Inflating balls is NP-hard

Guillaume Batog, Xavier Goaoc

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Abstract

A collection $\mathcal{C}$ of balls in $\mathbb{R}^d$ is δ-*inflatable* if it is isometric to the intersection $\mathcal{U} \cap E$ of some d-dimensional affine subspace E with a collection $\mathcal{U}$ of $(d+\delta)$-dimensional balls that are disjoint and have equal radius. We give a quadratic-time algorithm to recognize 1-inflatable collections of balls in any fixed dimension, and show that recognizing δ-inflatable collections of d-dimensional balls is NP-hard for $\delta \geq 2$ and $d \geq 3$ if the balls' centers and radii are given by numbers of the form $a + b\sqrt{c + d\sqrt{e}}$, where $a, \ldots, e$ are integers.

1 Introduction

Let $\mathcal{U}$ be a family of disjoint balls of equal radius – *unit balls*, for short – in $\mathbb{R}^{d+\delta}$ and $E \subset \mathbb{R}^{d+\delta}$ be a d-dimensional affine subspace – d-*flat*, for short. The intersection $\mathcal{C} = \mathcal{U} \cap E$ is a collection of d-dimensional balls that may no longer be unit, but still are not arbitrary: not every collection of balls in $\mathbb{R}^d$ can be obtained as sections of higher dimensional disjoint unit balls. We say that $\mathcal{C}$ is *inflatable* and $\mathcal{U}$ is an *inflation* of dimension δ of $\mathcal{C}$. In this paper, we study the δ-*inflation problem for balls in* $\mathbb{R}^d$:

> δ-*inflation problem*: given a collection of n balls in $\mathbb{R}^d$, decide if it admits an inflation of dimension δ.

Motivation. This problem originates from geometric transversal theory, an area that investigates properties of geometric transversals to collections of subsets of $\mathbb{R}^d$ (see e.g. the survey of Wenger [20]), and in particular of line transversals to disjoint convex objects. The topology and combinatorics of the set of line transversals to disjoint objects depend on the geometry of these objects, typical examples being

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the existence of Helly-type theorems [1, 6, 7, 13, 14] or the maximum number of geometric permutations [2, 3, 8, 10, 16, 17, 18, 19]. Understanding the hierarchy among classes of convex sets induced by such properties is a central question in geometric transversal theory.

In this hierarchy, the class of collections of disjoint unit balls is in a very particular situation. For example, the set of line transversals to any collection of disjoint unit balls has at most 3 connected components [7, 8] while this number can be $\Omega(n^{d-1})$ for n disjoint balls of arbitrary radii in $\mathbb{R}^d$ [18] and $\Omega(n)$ for n disjoint translates of a convex set [3]. Another particularity is that any family of disjoint unit balls in $\mathbb{R}^d$ has a line transversal if every $4d-1$ members have a line transversal [7, 13] but no similar Helly-type theorem exists for collections of disjoint balls with arbitrary radii [9, 12] or collections of disjoint translates of a convex set [14].

For these two examples, the properties of unit balls do generalize to inflatable collections of balls [7, 8] and it is unclear whether further extension of this class is possible. Looking for a working characterization of inflatable families of balls is a natural approach to this question.

Related work. In the context of geometric transversal theory, the notion of inflatability was first introduced by Cheong et al. [7] as a tool to generalize properties of line transversals to 4-dimensional disjoint unit balls to arbitrary dimension. In the same paper, they also proved that two balls are inflatable if and only if the squared distance between their centers is at least twice the sum of their squared radii (c.f. Lemma 2). To the best of our knowledge, this was the first use of the notion of inflatability and with the exception of the case of 2 balls, the present paper is the first investigation of the problem of recognizing inflatable families of balls.

Several other results also aim at bridging the gap between unit and non unit balls with respect to properties of line transversals. Zhou and Suri [22] proved that disjoint balls with radii in the range $[1, \gamma]$ have at most $O(\gamma^{\log \gamma})$ geometric permutations, and Hadwiger [11] proved a Helly-type theorem for "thinly distributed" families of balls, i.e. families such that the distance between the centers of any two balls is at least twice the sum of their radii.

Results. First, we show that the 1-inflation problem can be solved in quadratic time:

Theorem 1. *We can recognize 1-inflatable families of n balls in d dimensions using $O(n^2)$ arithmetic operations on the balls' radii and center coordinates.*

Our proof gives an algorithm to compute such an inflation, when it exists, using $O(n^2)$ operations $+, -, *, /, \sqrt[2]{\cdot}$ or comparison on the center coordinates and radii of the balls.

Recall that an algebraic number is a root of some polynomial with integer coefficients. Any algebraic number admits a finite representation such as the isolating interval representation [21] that allows standard arithmetic computations. A collection of balls with algebraic coordinate centers and radii can thus be described by a finite string of bits and the restriction of the inflation problem to this type of input can be studied in the bit model. We prove that this version of the problem is NP-hard:

Theorem 2. *For $\delta \geq 2$ and $d \geq 3$, deciding if a collection of balls in $\mathbb{R}^d$ with algebraic coordinate centers and radii has an inflation of dimension δ is NP-hard.*

More precisely, we show that the problem is NP-hard for collections of balls with integer radii and centers with coordinates of the form $a + b\sqrt{c + d\sqrt{e}}$, where $a, \dots, e$ are integers.

Paper organization. After some preliminary remarks on the inflation of pairs of balls (Section 2), we prove Theorem 1 by reducing the 1-inflation problem to the 2-coloring problem on some appropriate graph (Section 3). Then, we prove Theorem 2 by giving a polynomial reduction from the graph 6-coloring problem to the δ-inflation of d-dimensional balls with algebraic centers and radii (Section 4).

2 Inflations of pairs of balls

We denote by $\mathbb{R}$ the set of reals, by $\mathbb{S}^d$ the set of unit vectors in $\mathbb{R}^{d+1}$ and identify $\mathbb{S}^0$ with $\{-1, 1\}$ and $\mathbb{S}^1$ with the set of angles $[0, 2\pi[$. We consider the balls to be open and say that two balls are *tangent* if their bounding spheres are externally tangent. Let $\mathcal{C} = \{B_1, \dots, B_n\}$ be a collection of balls in $\mathbb{R}^d$. An *inflation* of $\mathcal{C}$ is a collection $\mathcal{U}$ of disjoint balls with equal radius ρ in some $\mathbb{R}^{d+\delta}$ such that $E \cap \mathcal{U}$ is isometric to $\mathcal{C}$ for some d-flat E. The *dimension* of the inflation is δ and its *radius* is ρ. A collection of balls is inflatable if its admits an inflation, of any dimension or radius. In the rest of the paper, we do not consider the trivial inflation where $\delta = 0$ and assume that $\delta \geq 1$.

We identify each ball $B_i \in \mathcal{C}$ with its image in $E \cap \mathcal{U}$ and denote by B'_i the corresponding ball of radius ρ in $\mathcal{U}$. Let v_i denote the vector from the center of B_i to that of B'_i, r_i denote the radius of B_i and d_{ij} denote the distance between the center of B_i and that of B_j. $\mathcal{U}$ is an inflation if and only if:

$$\forall i \neq j, \quad d_{ij}^2 + ||v_i||^2 + ||v_j||^2 - 2v_i \cdot v_j \geq 4\rho^2.$$

Since all vectors v_i are orthogonal to E, we can write $\overrightarrow{v_i} = ||v_i|| \overrightarrow{x_i}$ with $\overrightarrow{x_i} \in \mathbb{S}^{\delta-1}$. We call $\overrightarrow{x_i}$ the *inflation vector* of B_i. From $||v_i||^2 = \rho^2 - r_i^2$ we deduce that a family of vectors $\overrightarrow{x_1}, \ \ldots, \ \overrightarrow{x_n}$ defines an inflation of radius ρ if and only if:

$$\forall i \neq j, \quad 2\,(\overrightarrow{x_i} \cdot \overrightarrow{x_j})\sqrt{(\rho^2 - r_i^2)(\rho^2 - r_j^2)} \leq d_{ij}^2 - r_i^2 - r_j^2 - 2\rho^2. \qquad (1)$$

Lemma 1. *Two inflatable balls admit an inflation of dimension* 1 *with opposite inflation vectors.*

Proof. If two balls are inflatable then there exists a triple $(\rho, \overrightarrow{x_i}, \overrightarrow{x_j})$ satisfying Inequation (1). Since

$$-1 = -\overrightarrow{x_i} \cdot \overrightarrow{x_i} \leq \ \overrightarrow{x_i} \cdot \overrightarrow{x_j}$$

the triple $(\rho, \overrightarrow{x_i}, -\overrightarrow{x_i})$ also satisfies this inequation. The space E spanned by the d-space containing the original balls and the vector x_i intersects the two inflated balls in disjoint balls of equal radius. Since E has dimension $d+1$, this is an inflation of dimension 1. □

Consider two disjoint balls B_i and B_j with radii r_i , r_j and centers d_{ij} apart. We define ρ_{ij} as follows:

$$\rho_{ij} = \begin{cases} \max(r_i, r_j) & \text{if } d_{ij}^2 \geq r_i^2 + r_j^2 + 2\max(r_i^2, r_j^2) \\ \frac{1}{2}\sqrt{\frac{(d_{ij}^2 - r_i^2 - r_j^2)^2 - 4r_i^2 r_j^2}{d_{ij}^2 - 2r_i^2 - 2\,r_j^2}} & \text{otherwise.} \end{cases} \qquad (2)$$

Observe that for two disjoint balls with equal radii then $\rho_{ij} = r_i = r_j$.

Lemma 2. *Two disjoint balls* B_i *and* B_j *are inflatable if and only if* $d_{ij}^2 > 2(r_i^2 + r_j^2)$ *when* $r_i \neq r_j$ *and if and only if* $d_{ij}^2 \geq 2(r_i^2 + r_j^2)$ *when* $r_i = r_j$. *The set of radius to which they are inflatable is the interval* $[\rho_{ij}, +\infty[$.

The first statement was previously observed by Cheong et al. [7, Lemma 6].

Proof. By Lemma 1 and Equation (1), the two balls are inflatable if and only if there exists $\rho \geq \max(r_i, r_j)$ such that

$$d_{ij}^2 - r_i^2 - r_j^2 - 2\rho^2 \geq -2\sqrt{(\rho^2 - r_i^2)(\rho^2 - r_j^2)}.$$

With

$$\mathcal{M}_{ij}(\rho) = \left(\sqrt{\rho^2 - r_i^2} - \sqrt{\rho^2 - r_j^2}\right)^2 \qquad (3)$$

we have that the two balls are inflatable with radius $\rho \geq \max(r_i, r_j)$ if and only if:

$$d_{ij}^2 \geq 2(r_i^2 + r_j^2) + \mathcal{M}_{ij}(\rho).$$

If $r_i = r_j$, $\mathcal{M}_{ij}$ is constant equal to 0 and the equivalence holds and $\rho_{ij} = r_i$. Now if $r_i \neq r_j$, for any $\rho \geq \max(r_i, r_j)$ we have $\mathcal{M}_{ij}(\rho) > 0$ and it follows that the two balls are inflatable only if $d_{ij}^2 > 2(r_i^2 + r_j^2)$. A first-order Taylor expansion of the square root function around 1 yields that

$$\mathcal{M}_{ij}(\rho) \underset{\rho \to \infty}{\sim} \frac{(r_j^2 - r_i^2)^2}{4\rho^2},$$

and we obtain that $\lim_{\rho \to \infty} \mathcal{M}_{ij}(\rho) = 0$. It follows that if $d_{ij}^2 > 2(r_i^2 + r_j^2)$ the two balls are inflatable.

The two balls are inflatable only to radius larger or equal to $\max(r_i, r_j)$. Also, $\mathcal{M}_{ij}$ is decreasing on $[\max(r_i, r_j), +\infty[$, as can be observed from its derivative:

$$\frac{d}{d\rho}\mathcal{M}_{ij}(\rho) = \rho \left(\sqrt{\rho^2 - r_i^2} - \sqrt{\rho^2 - r_j^2} \right) \left(\frac{1}{\sqrt{\rho^2 - r_i^2}} - \frac{1}{\sqrt{\rho^2 - r_j^2}} \right). \quad (4)$$

The set of radius to which the balls are inflatable is thus an interval $[\alpha, +\infty[$. By symmetry, we assume that $r_j > r_i$ and, from $\mathcal{M}_{ij}(r_j) = r_j^2 - r_i^2$, we get:

$$\alpha = r_j \Leftrightarrow d_{ij}^2 \geq r_i^2 + 3r_j^2.$$

If $\alpha \neq r_j$ we have

$$\mathcal{M}_{ij}(\alpha) = d_{ij}^2 - 2(r_i^2 + r_j^2)$$

which reduces to

$$d_{ij}^2 - r_i^2 - r_j^2 - 2\alpha^2 = -2\sqrt{(\alpha^2 - r_i^2)(\alpha^2 - r_j^2)}.$$

This equality is equivalent to

$$4(\alpha^2 - r_i^2)(\alpha^2 - r_j^2) = (2\alpha^2 + r_i^2 + r_j^2 - d_{ij}^2)^2$$

and rewrites:

$$4(d_{ij}^2 - 2r_i^2 - 2r_j^2)\alpha^2 = (d_{ij}^2 - r_i^2 - r_j^2)^2 - 4r_i^2 r_j^2.$$

We finally get that

$$\alpha = \frac{1}{2}\sqrt{\frac{(d_{ij}^2 - r_i^2 - r_j^2)^2 - 4r_i^2 r_j^2}{d_{ij}^2 - 2r_i^2 - 2\,r_j^2}}$$

when $d_{ij}^2 < r_i^2 + 3r_j^2$, which concludes the proof. □

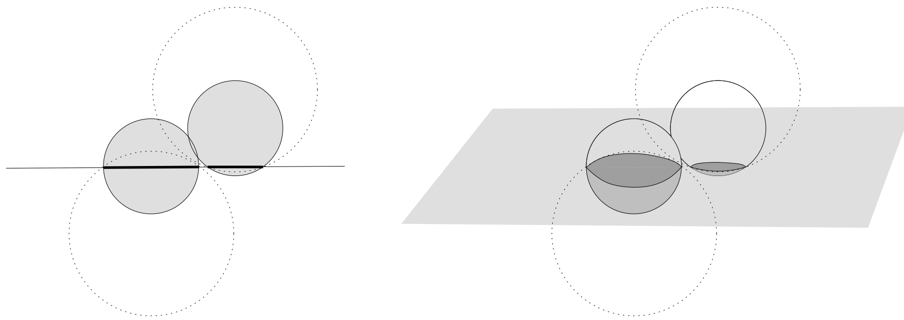

Figure 1: Two balls with radii $r_1 < r_2$ inflatable only to radii strictly larger than r_2: in dimension 1 (left) and 2 (right).

It may be surprising that ρ_{ij} can be strictly greater than $\max(r_i, r_j)$. An example of this situation is given in Figure 1.

Lemma 3. *Two tangent balls are inflatable if and only if they have equal radii. In any inflation they have opposite inflation vectors.*

Proof. It follows from Lemma 2 that two tangent balls are inflatable if and only if they have equal radii. Consider an inflation of radius ρ of two tangent balls of equal radius r with inflation vectors $\overrightarrow{x}$ and $\overrightarrow{y}$. Following Condition (1) the inflated balls are disjoint if and only if

$$(\overrightarrow{x} \cdot \overrightarrow{y})\,(\rho^2 - r^2) \leq r^2 - \rho^2 \quad \Longleftrightarrow \quad \overrightarrow{x} \cdot \overrightarrow{y} \leq -1 \quad \text{or} \quad \rho = r$$

that is, if and only if $\overrightarrow{x}$ and $\overrightarrow{y}$ are opposite (or null if $\rho = r$, in which case they can also be considered opposite). □

3 Inflations of dimension 1

Let $\mathcal{C} = \{B_1, \ldots, B_n\}$ be a collection of balls in $\mathbb{R}^d$. We denote by ρ_{ij} the smallest radius of an inflation of the balls (B_i, B_j) as defined in Equation (2) and by $R_{\mathcal{C}}$ the maximum of ρ_{ij} over all pairs $\{i, j\}$. Let $G_\rho(\mathcal{C}) = (\mathcal{C}, E)$ denote the graph where E contains an edge (B_i, B_j) if and only if

$$2\sqrt{(\rho^2 - r_i^2)(\rho^2 - r_j^2)} > d_{ij}^2 - r_i^2 - r_j^2 - 2\rho^2, \tag{5}$$

that is, according to Condition (1), if these balls cannot be inflated to radius ρ using equal inflation vectors.

Lemma 4. *$\mathcal{C}$ admits an inflation of dimension 1 and radius ρ if and only if $\rho \geq R_{\mathcal{C}}$ and $G_\rho(\mathcal{C})$ admits a 2-coloring.*

Proof. Consider a 2-coloring ϕ of $G_\rho(\mathcal{C})$ for $\rho \geq R_{\mathcal{C}}$; since $\mathbb{S}^0$ has cardinality 2, we can assume that ϕ colors $G_\rho(\mathcal{C})$ by vectors in $\mathbb{S}^0$. Let $\mathcal{U}$ denote the collection of $(d+1)$-dimensional balls obtained by inflating each ball B_i to radius ρ using $\phi(B_i)$ as inflation vector. Lemma 2 yields that $\mathcal{C}$ is pairwise inflatable with radius ρ, and so Lemma 1 implies that pairs inflated using opposite inflation vectors do not intersect. By construction, pairs inflated using equal inflation vectors are not connected by an edge in $G_\rho(\mathcal{C})$ and, thus, do not intersect. Therefore, $\mathcal{U}$ is an inflation of dimension 1 of $\mathcal{C}$.

Conversely, consider an inflation $\mathcal{U}$ of dimension 1 and radius ρ of $\mathcal{C}$. Lemma 2 implies that $\rho \geq R_{\mathcal{C}}$. Let $\phi : \mathcal{C} \to \mathbb{S}^0$ map a ball in $\mathcal{C}$ to its inflation vector if that vector is not null, and to an arbitrary vector in $\mathbb{S}^0$ otherwise. By construction, two balls connected in $G_\rho(\mathcal{C})$ cannot be inflated using the same inflation vector. If a ball B_i has inflation vector $\overrightarrow{0}$, then its radius is $r_i = \rho$ and it is connected in $G_\rho(\mathcal{C})$ only to balls B_j satisfying:

$$d_{ij}^2 - r_i^2 - r_j^2 - 2r_i^2 < 0.$$

Condition (1) then implies that balls B_i and B_j are not inflatable to radius ρ. Since $\rho \geq R_{\mathcal{C}}$, any ball B_i with inflation vector $\overrightarrow{0}$ is isolated in $G_\rho(\mathcal{C})$, and ϕ is a 2-coloring of $G_\rho(\mathcal{C})$. □

The family of graphs $G_\rho(\mathcal{C})$ is monotone for $\rho \geq R_{\mathcal{C}}$:

Lemma 5. *$G_{\rho_1}(\mathcal{C}) \subset G_{\rho_2}(\mathcal{C})$ for any $R_{\mathcal{C}} \leq \rho_1 \leq \rho_2$.*

Proof. Using the function $\mathcal{M}_{i,j}$ defined in Equation (3), Condition (5) rewrites as:

$$d_{ij}^2 < 4\rho^2 - \mathcal{M}_{i,j}(\rho).$$

The right-hand term is increasing for $\rho \geq R_{\mathcal{C}}$ as the sum of two increasing functions (c.f. Equation (4)), and the statement follows. □

Combining Lemmas 4 and 5, we get that $\mathcal{C}$ is 1-inflatable if and only if $G_{R_{\mathcal{C}}}(\mathcal{C})$ has a 2-coloring. Computing $G_{R_{\mathcal{C}}}(\mathcal{C})$ takes $O(n^2)$ arithmetic operations on the balls' parameters (squaring Condition (5) to avoid manipulating square roots), and deciding if it is 2-colorable can also be done in $O(n^2)$ time. Theorem 1 follows.

4 Inflation of arbitrary dimension

In this section we prove Theorems 2. In fact, we prove that the δ-inflation problem on d-dimensional balls with integer radii and center coordinates of the form $a + b\sqrt{c + d\sqrt{e}}$, where $a, \ldots, e$ are integers, is NP-hard. Recall that the *contact graph* of a collection $\mathcal{C}$ of balls is the graph whose vertices are the balls and whose edges connect two vertices if and only if the corresponding balls are tangent.

4.1 Inflation of dimension 2 of balls in $\mathbb{R}^3$

Let $G = (V, E)$ be a graph with $V = \{v_1, \ldots, v_n\}$ and let us equip the affine space $\mathbb{R}^3$ with an orthonormal frame (O, x, y, z). We construct a family $B_2(G)$ of balls in $\mathbb{R}^3$ such that 6-colorings G correspond to inflations of dimension 2 of $B_2(G)$. We proceed in five steps:

1. For every edge $(v_i, v_j) \in E$ we place two balls of radius 25, called *poles*, at

   $$(500n(i-1) + 500j + 29, 0, 0) \quad \text{and} \quad (500n(i-1) + 500j - 29, 0, 0),$$

   one labelled v_i and the other v_j. These two balls form a *dipole*.

2. For $i = 1, \ldots, n$ we place a series of balls of radius 25 at positions

   $$\{(50t, 200, 100i) | t = 0, \ldots, 10(n^2 + 1)\}$$

   and call it the *spine* of v_i (or i^{th} spine).

3. We connect each ball labelled v_i to the spine associated to v_i as depicted in Figure 2. More precisely, from a pole ball centered at $(x \pm 29, 0, 0)$ we draw a chain of tangent balls of radius 25 at positions $(x \pm 79, 0, 50t)$ for all integers t between 0 and $2i$. To connect the last of these balls to the spine we cannot use a “straight” chain of balls and recourse to an ad-hoc construction explicited in Figure 2 for $x = 100$ (and that can be translated for any x).

4. Using local adjustment along the spines, as depicted on Figure 3, we make sure that the distance between any two poles with the same label in the contact graph of $B_2(G)$ is even.

5. We finally place three pairwise tangent balls of radius 37 centered respectively at:

   $$\begin{pmatrix} 0 \\ 0 \\ -100 \end{pmatrix}, \begin{pmatrix} 74 \\ 0 \\ -100 \end{pmatrix}, \text{ and } \begin{pmatrix} 37 \\ 37\sqrt{3} \\ -100 \end{pmatrix}.$$

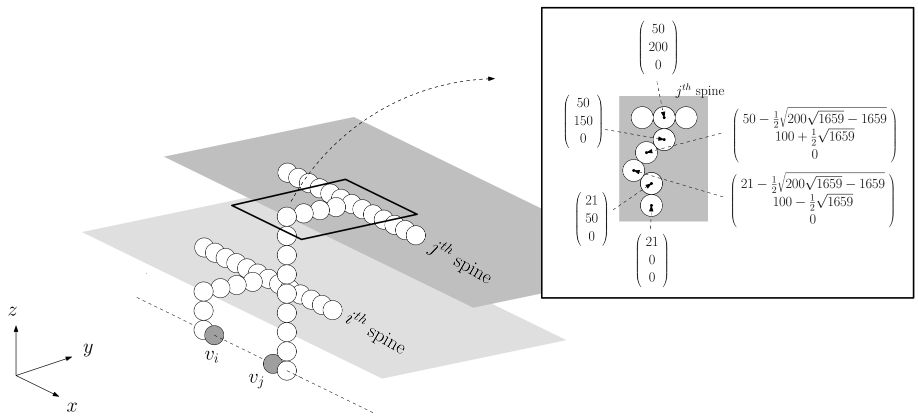

Figure 2: Connecting the poles to the spines (the distance between the poles has been increased for readability). The detailed view explicits the connection of the last ball from the vertical chain starting at v_j to the j^{th} spine, in the plane $z = 100j$.

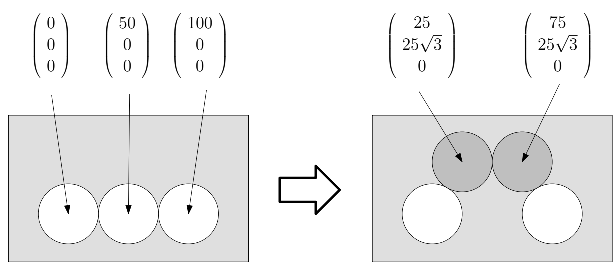

Figure 3: Adjusting the parity in the number of balls of a sequence.

Note that these balls are "isolated" in the sense that their centers are at distance at least 100 from the center of any other ball of $B_2(G)$.

Since the construction of $B_2(G)$ is polynomial in the size of G and deciding if a graph is 6-coloriable is NP-hard [15], the next lemma implies Theorem 2 for the case $d = 3$ and $\delta = 2$.

Lemma 6. *G has a 6-coloring if and only if $B_2(G)$ is 2-inflatable.*

Proof. The contact graph of $B_2(G)$ consists of a triangle of balls of radius 37 and one tree per vertex of G, each tree containing a spine and all the poles with the same label. Lemma 3 implies that any inflation of three pairwise tangent balls uses null inflation vectors. Thus, any inflation of $B_2(G)$ has radius 37.

Let T_i denote the tree containing the poles labelled v_i. By iterating Lemma 3 we get that in any inflation of $B_2(G)$ the balls in T_i use only two opposite inflation vectors. More precisely, two balls use equal inflation vectors if their distance in T_i is even and opposite inflation vectors if that distance is odd. Thus, step 5 in our construction implies that all poles in a given tree use equal inflation vectors. By construction two balls that are not tangent have centers at least $50\sqrt{2}$ apart. Thus, Condition (1) guarantees that the inflations to radius 37 of two non-tangent balls from the same tree can never intersect. Finally, we get that all balls from a given tree can be inflated and that in any such inflation the poles use the same inflation vector.

Let $\overrightarrow{u_1}, \ldots, \overrightarrow{u_n}$ be a collection of vectors in $\mathbb{S}^1$. We claim that $B_2(G)$ admits an inflation of dimension 2 where the poles labelled v_i have inflation vector $\overrightarrow{u_i}$ if and only if:

$$\forall (v_i, v_j) \in E, \quad \angle(\overrightarrow{u_i}, \overrightarrow{u_j}) \geq \frac{\pi}{3}. \tag{6}$$

Since two balls in a dipole are distance 58 apart, Condition (1) yields that their inflation vectors must form an angle of at least $\frac{\pi}{3}$ and Condition (6) is necessary. Conversely, the distance between the centers of two balls that do not form a dipole is at least $50\sqrt{2}$ and, following Condition (1), only the dipoles enforce constraints on the $\overrightarrow{u_i}$. Thus, Condition (6) is also sufficient.

For $k = 0, \ldots, 5$ let R_k denote the interval $[k\frac{\pi}{3}, (k+1)\frac{\pi}{3}) \subset \mathbb{S}^1$. Assume that $B_2(G)$ has an inflation of dimension 2 and let $\overrightarrow{u_i}$ denote the common inflation vector of the poles labelled v_i. Condition (6) implies that if v_i and v_j are connected by an edge in G then $\overrightarrow{u_i}$ and $\overrightarrow{u_j}$ belong to distinct regions R_k. Thus, if $B_2(G)$ has an inflation of dimension 2 then G has a coloring by $\{R_0, \ldots, R_5\}$. Conversely, consider a coloring of G by $\{R_0, \ldots, R_5\}$. If v_i has color R_k then let $\overrightarrow{u_i}$ denote the endpoint of R_k contained in that interval. Since for any $(v_i, v_j) \in E$ the angle between $\overrightarrow{u_i}$ and $\overrightarrow{u_j}$ is at least $\frac{\pi}{3}$, these vectors induce an inflation of $B_2(G)$ of dimension 2. □

4.2 Inflation of dimension δ of balls in $\mathbb{R}^3$

We now build on the previous construction to extend the proof to $d = 3$ and $\delta \geq 3$. The idea is to first add a "U-shape" gadget, a chain of tangent balls of radius 25, that can only be inflated using vectors orthogonal to those used to inflate the spines. Thus, if we add one gadget to $B_2(G)$ we will get a new collection of balls that is 3-inflatable if and only if $B_2(G)$ is 2-inflatable. The second gadget bootstraps this idea by enforcing two U-shapes to also use orthogonal inflation vectors. From there, $B_\delta(G)$ is simply obtained by adding $\delta - 2$ U-shapes to $B_2(G)$. An overview of the construction is given in Figure 4.

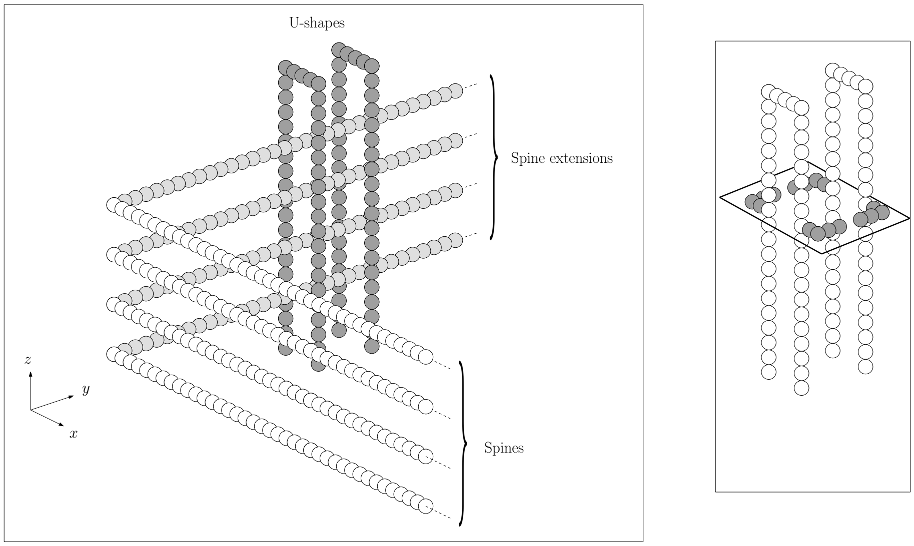

Figure 4: Turning $B_2(G)$ into $B_\delta(G)$: extensions of the spines and U-shapes (left) and gadgets between U-shapes, detailed on Figure 6 (right)

We proceed in three steps:

1. For $i = 1, \ldots, n$ we extend the spine of v_i by a series of balls of radius 25 at positions:
$$\{(0, 200 + 50t, 100i) | t = 1, \ldots, 4\delta + 10\}.$$

2. For $j = 1, \ldots, \delta - 2$ we place a U-shape in the plane $y = 300 + 200j$. This shape, described in Figure 5, satisfies in particular the following constraints: (i) it is a single chain of tangent balls of radius 25, (ii) the ball from the spine of v_i in the plane of the U-shape is at distance exactly $2\sqrt{997}$ from two of the balls of the U-shape, (iii) in the contact graph, the distance between these two balls is odd, and (iv) all distances that we do not specify are larger than $50\sqrt{2}$.

3. For every pair of U-shapes we add a gadget forcing their inflation vectors to be orthogonal. The principle of the construction is described in Figure 6, the gadget associated to U_i and U_j is placed in the plane $z = 500n(i-1)+500j$. All balls have radius 25, in each chain, balls that are not tangent have centers at least $50\sqrt{2}$ apart, and centers have coordinates of the form $a+b\sqrt{c+d\sqrt{e}}$ where $a, \ldots, e$ are integers. For the sake of the presentation, we omit the precise coordinates.

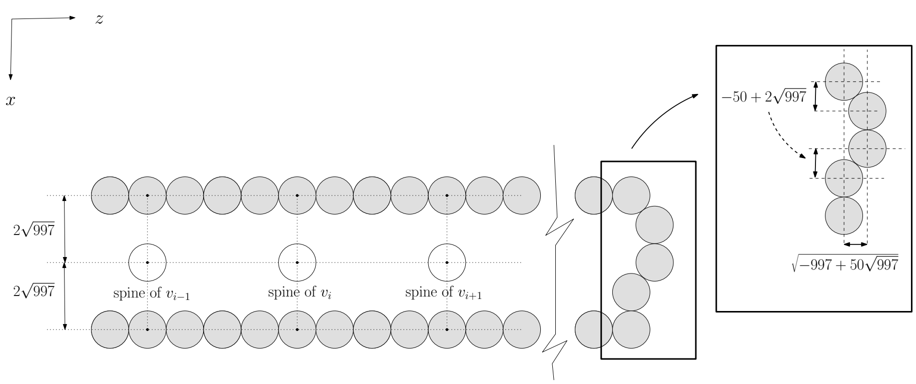

Figure 5: A U-shape. The distance constraint the spine balls to use inflation vectors orthogonal to those of the U-shape.

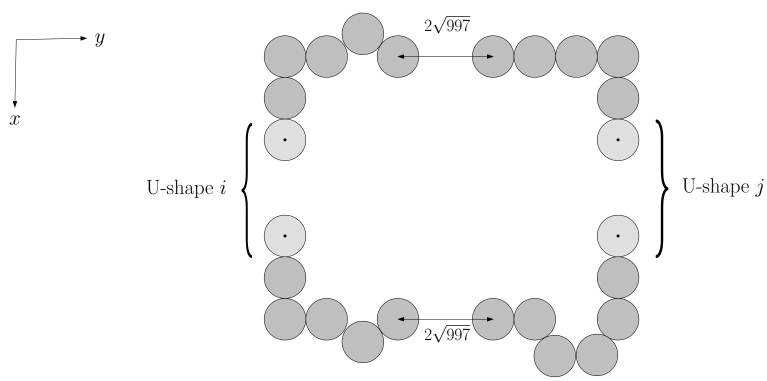

Figure 6: Connections between the U-shapes i and j.

Lemma 7. *For any fixed $\delta \geq 3$, G has a 6-coloring if and only if $B_\delta(G)$ has an inflation of dimension δ.*

Proof. By construction, each U-shape alone can be inflated to radius 37 and in each such inflation it only uses a pair of opposite inflation vectors $(\overrightarrow{u}, -\overrightarrow{u})$. The interactions between a U-shape and a spine imply that their respective inflation vectors are orthogonal. Similarly, step 4 forces the inflation vectors of any pair of U-shapes to be orthogonal.

If $B_2(G)$ has an inflation of dimension 2 then $B_\delta(G)$ has an inflation of dimension δ: we simply inflate the U-shapes using $\delta - 2$ pairwise orthogonal vectors that are also orthogonal to those used in the inflation of $B_2(G)$. Conversely, assume that $B_\delta(G)$ has an inflation of dimension δ. Then its U-shapes require $\delta - 2$ pairwise orthogonal inflation vectors. By construction, $B_\delta(G)$ contains a copy of $B_2(G)$ that can only be inflated using vectors orthogonal to the inflation vectors of its U-shapes. Thus, any inflation of $B_\delta(G)$ of dimension δ induces an inflation of $B_2(G)$ of dimension 2. The result then follows from Lemma 6. □

4.3 Proof of Theorem 2

We can now put together a complete proof of the main result of this section:

Proof of Theorem 2. For $d = 3$ the Theorem follows from our polynomial-time reductions of the 6-colorability of an arbitrary graph G, which is NP-hard, to the δ-inflatability of the collection of balls $B_\delta(G)$. Note that $B_\delta(G)$ can be considered as a collection $B'_\delta(G)$ of d-dimensional balls, whose centers lie on a 3-flat, and that the δ-inflatability of these two collections are equivalent. Thus, the case $d = 3$ implies the case $d \geq 3$. □

5 Conclusion

The original question was whether there exists a *working description* of the space of families of d-dimensional inflatable balls. We showed that the algorithmic question of recognizing if a given instance is δ-inflatable is easy if $\delta = 1$ and likely to be difficult if $\delta \geq 2$ and $d \geq 3$. Although these results strongly suggest that a "working description" of this space is unlikely to be achieved, they do not completely answer the original question as we discuss in the following remarks.

Remark 1. The original question is phrased over the reals and our negative answer is phrased in the bit model. Arguably, the natural computational model to study this question would rather be that introduced by Blum et al. [4, 5], but our Theorem 2 does not trivially extend to a hardness result in this model.

Remark 2. Clearly, if n balls are inflatable they admit an inflation of dimension at most n. However, since the number of balls in $B_\delta(G)$ increases with δ, our reduction does not apply to the problem of recognizing inflatable families of balls when no restriction is put on the dimension of the inflation.

Remark 3. Realizing our construction $B_2(G)$ in the plane raises non-trivial graph drawing issues, as one has to control the interactions between non-tangent balls. We would still not be surprised if the 2-inflation of families of disks is already be NP-hard but, however, expect that the case $d = 1$ behaves differently as the ordering of the original balls may drastically change the nature of the problem.

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