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Abstract

We present a new non conforming space-time mesh refinement method for symmetric first order hyperbolic system. This method is based on the one hand on the use of a conservative higher order discontinuous Galerkin approximation for space discretization and a finite difference scheme in time, on the other hand on appropriate discrete transmission conditions between the grids. We use a discrete energy technique to drive the construction of the matching procedure between the grids and guarantee the stability of the method.

Key words: Local time stepping, discontinuous Galerkin, first-order hyperbolic problem, energy conservation, finite difference scheme, explicit scheme.
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1 Introduction

This work has been motivated by the construction of a non conforming space-time mesh refinement method for wave propagation in aeroacoustics, in the spirit of the previous work for the wave equation [3,4], Maxwell's equations [5] or elastodynamics [1,6], when finite elements are used for the space discretization of the equations. The novelty of this work lies on the fact that a discontinuous Galerkin (DG) approximation is used for space discretization. This is specifically interesting for aeroacoustic applications. Since, to our knowledge, there is no existing finite element method for the linearized Euler equations.

The inherent flexibility of this type of method allows us ,"naturally", to deal with non matching grids in space. However, when time discretization (using finite differences) is concerned, the use of a local time step, which is for instance highly desirable in case of refined space meshes, remains a difficult question: particularly, the stability of the resulting numerical method is a delicate issue. Let us mention that the question of local time stepping with DG methods has been considered in the particular case of Maxwell's equations in [12] (where the stability question is not completely clarified) or in [8] for elastic waves. More recently, in [7], a method is proposed for the second order equation where the objective (energy conservation) is close to ours but the method consists more in working on the time discretization, typically in the fine grid. In this paper, we develop a rather general method which is applicable to zero order perturbations of symmetric hyperbolic systems in the sense of Friedrichs (Linearized Euler equations or Maxwell's equations are of this type). The key point is the derivation of appropriate discrete transmission conditions between two space-time grids. We use a discrete energy technique to construct such conditions, and guarantee the stability of the matching procedure under the same CFL condition that should be used if the two grids were considered separately. The object of this paper is to present the construction of the method and to emphasize the stability analysis. Some numerical tests in 2D are presented in the last section.

2 First order symmetric hyperbolic systems

Let Ω be a domain in $\mathbb{R}^d$, $d = 1, 2, 3$, we consider a zero order perturbation of symmetric hyperbolic system in the sense of Friedrichs [9] :

$$\begin{cases} \text{find } \mathbf{u} : \Omega \times \mathbb{R}^+ \mapsto \mathbb{R}^m \text{ such that:} \\ M\partial_t \mathbf{u} + \sum_{j=1}^{d} A_j \partial_{x_j} \mathbf{u} + C\mathbf{u} = \mathbf{f}, \qquad \text{in } \Omega \times \mathbb{R}^+ \setminus \{0\}, \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), \qquad \text{in } \Omega, \end{cases} \tag{1}$$

where M, A_j, $j = 1, \cdots, d$ and C are square matrices of dimension $m \times m$, uniformly bounded functions of $x \in \Omega$. We assume that M is symmetric positive definite (uniformly in x) and that each A_j, $j = 1, \cdots, d$ is symmetric. We complete the system (1) by the boundary condition on $\Gamma = \partial\Omega$ ($\mathbf{n} = (n_1, \cdots, n_d)^t$ being the unit outward normal to Γ):

$$(A(\mathbf{n}) - N)\mathbf{u} = 0 \quad \text{on } \Gamma \times \mathbb{R}^+ \setminus \{0\}, \quad A(\mathbf{n}) = \sum_{j=1}^{d} n_j A_j, \tag{2}$$

where, for each $x \in \Gamma$, $N = N(x)$ is a $m \times m$ matrix satisfying

$$N + N^t \geq 0, \quad \ker(A(\mathbf{n}) - N) + \ker(A(\mathbf{n}) + N) = \mathbb{R}^m. \tag{3}$$

The well-posedness of the initial boundary value problem (1, 2), under conditions (3), is well known since the work of Friedrichs [9], when the A_j's satisfy:

$$\mathbf{div}\, A = \sum_{j=1}^{d} \partial_{x_j} A_j \in L^\infty(\mathbb{R}^d; \mathbb{R}^{m\times m}) \tag{4}$$

Let us emphasize here the related energy identity and energy estimates. Denoting by $(\cdot\,,\cdot)$ (respectively $<\cdot\,,\cdot>$) the inner product in $L^2(\Omega)^m$ (respectively $L^2(\Gamma)^m$), we define the energy of $\mathbf{u}$ at time t by:

$$E(t) = \Big(M\,\mathbf{u}(t), \mathbf{u}(t)\Big).$$

One easily sees that this quantity satisfies

$$\frac{1}{2}\,\frac{dE}{dt} = (\mathbf{f}, \mathbf{u}) - \Big(\Big(C - \frac{1}{2}\mathbf{div}A\Big)\,\mathbf{u}, \mathbf{u}\Big) - < \frac{N+N^t}{4}\,\mathbf{u}\,, \mathbf{u} > . \tag{5}$$

Indeed, multiplying the equation in (1) by $\mathbf{u}$, we get after integration on Ω:

$$\frac{1}{2}\,\frac{d}{dt}\int_\Omega M\,\mathbf{u}\cdot\mathbf{u}\,dx + \sum_{j=1}^{d}\int_\Omega (A_j\,\partial_{x_j}\mathbf{u})\cdot\mathbf{u}\,dx + \int_\Omega C\,\mathbf{u}\cdot\mathbf{u}\,dx = \int_\Omega \mathbf{f}\cdot\mathbf{u}\,dx. \tag{6}$$

We integrate by parts the second term. By symmetry of the A_j's, we get:

$$\sum_{j=1}^{d}\int_\Omega (A_j\,\partial_{x_j}\mathbf{u})\cdot\mathbf{u}\,dx = -\sum_{j=1}^{d}\int_\Omega (A_j\,\partial_{x_j}\mathbf{u})\cdot\mathbf{u}\,dx - \int_\Omega \mathbf{div}A\,\mathbf{u}\cdot\mathbf{u}\,dx + \int_\Gamma A(\mathbf{n})\mathbf{u}\cdot\mathbf{u}\,d\sigma,$$

which gives us

$$\left|\begin{aligned}\sum_{j=1}^{d}\int_\Omega (A_j\,\partial_{x_j}\mathbf{u})\cdot\mathbf{u}\,dx &= \frac{1}{2}\Big[-\int_\Omega \mathbf{div}A\,\mathbf{u}\cdot\mathbf{u}\,dx + \int_\Gamma A(\mathbf{n})\mathbf{u}\cdot\mathbf{u}\,d\sigma\Big],\\ &= \frac{1}{2}\Big[-\int_\Omega \mathbf{div}A\,\mathbf{u}\cdot\mathbf{u}\,dx + \frac{1}{2}\int_\Gamma (N+N^t)\mathbf{u}\cdot\mathbf{u}\,d\sigma\Big].\end{aligned}\right. \tag{7}$$

where we have used the boundary condition (2). Finally, substituting (7) into (6) leads to (5). The reader will notice that:

- Using Gronwall's lemma, it is easy to get L^2 estimates of the solution. This is where the positivity property of N plays a major role. When $\mathbf{f}$ belongs to $C^0(L^2)$, one obtains an upper bound in $\exp\alpha t$ where α is related to $\|\mathbf{div}A\|_\infty$ and $\|C\|_\infty$ and the lower bound for the eigenvalues of M.

- When the symmetric part of $2C - \mathbf{div}A$ is positive, one sees that the energy $E(t)$ is a decreasing function of time as soon as $\mathbf{f} = 0$: the system is dissipative and one gets uniform (in time) L^2 estimates of the solution.

Remark 1 *An important application is sound propagation in a homogeneous fluid in stationnary flow (aeroacoustics). The unknown is* $\mathbf{u} = (\mathbf{v}, p)^t$ *where* $\mathbf{v}$ *is the velocity field and* p *the pressure, governed by the compressible Euler equations (after appropriate scaling) :*

$$\begin{cases} \partial_t \mathbf{v} + \Big(\mathbf{M} \cdot \nabla\Big)\mathbf{v} + \Big(\mathbf{v} \cdot \nabla\Big)\mathbf{M} + \nabla p = 0, \\ \\ \partial_t p + \Big(\mathbf{M} \cdot \nabla\Big)p + \nabla \cdot \mathbf{v} = 0, \end{cases} \tag{8}$$

where $\mathbf{M}(x)$ *is the given Mach vector distribution for the reference flow. The reader will easily check that (8) is a particular case of (1) and that the condition (4) amounts to saying that* $\mathbf{M}(x)$ *is Lipschitz-continuous. Finally the slipping condition* $\mathbf{v} \cdot \mathbf{n} = 0$ *is of type (2) and satisfies (3).*

3 Conservative discontinuous Galerkin method

The method developed here is applicable to the general problem (1, 2)(see section 5) but, for the sake of clarity, we first consider the case where $\Omega = \mathbb{R}^d$ (no boundary), the matrices A_j, $j = 1, \cdots, d$ are constant, $C = 0$ and $\mathbf{f} = 0$. Note that this is a case where the energy is conserved.

A conservative variational formulation. A particularity of DG methods (see [2,10]) is that a mesh of the domain is introduced before the space discretization. Thus we consider a (family of) mesh(es) $\mathcal{T}_h,\ h > 0$, of $\Omega = \mathbb{R}^d$

$$\mathcal{T}_h = \{K \in \mathcal{T}_h\}, \quad \overline{\Omega} = \cup_{K \in \mathcal{T}_h} K,$$

whose stepsize $h = \sup \operatorname{diam} K$ is devoted to tend to 0. Here,we do not make particular assumptions on the shape of the "elements" K. However, in practice, these will be triangles or quadrilaterals in 2D, tetrahedra or hexahedra in 3D. The point to emphasize is that to the conformity is not required (i. e. hanging nodes are allowed). We assume in the sequel that the solution $\mathbf{u}(t)$ of (1, 2) satisfies

$$\forall\, h, \quad \mathbf{u}(t) \in C^0(\mathbb{R}^+; \mathbb{V}^h), \tag{9}$$

where we have introduced the space:

$$\mathbb{V}^h = \{\mathbf{v} \in L^2(\Omega)^m \ / \ \forall\, K \in \mathcal{T}_h,\ \mathbf{v}_K := \mathbf{v}_{|K} \in H^1(K)^m\} \tag{10}$$

Denote $\mathbf{n}_K$ the unit outward normal to ∂K and introducing the interfaces $\Sigma_{KL} = \partial K \cap \partial L$ (possibly empty) for all $(K, L) \in \mathcal{T}_h^2$, we notice that (1) is equivalent to finding $\mathbf{u}(t) : \mathbb{R}^+ \mapsto \mathbb{V}^h$ such that:

$$M \partial_t \mathbf{u}_K + \sum_{j=1}^{d} A_j \partial_{x_j} \mathbf{u}_K = 0, \quad \text{in } K, \quad \forall\, K \in \mathcal{T}_h, \tag{11a}$$

$$A(\mathbf{n}_k)\mathbf{u}_k + A(\mathbf{n}_L)\mathbf{u}_L = 0, \quad \text{on } \Sigma_{KL}, \quad \forall\, (K, L) \in \mathcal{T}_h^2, \tag{11b}$$

To obtain a variational formulation, we multiply (11a) by some $\mathbf{v}_K \in [H^1(K)]^m$ and then integrate on K to obtain:

$$\frac{d}{dt}\int_K M\mathbf{u}_K \cdot \mathbf{v}_K \, dx + \sum_{j=1}^{d} \int_K A_j \partial_{x_j} \mathbf{u}_K \cdot \mathbf{v}_K \, dx = 0.$$

After an integration by parts, we have:

$$\left| \begin{array}{l} \dfrac{d}{dt}\displaystyle\int_K M\mathbf{u}_K \cdot \mathbf{v}_K \, dx - \sum_{j=1}^{d} \int_K A_j \mathbf{u}_K \cdot \partial_{x_j} \mathbf{v}_K \, dx \\ \qquad\qquad + \displaystyle\sum_L \int_{\Sigma_{KL}} A(\mathbf{n}_K)\mathbf{u}_K \cdot \mathbf{v}_K \, d\sigma = 0. \end{array} \right. \tag{12}$$

The general principle of DG methods is to replace on Σ_{KL}, the quantity $A(\mathbf{n}_K)\mathbf{u}_K$ by some numerical flux $\mathcal{F}(\mathbf{u}_K, \mathbf{u}_L)$ on Σ_{KL}, in order to take into account the transmission conditions (11b). There are various possible choices for the fluxes $\mathcal{F}$ (this is where DG approximations may differ). As in [12], we consider here the centered fluxes, which will lead to an energy preserving method. Using (11b) and $\mathbf{n}_L = -\mathbf{n}_K$ on Σ_{KL}, we have, on Σ_{KL}:

$$A(\mathbf{n}_K)\,\mathbf{u}_K = \frac{1}{2}\left[A(\mathbf{n}_K)\,\mathbf{u}_K - A(\mathbf{n}_L)\,\mathbf{u}_L\right] \quad \Big(\equiv A(\mathbf{n}_K)\,\frac{\mathbf{u}_K + \mathbf{u}_L}{2} \Big) \tag{13}$$

We choose $\mathcal{F}(\mathbf{u}_K, \mathbf{u}_L)$ as the right hand side of (13) to obtain:

$$\left| \begin{array}{l} \dfrac{d}{dt}\displaystyle\int_K M\mathbf{u}_K \cdot \mathbf{v}_K \, dx - \sum_{j=1}^{d} \int_K A_j \mathbf{u}_K \cdot \partial_{x_j} \mathbf{v}_K \, dx \\ \quad + \dfrac{1}{2}\displaystyle\sum_L \int_{\Sigma_{KL}} \left[A(\mathbf{n}_K)\,\mathbf{u}_K - A(\mathbf{n}_L)\,\mathbf{u}_L\right] \cdot \mathbf{v}_K d\sigma = 0. \end{array} \right.$$

After summation over K, we see that $\mathbf{u} \equiv \mathbf{u}^h$ is the solution of the problem

$$\left\{ \begin{array}{l} \text{find } \mathbf{u}^h(t) : \mathbb{R}^+ \mapsto \mathbb{V}^h \text{ such that :} \\ \dfrac{d}{dt}\, m(\mathbf{u}^h, \mathbf{v}_h) + a_h(\mathbf{u}^h, \mathbf{v}_h) = 0, \quad \forall\, \mathbf{v}_h \in \mathbb{V}^h, \end{array} \right. \tag{14}$$

where we have defined the bilinear forms in $\mathbb{V}_h$:

$$\left\{ \begin{array}{l} m(\mathbf{u}_h, \mathbf{v}_h) \;=\; \displaystyle\int_\Omega M(x)\mathbf{u}_h \cdot \mathbf{v}_h \, dx \quad \Big(\equiv \sum_K \int_K M(x)\mathbf{u}_K \cdot \mathbf{v}_K \, dx \Big), \\ \\ a_h(\mathbf{u}_h, \mathbf{v}_h) = \dfrac{1}{2}\displaystyle\sum_{K,L} \int_{\Sigma_{KL}} \left[A(\mathbf{n}_K)\,\mathbf{u}_K - A(\mathbf{n}_L)\,\mathbf{u}_L\right] \cdot \mathbf{v}_K \, d\sigma \\ \qquad\qquad - \displaystyle\sum_{j,K} \int_K A_j\, \mathbf{u}_K \cdot \partial_{x_j} \mathbf{v}_K \, dx. \end{array} \right. \tag{15}$$

The fundamental property of $a_h(\cdot,\cdot)$ is given by the following theorem:

Theorem 1 *For all* $(\mathbf{u}_h, \mathbf{v}_h) \in \mathbb{V}_h^2$, $\quad a_h(\mathbf{u}_h, \mathbf{v}_h) = -a_h(\mathbf{v}_h, \mathbf{u}_h)$.

Proof. We integrate by parts the second term in $a_h(\mathbf{u}_h, \mathbf{v}_h)$ (cf. 15) to get:

$$\left| \begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) = & -\frac{1}{2} \sum_{K,L} \int_{\Sigma_{KL}} [A(\mathbf{n}_K)\, \mathbf{u}_K + A(\mathbf{n}_L)\, \mathbf{u}_L] \cdot \mathbf{v}_K \, d\sigma. \\ & + \sum_{j,K} \int_K A_j\, \mathbf{u}_K \cdot \partial_{x_j} \mathbf{v}_K \, dx \end{aligned} \right.$$

Taking half of the sum of the two expressions we got for $a_h(\mathbf{u}_h, \mathbf{v}_h)$, we obtain

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = \frac{1}{2} \sum_{j,K} \int_K \left[A_j\, \partial_{x_j} \mathbf{u}_K \cdot \mathbf{v}_K - A_j\, \mathbf{u}_K \cdot \partial_{x_j} \mathbf{v}_K \right] dx - \frac{1}{2}\, \gamma_h(\mathbf{u}_h, \mathbf{v}_h)$$

where $\gamma_h(\mathbf{u}_h, \mathbf{v}_h) = \sum_{K,L} \int_{\Sigma_{KL}} A(\mathbf{n}_L)\, \mathbf{u}_L \cdot \mathbf{v}_K \, d\sigma$.

To conclude, it remains to check that $\gamma_h(\cdot, \cdot)$ is skew-symmetric. Interchanging the roles of K and L in the summation, we have, since $\Sigma_{KL} = \Sigma_{LK}$,

$$\gamma_h(\mathbf{u}_h, \mathbf{v}_h) = \sum_{K,L} \int_{\Sigma_{KL}} A(\mathbf{n}_K)\, \mathbf{u}_K \cdot \mathbf{v}_L \, d\sigma.$$

Using first $A(\mathbf{n}_K) = -A(\mathbf{n}_L)$ on Σ_{KL}, next the symmetry of $A(\mathbf{n}_L)$, we get

$$\gamma_h(\mathbf{u}_h, \mathbf{v}_h) = -\sum_{K,L} \int_{\Sigma_{KL}} A(\mathbf{n}_L)\, \mathbf{u}_K \cdot \mathbf{v}_L \, d\sigma = -\sum_{K,L} \int_{\Sigma_{KL}} A(\mathbf{n}_L)\, \mathbf{v}_L \cdot \mathbf{u}_K \, d\sigma,$$

that is to say $\gamma_h(\mathbf{u}_h, \mathbf{v}_h) = -\gamma_h(\mathbf{v}_h, \mathbf{u}_h)$. $\square$

Remark 2 *Writing* $\gamma_h(\mathbf{u}_h, \mathbf{v}_h) = \Big(\gamma_h(\mathbf{u}_h, \mathbf{v}_h) - \gamma_h(\mathbf{v}_h, \mathbf{u}_h)\Big)/2$, *we get another expression of* $a_h(\mathbf{u}_h, \mathbf{v}_h)$ *which we shall use in the sequel, namely:*

$$\left| \begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) = & \frac{1}{2} \sum_K \sum_{j=1}^d \int_K \left[A_j\, \partial_{x_j} \mathbf{u}_K \cdot \mathbf{v}_K - A_j\, \mathbf{u}_K \cdot \partial_{x_j} \mathbf{v}_K \right] dx \\ & + \frac{1}{4} \sum_{K,L} \int_{\Sigma_{KL}} [A(\mathbf{n}_K)\, \mathbf{u}_L \cdot \mathbf{v}_K - A(\mathbf{n}_K)\, \mathbf{v}_L \cdot \mathbf{u}_K] \, d\sigma. \end{aligned} \right. \tag{16}$$

Space discretization. To each $K \in \mathcal{T}_h$, we associate a finite-dimensional subspace $\mathbb{V}_K$ of $[H^1(K)]^m$, (typically $\mathbb{V}_k = [P_r(K)]^m$, where $P_r(K)$ the set polynomials of degree $\leq r$) and define "discrete" subspace of $\mathbb{V}^h$

$$\mathbb{V}_h = \{\mathbf{v} \in L^2(\Omega)^m \;/\; \forall\, K \in \mathcal{T}_h,\ \mathbf{v}_K := \mathbf{v}_{|K} \in \mathbb{V}_K\} \tag{17}$$

The semi discrete problem consists in finding $\mathbf{u}_h(t) : \mathbb{R}^+ \mapsto \mathbb{V}_h$ such that

$$\frac{d}{dt} m(\mathbf{u}_h, \mathbf{v}_h) + a_h(\mathbf{u}_h, \mathbf{v}_h) = 0, \quad \forall\, \mathbf{v}_h \in \mathbb{V}_h, \tag{18}$$

From theorem 1, we deduce that the energy of the semi-discrete solution $\mathbf{u}_h$, namely $E_h(t) = m(\mathbf{u}_h, \mathbf{u}_h)$, is conserved in time.

Time discretization and stability analysis. We construct a fully explicit (at most local mass matrices have to be inverted, element by element) leap-frog scheme.We choose to compute the discrete solutions at times $t^{n+1/2} = (n + 1/2)\Delta t$. This is a priori a strange choice but we make it for convenience. It will allow us to simplify the notation for the presentation of the local time stepping procedure: find $\mathbf{u}_h^{n+1/2} \in \mathbb{V}_h, \quad n\mathbb{N}$ such that

$$m(\frac{\mathbf{u}_h^{n+3/2} - \mathbf{u}_h^{n-1/2}}{2\Delta t}\ , \mathbf{v}_h) + a_h(\mathbf{u}_h^{n+1/2}, \mathbf{v}_h) = 0, \quad \forall\, \mathbf{v}_h \in \mathbb{V}_h \quad n \geq i. \tag{19}$$

Choosing $\mathbf{v}_h = \mathbf{u}_h^{n+3/2} + \mathbf{u}_h^{n-1/2}$ in (19), and using again theorem 1, one deduces the conservation in time of the discrete energy

$$E_h^n = \frac{1}{2}\Big[\, \|\mathbf{u}_h^{n+1/2}\|^2 + \|\mathbf{u}_h^{n-1/2}\|^2 \,\Big] + \Delta t\ a_h(\mathbf{u}_h^{n+1/2}, \mathbf{u}_h^{n-1/2}), \tag{20}$$

where $\|\mathbf{v}\|^2 = m(\mathbf{v}, \mathbf{v}),\ \forall\, \mathbf{v} \in \mathrm{L}^2(\Omega)^m$. To get a sufficient L^2 stability condition, it suffices to show the positivity the energy (20), which leads to

$$\Delta t\ \|a_h\| \leq 1 \ \text{ where } \ \|a_h\| = \sup_{\mathbf{u}_h, \mathbf{v}_h \in \mathbb{V}_h \setminus \{0\}} \frac{a_h(\mathbf{u}_h, \mathbf{v}_h)}{\|\mathbf{u}_h\|\, \|\mathbf{v}_h\|}\ . \tag{21}$$

Remark 3 *(21) is an abstract CFL condition. In practice, since one deals with a first order differential operator one has $\|a_h\| = O(h^{-1})$ and (21) means that the ratio $\Delta t/h$ much remain bounded.*

4 A conservative local time stepping procedure

We suppose that Ω is a union of two domain Ω_c and Ω_f separated by an interface $\Sigma = \bar{\Omega}_c \cap \bar{\Omega}_f$. We suppose that, for some reason (see [5,1] for some motivating examples), the mesh $\mathcal{T}_h$ is the union of a (typically coarse) mesh $\mathcal{T}_h^c$ for the domain Ω_c and a (typically fine) mesh $\mathcal{T}_h^f$ for the domain Ω_f. Our goal is to use in each grid a different time step, Δt_f in Ω_f and $\Delta t_c > \Delta t_f$ in Ω_c. The motivation for doing this can be dictated by various reasons [5,1]: for instance, according to the hyperbolic nature of the problem, it is natural to choose Δt proportional to the local mesh size, which is in agreement with the CFL condition (see remark 3).

The method that we propose in this paper is valid when the ratio between the

two time steps is a rational number. However, for the sake of simplicity, we explain its construction in the case where this ratio is 2:

$$\Delta t_c = \Delta t, \quad \Delta t_f = \Delta t/2. \tag{22}$$

We begin by rewriting the semi-discrete formulation associated to the mesh $\mathcal{T}_h = \mathcal{T}_h^c \cup \mathcal{T}_h^f$ by a 2×2 system whose unknowns are the respective solutions of the semi-discrete solution $\mathbf{u}_h$ on Ω_c and Ω_f. For this, we need the additional notation:

$$\mathcal{I}_\ell = \{(K,L) \,/\, \Sigma_{KL} \neq \emptyset,\, \Sigma_{KL} \subset \Omega_\ell\}, \;\; \forall\, \ell \in \{c,f\},$$

$$\mathcal{I} = \{(K,L) \,/\, \Sigma_{KL} \neq \emptyset,\, \Sigma_{KL} \subset \Sigma\}.$$

Moreover, we define:

$$\mathbb{V}_h^\ell = \{(\mathbf{v}_h)_{|\Omega_\ell}, \mathbf{v}_h \in \mathbb{V}_h\} \equiv \{\mathbf{v}_{\ell,h} \in L^2(\Omega_\ell) \,/\, \forall\, K \in \mathcal{T}_h^\ell, (\mathbf{v}_{\ell,h})_{|K} \in \mathbb{V}_K\}. \tag{23}$$

Denoting $\mathbf{u}_{\ell,h} = \mathbf{u}_h|_{\Omega_\ell}$ and using (16), it is easy to see that we can rewrite the variational formulation (14) as find $\mathbf{u}_{\ell,h}(t) : \mathbb{R}^+ \mapsto \mathbb{V}_h^\ell$ such that :

$$\begin{cases} \dfrac{d}{dt} m^c(\mathbf{u}_{c,h}, \mathbf{v}_{c,h}) + a_h^c(\mathbf{u}_{c,h}, \mathbf{v}_{c,h}) + b_h(\mathbf{u}_{f,h}, \mathbf{v}_{c,h}) = 0, & \forall\, \mathbf{v}_{c,h} \in \mathbb{V}_h^c, \\ \dfrac{d}{dt} m^f(\mathbf{u}_{f,h}, \mathbf{v}_{f,h}) + a_h^f(\mathbf{u}_{f,h}, \mathbf{v}_{f,h}) - b_h(\mathbf{v}_{f,h}, \mathbf{u}_{c,h}) = 0, & \forall\, \mathbf{v}_{f,h} \in \mathbb{V}_h^f, \end{cases} \tag{24}$$

where m^ℓ and a_h^ℓ are the following local bilinear forms on $\mathbb{V}_h^\ell \times \mathbb{V}_h^\ell$:

$$\left| \begin{array}{l} m^\ell(\mathbf{u}_h, \mathbf{v}_h) = \displaystyle\int_{\Omega_\ell} \mathbf{u}_K \cdot \mathbf{v}_K \, dx \quad \Big(\equiv \sum_{K \in \mathcal{T}_h^\ell} \int_K M(x)\, \mathbf{u}_K \cdot \mathbf{v}_K \, dx \Big), \\ \\ a_h^\ell(\mathbf{u}_h, \mathbf{v}_h) \;= \dfrac{1}{2} \displaystyle\sum_{K \in \mathcal{T}_h^\ell} \sum_{j=1}^{d} \int_K \Big[A_j\, \partial_{x_j} \mathbf{u}_K \cdot \mathbf{v}_K - A_j\, \mathbf{u}_K \cdot \partial_{x_j} \mathbf{v}_K \Big] \, dx \\ \qquad\qquad + \dfrac{1}{4} \displaystyle\sum_{(K,L) \in I_\ell} \int_{\Sigma_{KL}} \left[A(\mathbf{n}_K)\, \mathbf{u}_L \cdot \mathbf{v}_K - A(\mathbf{n}_K)\, \mathbf{v}_L \cdot \mathbf{u}_K \right] \, d\sigma, \end{array} \right. \tag{25}$$

and b_h is the interface bilinear form on $\mathbb{V}_h^f \times \mathbb{V}_h^c$:

$$b_h(\mathbf{u}_{f,h}, \mathbf{u}_{c,h}) = \frac{1}{4} \sum_{(K,L) \in I} \int_{\Sigma_{KL}} A(\mathbf{n}_k) \mathbf{u}_L^f \cdot \mathbf{u}_K^c \, d\sigma. \tag{26}$$

For the time discretization, our aim is to determine a numerical scheme which coincides inside each grid with the scheme (19) (up to the change of time step) and guarantees the conservation of an appropriate discrete energy. To be more precise, we need to give:

(i) the discrete instants at which the solution will be computed in each grid,

(ii) the choice of the discrete energy we wish to conserve.

The choice of (i) and (ii) is not unique. Another choice could result into another scheme. In this paper, we use a "staggered" grid in time and propose to compute the solution in Ω_c at times $t^{n+1/2} = (n+1/2)\Delta t$ and the solution in Ω_f at times $t^{n+3/4} = (n+3/4)\Delta t$ and $t^{n+1/4} = (n+1/4)\Delta t$. According to the leap-frog procedure, this leads us to propose the following scheme for the time interval $]t^{n+1/2}, t^{n+3/2}]$ (with obvious notation):

$$\begin{cases} m^c(\dfrac{\mathbf{u}_{c,h}^{n+\frac{3}{2}} - \mathbf{u}_{c,h}^{n-\frac{1}{2}}}{2\Delta t}, \mathbf{v}_{c,h}) + a_h^c(\mathbf{u}_{c,h}^{n+\frac{1}{2}}, \mathbf{v}_{c,h}) + b_h([\mathbf{u}_{f,h}]^{n+\frac{1}{2}}, \mathbf{v}_{c,h}) = 0, \\ m^f(\dfrac{\mathbf{u}_{f,h}^{n+\frac{5}{4}} - \mathbf{u}_{f,h}^{n+\frac{1}{4}}}{\Delta t}, \mathbf{v}_{f,h}) + a_h^f(\mathbf{u}_{f,h}^{n+\frac{3}{4}}, \mathbf{v}_{f,h}) - b_h(\mathbf{v}_{f,h}, [\mathbf{u}_{c,h}]^{n+\frac{3}{4}}) = 0, \\ m^f(\dfrac{\mathbf{u}_{f,h}^{n+\frac{3}{4}} - \mathbf{u}_{f,h}^{n-\frac{1}{4}}}{\Delta t}, \mathbf{v}_{f,h}) + a_h^f(\mathbf{u}_{f,h}^{n+\frac{1}{4}}, \mathbf{v}_{f,h}) - b_h(\mathbf{v}_{f,h}, [\mathbf{u}_{c,h}]^{n+\frac{1}{4}}) = 0, \\ \forall\, (\mathbf{v}_{c,h}, \mathbf{v}_{f,h}) \in \mathbb{V}_h^c \times \mathbb{V}_h^f. \end{cases} \tag{27}$$

Here, $[\mathbf{u}_{f,h}]^{n+1/2}$ (respectively $[\mathbf{u}_{c,h}]^{n+3/4}$ and $[\mathbf{u}_{f,h}]^{n+1/4}$) is an approximation of $\mathbf{u}_{f,h}$ (respectively $\mathbf{u}_{c,h}$) at time $t^{n+1/2}$ (respectively $t^{n+3/4}$ and $t^{n+1/4}$) to be determined. According to (20), it is natural to define as a discrete energy in Ω_c at time t^n, the quantity (setting $\|\mathbf{v}_c\|_c^2 = m^c(\mathbf{v}_c, \mathbf{v}_c)$)

$$E_{c,h}^n = \frac{1}{2}\left[\, \|\mathbf{u}_{c,h}^{n+1/2}\|_c^2 + \|\mathbf{u}_{c,h}^{n-1/2}\|_c^2 \,\right] + \Delta t\; a_h^c(\mathbf{u}_{c,h}^{n+1/2}, \mathbf{u}_{c,h}^{n-1/2})\;. \tag{28}$$

Analogously, the discrete energy in Ω_f can be defined at all instants $k\Delta t/2, k \in \mathbb{N}$, in particular at times t^n (with $\|\mathbf{v}_f\|_f^2 = m^f(\mathbf{v}_f, \mathbf{v}_f)$):

$$E_{f,h}^n = \frac{1}{2}\left[\, \|\mathbf{u}_{f,h}^{n+1/4}\|_f^2 + \|\mathbf{u}_{f,h}^{n-1/4}\|_f^2 \,\right] + \frac{\Delta t}{2}\; a_h^f(\mathbf{u}_{f,h}^{n+1/4}, \mathbf{u}_{f,h}^{n-1/4})\;. \tag{29}$$

Therefore, the total discrete energy can be defined at times t^n by

$$E_h^n = E_{c,h}^n + E_{f,h}^n \tag{30}$$

To obtain a discrete energy identity, we first choose

$$\mathbf{v}_{c,h} = \mathbf{u}_{c,h}^{n+3/2} + \mathbf{u}_{c,h}^{n-1/2}$$

in the first equation of (27) to compute $E_{c,h}^{n+1} - E_{c,h}^n$. Next, we take

$$\mathbf{v}_{f,h} = \mathbf{u}_{f,h}^{n+5/4} + \mathbf{u}_{f,h}^{n+1/4} \quad \text{and} \quad \mathbf{v}_{f,h} = \mathbf{u}_{f,h}^{n+3/4} + \mathbf{u}_{f,h}^{n-1/4}$$

respectively in the second and third equations of (27) and add the two resulting equalities to compute

$$E_{f,h}^{n+1} - E_{f,h}^n.$$

After summation, we get the following energy identity:

$$\left|\begin{array}{l} \dfrac{E_h^{n+1}-E_h^n}{\Delta t} = b_h\Big([\mathbf{u}_{f,h}]^{n+1/2}, \dfrac{\mathbf{u}_{c,h}^{n+3/2}+\mathbf{u}_{c,h}^{n-1/2}}{2}\Big) \\ \quad -\dfrac{1}{2}\Big[\, b_h\Big(\dfrac{\mathbf{u}_{f,h}^{n+5/4}+\mathbf{u}_{f,h}^{n+1/4}}{2}, [\mathbf{u}_{c,h}]^{n+3/4}\Big) + b_h\Big(\dfrac{\mathbf{u}_{f,h}^{n+3/4}+\mathbf{u}_{f,h}^{n-1/4}}{2}, [\mathbf{u}_{c,h}]^{n+1/4}\Big)\,\Big] \end{array}\right.$$

Thus, we have conservation of the energy, i. e. $E_h^{n+1} = E_h^n$, as soon as

$$\left|\begin{array}{l} [\mathbf{u}_{c,h}]^{n+3/4} = [\mathbf{u}_{c,h}]^{n+1/4} = \dfrac{\mathbf{u}_{c,h}^{n+3/2}+\mathbf{u}_{c,h}^{n-1/2}}{2}, \\ [\mathbf{u}_{f,h}]^{n+1/2} = \dfrac{\mathbf{u}_{f,h}^{n+5/4}+\mathbf{u}_{f,h}^{n+3/4}+\mathbf{u}_{f,h}^{n+1/4}+\mathbf{u}_{f,h}^{n-1/4}}{4}. \end{array}\right. \tag{31}$$

The scheme we propose is finally (27, 31). The energy conservation property guarantees of course the L^2 stability of this scheme under the CFL condition

$$\Delta t \, \max\Big(\, \|a_h^c\|, \frac{\|a_h^f\|}{2} \, \Big) \leq 1. \tag{32}$$

i. e. the same CFL condition as if the computations in the two grids were decoupled. After decomposition of the discrete unknowns on appropriate bases of the spaces $\mathbb{V}_h^c$ and $\mathbb{V}_h^f$ (constructed from bases of the local spaces $\mathbb{V}_K$), we end up with the following algebraic problem (with again obvious notation)

$$\left\{\begin{array}{l} M_h^c \dfrac{\mathbf{U}_c^{n+\frac{3}{2}}-\mathbf{U}_c^{n-\frac{1}{2}}}{2\Delta t} + A_h^c\,\mathbf{U}_c^{n+\frac{1}{2}} + B_h \dfrac{\mathbf{U}_f^{n+\frac{5}{4}}+\mathbf{U}_f^{n+\frac{3}{4}}+\mathbf{U}_f^{n+\frac{1}{4}}+\mathbf{U}_f^{n-\frac{1}{4}}}{4} = 0, \\ M_h^f \dfrac{\mathbf{U}_f^{n+\frac{5}{4}}-\mathbf{U}_f^{n+\frac{1}{4}}}{\Delta t} + A_h^f\,\mathbf{U}_f^{n+\frac{3}{4}} - B_h^t \dfrac{\mathbf{U}_c^{n+\frac{3}{2}}+\mathbf{U}_c^{n-\frac{1}{2}}}{2} = 0, \\ M_h^f \dfrac{\mathbf{U}_f^{n+\frac{3}{4}}-\mathbf{U}_f^{n-\frac{1}{4}}}{\Delta t} + A_h^f\,\mathbf{U}_f^{n+\frac{1}{4}} - B_h^t \dfrac{\mathbf{U}_c^{n+\frac{3}{2}}+\mathbf{U}_c^{n-\frac{1}{2}}}{2} = 0, \end{array}\right. \tag{33}$$

where M_h^c and M_h^f are the local mass matrices, A_h^c and A_h^f are the local stiffness matrices, and B_h is the coupling stiffness matrix (transmission matrix).

This scheme is of order two "inside" each grid, but at the interface, looking at equations (31), we see that, if the second equality is consistent in $O(\Delta t^2)$, the first one is only in $O(\Delta t)$. By adapting the analysis of [4] and [11], one can see that, in 1D, the L^2- global accuracy of the method is $O(\Delta t^{3/2})$ (this is also confirmed by numerical experiments in 2D). Following [13], we propose a post-processing to restore the second order accuracy, considering

$$\tilde{\mathbf{U}}_c^{n+\frac{1}{2}} = \frac{\mathbf{U}_c^{n+\frac{3}{2}}+\mathbf{U}_c^{n-\frac{1}{2}}}{2}, \quad \tilde{\mathbf{U}}_f^{k} = \frac{\mathbf{U}_f^{k+\frac{5}{4}}+\mathbf{U}_f^{k+\frac{3}{4}}+\mathbf{U}_f^{k+\frac{1}{4}}+\mathbf{U}_f^{k-\frac{1}{4}}}{4}\,, \tag{34}$$

with k equal to n or $n+1/2$. We can write directly the scheme satisfied by $\tilde{\mathbf{U}}_c^n$ and $\tilde{\mathbf{U}}_f^n$, which is of second order (all approximations are centered).

$$\begin{cases} M_h^c \, \dfrac{\tilde{\mathbf{U}}_c^{n+3/2} - \tilde{\mathbf{U}}_c^{n-1/2}}{2\Delta t} + A_h^c \, \tilde{\mathbf{U}}_c^{n+1/2} + B_h \dfrac{\tilde{\mathbf{U}}_f^{n+3/2} + \tilde{\mathbf{U}}_f^{n-1/2}}{2} = 0, \\ M_h^f \, \dfrac{\tilde{\mathbf{U}}_f^{n+3/2} - \tilde{\mathbf{U}}_f^{n+1/2}}{\Delta t} + A_h^f \, \tilde{\mathbf{U}}_f^{n+1} - B_h^t \dfrac{\tilde{\mathbf{U}}_c^{n+3/2} + \tilde{\mathbf{U}}_c^{n+1/2}}{2} = 0, \\ M_h^f \, \dfrac{\tilde{\mathbf{U}}_f^{n+1} - \tilde{\mathbf{U}}_f^{n}}{\Delta t} + A_h^f \, \tilde{\mathbf{U}}_f^{n+1/2} - B_h^t \dfrac{\tilde{\mathbf{U}}_c^{n+3/2} + 2\tilde{\mathbf{U}}_c^{n+1/2} + \tilde{\mathbf{U}}_c^{n-1/2}}{4} = 0, \end{cases} \tag{35}$$

This scheme is stable under condition (32) since it is equivalent to (33). However, the direct stability analysis of (35) would be far from trivial.

For computing the discrete unknowns in the interval $]t^{n+1/2}, t^{n+3/2}]$, one has to invert the (sparse) linear system:

$$\begin{bmatrix} M_h^c & \Delta t B_h & 0 \\ -\frac{\Delta t}{2} B_h^t & M_h^f & \Delta t A_h^f \\ -\frac{\Delta t}{4} B_h^t & 0 & M_h^f \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{U}}_c^{n+3/2} \\ \tilde{\mathbf{U}}_f^{n+3/2} \\ \tilde{\mathbf{U}}_f^{n+1} \end{bmatrix} = F^n \tag{36}$$

with F^n computed from previous instants. Since the matrix B_h only "sees" the unknowns attached to elements that touch the interface Σ, the scheme remains essentially explicit for interior unknowns. It is only implicit for the unknowns in the neighborhood of the interface.

Remark 4 *The invertibility of (36) is a consequence of the stability analysis. It can also be checked after elimination of $\tilde{\mathbf{U}}_f^{n+3/2}$ and $\tilde{\mathbf{U}}_f^{n+1}$, which leads to*

$$\mathcal{M}_h \, \tilde{\mathbf{U}}_c^{n+3/2} = \mathcal{F}, \tag{37}$$

where the symmetric matrix

$$\mathcal{M}_h = M_h^c + \frac{\Delta t^2}{2} \left[B_h (M_h^f)^{-1} B_h^t - \frac{\Delta t}{2} B_h (M_h^f)^{-1} A_h^f (M_h^f)^{-1} B_h^t \right]$$

is shown to be positive definite under the condition (32). Solving (37) can be used for the practical implementation. In particular, if, on the interface Σ, the mesh of Ω_f is a sub-mesh of the mesh of Ω_c, one can show that $\mathcal{M}_h$ is block diagonal element by element and the scheme is thus completely explicit.

5 Extensions of the method.

Variable coefficients, lower order perturbation and boundary conditions. The method we have presented in section 4 easily extends to the general problem

(1, 2). In fact, the major modification concerns the semi-discrete problem in space, i. e. we have simply to explain how (14) is modified. The discontinuous Galerkin formulation of (1) can be written

$$\begin{cases} \text{find } \mathbf{u}_h(t) : \mathbb{R}^+ \mapsto \mathbb{V}^h \text{ such that :} \\ \dfrac{d}{dt}\, m(\mathbf{u}_h, \mathbf{v}_h) + a_h(\mathbf{u}_h, \mathbf{v}_h) + \tilde{a}(\mathbf{u}_h, \mathbf{v}_h) = 0, \ \ \forall\, \mathbf{v}_h \in \mathbb{V}^h, \end{cases} \tag{38}$$

where the bilinear forms $m(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$ are still defined by (15), the only differences being that the matrices A_j and $A(n_K)$ (the reader will notice that this matrix remains well defined thanks to the regularity assumption (4)) vary in space and that we consider a mesh of the domain Ω. The additional bilinear form $\tilde{a}(\cdot,\cdot)$ is given by

$$\tilde{a}(\mathbf{u}_h, \mathbf{v}_h) = \int_\Omega (C - \frac{1}{2}\,\mathbf{div} A)\mathbf{u}_h \cdot \mathbf{v}_h \, dx + \int_\Gamma N\mathbf{u}_h \cdot \mathbf{v}_h \, d\sigma. \tag{39}$$

and corresponds to a block diagonal matrix (it does not contain any interface term). All what concerns the time discretization remains essentially unchanged except that, in order to preserve the stability condition, the term $\tilde{a}(\mathbf{u}_h, \mathbf{v}_h)$ should be approximated by the classical Crank-Nicolson procedure. This does not perturb the explicit nature of the scheme thanks to the properties of $\tilde{a}(\cdot,\cdot)$.

A more general time-step ratio. We keep here the notation of section 4. We assume that the time steps Δt_c and Δt_f have a common multiple Δt:

$$\Delta t_c = \Delta t / q_c, \quad \Delta t_f = \Delta t / q_f, \tag{40}$$

where q_c and $q_f > q_c$ are two relative prime integers (in section 4, we considered $q_c = 1$, q_f=2). The solution in Ω_ℓ in the interval $]t^{n+1/2}, t^{n+3/2}]$ is computed at q_ℓ equally distributed instants, distant from Δt_l:

$$t^{n+\frac{2k+3}{2q_\ell}} = (n + \frac{2k+3}{2q_\ell})\ \Delta t, \quad k = 0, \cdots, q_\ell - 1\ .$$

We present the scheme directly in algebraic form ($B_h^c = B_h$ and $B_h^f = -B_h^t$):

$$\begin{cases} M_h^\ell \dfrac{\mathbf{U}_\ell^{n+\frac{2k+3}{2q_\ell}} - \mathbf{U}_\ell^{n+\frac{2k-1}{2q_\ell}}}{2\Delta t_\ell} + A_h^\ell \mathbf{U}_\ell^{n+\frac{2k+1}{2q_\ell}} + B_h^\ell [\mathbf{U}_m]^{n+\frac{2k+1}{2q_\ell}} = 0, \\ \\ \ell \in \{c, f\}, \quad m \in \{c, f\} \setminus \{\ell\}, \quad k = 0, \cdots, q_\ell - 1. \end{cases} \tag{41}$$

$$\text{with} \qquad [\mathbf{U}_m]^{n+\frac{2k+1}{2q_\ell}} = \sum_{j=0}^{q_m - 1} \frac{\mathbf{U}_m^{n+\frac{2j+3}{2q_m}} + \mathbf{U}_m^{n+\frac{2j-1}{2q_m}}}{2q_m}\ , \tag{42}$$

a choice which is shown to imply the conservation of the discrete energy:

$$E_h^n = E_{c,h}^n + E_{f,h}^n, \tag{43}$$

$$E^n_{\ell,h} = \frac{1}{2}\left[\|\mathbf{u}_{\ell,h}^{n+\frac{1}{2q_\ell}}\|^2_\ell + \|\mathbf{u}_{\ell,h}^{n-\frac{1}{2q_\ell}}\|^2_\ell \right] + \Delta t_\ell \; a^\ell_h(\mathbf{u}_{\ell,h}^{n+\frac{1}{2q_\ell}}, \mathbf{u}_{\ell,h}^{n-\frac{1}{2q_\ell}}). \qquad (44)$$

which implies the L^2 stability of (41, 42) under the condition

$$\Delta t \;\; \max\Big(\frac{\|a^c_h\|}{q_c}, \frac{\|a^f_h\|}{q_f} \Big) \leq 1. \qquad (45)$$

As in simple case, a post-processing of (41) is necessary to have a scheme of order two everywhere, considering for all $k = 0, \cdots, q_\ell - 1$

$$\tilde{\mathbf{U}}_\ell^{n+\frac{1}{2}+\frac{k}{q_\ell}} = \sum_{j=0}^{q_\ell-1} \frac{\mathbf{U}_\ell^{n+\frac{2k+2j+3}{2q_\ell}} + \mathbf{U}_\ell^{n+\frac{2k+2j-1}{2q_\ell}}}{2q_\ell}, \; \ell \in \{c, f\} \; . \qquad (46)$$

We can then write the scheme satisfied by $\tilde{\mathbf{U}}_\ell$ as:

$$M^\ell_h \frac{\tilde{\mathbf{U}}_\ell^{n+\frac{1}{2}+\frac{k}{q_\ell}} - \tilde{\mathbf{U}}_\ell^{n+\frac{1}{2}+\frac{k-2}{q_\ell}}}{2\Delta t_\ell} + A^\ell_h \tilde{\mathbf{U}}_\ell^{n+\frac{1}{2}+\frac{k-1}{q_\ell}} + B^\ell_h [\tilde{\mathbf{U}}_m]^{n+\frac{1}{2}+\frac{k-1}{q_\ell}} = 0 \; , \qquad (47)$$

which is clearly of second order in time under the following equality:

$$[\tilde{\mathbf{U}}_m]^{n+\frac{1}{2}+\frac{k-1}{q_\ell}} = \begin{cases} \dfrac{k\,\tilde{\mathbf{U}}_m^{n+3/2} + (2q_\ell - 2)\,\tilde{U}_m^{n+1/2} + (2-k)\,\tilde{U}_m^{n-1/2}}{2q_\ell}, & \text{if } 0 \leq k \leq 2, \\ \dfrac{(2k-2)\,\tilde{U}_m^{n+3/2} + (2q_\ell - 2k + 2)\,\tilde{\mathbf{U}}_m^{n+1/2}}{2q_\ell}, & \text{if } k \geq 2. \end{cases}$$

6 Numerical results

Numerical 1D test experiments permit to check the convergence and second order accuracy of our method.

We have chosen to represent here the results if an experiment of "artificial" mesh refinement in 2D. The goal of the experiment is to check that no numerical artefact is produced by the change of mesh. We consider a waveguide geometry. The computational domain $\Omega = \Omega_c \cup \Omega_f$, with $\Omega_c = [0, 20] \times [0, 10]$ and $\Omega_f = [20, 25] \times [0, 10]$, we take the space step $h_c = 0.05$ and Δt_c satisfies the stability condition (32) (resp. $h_f = h_c/2$ and $\Delta t_f = \Delta t_c/2$) in the coarse mesh (resp. in the fine mesh) which correspond to a mesh refinement $(1, 2)$. We consider the linearized Euler equations (8) with the initial conditions $\mathbf{v}(x, 0) = 0$; $p(x, 0) = \exp(-\log(2.)r/100)$, $r = ((x_1 - 10)^2 + (x_2 - 5)^2)^{1/2}$ and the boundary condition $\mathbf{v} \cdot \mathbf{n} = 0$ on Γ. In the figure 1 (resp. figure 2) we present the snapshots of the pressure p for wave equation; $\mathbf{M} = 0$ in (8) (resp. for aeroacoustic application with $\mathbf{M} = (0.7, 0)$). For the both simulations, we can observe that there is no artificial reflected or transmitted wave at the interface Σ between the the domains Ω_c and Ω_f.

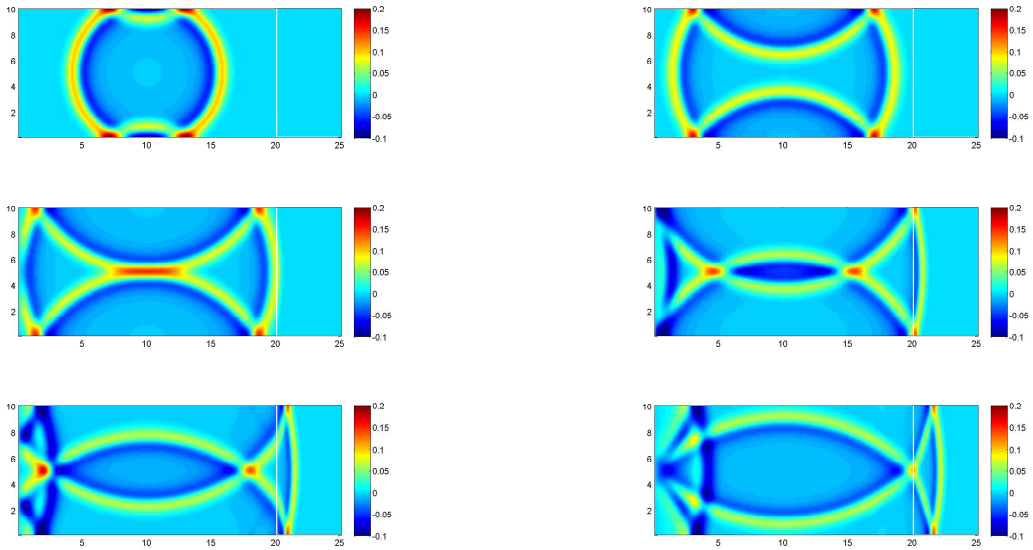

Fig. 1. Numerical simulations for the wave equation

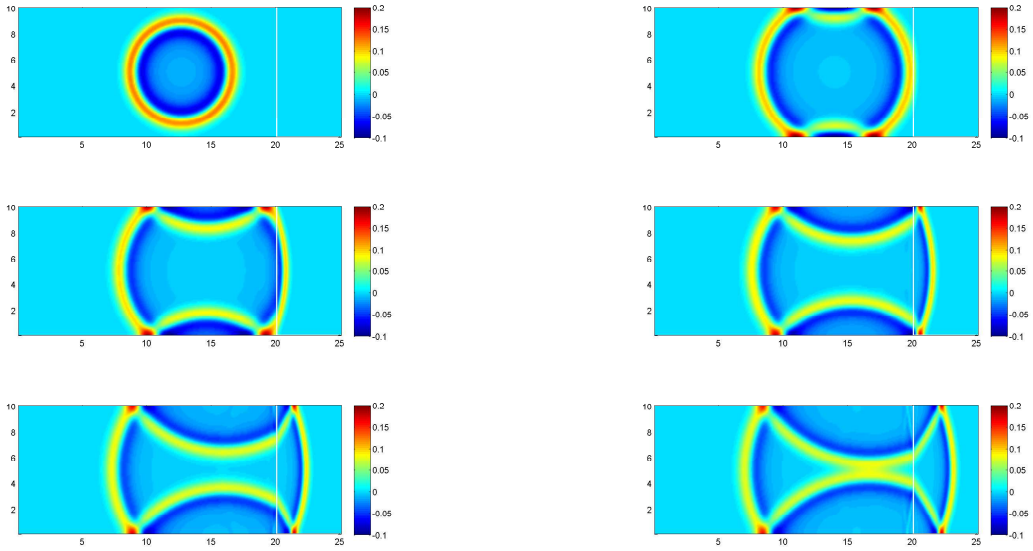

Fig. 2. Numerical simulations for the linearized Euler equations

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