the

two states should receive the `top` value, and every variable that occurs in both states should receive the join of the two values found in each state. We describe this by writing a function that studies all the variables that occur in one of the lists: it is guaranteed to perform the right behavior for all the variables in both lists, it naturally associates the `top` value to the variables that do not occur in the first list (because no pair is added for these variables), and it naturally associates the `top` value to the variables that do not occur in the second list, because `top` is the value found in the second list and `join` preserves `top`.

```
Fixpoint join_state (s1 s2:state) : state :=
  match s1 with
  | nil => nil
  | (x,v)::t1 => a_upd x (join v (lookup s2 x)) (join_state t1 s2)
  end.
```

Because we sometimes detect that some instruction will not be executed we occasionally have to consider situation where we are not given a state after executing a while loop. In this case, we have to combine together a state and the absence of a state. But because the absence of state corresponds to a false assertion, the other state is enough to describe the required invariant. We encode this in an auxiliary function.

```
Definition join_state' (s: state)(s':option state) : state :=
  match s' with
  | Some s' => join_state s s'
  | None => s
  end.
```

### 5.3 Running the body a few times

In our general description of the abstract interpretation of loops, we need to execute the body of loops in two different modes: one mode is a *widening* mode the other is a *narrowing* mode. In the narrowing mode, after executing the body of the loop needs to be joined with the initial state before executing the body of the loop, so that the result state is less precise than both the state before executing the body of the loop and the state after executing the body of the loop. In the *narrowing* mode, we start the execution with an environment that is guaranteed to be large enough, hoping to narrow this environment to a more precise value. In this case, the `join` operation must not be done with the state that is used to start the execution, but with another state which describes the information known about variables before considering the loop. To accommodate these two modes of abstract execution, we use a function that takes two states as input: the first state is the one with which the result must be joined, the second state is the one with which execution must start. In this function, the argument `ab` is the function that describes the abstract interpretation on the instruction inside the loop, the argument `b` is the test of the loop. The function `ab` returns an optional state and an annotated instruction. The optional state is `None` when the

abstract interpreter can detect that the execution of the program from the input state will never terminate. When putting all elements together, the argument `ab` will be instantiated with the recursive call of the abstract interpreter on the loop body.

```

Definition step1 (ab: state -> a_instr * option state)
  (b:bexpr) (init s:state) : state :=
  match learn_from_success s b with
  | Some s1 => let (_, s2) := ab s1 in join_state' init s2
  | None => s
  end.

```

We then construct a function that repeats `step1` a certain number of times. This number is denoted by a natural number `n`. In this function, the constant `0` is a natural number and we need to make it precise to Coq's parser, by expressing that the value must be interpreted in a parsing scope for natural numbers instead of integers, using the specifier `%nat`.

```

Fixpoint step2 (ab: state -> a_instr * option state)
  (b:bexpr) (init s:state) (n:nat) : state :=
  match n with
  | 0%nat => s
  | S p => step2 ab b init (step1 ab b init s) p
  end.

```

The complexity of these functions can be improved: there is no need to compute all iterations if we can detect early that a fixed point was reached. In this paper, we prefer to keep the code of the abstract interpreter simple but potentially inefficient to make our formal verification work easier.

#### 5.4 Verifying that a state is more precise than another

To verify that we have reached an invariant, we need to check for a state `s`, so that running this state through `step1 ab b s s` returns a new state that is not less precise than `s`. For this, we assume that there exist a function that makes it possible to compare two abstract values:

- `thinner : A -> A -> bool`, this function returns `true` when the first abstract value gives more precise information than the second one.

Using this basic function on abstract values, we define a new function on states:

```

Fixpoint s_stable (s1 s2 : state) : bool :=
  match s1 with
  | nil => true
  | (x,v)::t1 => thinner (lookup s2 x) v && s_stable t1 s2
  end.

```

This function traverses the first state to check that the abstract value associated to each variable is less precise than the information found in the second state. This function is then easily used to verify that a given state is an invariant through the abstract interpretation of a loop's test and body.

```
Definition is_inv (ab:state-> a_instr * option state)
  (s:state)(b:bexpr):bool := s_stable s (step1 ab b s s).
```

## 5.5 Narrowing a state

The `step2` function receives two arguments of type `state`. The first argument is solely used for join operations, while the second argument is used to start a sequence of abstract states that correspond to iterated interpretations of the loop test and body. When the start state is not stable through interpretation, the resulting state is larger than both the first argument and the start argument. When the start state is stable through interpretation, there are cases where the resulting state is smaller than the start state.

For instance, in the cases where the abstract values are `even` and `odd`, if the first state argument maps the variable `y` to `even` and the variable `z` to `odd`, the start state maps `y` and `z` to the top abstract value (the abstract value that gives no information) and the while loop is the following:

```
while (x < 10) do x := x + 1; z:= y + 1; y := 2 done
```

Then, after abstractly executing the loop test and body once, we obtain a state where `y` has the value `even` and `z` has the top abstract value. This state is more precise than the start state. After abstractly executing the loop test and body a second time, we obtain a state where `z` has the value `odd` and `y` has the value `even`. This state is more precise than the one obtained only after the first abstract run of the loop test and body.

The example above shows that over-approximations are improved by running the abstract interpreter again on them. This phenomenon is known as *narrowing*. It is worth forcing a narrowing phase after each phase that is likely to produce an over-approximation of the smallest fixed-point of the abstract interpreter. This is used in the abstract interpreter that we describe below.

## 5.6 Allowing for over-approximations

In general, the finite amount of abstract computation performed in the `step2` function is not enough to reach the smallest stable abstract state. This is related to the undecidability of the halting problem: it is often possible to write a program where a variable will receive a precise value exactly when some other program terminates. If we were able to compute the abstract value for this variable in a finite amount of time, we would be able to design a program that solves the halting problem.

Even if we are facing a program where finding the smallest state can be done in a finite amount of time, we may want to accelerate the process by taking over-approximations. For instance, if we consider the following loop:

```
while x < 10 do x := x + 1 done
```

If the abstract values we are working with are intervals and we start with the interval  $[0,0]$ , after abstractly interpreting the loop test and body once, we obtain that the value for  $x$  should contain at least  $[0,1]$ , after abstractly interpreting 9 times, we obtain that the value for  $x$  should contain at least  $[0,9]$ . Until these 9 executions, we have not seen a stable state. At the 10th execution, we obtain that the value for  $x$  should contain at least  $[0, 10]$  and the 11th execution shows that this value actually is stable.

At any time before a stable state is reached, we may choose to replace the current unstable state with a state that is “larger”. For instance, we may choose to replace  $[0,3]$  with  $[0,100]$ . When this happens, the abstract interpreter can discover that the resulting state after starting with the one that maps  $x$  to  $[0,100]$  actually maps  $x$  to  $[0,10]$ , thus  $[0,100]$  is stable and is good candidate to enter a narrowing phase. This narrowing phase actually converges to a state that maps  $x$  to  $[0,10]$ .

The choice of over-approximations is arbitrary and information may actually be lost in the process, because over-approximated states are less precise, but this is compensated by the fact that the abstract interpreter gives quicker answers. The termination of the abstract interpreter can even be guaranteed if we impose that a guaranteed over-approximation is taken after a finite amount of steps. An example of a guaranteed over-approximation is a state that maps every variable to the top abstract value. In our Coq encoding, such a state is represented by the `nil` value.

The choice of over-approximation strategies varies from one abstract domain to the other. In our Coq encoding, we chose to let this over-approximation be represented by a function with the following signature:

- `over_approx : nat -> state -> state -> state` When applied to `n`, `s`, and `s'`, this function computes an over approximation of `s'`. The value `s` is supposed to be a value that comes before `s'` in the abstract interpretation and can be used to choose the over-approximation cleverly, as it gives a sense of direction to the current evolution of successive abstract values. The number `n` should be used to fine-tune the coarseness of the over-approximation: the lower the value of `n`, the coarser the approximation.

For instance, when considering the example above, knowing that  $s = [0, 1]$  and  $s' = [0, 2]$  are two successive unstable values reached by the abstract interpreter for the variable  $x$  can suggest to choose an over-approximation where the upper bound changes but the lower bound remains unchanged. In this case, we expect the function `over_approx` to return  $[0, +\infty]$ , for example.

## 5.7 The main invariant searching function

We can now describe the function that performs the process described in section 5.1. The code of this function is as follows:

```

Fixpoint find_inv ab b init s i n : state :=
  let s' := step2 ab b init s (choose_1 s i) in
  if is_inv ab s' b then s' else
  match n with
  | 0%nat => nil
  | S p => find_inv ab b init (over_approx p s s') i p
  end.

```

The function `choose_1` is provided at the same time as all other functions that are specific to the abstract domain `A`, such as `join`, `a_add`, etc.

The argument function `ab` is supposed to be the function that performs the abstract interpretation of the loop inner instruction `i` (also called the loop body), the boolean expression `b` is supposed to be the loop test. The state `init` is supposed to be the initial input state at the first invocation of `find_inv` on this loop and `s` is supposed to be the current over-approximation of `init`, `n` is the number of over-approximations that are still allowed before the function should switch to the `nil` state, which is a guaranteed over-approximation. This function systematically runs the abstract interpreter on the inner instruction an arbitrary number of times (given by the function `choose_1`) and then tests whether the resulting state is an invariant. Narrowing steps actually take place if the number of iterations given by `choose_1` is large enough. If the result of the iterations is an invariant, then it is returned. When the result state is not an invariant, the function `find_inv` is called recursively with a larger approximation computed by `over_approx`. When the number of allowed recursive calls is reached, the `nil` value is returned.

## 5.8 Annotating the loop body with abstract information

The `find_inv` function only produces a state, while the abstract interpreter is also supposed to produce an annotated version of the instruction. Once we know the invariant, we can annotate the while loop with this invariant and obtain an annotated version of the loop body by re-running the abstract interpreter on this instruction. This is done with the following function:

```

Definition do_annot (ab:state-> a_instr * option state)
  (b:bexpr) (s:state) (i:instr) : a_instr :=
  match learn_from_success s b with
  | Some s' => let (ai, _) := ab s' in ai
  | None => mark i
  end.

```

In this function, `ab` is supposed to compute the abstract interpretation of the loop body. When the function `learn_from_success` returns a `None` value, this means that the loop body is never executed and it is marked as dead code by the function `mark`.

## 5.9 The abstract interpreter's main function

With the function `find_inv`, we can now design a new abstract interpreter. Its main structure is about the same as for the naive one, but there are two important differences. First, the abstract interpreter now uses the `find_inv` function to compute an invariant state for the while loop. Second, this abstract interpreter can detect cases where instructions are guaranteed to not terminate. This is a second part of dead code detection: when a good invariant is detected for the while loop, a comparison between this invariant and the loop test may give the information that the loop test can never be falsified. If this is the case, no state is returned and the instructions following this while loop in sequences must be marked as dead code. This is handled by the fact that the abstract interpreter now returns an optional state and an annotated instruction. The case for the sequence is modified to make sure instructions are marked as dead code when receiving no input state.

```
Fixpoint ab2 (i:instr)(s:state) : a_instr*option state :=
  match i with
  | assign x e =>
    (pre (s_to_a s) (a_assign x e), Some (a_upd x (a_af s e) s))
  | seq i1 i2 =>
    let (a_i1, s1) := ab2 i1 s in
    match s1 with
    | Some s1' =>
      let (a_i2, s2) := ab2 i2 s1' in
      (a_seq a_i1 a_i2, s2)
    | None => (a_seq a_i1 (mark i2), None)
    end
  | while b i =>
    let inv := find_inv (ab2 i) b s s i (choose_2 s i) in
    (a_while b (s_to_a inv)
     (do_annot (ab2 i) b inv i),
     learn_from_failure inv b)
  end.
```

This function relies on an extra numeric function `choose_2` to decide the number of times `find_inv` will attempt progressive over-approximations before giving up and falling back on the `nil` state. Like `choose_1` and `over_approx`, this function must be provided at the same time as the type for abstract values.

## 6 Proving the correctness of the abstract interpreter

To prove the correctness of our abstract interpreter, we adapt the correctness statements that we already used for the naive interpreter. The main change is that the resulting state is optional, with a `None` value corresponding to non-termination. This means that when a `None` value is obtained we can take the post-



condition as the false assertion. This is expressed with the following function, mapping an optional state to an assertion.

```
Definition s_to_a' (s':option state) : assert :=
  match s' with Some s => s_to_a s | None => a_false end.
```

The main correctness statement thus becomes the following one:

```
Theorem ab2_correct : forall i i' s s', consistent s ->
  ab2 i s = (i', s') -> valid m (vc i' (s_to_a' s')).
```

By comparison with the similar theorem for `ab1`, we removed the part about the final state satisfying the `consistent`. This part is actually proved in a lemma beforehand. The reason why we chose to establish the two results at the same time for `ab1` and in two stages for `ab2` is anecdotal.

As for the naive interpreter this theorem is paired with a lemma asserting that cleaning up the resulting annotated instruction `i'` yields back the initial instruction `i`. We actually need to prove two lemmas, one for the `mark` function (used to mark code as dead code) and one for `ab2` itself.

```
Lemma mark_clean : forall i, cleanup (mark i) = i.
```

```
Theorem ab2_clean : forall i i' s s',
  ab2 i s = (i', s') -> cleanup i' = i.
```

These two lemmas are proved by induction on the structure of the instruction `i`.

## 6.1 Hypotheses about the auxiliary functions

The abstract interpreter relies on a collection of functions that are specific to the abstract domain being handled. In our Coq development, this is handled by defining the function inside a section, where the various components that are specific to the abstract domain of interpretation are given as section variables and hypotheses. When the section is closed, the various functions defined in the section are abstracted over the variables that they use. Thus, the function `ab2` becomes a 16-argument function. The extra twelve arguments are as follows:

1. `A` : `Type`, the type containing the abstract values,
2. `from_Z` : `Z -> A`, a function mapping integer values to abstract values,
3. `top` : `A`, an abstract value representing lack of information,
4. `a_add` : `A -> A -> A`, an addition operation for abstract values,
5. `to_pred` : `A -> aexpr -> assert`, a function mapping abstract values to their interpretations as assertions on arithmetic expressions,
6. `learn_from_success` : `state A -> bexpr -> state A`, a function that is able to improve a state, knowing that a boolean expression's evaluation returns `true`,

7. `learn_from_failure` : `state A -> bexpr -> state A`, similar to the previous one, but using the knowledge that the boolean expression's evaluation returns `false`,
8. `join` : `A -> A -> A`, a binary function on abstract values that returns an abstract value that is coarser than the two inputs,
9. `thinner` : `A -> A -> bool`, a comparison function that succeeds when the first argument is more precise than the second,
10. `over_approx` : `nat -> state A -> state A -> state A`, a function that implements heuristics to find over-approximations of its arguments,
11. `choose_1` : `state A -> instr -> nat`, a function that returns the number of times a loop body should be executed with a given start state before testing for stabilisation,
12. `choose_2` : `state A -> instr -> nat`, a function that returns the number of times over-approximations should be attempted before giving up and using the coarsest state.

Most of these functions must satisfy a collection of properties to ensure that the correctness statement will be provable. There are fourteen such properties, which can be sorted in the following way:

1. Three properties are concerned with the assertions created by `to_pred`, with respect to their logical interpretation and to substitution.
2. Two properties are concerned with the consistency of interpretation of abstract values obtained through `from_Z` and `a_add` as predicates over integers.
3. Two properties are concerned with the logical properties of abstract states computed with the help of `learn_from_success` and `learn_from_failure`.
4. Four properties are concerned with ensuring that `over_approx`, `join`, and `thinner` do return or detect over-approximations correctly,
5. Three properties are concerned with ensuring that the `consistent` property is preserved through `learn_from...` and `over_approx`.

## 6.2 Maintaining the consistent property

For this abstract interpreter, we need again to prove that it maintains the property that all states are duplication-free. It is first established for the `join_state` operation. Actually, since the `join_state` operation performs repetitive updates from the `nil` state, the result is duplication-free, regardless of the duplications in the inputs. This is easily obtained with a proof by induction on the first argument. For once, we show the full proof script.

```

Lemma join_state_consistent :
  forall s1 s2, consistent (join_state s1 s2).
intros s1 s2; induction s1 as [ | [x v] s1 IHs1]; simpl; auto.
apply consistent_update; auto.
Qed.

```

The first two lines of this Coq excerpt give the theorem statement. The line `intros ...` explains that a proof by induction should be done. This proof raises two cases, and the `as ...` fragment states that in the step case (the second case), one should consider a list whose tail is named `s1` and whose first pair contains a variable `x` and an abstract value `v`, and we have an induction hypothesis, which should be named `IHs1`: this induction hypothesis states that `s1` already satisfies the `consistent` property. The `simpl` directive expresses that the recursive function should be simplified if possible, and `auto` attempts to solve the goals that are generated. Actually, the computation of recursive functions leads to proving `true = true` in the base case and `auto` takes care of this. For the step case, we simply need to rely on the theorem `consistent_update` (see section 4.6). The premise of this theorem actually is `IHs1` and `auto` finds it.

### 6.3 Relating input abstract states and pre-conditions

Similarly to what was done for the naive abstract interpreter, we want to ensure that the interpretation of the input abstract state as a logical formula implies the pre-condition for the generated annotated instruction and the generated post-condition. For the while loop, this relies on the fact that the selected invariant is obtained after repetitive joins with the input state. We first establish two monotonicity properties for the `join_state` function, we show only the first one:

```
Lemma join_state_safe_1 : forall g s1 s2,
  ia m g (s_to_a s1) -> ia m g (s_to_a (join_state s1 s2)).
```

Then, we only need to propagate the property up from the `step1` function. Again, we show only the first one but there are similar lemmas for `step2`, `find_inv`; and we conclude with the property for `ab2`:

```
Lemma step1_pc : forall g ab b s s',
  ia m g (s_to_a s) -> ia m g (s_to_a s') ->
  ia m g (s_to_a (step1 ab b s s')).
```

```
Lemma ab2_pc :
  forall i i' s s', ab2 i s = (i', s') ->
  forall g a, ia m g (s_to_a s) -> ia m g (pc i' a).
```

The proof for `step1_pc` is a direct consequence of the definition and the properties of `join_state`. The proofs for `step2` and `find_inv` are done by induction on `n`. The proof for `ab2` is an easy induction on the instruction `i`. In particular, the two state arguments to the function `find_inv` are both equal to the input state in the case of `while` loops.

### 6.4 Validity of the generated conditions

The main theorem is about ensuring that all verification conditions are provable. A good half of this problem is already taken care of when we prove the theorem

ab2\_pc, which expresses that at each step the state is strong enough to ensure the validity of the pre-condition for the instruction that follows. The main added difficulty is to verify that the invariant computed for each while loop actually is invariant. This difficulty is taken care of by the structure of the function `find_inv`, which actually invokes the function `is_inv` on its expected output before returning it. Thus, we only need to prove that `is_inv` correctly detects states that are invariants:

```
Lemma is_inv_correct :
  forall ab b g s s' s2 ai,
    is_inv ab s b = true -> learn_from_success s b = Some s' ->
      ab s' = (ai, s2) -> ia m g (s_to_a' s2) -> ia m g (s_to_a s).
```

We can then deduce that `find_inv` is correct: the proof proceeds by showing that the value this function returns is either verified using `is_inv` or the `nil` state. The correctness statement for `find_inv` has the following form:

```
Lemma find_inv_correct : forall ab b g i n init s s' s2 ai,
  learn_from_success (find_inv ab b init s i n) b = Some s' ->
    ab s' = (s2, ai) -> ia m g (s_to_a' s2) ->
      ia m g (s_to_a (find_inv ab b init s i n)).
```

This can then be combined with the assumptions that `learn_from_success` and `learn_from_failure` correctly improve the information given in abstract state to show that the value returned for while loops in `ab2` is correct. These assumptions have the following form (the hypothesis for the `learn_from_failure` has a negated third assumption).

```
Hypothesis learn_from_success_sem :
  forall s b g, consistent s ->
    ia m g (s_to_a s) -> ia m g (a_b b) ->
      ia m g (s_to_a' (learn_from_success s b)).
```

## 7 An interval-based instantiation

The abstract interpreters we have described so far are generic and are ready to be instantiated on specific abstract domains. In this section we describe an instantiation on an abstract domain to represent intervals. This domain of intervals contains intervals with finite bounds and intervals with infinite bounds. The interval with two infinite bounds represents the whole type of integers. We describe these intervals with an inductive type that has four variants:

```
Inductive interval : Type :=
  | above : Z -> interval
  | below : Z -> interval
  | between : Z -> Z -> interval
  | all_Z : interval.
```

For instance, the interval containing all values larger than or equal to 10 is represented by `above 10` and the whole type of integers is represented by `all_Z`.

The interval associated to an integer is simply described as the interval with two finite bounds equal to this integer.

```
Definition i_from_Z (x:Z) := between x x.
```

When adding two intervals, it suffices to add the two bounds, because addition preserves the order on integers. Coping with all the variants of each possible input yields a function with many cases.

```
Definition i_add (x y:interval) :=
  match x, y with
  | above x, above y => above (x+y)
  | above x, between y z => above (x+y)
  | below x, below y => below (x+y)
  | below x, between y z => below (x+z)
  | between x y, above z => above (x+z)
  | between x y, below z => below (y+z)
  | between x y, between z t => between (x+z) (y+t)
  | _, _ => all_Z
  end.
```

The assertions associated to each abstract value can rely on only one, as we can re-use the same comparison predicate for almost all variants. This is described in the `to_pred` function.

```
Definition i_to_pred (x:interval) (e:aexpr) : assert :=
  match x with
  | above a => pred "leq" (anum a::e::nil)
  | below a => pred "leq" (e::anum a::nil)
  | between a b => a_conj (pred "leq" (anum a::e::nil))
    (pred "leq" (e::anum b::nil))
  | all_Z => a_true
  end.
```

Of course, the meaning attached to the string "leq" must be correctly fixed in the corresponding instantiation for the `m` parameter:

```
Definition i_m (s : string) (l: list Z) : Prop :=
  if string_dec s "leq" then
    match l with x::y::nil => x <= y | _ => False end
  else False.
```

## 7.1 Learning from comparisons

The functions `i_learn_from_success` and `i_learn_from_failure` used when processing while loops can be made arbitrarily complex. For the sake of conciseness, we have only designed a pair of functions that detect the case where

the boolean test has the form  $x < e$ , where  $e$  is an arbitrary arithmetic expression. In this case, the function `i_learn_from_success` updates only the value associated to  $x$ : the initial interval associated with  $x$  is intersected with the interval of all values that are less than the upper bound of the interval computed for  $e$ . An impossibility is detected when the lowest possible value for  $x$  is larger than or equal to the upper bound for  $e$ . Even this simple strategy yields a function with many cases, of which we show only the cases where both  $x$  and  $e$  have interval values with finite bounds:

```

Definition i_learn_from_success s b :=
  match b with
  | blt (avar x) e =>
    match a_af _ i_from_Z all_Z i_add s e,
          lookup _ all_Z s x with
    ...
    | between _ n, between m p =>
      if Z_le_dec n m then None else
      if Z_le_dec n p
      then Some (a_upd _ x (between m (n-1)) s)
      else Some s
    ...
  | _ => Some s
end.

```

In the code of this function, the functions `a_af`, `lookup`, and `a_upd` are parameterized by the functions from the datatype of intervals that they use: `i_from_Z`, `all_Z` and `i_add` for `a_af`, `all_Z` for `lookup`, etc.

The function `i_learn_from_failure` is designed similarly, looking at upper bounds for  $x$  and lower bounds for  $e$  instead.

## 7.2 Comparing and joining intervals

The treatment of loops also requires a function to find upper bounds of pairs of intervals and a function to compare two intervals. These functions are simply defined by pattern-matching on the kind of intervals that are encountered and then comparing the upper and lower bounds.

```

Definition i_join (i1 i2:interval) : interval :=
  match i1, i2 with
  | above x, above y =>
    if Z_le_dec x y then above x else above y
  ...
  | between x y, between z t =>
    let lower := if Z_le_dec x z then x else z in
    let upper := if Z_le_dec y t then t else y in
    between lower upper

```

```

| _, _ => all_Z
end.

```

```

Definition i_thinner (i1 i2:interval) : bool :=
  match i1, i2 with
  | above x, above y => if Z_le_dec y x then true else false
  | above _, all_Z => true
  ...
  | between x _, above y => if Z_le_dec y x then true else false
  | between _ x, below y => if Z_le_dec x y then true else false
  | _, all_Z => true
  ...
end.

```

### 7.3 Finding over-approximations

When the interval associated to a variable does not stabilize, an over-approximation must be found for this interval. We implement an approach where several steps of over-approximation can be taken one after the other. For intervals, finding over-approximations can be done by pushing one of the bounds of each interval to infinity. We use the fact that the generic abstract interpreter calls the over-approximation with two values to choose the bound that should be pushed to infinity: in a first round of over-approximation, only the bound that does not appear to be stable is modified. This strategy is particularly well adapted for loops where one variable is increased or decreased by a fixed amount at each execution of the loop's body.

The strategy is implemented in two functions, the first function over-approximates an interval, the second function applies the first to all the intervals found in a state.

```

Definition open_interval (i1 i2:interval) : interval :=
  match i1, i2 with
  | below x, below y => if Z_le_dec y x then i1 else all_Z
  | above x, above y => if Z_le_dec x y then i1 else all_Z
  | between x y, between z t =>
    if Z_le_dec x z then if Z_le_dec t y then i1 else above x
    else if Z_le_dec t y then below y else all_Z
  | _, _ => all_Z
end.

```

```

Definition open_intervals (s s':state interval) : state interval :=
  map (fun p:string*interval =>
    let (x, v) := p in
    (x, open_interval v (lookup _ all_Z s' x))) s.

```

The result of `open_interval i1 i2` is expected to be an over-approximation of `i1`. The second argument `i2` is only used to choose which of the bounds of `i1` should be modified.

The function `i_over_approx` receives a numeric parameter to indicate the strength of over-approximation that should be applied. Here, there are only two strengths: at the first try (when the level is larger than 0), the function applies `open_intervals`; at the second try, it simply returns the `nil` state, which corresponds to the `top` value in the domain of abstract states.

```
Definition i_over_approx n s s' :=
  match n with
  | S _ => open_intervals s s'
  | _ => nil
  end.
```

The abstract interpreter also requires two functions that compute the number of attempts at each level of repetitive operation. We define these two functions as constant functions:

```
Definition i_choose_1 (s:state interval) (i:instr) := 2%nat.
Definition i_choose_2 (s:state interval) (i:instr) := 3%nat.
```

Once the type `interval` and the various functions are provided we obtain an abstract interpreter for computing with intervals.

```
Definition abi :=
  ab2 interval i_from_Z all_Z i_add i_to_pred
    i_learn_from_success i_learn_from_failure
    i_join i_thinner i_over_approx i_choose_1 i_choose_2.
```

We can already run this instantiated interpreter inside the Coq system. For instance, we can run the interpreter on the instruction:

```
while x < 10 do x := x + 1 done
```

This gives the following dialog (where the answer of the Coq system is written in italics):

```
Eval vm_compute in
  abi (while (blt (avar "x") (anum 10))
    (assign "x" (aplus (avar "x") (anum 1))))
    (("X", between 0 0)::nil).
  = (a_while (blt (avar "x") (anum 10)))
    (a_conj
      (a_conj (pred "leq" (anum 0 :: avar "x" :: nil)))
      (pred "leq" (avar "x" :: anum 10 :: nil))) a_true)
    (pre
      (a_conj
        (a_conj (pred "leq" (anum 0 :: avar "x" :: nil))))
```



```

      (pred "leq" (avar "x" :: anum 9 :: nil)) a_true)
    (a_assign "x" (aplus (avar "x") (anum 1))))),
  Some (("x", between 10 10) :: nil))
: a_instr * option (state interval)

```

## 8 Conclusion

This paper describes how the functional language present in a higher-order theorem prover can be used to encode a tool to perform a static analysis on an arbitrary programming language. The example programming language is chosen to be extremely simple, so that the example can be described precisely in this tutorial paper. The static analysis tool that we described is inspired by the approach of abstract interpretation. However this work is not a comprehensive introduction to abstract interpretation, nor does it cover all the aspects of encoding abstract interpretation inside a theorem prover. Better descriptions of abstract interpretation and its formal study are given in [11,5,12].

The experiment is performed with the Coq system. More extensive studies of programming languages using this system have been developed over the last years. In particular, experiments by the CompCert team show that not only static analysis but also efficient compilation can be described and proved correct [4,10,6]. Coq is also used extensively for the study of functional programming languages, in particular to study the properties of type systems and there are a few Coq-based solutions to the general landmark objective known as POPLMark [1].

The abstract interpreter we describe here is inefficient in many respects: when analysing the body of a loop, this loop needs to be executed abstractly several times, the annotations computed each time are forgotten, and then when an invariant is discovered, the whole process needs to be done again to produce the annotated instruction. A more efficient interpreter could be designed where computed annotations are kept in memory long enough to avoid recomputation when the invariant is found. We did not design the abstract interpreter with this optimisation, thinking that the sources of inefficiency could be calculated away through systematic transformation of programs, as studied in another paper in this volume. The abstract interpreter provided with the paper [2] contains some of these optimisations.

An important remark is that program analyses can be much more efficient when they consider the relations between several variables at a time, as opposed to the experiment described here where the variables are considered independently of each other. More precise work where relations between variables can be tracked is possible, on the condition that abstract values are used to describe complete states, instead of single variables as in [4], where the result of the analysis is used as a basis for a compiler optimisation known as *common subexpression elimination*.

We have concentrated on a very simple while language in this paper, for didactical purposes. However, abstract interpreters have been applied to much

more complete programming languages. For instance, the Astree [8] analyser covers most of the C programming language. On the other hand, the foundational papers describe abstract interpretation in terms of analyses on control flow graphs. The idea of abstract interpretation is general enough that it should be possible to apply it to any form of programming language.

## References

1. B. Aydemir, A. Bohannon, M. Fairbairn, J. Foster, B. Pierce, P. Sewell, D. Vytinotis, G. Washburn, S. Weirich, and S. Zdancewic. Mechanized metatheory for the masses: The POPLmark challenge. In *Proceedings of the Eighteenth International Conference on Theorem Proving in Higher Order Logics (TPHOLs 2005)*, 2005.
2. Yves Bertot. Theorem proving support in programming language semantics. Technical Report 6242, INRIA, 2007. to appear in a book in memory of Gilles Kahn.
3. Yves Bertot and Pierre Castéran. *Interactive Theorem Proving and Program Development, Coq'Art:the Calculus of Inductive Constructions*. Springer-Verlag, 2004.
4. Yves Bertot, Benjamin Grégoire, and Xavier Leroy. A structured approach to proving compiler optimizations based on dataflow analysis. In *Types for Proofs and Programs, Workshop TYPES 2004*, volume 3839 of *Lecture Notes in Computer Science*, pages 66–81. Springer, 2006.
5. Frédéric Besson, Thomas Jensen, and David Pichardie. Proof-carrying code from certified abstract interpretation to fixpoint compression. *Theoretical Computer Science*, 364(3):273–291, 2006.
6. Sandrine Blazy, Zaynah Dargaye, and Xavier Leroy. Formal verification of a C compiler front-end. In *FM 2006: Int. Symp. on Formal Methods*, volume 4085 of *Lecture Notes in Computer Science*, pages 460–475. Springer, 2006.
7. Patrick Cousot and Radhia Cousot. Abstract interpretation: a unified lattice model for static analysis of programs by construction or approximation of fixpoints. In *Conference Record of the Fourth ACM Symposium on Principles of Programming Languages, POPL'77*, pages 238–252. ACM Press, 1977.
8. Patrick Cousot, Radhia Cousot, Jérôme Feret, Antoine Mine Laurent Mauborgne, David Monniaux, and Xavier Rival. The Astrée analyzer. In *European Symposium on Programming, ESOP'XIV*, volume 3444 of *LNCS*, pages 21–30. Springer, 2005.
9. Edsger W. Dijkstra. *A discipline of Programming*. Prentice Hall, 1976.
10. Xavier Leroy. Formal certification of a compiler back-end, or: programming a compiler with a proof assistant. In *33rd symposium Principles of Programming Languages*, pages 42–54. ACM Press, 2006.
11. David Pichardie. *Interprétation abstraite en logique intuitionniste : extraction d'analyseurs Java certifiés*. PhD thesis, Université Rennes 1, 2005. In french.
12. David Pichardie. Building certified static analysers by modular construction of well-founded lattices. In *Proc. of the 1st International Conference on Foundations of Informatics, Computing and Software (FICS'08)*, Electronic Notes in Theoretical Computer Science, 2008.
13. The Coq development team. The coq proof assistant, 2008. <http://coq.inria.fr>.