Stochastic spectral formulations for elliptic problems

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Summary. We describe new stochastic spectral formulations with very good properties in terms of conditioning. These formulations are built by combining Monte Carlo approximations of the Feynman-Kac formula and standard deterministic approximations on basis functions. We give error bounds on the solutions obtained using these formulations in the case of linear approximations. Some numerical tests are made on an anisotropic diffusion equation using a tensor product Tchebychef polynomial basis and one random point schemes quantified or not.

1 Introduction

The Feynman-Kac formula is a very powerful tool to achieve stochastic representations of the pointwise solution of numerous partial differential equations like diffusion or transport equations [5, 11]. If we consider for example the Dirichlet boundary value problem in a domain $D \subset \mathbb{R}^d$ with a sufficiently smooth boundary ∂D

$$\begin{cases} Lu = -f & x \in D \\ u = g & x \in \partial D \end{cases} \tag{1}$$

We have the classical representation of the solution: $\forall x \in D$

$$u(x) = \mathbb{E}_x \left[g(X_{\tau_D}) + \int_0^{\tau_D} f(X_s) ds \right], \tag{2}$$

where X_t is a stochastic process solution of the stochastic differential equation relative to the operator L and where τ_D is the exit time of this process from

the domain D. Gobet and Maire have introduced in [7] sequential Monte Carlo algorithms to compute global approximations of the solutions combining this formula and deterministic linear approximations. This has led to a geometric reduction up to threshold of both the bias and the variance involved in the Monte Carlo computation of the Feynman-Kac representations [8]. In order to improve the speed of convergence of the sequential algorithms, we have described new schemes for the evaluation of the source terms based on the one random point method and quantization techniques [14]. In the case of the Poisson equation, we have made a new interpretation of the algorithm which has led to a direct spectral formulation with almost perfect properties in terms of conditioning. Our goal is to show that this formulation can be extended in the case of a general elliptic operator and to give some error bounds on the solution if the approximations of this solution are linear. In Section 4, we give some numerical results on an anisotropic diffusion over a square domain using an approximation based on tensor product Tchebychef polynomial interpolation and either Monte Carlo simulations or quantization tools.

2 The stochastic spectral formulation

2.1 One random step schemes

The goal of this section is to remind the tools introduced in [14] to compute Feynman-Kac representations at a numerical cost which is similar for the boundary and source terms. We also assume that f and g are bounded. Representation (2) is computed using a Monte Carlo method which requires the simulation of the process X_t using an approximation scheme like the Euler scheme [2] with a time step Δt. If there are n steps until absorption, the approximation of $g(X_{\tau_D})$ is computed using a projection of $X_{n\Delta t}$ on the boundary ∂D. The standard approximation of $\int_0^{\tau_D} f(X_s^x)ds$ by the rectangle method is $\Delta t \sum_{i=1}^{n} f(X_{i\Delta t})$. For each simulated trajectory, we can see that many evaluations of the function f are required and only one of the function g. Thanks to the representation

$$\mathbb{E}_x\left[\int_0^{\tau_D} f(X_s)ds\right] = \mathbb{E}_x\left[\int_0^1 \tau_D f(X_{y\tau_D})dy\right] = \mathbb{E}_x\left[\tau_D f(X_{U\tau_D})\right]$$

introduced in [14], we can rewrite the Feynman-Kac formula as

$$u(x) = \mathbb{E}_x\left[g(X_{\tau_D}) + \tau_D f(X_{U\tau_D})\right]$$

where U is a random variable with uniform law on $[0,1]$. We replace the standard approximation by $n\Delta t f(X_{J\Delta t})$ where J is a discrete uniform random variable on the set $1..n$. This new estimator uses now only one evaluation of f and we have showed in [14] that in most situations the increase of its variance

is compensated by the decay of its computational cost. This is especially true when Δt is small, x is away from the boundary and the evaluation of f is costly.

When the operator L is $\frac{1}{2}\Delta$, the stochastic process to simulate is the Brownian motion B_t for which different methods of simulation are avalaible. The Euler scheme with discretization parameter Δt writes

$$B_0 = x,\ B_{n+1} = B_n + \sqrt{\Delta t} Y_n$$

where the Y_n are independent standard Gaussian random variables. The crude version makes the simulation stops once $B_{n+1} \in D^C$. This leads to approximations that are of weak order $\sqrt{\Delta t}$. It is possible to take into account the possibility for the Brownian motion to leave the domain between step n and $n+1$ and be back into it at time $(n+1)\Delta t$ to obtain a scheme of weak order Δt using the half-space approximation [6]. Some faster schemes can be used like the walk on rectangles [4] or the walk on spheres method [17].

We have developed in [14] a one random point version of the walk on spheres method using the modified walk on spheres method introduced in [10]. This method has been tested in [14] and appeared as the most efficient in all the examples we have tried. The one random point method has been also used successfully in [16] for the exact simulation of prices and hedges in the financial mathematic context.

2.2 Quantization

In some situations like spectral methods [3] or in the sequential Monte Carlo methods developed in earlier works [8, 9], the points where the solution is computed are fixed. We shall describe what can be done in the case of a diffusion equation in a general bounded domain D in dimension two. For a fixed point (x, y), we can already use the N Monte Carlo simulations to build a quadrature formula

$$u(x,y) \simeq \sum_{i=1}^{N} \frac{1}{N} g(x_i^b, y_i^b) + \frac{\tau_D^{(i)}}{N} f(x_i^s, y_i^s)$$

at some random points (x_i^b, y_i^b) of the boundary and (x_i^s, y_j^s) of the interior of the domain. In order to increase the rate of convergence of this kind of formula, we can furthermore optimize the locations of the points of evaluations of both f and g by using quantization techniques [15]. For the quantization of p points of the boundary term

$$\mathbb{E}_{x,y}(g(X_{\tau_D})) = \int_{\partial D} g(s) w_{x,y}^b(s) ds,$$

where $w_{x,y}^b(s)$ is the the law of the exit position of the stochastic process X_s starting at the point (x, y), we only need to define a distance on ∂D which is

just the geodesic distance on this set. For the source term, the quantization problem consists in the minimization of

$$J_{x,y}(p) = \min(\int_0^\infty \int_D \inf_{1\leq i\leq M} d^2(z-z_i) w^s_{x,y}(z)dz : \{z_1, z_2 ... z_p \in D\times]0,\infty[\})$$

where $w^s_{x,y}(z)$ is the joint law of $(\tau_D^{x,y}, X_{U_{\tau_D^{x,y}}})$ and $d^2(z-z_i) = (x-x_i)^2 + (y-y_i)^2 + (t-t_i)^2$. After convergence, we obtain an approximation of the solution at a given point (x,y)

$$u(x,y) \simeq \sum_{i=1}^q (\int_{C_i} w^b_{x,y}(s)ds) g(x_i,y_i) + \sum_{i=1}^p (\int_{C_i} w^s_{x,y}(z)dz) t_i f(x_i,y_j)$$

which can be written as a quadrature formula of the form

$$u(x,y) \simeq \sum_{i=1}^q a_i g(x_i^b, y_i^b) + \sum_{j=1}^p b_j f(x_j^s, y_j^s).$$

2.3 Formulation and asymptotic properties

We want to compute a global approximation of the solution u and we assume that it can be written in a linear form

$$P_N u(x) = \sum_{i=1}^N u(x_i)\Psi_i(x)$$

for some functions $\Psi_i(x)$ that are at least twice continuously differentiable and some points $x_i \in D$. We also assume that for every point x_i, we can approximate $u(x_i)$ via for instance a numerical approximation of the Feynman-Kac formula by

$$\widetilde{u}(x_i) = \sum_{k=1}^q a_{i,k} g(z^b_{i,k,\Delta}) + \sum_{l=1}^p b_{i,l} f(z^s_{i,l,\Delta})$$

where this approximation is such that

$$\lim_{p,q\to\infty,\Delta\to 0} \widetilde{u}(x_i) = u(x_i).$$

The points $z^b_{i,k,\Delta}$ are located on the boundary ∂D and the points $z^s_{i,l,\Delta}$ in D. We now let $r_N(x) = u(x) - P_N u(x)$ and write the partial differential equation solved by $r_N(x)$. We have

$$Lr_N = Lu - LP_N u = f - LP_N u$$

in D with boundary conditions

$$r_N = g - P_N u.$$

We have

$$r_N(x) = \mathbb{E}_x[(g - P_N u)(X_{\tau_D}) + \tau_D(f - LP_N u)(X_{U\tau_D})]$$

and hence the approximation

$$r_N(x_i) = \sum_{k=1}^{q} a_{i,k}(g(z^b_{i,k,\Delta}) - P_N u(z^b_{i,k,\Delta})) + \sum_{l=1}^{p} b_{i,l}(f(z^s_{i,l,\Delta}) - LP_N u(z^s_{i,l,\Delta}))$$

which leads to the linear system $Cu = d$ with

$$c_{i,i} = \sum_{k=1}^{q} a_{i,k}\Psi_i(z^b_{i,k,\Delta}) + \sum_{l=1}^{p} b_{i,l}L\Psi_i(z^s_{i,l,\Delta}) + 1 - \Psi_i(x_i)$$

and for $i \neq j$,

$$c_{i,j} = \sum_{k=1}^{q} a_{i,k}\Psi_j(z^b_{i,k,\Delta}) + \sum_{l=1}^{p} b_{i,l}L\Psi_j(z^s_{i,l,\Delta}) - \Psi_j(x_i)$$

and

$$d_i = \sum_{k=1}^{q} a_{i,k}g(z^b_{i,k,\Delta}) + \sum_{l=1}^{p} b_{i,l}f(z^s_{i,l,\Delta}).$$

As we have done in [14], we can look at the asymptotic system we obtain when $p, q \to \infty$ and $\Delta \to 0$. The term

$$\sum_{k=1}^{q} a_{i,k}\Psi_j(z^b_{i,k,\Delta}) + \sum_{l=1}^{p} b_{i,l}L\Psi_j(z^s_{i,l,\Delta})$$

is the approximation at point x_i of the solution of the equation

$$Lu = L\Psi_j$$

with boundary conditions

$$u = \Psi_j$$

on ∂D that is $\Psi_j(x_i)$. We deduce immediately that the matrix of the asymptotic system converges toward the identity matrix of size N. Our goal is now to give, for fixed values of p, q and Δ, a bound on the error we get using this stochastic spectral formulation.

3 Error bounds on the solution

3.1 The unbiased Monte Carlo case

We first consider that we use Monte Carlo estimators with exact simulations schemes (which is equivalent to take the parameter $\Delta = 0$) for the two terms of the Feynman-Kac representations. The Monte Carlo estimator of

$$\mathbb{E}_{x_i}\left[(g - P_N u)(X_{\tau_D})\right] + \mathbb{E}_{x_i}\left[\tau_D (f - LP_N u)(X_{U\tau_D})\right] - u(x_i) + P_N u(x_i)$$

using q_i independent sample values for the first term and p_i independent sample values for the second term writes

$$\overline{Y_i^{(b)}} + \overline{Y_i^{(s)}} - u(x_i) + P_N u(x_i)$$

where

$$\overline{Y_i^{(b)}} = \frac{1}{q_i}\sum_{k=1}^{q_i}(g(z_{i,k}^b) - P_N u(z_{i,k}^b)),\ \overline{Y_i^{(s)}} = \frac{1}{p_i}\sum_{l=1}^{p_i}\tau_D^{(i,l)}(f(z_{i,l}^s) - LP_N u(z_{i,l}^s)),$$

the random points $z_{i,k}^b$ are located on the boundary ∂D and the random points $z_{i,l}^s$ in D. We are exactly in the situation of the previous linear system $Cu = d$ which solution is, if C is non-singular, $u_i = u(x_i) = C^{-1}d$ by letting $a_{i,k} = \frac{1}{q_i}$ and $b_{i,l} = \frac{1}{p_i}\tau_D^{(i,l)}$. Indeed, the law of large numbers shows that the random matrix C converges to the identity matrix. To give a confidence interval for the solution based on the central limit theorem, we give confidence intervals for each of the equations of the linear system. To do this for the equation relative to index i, we define the confidence interval

$$A_i = [\overline{Y_i^{(b)}} + \overline{Y_i^{(s)}} - r_N(x_i) - C_\alpha(\frac{\sigma_i^{(b)}}{\sqrt{q_i}} + \frac{\sigma_i^{(s)}}{\sqrt{p_i}}),\ \overline{Y_i^{(b)}} + \overline{Y_i^{(s)}} - r_N(x_i) + C_\alpha(\frac{\sigma_i^{(b)}}{\sqrt{q_i}} + \frac{\sigma_i^{(s)}}{\sqrt{p_i}})]$$

with $\mathbb{P}(A_i) \geq (1 - \alpha - \frac{\beta_i}{\sqrt{q_i}})(1 - \alpha - \frac{\gamma_i}{\sqrt{p_i}})$, where

$$(\sigma_i^{(b)})^2 = Var((g - P_N u)(X_{\tau_D}^{x_i})),\ (\sigma_i^{(s)})^2 = Var(\tau_D (f - LP_N u)(X_{U\tau_D}^{x_i})),$$

C_α correspond to the level of confidence α in the Gaussian case and

$$\beta_i = \frac{0.7655\mathbb{E}(\left|(g - P_N u)(X_{\tau_D}^{x_i})\right|^3)}{(\sigma_i^{(b)})^{\frac{3}{2}}},\ \gamma_i = \frac{0.7655\mathbb{E}(\left|(f - LP_N u)(X_{U\tau_D}^{x_i})\right|^3)}{(\sigma_i^{(s)})^{\frac{3}{2}}}.$$

The constants β_i and γ_i are obtained thanks to the Berry-Esseen inequality [18] which holds true even if p_i and q_i are small. This inequality requires the existence of third moments for $(g - P_N u)(X_{\tau_D}^{x_i})$ and $\tau_D (f - LP_N u)(X_{U\tau_D}^{x_i})$ which is verified since $g - P_N u$, $f - LP_N u$ are bounded and $\mathbb{E}(\tau_D^3) < \infty$. Note

that for sake of simplicity, we have included in the sets A_i the contribution of both source and boundary terms. We should mention that there is no guaranty that the constants β_i and γ_i are for instance lower than one. So if p_i and q_i are very small, our bounds may not be meaningful. Hence we obtain with probability

$$\mathbb{P}(A_1 A_2 ... A_n) \geq \prod_{i=1}^{N} (1 - \alpha - \frac{\beta_i}{\sqrt{q_i}})(1 - \alpha - \frac{\gamma_i}{\sqrt{p_i}}) = p_r$$

a system of inequalities

$$d - \delta d \leq Cv \leq d + \delta d$$

where C and d are defined as previously and where

$$\delta d_i = C_\alpha (\frac{\sigma_i^{(b)}}{\sqrt{q_i}} + \frac{\sigma_i^{(s)}}{\sqrt{p_i}}).$$

If C is non-singular, we have

$$\|v - u\| \leq \left\|C^{-1}\right\| \|Cv - Cu\| \leq \left\|C^{-1}\right\| \|\delta d\|$$

with probability at least p_r for any matrix norm. Furthermore

$$(\delta d_i)^2 \leq 2C_\alpha^2 (\frac{\mathbb{E}_{x_i}[(g - P_N u)^2(X_{\tau_D})]}{q_i} + \frac{\mathbb{E}_{x_i}[\tau_D^2 (f - LP_N u)^2(X_{U\tau_D})]}{p_i})$$

which gives

$$(\delta d_i)^2 \leq 2C_\alpha^2 (\frac{\sup_{x\in\partial D}(g - P_N u)^2(x)}{q_i} + \frac{\mathbb{E}_{x_i}[\tau_D^2] \sup_{x\in D}(f - LP_N u)^2(x)]}{p_i})$$

and

$$\|\delta d\|_2^2 \leq 2C_\alpha^2 (\frac{N \sup_{x\in\partial D}(g - P_N u)^2(x)}{q_i} + \frac{\sum_{i=1}^{N} \mathbb{E}_{x_i}[\tau_D^2] \sup(f - LP_N u)^2(x)]}{p_i}).$$

From a practical point of view, it can be efficient to choose the values p_i and q_i to be different and adapted to the variances of the source and boundary terms. For the theoretical study, we assume now that $p_i = q_i = M$, and we have furthermore

$$\|\delta d\|_2^2 \leq \frac{2NC_\alpha^2}{M} [\sup_{x\in\partial D} (g - P_N u)^2(x) + \max_{i=1,N} \mathbb{E}_{x_i}[\tau_D^2] \sup_{x\in D} (f - LP_N u)^2(x)].$$

We have finally with probability p_r,

$$\|v - u\|_2 \leq \left\|C^{-1}\right\|_2 \|\delta d\|_2 .$$

We can now study when we can ensure that C is regular and if so to find a bound on $\left\|C^{-1}\right\|_2$. We write $C = Id - F$ with

$$F_{i,j} = -\sum_{k=1}^{q_i} a_{i,k}\Psi_j(z^b_{i,k,\Delta}) - \sum_{l=1}^{p_i} b_{i,l}L\Psi_j(z^s_{i,l,\Delta}) + \Psi_j(x_i)$$

and $\forall i, j$ we obviously have $E[F_{i,j}] = 0$. We choose $0 < \beta < 1$ and we take M large enough such that $P(\|F\|_1 \leq \beta)$ with probability p_r. We have with the same probability

$$\left\|C^{-1}\right\|_1 \leq \frac{1}{1-\beta}$$

and finally

$$\|v - u\|_2 \leq \frac{\sqrt{N}}{1-\beta} \|\delta d\|_2 .$$

3.2 A basic one dimensional example

The goal of this section is to explain on a trivial example what mean the error bounds we have just obtained. We consider the Laplace equation $u'' = 0$ on the interval $[0, 1]$ with boundary conditions $u(0) = 0$, $u(1) = 1$ which solution is $u(x) = x$. The solution is computed at points $\frac{1}{3}$ and $\frac{2}{3}$ using an exact simulation via Bernoulli random variables W and Z such that $P(W = 0) = \frac{2}{3}$ and $P(Z = 0) = \frac{1}{3}$. We denote by p and q the Monte Carlo approximations of these probabilities using N samples. We choose for basis functions the Lagrange polynomials $\Psi_1(x) = -3(x - \frac{2}{3})$ and $\Psi_2(x) = 3(x - \frac{1}{3})$. The exact solution is in the approximation space so if the spectral matrix is regular, there is no error on the solution. As we have $\Psi_1(0) = 2, \Psi_1(1) = -1$ and $\Psi_2(0) = -1, \Psi_2(1) = 2$, the linear system to solve is $Cu = d$ with $C = \begin{pmatrix} 3p-1 & 2-3p \\ 3q-1 & 2-3q \end{pmatrix}$ and $d = \begin{pmatrix} 1-p \\ 1-q \end{pmatrix}$. The vector $(\frac{1}{3}, \frac{2}{3})$ is always solution of this system but is the unique solution only if C is regular that is when $det(C) = 3(p-q) \neq 0$. When N increases, $p \to \frac{2}{3}$ and $q \to \frac{1}{3}$ at a Monte Carlo speed. So the probability that $p = q$ decreases quickly with N. Moreover, even if $N = 1$, the solution is unique as soon as $W \neq Z$. The probability the matrix is singular is

$$p_N = \mathbb{P}_N(W = Z) = \sum_{k=0}^{N} C_N^k C_N^k (\frac{1}{3})^{2k} (\frac{2}{3})^{2N-2k} = (\frac{2}{9})^N \sum_{k=0}^{N} (C_N^k)^2$$

For instance, we have $p_1 = \frac{4}{9}$, $p_{10} \simeq 0,054$, $p_{20} \simeq 0.01$ and $p_{50} \simeq 0.0002$. We can conclude that we obtain an exact solution with a probability $p_N \geq 0.99$ as soon as $N \geq 20$.

3.3 The biased Monte Carlo case

We now assume that the process X_s is approximated by another process X_s^Δ built using a simulation scheme like the Euler scheme or the walk on spheres method. We also assume that $p_i = q_i = M$. In this situation we have to compute error bounds for expressions of the form

$$\mathbb{E}_x[g(X_{\tau_D}) + \tau_D f(X_{U\tau_D})] - \mathbb{E}_x[g(X^\Delta_{\tau_D}) + \tau^\Delta_D f(X^\Delta_{U\tau_D})] = e_1 + e_2$$

letting

$$e_1 = \mathbb{E}_x[g(X_{\tau_D}) - g(X^\Delta_{\tau_D})],\ e_2 = \mathbb{E}_x[\tau_D f(X_{U\tau_D}) - \tau^\Delta_D f(X^\Delta_{U\tau_D})].$$

First, we can notice that for any process X_s^Δ, we always have

$$|e_1| \leq 2 \sup_{x\in\partial D} |g(x)|$$

and if $\mathbb{E}_x[\tau_D^\Delta] < \infty$,

$$|e_2| \leq (\mathbb{E}_x[\tau_D] + \mathbb{E}_x[\tau_D^\Delta]) \sup_{x\in D} |f(x)|\,.$$

If we now really use that X_s^Δ is an approximation of X_s,we can expect to have error bounds of the form

$$|e_1| \leq C\Delta^\alpha \sup_{x\in\partial D} |g(x)|$$

and also

$$|e_2| \leq C_1\Delta^\beta \sup_{x\in D} |f(x)|$$

where α, β, C and C_1 are positive constants. In both cases, we have

$$|e_1 + e_2| \leq \mu_\Delta \sup_{x\in\partial D} |g(x)| + \nu_\Delta \sup_{x\in D} |f(x)|$$

where μ_Δ and ν_Δ are positive constants which may or may not go to zero as $\Delta \to 0$. If we go back to our problem, we obtain a new system of inequalities

$$d_\Delta - \delta d_\Delta \leq C_\Delta v_\Delta \leq d_\Delta + \delta d_\Delta$$

where C_Δ and d_Δ are defined using X_s^Δ and where

$$\|\delta d_\Delta\|_2^2 \leq \theta(\Delta, N, M) \sup_{x\in\partial D} (g - P_N u)^2(x) + \kappa(\Delta, N, M) \sup_{x\in D} (f - LP_N u)^2(x)$$

with

$$\theta(\Delta, N, M) = 4C_\alpha^2(\frac{N}{M} + \mu_\Delta),\ \kappa(\Delta, N, M) = 4C_\alpha^2(\frac{N}{M} \max_{i=1,N} \mathbb{E}_{x_i}[(\tau_D^\Delta)^2] + \nu_\Delta).$$

We have finally with probability p_r,

$$\|v_\Delta - u\|_2 \leq \left\|C_\Delta^{-1}\right\|_2 \|\delta d_\Delta\|_2 .$$

We observe that in fact the quality of the simulation scheme and the number of simulations have not such a big impact on $\|\delta d_\Delta\|_2$ as they influence only the constants $\theta(\Delta, N, M)$ and $\kappa(\Delta, N, M)$. This means that we can have a good enough control on $\|\delta d_\Delta\|_2$ even with a very bad simulation scheme and few simulations. On the contrary, we cannot expect a good convergence of C_Δ towards the identity matrix in this last situation so there is a lack of control on $\left\|C_\Delta^{-1}\right\|_2$. This can lead to very large values for $\left\|C_\Delta^{-1}\right\|_2$ when using polynomial bases of high degree in the approximation of f and g. This is similar to the bad conditioning of spectral methods for elliptic problems. We could also find an upper bound for $\left\|C_\Delta^{-1}\right\|_2$ as we did in the unbiased case when M is large enough and Δ is small enough.

3.4 Other cases

Instead of making Monte Carlo approximations of the Feynman-Kac representations, it might be possible to use other approximation methods like quasi-Monte Carlo methods or quantization which may have increased rates of convergence. In such cases, the error bounds are no more relying on the central limit theorem via the variance but on other estimates via the discrepancy or the distortion. Some work has been done to simulate diffusions at a quasi-Monte Carlo speed first for the heat equation in $\mathbb{R}^d$ see [12] and then for elliptic problems in bounded domains in the context of domain decomposition [1]. This last approach is very promising but we do not know yet how to combine it with the one random point approximation of the source term. The remaining problem is how to compute a quasi-Monte Carlo approximation of $\mathbb{E}_x[\tau_D f(X_{U\tau_D})]$. In Section 2.2, we have described how the quantization method works in our context and especially how to deal with the source term evaluated by the one random point method. In both situations, error bounds in the case of zero-bias schemes will be deterministic and the speed of convergence is likely to be faster than the Monte Carlo speed.

4 Numerical results

We describe our method on equation (1) in the unit square $D = [-1, 1]^2$ where

$$L = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + 4\frac{\partial^2}{\partial x \partial y} + 5\frac{\partial^2}{\partial y^2}\right).$$

We solve this equation for three different source terms f_i and boundary conditions g_i for which the exact solutions are $u_1(x, y) = (1 - x^2)(1 - y^2)$,

$u_2(x,y) = (1-x^3)(1-y^3)$ and $u_3(x,y) = \sin\big((1-x^2)(1-2y^2)\big)$. For example, $f_1(x,y) = 6 - 5x^2 - y^2 + 8xy$ and $g_1(x,y) = 0$.

The stochastic process associated to L is solution to the SDE

$$\begin{cases} dX_t = dB_t^1 \\ dY_t = 2dB_t^1 + dB_t^2. \end{cases}$$

In all the following results, we use an Euler scheme with time step Δ and the half-space approximation [6]. For either Monte Carlo simulations or quantization and for each of the grid points, we take the same number of points M on the boundary and in the domain that is $p_i = q_i = M$. We approximate our solution using Tchebychef interpolation polynomials. Hence the N basis functions are the 2D Lagrange polynomials φ_i associated to the Tchebychef grid. We use either Monte Carlo simulations or quantization tools.

We give some criteria to study our method on these examples. These criteria are the error on the solution $err(i) = \sup|u_i - \hat{u}_i|$ (where the supremum is taken over the points of the Tchebychef grid), the condition number $\kappa(C)$ and the spectral radius of the Jacobi $\rho(J)$ and Gauss-Seidel $\rho(GS)$ iteration matrices. We summarize the results in Tables 1 and 2.

N	err_1	err_2	err_3	$\kappa(C)$	$\rho(J)$	$\rho(GS)$
9	3.9×10^{-16}	3.8×10^{-2}	2.7×10^{-2}	1.1	4.9×10^{-3}	4.4×10^{-2}
16	1.5×10^{-15}	3.8×10^{-15}	1.3×10^{-2}	1.4	1.2×10^{-1}	1.6×10^{-2}
121	8.5×10^{-14}	1.1×10^{-13}	1.4×10^{-4}	974	1.74	3.39

Table 1. Numerical Results for the Monte Carlo procedure with 1000 realisations and a time step $\Delta = 10^{-3}$

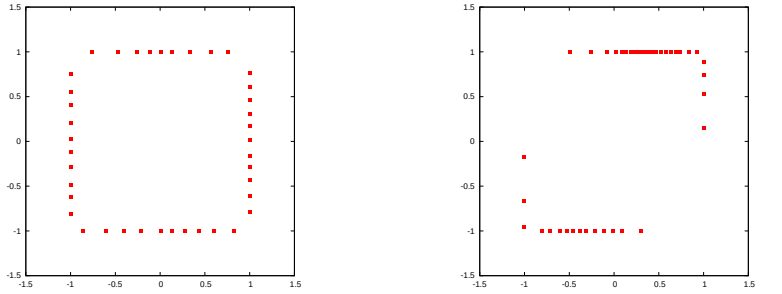
In Figure 1, we plot the quantization points on the boundary for point $(x_0, y_0) = (0,0)$ and we compare them to the ones obtained in the Brownian case.

N	M	err_1	err_2	err_3	$\kappa(C)$	$\rho(J)$	$\rho(GS)$
9	80	4.4×10^{-16}	2.1×10^{-2}	6.3×10^{-3}	1.06	1.5×10^{-2}	2.4×10^{-3}
16	80	5.6×10^{-16}	2.6×10^{-15}	1.7×10^{-2}	1.13	3.2×10^{-2}	7.3×10^{-3}
121	200	2.6×10^{-15}	8.4×10^{-15}	1.3×10^{-5}	8.5	0.46	0.28

Table 2. Numerical Results for quantization procedure with M quantifiers and a time step $\Delta = 10^{-4}$

We can notice that when the solution is in the approximation space there is almost no error on the solution. The condition number of the corresponding deterministic collocation methods is a $O(N^4)$ [3]. Here the condition number

is very small especially when using quantization points. When N=121, we observe that the quantization method is a lot more efficient: it provides a more accurate solution with a smaller condition number and the Jacobi and Gauss-Seidel method are convergent.



(a) Brownian Motion (b) Anisotropic Process

Fig. 1. Quantization points $(x_0, y_0) = (0, 0)$

5 Conclusion

We have introduced and studied stochastic versions of the collocation method for the solution of elliptic problems in a bounded domain. We have given asymptotic properties of the stochastic spectral matrix and error bounds on the approximate solutions in the very general context of linear approximations. We have proved the convergence of the spectral matrix toward the identity matrix when increasing the number of simulations and decreasing the stepsize of the simulation schemes involved in the approximations of the Feynman-Kac representations. We have also proved that very accurate solutions can be obtained even when using a small number of simulations with a poor simulation scheme. Numerical results have confirmed the efficiency of the method on the Poisson equation [14] and on an anisotropic diffusion in the unit square. We have also paid a special attention to the optimization of the computation of the Feynman-Kac formula via one random step schemes and quantization tools. In the spirit of what has been done in [13] for numerical integration, the combination of stochastic tools and deterministic approximations has led to stochastic spectral methods which are asymptotically perfect in terms of conditioning. Further numerical examples should be performed on more complex domains or partial differential equations to emphasize the simplicity and efficiency of this new approach.

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