On The Topology of Planar Algebraic Curves

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Abstract

We introduce a method to compute the topology of planar algebraic curves. The curve may not be in generic position and may have vertical asymptotes. The algebraic tools are rational univariate representation for zero-dimentional ideals and multiplicities in these ideals. Experiments show the efficiency of our algorithm.

1 Introduction

We consider the problem of computing a geometric representation of a planar real algebraic curve $\mathcal{C}$, defined in a Cartesian coordinate system by a bivariate polynomial f with rational coefficients. More precisely, we address the problem of computing a planar graph whose vertices are mapped to points in the plane (possibly at infinity) and such that drawing the arcs as line segments gives a drawing isotopic to the input curve (see Figure 1). We assume that $\mathcal{C}$ is square-free and has no vertical lies. The assumption is a standard one in the context of this type of problems, while treating vertical lines is easy to deal with. There is no other assumptions on $\mathcal{C}$.

There have been many papers addressing the problem of computing the topology of algebraic plane curves [2, 3, 4, 8]. All algorithms use a sheared curve if it is not in generic position.

All these algorithms perform the following phases. (1) Project the x-critical points of the curve on the x-axis, using resultants or Sturm-Habicht sequences, and isolate the real roots of the resulting univariate polynomial in x. This gives the x-coordinates of all the x-critical points. (2) For each such value x_i, compute the intersection points between the curve $\mathcal{C}$ and the vertical line $x = x_i$. (3) Through each of these points, determine the number of branches of $\mathcal{C}$ coming from the left and going to the right. (4) Connect all these points appropriately.

The main difficulty in all these algorithms is to compute efficiently all the critical points in Phase 2

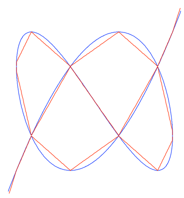

Figure 1: $\mathcal{C} : 16\,x^5 - 20\,x^3 + 5\,x - 4\,y^3 + 3\,y = 0$ plotted in MAPLE and its isotopic graph computed by ISOTOP.

because the x-critical values in Phase 1 are, a priori, non-rational thus computing the corresponding y-coordinates in Phase 2 amounts, in general, to solving a univariate polynomial with non-rational coefficients and at least a multiple root (corresponding to the critical point).

Our contributions. Unlike most previous algorithms, this algorithm handles curves in non-generic positions in the given Cartesian coordinate system without shearing. The other originality of our approach is that we succeed to avoid the computation of sub-resultant sequences. As a result, our MAPLE implementation, ISOTOP, substantially outperforms previous available MAPLE implementations, TOP [3] and INSULATE [8]. Recall that TOP requires an initial precition to be set, thus the results may not correct if the precision is not set appropriately. INSULATE is certified but the critical points may be computed in a sheared coordinate system. We observe on preliminary tests that the running time of our implementation is remarkably stable. Furthermore, we observe, on 24 curves of degree between 4 and 13, that ISOTOP is, on average, 8.34 times faster than INSULATE with a median of 4.3 and extrema of 0.8 and 43.9. Compared to TOP using 60 (resp. 500) digits of initial precision, ISOTOP is, on average, 10.36 (resp. 26.7) times faster with a median of 1.2 (resp. 4.25) and extrema of 0.2 and 103.3 (resp. 0.7 and 193.2).

The novelty of our algorithm mainly relies upon the use of three new ingredients for this problem. First, we use a formula of Teissier relating the multiplicities of the roots of a polynomial or a system, which avoids

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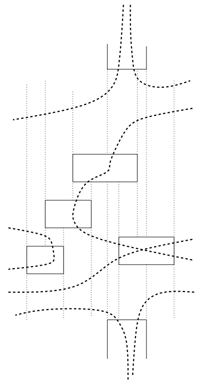

Figure 2: Example of rectangle decomposition of the plane induced by the isolating boxes (critical and asymptotes).

computing Sturm-Habicht type sequences.

Second, we isolate the roots of bivariate systems using (i) Gröbner bases, (ii) Rational Univariate Representations (RUR) , and (iii) a subdivision technique based on Descartes' rule for isolating the roots of the univariate polynomials.

Third, we present a variant of the standard combinatorial part of the algorithm for computing the topology. This variant avoids computing points of the curve on (rational) vertical lines between the critical points. Alternatively, we compute a decomposition of the plane by rectangular boxes containing at most one critical point. To achieve the connection in each box, we use their multiplicities in the system of critical points.

We also present an output-sensitive bound on the worst-case complexity of our algorithm. To the best of our knowledge, this is, for the problem considered here, the first time that the complexity of a (certified) algorithm based on refinements and approximations is analyzed. Our technique, even though not novel, could presumably also be used to analyze the complexity of some previous algorithms.

2 Notations and Preliminaries.

Let $\mathcal{C}_f$ be a real algebraic plane curve defined by a bivariate polynomial f in $\mathbb{Q}[x, y]$.

Let f_x denote the derivative of f with respect to x and f_{y^k} (sometimes also f_k) denote the k^{th} derivative with respect to y. A point $\mathbf{p} = (\alpha, \beta) \in \mathbb{C}^2$ is called x-critical (simply called critical) if $f(\mathbf{p}) = f_y(\mathbf{p}) = 0$, singular if $f(\mathbf{p}) = f_x(\mathbf{p}) = f_y(\mathbf{p}) = 0$, and x-extreme (simply called extreme) if $f(\mathbf{p}) = f_y(\mathbf{p}) = 0$ and $f_x(\mathbf{p}) \neq 0$. Similarly are defined y-critical and y-extreme points.

The ideal generated by polynomials $P_1, \ldots, P_i$ is denoted $\mathbb{I}(P_1, \ldots P_i)$. In the following, we often identify the ideal and the system of equations $\{P_1 = 0, \ldots, P_i = 0\}$ (or any equivalent system induced by a set of generators of the ideal). Let $I_c = \mathbb{I}(f, f_y)$ and $I_s = \mathbb{I}(f, f_x, f_y)$. Their roots are, respectively, the x-critical and singular points of $\mathcal{C}$. We also consider the Jacobian ideal $I_j = \mathbb{I}(f_x, f_y)$ and its associated ideal $I_m = \mathbb{I}(f_x/\gcd(f_x, f_y), f_y/\gcd(f_x, f_y))$ called the Milnor ideal.

We now recall the notion of multiplicity of the roots of an ideal, then we state two lemmas using this notion for studying the local topology at singular points. Geometrically, the notion of multiplicity of intersection of two regular curves is intuitive. If the intersection is transverse, the multiplicity is one; otherwise, it is greater than one and it measures the level of degeneracy of the tangential contact between the curves. Defining the multiplicity of the intersection of two curves at a point that is singular for one of them (or possibly both) is more involved and an abstract and general concept of multiplicity in an ideal is needed.

Definition. 1 ([1]) *Let I be an ideal of $\mathbb{Q}[x, y]$ and $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$. To each zero (α, β) of I corresponds a local ring $(\overline{\mathbb{Q}}[x, y]/I)_{(\alpha,\beta)}$ obtained by localizing the ring $\overline{\mathbb{Q}}[x, y]/I$ at the maximal ideal $\mathbb{I}(x-\alpha, y-\beta)$. When this local ring is finite dimensional as $\overline{\mathbb{Q}}$-vector space, we say that (α, β) is an isolated zero of I and this dimension is called the* **multiplicity of (α, β) as a zero of I.**

Let $f, g \in \mathbb{Q}[x, y]$ be such that the intersection of $\mathcal{C}_f$ and $\mathcal{C}_g$ in $\mathbb{C}^2$ contains a zero-dimensional component equal to point $\mathbf{p} = (\alpha, \beta)$. Then (α, β) is an isolated zero of $\mathbb{I}(f, g)$ and its multiplicity, denoted by $Int(f, g, \mathbf{p})$, is called the **intersection multiplicity of the two curves at this point.** *A singular point of a curve $\mathcal{C}_f$ is an isolated zero of the Jacobian ideal $I_j = \mathbb{I}(f_x, f_y)$ and the multiplicity of this point as a zero of this system is called the* **Milnor number** *of the singular point.*

We call a **fiber** *a vertical line of equation $x = \alpha$. For a point $\mathbf{p} = (\alpha, \beta)$ on the curve $\mathcal{C}_f$, we call the multiplicity of β in the univariate polynomial $f(\alpha, y)$ the* **multiplicity of p in its fiber** *and denote it as $mult(f(\alpha, y), \beta)$.*

Lemma 2 ([7]) *For a singular point $\mathbf{p} = (\alpha, \beta)$ of the curve $\mathcal{C}_f$, we have*

$$mult(f(\alpha, y), \beta) = Int(f, f_y, \mathbf{p}) - Int(f_x, f_y, \mathbf{p}) + 1. \tag{1}$$

For an x-extreme point, the same formula holds and, as the Milnor number vanishes, it simplifies into

$$mult(f(\alpha, y), \beta) = Int(f, f_y, \mathbf{p}) + 1. \tag{2}$$

Lemma 3 ([8]) *Let $\mathbf{p} = (\alpha, \beta)$ be a real singular point of the curve $\mathcal{C}_f$ of multiplicity k in its fiber. Let*

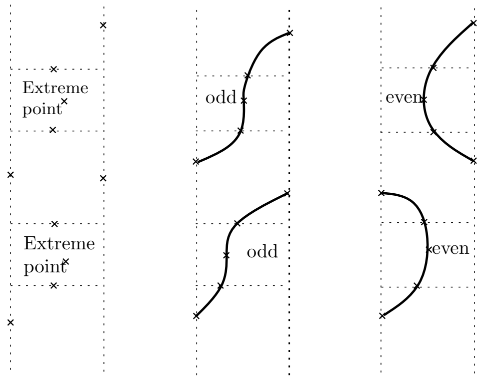

Figure 3: Different connections according to multiplicities for the same crossing pattern.

B be a box satisfying (i) B contains $\mathbf{p}$ and no other x-critical point, (ii) the function f_{y^k} does not vanish on B, and (iii) the curve $\mathcal{C}_f$ crosses the border of B only on the left or the right sides. Then the topology of the curve in B is given by connecting the singular point with all the intersections on the border.

Given a zero-dimensional ideal $I = \mathbb{I}(P_1, \ldots, P_s)$ where the $P_i \in \mathbb{Q}[x_1, \ldots, x_n]$, a Rational Univariate Representation (RUR) [6] of the solutions $\mathrm{V}(I)$ is given by $F(t) = 0, x_1 = \frac{G_1(t)}{G_0(t)}, \ldots, x_n = \frac{G_n(t)}{G_0(t)}$, where $F, G_0, \ldots, G_n$ are univariate polynomials in $\mathbb{Q}[t]$ (where t is a new variable). All these univariate polynomials, and thus the RUR, are uniquely defined with respect to a given polynomial $\gamma \in \mathbb{Q}[x_1, \ldots, x_n]$ which is injective on $V(I)$; γ is called the **separating polynomial** of the RUR. [1]

3 Algorithm

Input. A square free bivariate polynomial $f \in \mathbb{Q}[x, y]$ without factor in $\mathbb{Q}[x]$.

Step 1. *Isolating boxes of the singular points and of the x-extreme points.* First compute the RURs of I_c and I_m with the same separating polynomial γ. Let F_c and F_m denote the univariate polynomials of the RURs of I_c and I_m. Isolate the roots of F_c and those of the gcd of F_c and F_m and match the resulting intervals. The roots of F_c that are roots of $\gcd(F_c, F_m)$ map to the singular points of $\mathcal{C}$ and the other roots of F_c map to the x-extreme points of $\mathcal{C}$. Using the RUR of I_c, compute isolating boxes of the singular points and of the x-extreme points. Note that these boxes are pairwise disjoint since they isolate the x-critical points of $\mathcal{C}$ (see Figure 2).

[1]The polynomial F is the characteristic polynomial of m_γ, the multiplication operator by the polynomial γ, in $\mathbb{Q}[x_1, \ldots, x_n]/I$.

Step 2. *Refinement of the isolating boxes of the x-extreme points.* For each vertical or horizontal side of each such box, B, isolate its intersections with $\mathcal{C}$ and refine the box (by refining the corresponding isolating interval of F_c) until there are two intersection points on the border of B. We further refine until there is at most one crossing on the top (resp. bottom) side of B. Unlike comparable algorithms, we do not require that $\mathcal{C}$ intersects the boundary of B only on its vertical sides.

Step 3. *Refinement of the isolating boxes of the singular points.* We refine these boxes exactly as in [8] (see Lemma 3) except for the way the multiplicity k of each singular point in its fiber is computed. We deduce k from the Teissier formula (see Lemma 2) and from the RURs computed above.

Consider each singular point, $\mathbf{p}$, in turn. Compute the multiplicities k_c and k_m of the root associated to $\mathbf{p}$ in the univariate polynomials F_c and F_m, respectively. This gives the multiplicity of $\mathbf{p}$ in I_c and I_m because the RURs preserve multiplicities. By the Teissier formula, the multiplicity of $\mathbf{p}$ in its fiber is $k = k_c - k_m + 1$. Refine the box containing $\mathbf{p}$ until f_{y^k} does not vanish in the box. Further refine the x-coordinates of the box until $\mathcal{C}$ only intersects the vertical boundary of the box.

Step 4. *Vertical asymptotes.* If there exist vertical asymptotes, we can work as follows.

First, compute an upper bound M_y on the absolute value of the y-coordinates of the y-critical points (this is of course done without computing these critical points). Compute also a bound M_x on the absolute value of the y-coordinates of the x-critical points (which are already computed). Isolate the roots of the polynomial in x obtained as the leading coefficient of f seen as a polynomial in y. For each root α we have an isolating interval $[a, b]$. Substitute $x = a$ (resp. $x = b$) in f and deduce an upper bound, M_α, on the absolute value of the y-coordinates of the intersection of $\mathcal{C}$ and $x = a$ (resp. $x = b$). Set $M = \max(M_\alpha, M_x, M_y)$. Then, a branch crossing the segment $]a, b[\times M$ (resp. $]a, b[\times - M$) goes to $+\infty$ (resp. $-\infty$) with asymptote $x = \alpha$. Finally, we determine whether a given branch is to the left or to the right of the asymptote by comparing the x-coordinates of the asymptote and the crossing point (see Figure 2).

Step 5. *Connections.* For simplicity, all the boxes computed above are called critical boxes and the points at infinity on vertical asymptotes are also called critical. First compute, with a sweep-line algorithm, the vertical rectangular decomposition obtained by extending the vertical sides of the critical boxes either to infinity or to the first encountered critical box (see Figure 2). On each of the edges of the decomposition, isolate the intersections with $\mathcal{C}$. Create vertices in the graph corresponding to these intersection points and to the critical points. For describing the arcs connecting these vertices in the graph, we assimilate, for simplicity, the points and the graph vertices. In each critical box, connect the critical point to the points on the boundary of the box.

For computing the connections in the non-critical rectangles of the decomposition, we use the multiplicities of the extreme points and a greedy algorithm. According to Lemma 2, the multiplicity k of an extreme point $\mathbf{p}$ in its fiber is one plus the multiplicity of $\mathbf{p}$ in I_c. Recall that the multiplicity of $\mathbf{p}$ in I_c is the multiplicity of the corresponding root in the polynomial F_c of the RUR. All the multiplicities of the extreme points in their fibers can thus be efficiently computed.

The geometric meaning of the parity of this multiplicity is the following: if it is even, the curve makes a U-turn at the extreme point; else, the curve is x-monotone in the neighborhood of the extreme point. Still, there are some difficulties for connecting the vertices, as illustrated on Figure 3: on the left is the information we may have on the crossings for two extreme points with x-overlapping boxes; the second and third drawings are two possible connections *in the middle rectangle* for given parities of the multiplicities. To distinguish between these two situations we compute the connections in rectangles starting from the top such that the connections in a rectangle below a critical box are computed once the connections in all the rectangles above the box are done.

First, the connections in the unbounded rectangles above critical boxes are straightforward: the connections between the vertices on the two vertical walls are one-to-one starting from infinity and if a vertex remains on a vertical wall, there is a vertex on the horizontal wall which it has to be connected with. Now, once all the connections have been computed in the rectangle(s) above the box of an extreme point, these connections and the multiplicity of the extreme point yield the connections in the rectangle(s) below. Indeed, if there is a vertex on the bottom side of the critical box, it lies on the top side of a rectangle and, inside this rectangle, the vertex is connected to the topmost vertex on the left or right wall, depending on the multiplicity of the extreme point and on the side of the connection of the branch above the extreme point; the other connections in this rectangle and in the possible other rectangles below the critical box are performed similarly as for unbounded rectangles.

Note that the two unbounded rectangles that are vertical half-planes are treated separately in a straightforward manner: for each vertex on the vertical wall is associated an arc going to infinity.

Output. A graph isotopic to the curve is output. In addition x-critical, singular points and vertical asymptotes are identified and their position is approximated by boxes.

4 Complexity Analysis

Consider a real algebraic plane curve defined by a square-free polynomial $f \in \mathbb{Z}[x, y]$ of degree bounded by d and maximum coefficient bit-size bounded by τ. Let s_k be the bit-size of the separation bound of f and all its derivatives with respect to y, s_c be the bit-size of the separation bound between all the critical points of f, $s = \max\{s_c, s_k\}$ and R be the number of critical points. Let $\widetilde{\mathcal{O}}_B$ denote the bit complexity where the polylogarithmic factors are omitted. For reasons of space we omit the proof of the following theorem.

Theorem 4 *Our algorithm computes the topology in* $\widetilde{\mathcal{O}}_B\left(R(d^{12}\tau s + d^{12}s^2 + d^7\tau^2)\right)$ *time, which is in* $\widetilde{\mathcal{O}}_B(N^{22})$*, where* $N = \max\{d, \tau\}$.

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References

[1] D. Cox, J. Little, and D. O'Shea. *Using Algebraic Geometry.* Number 185 in GTM. Springer, New York, 2nd edition, 2005.

[2] A. Eigenwillig, M. Kerber, and N. Wolpert. Fast and exact geometric analysis of real algebraic plane curves. In C. W. Brown, editor, *Proc. Int. Symp. Symbolic and Algebraic Computation*, pages 151–158, Waterloo, Canada, 2007. ACM.

[3] L. González-Vega and I. Necula. Efficient topology determination of implicitly defined algebraic plane curves. *Computer Aided Geometric Design*, 19(9), 2002.

[4] H. Hong. An efficient method for analyzing the topology of plane real algebraic curves. *Mathematics and Computers in Simulation*, 42(4–6):571–582, 1996.

[5] B. Mourrain, S. Pion, S. Schmitt, J.-P. Técourt, E. P. Tsigaridas, and N. Wolpert. Algebraic issues in Computational Geometry. In J.-D. Boissonnat and M. Teillaud, editors, *Effective Computational Geometry for Curves and Surfaces*, Mathematics and Visualization, Chapter 3. Springer, 2006.

[6] F. Rouillier. Solving zero-dimensional systems through the rational univariate representation. *Journal of Applicable Algebra in Engineering, Communication and Computing*, 9(5):433–461, 1999.

[7] B. Teissier. Cycles évanescents, sections planes et conditions de whitney. (french). In *Singularités á Cargèse (Rencontre Singularités Géom. Anal., Inst. Études Sci., Cargèse, 1972)*, number 7–8 in Astерisque, pages 285–362. Soc. Math. France, Paris, 1973.

[8] N. Wolpert and R. Seidel. On the Exact Computation of the Topology of Real Algebraic Curves. In *Symposium of Computational Geometry.* ACM, 2005.