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Abstract: We prove convergence and optimal complexity of an adaptive finite element algorithm on quadrilateral meshes. The local mesh refinement algorithm is based on regular subdivision of marked cells, leading to meshes with hanging nodes. In order to avoid multiple layers of these, a simple rule is defined, which leads to additional refinement. We prove an estimate for the complexity of this refinement technique. As in former work, we use an adaptive marking strategy which only leads to refinement according to an oscillation term, if it is dominant. In comparison to the case of triangular meshes, the a posteriori error estimator contains an additional term which implicitly measure the deviation of a given quadrilateral from a parallelogram. The well-known lower bound of the estimator for the case of conforming P^1 elements does not hold here. We instead prove decrease of the estimator, in order to establish convergence and complexity estimates.

Key-words: Adaptive finite elements, convergence of adaptive algorithms, complexity estimates, quadrilateral meshes, hanging nodes

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Convergence d'une méthode éléments finis adaptative pour des maillages quadrilatéraux

Résumé : Nous démontrons la convergence d'un algorithme d'éléments finis adaptatifs sur un maillage formés de quadrilatéres. Le raffinement local du maillage consiste en une subdivision réguliére des mailles marquées, faisant ainsi apparaitres des noeuds flottants. De plus, nous interdisons que deux mailles voisines aient deux niveaux de raffinements d'cart, et pour cela nous sommes contraints d'introduire des raffinement supplémentaires. Nous donnons alors une estimation de la complexité de cette technique de raffinement. Par rapport au cas des maillages triangulaires l'estimateur d'erreur contient un terme supplémentaire mesurant la dformation des quadrilatéres par rapport à un parallélogramme. Le résultat classiqe en P^1 sur la borne inférieure de l'estimateur n'est plus vérifié dans ce cas et nous dḿontrons alors une décroissance de l'estimateur pour établir la convergence et analyser la complexité de la méthode

Mots-clés : Eléments finis adaptatifs, convergence d'algorithmes adaptatifs, estimation de la complexité, maillages quadrilatéraux, noeuds flottants

1 Introduction

We consider an adaptive finite element method for the Poisson problem: For a given bounded polygonal domain $\Omega \subset \mathbb{R}^2$ and $f \in L^2(\Omega)$ find $u \in H_0^1(\Omega)$ such that

$$\langle \nabla u, \nabla v \rangle = \langle f, v \rangle \quad \forall v \in H_0^1(\Omega), \tag{1}$$

and its discretization using the lowest-order conforming spaces on quadrilateral meshes. The standard adaptive algorithm consists of successive loops of the sequence

$$\textbf{SOLVE} \rightarrow \textbf{ESTIMATE} \rightarrow \textbf{MARK} \rightarrow \textbf{REFINE}. \tag{2}$$

Each step of the algorithm has to be designed with care. In the following we make the simplifying assumption that the discrete problems are solved without error.

The analysis of adaptive finite element methods has made important progress in recent years. The first convergence result has been proven in the one-dimensional case by Babuška and Vogelius (2). Based on classical residual-based a posteriori error estimators (1; 13; 16) it has been shown by Dörfler (12) and Morin, Nochetto, and Siebert (14) that an adaptive mesh refinement algorithm converges towards the solution of the Poisson equation. An important further result is the estimation of the dimension of the adaptively constructed discrete spaces by Binev, Dahmen, and DeVore in (6), and Stevenson (15). The importance of these contributions lays in the fact that they show optimal complexity of certain adaptive algorithms: if the solution of the problem can be approximated by a given discretization method on a given family of meshes at a certain rate, the iteratively constructed sequence of meshes will realize this rate up to a constant factor.

In this work, we present an adaptive finite element method on quadrilateral meshes. The above cited articles all deal with the conforming P^1 elements triangular meshes and the few known complexity estimates are based on refinement by the newest-vertex algorithm. These results do not immediately carry over to the case of quadrilateral meshes, since they rely on the properties of P^1 functions having constant normal derivative of on the edges and vanishing Laplacian in the interior of the elements, and on the properties of the special refinement algorithm.

Our approach is based on conforming iso-parametric finite elements on locally refined quadrilateral meshes with hanging nodes, widely used in practice. The use of hanging nodes allows for relatively simple refinement, but additional care is required in order to avoid several layers of hanging nodes. A simple refinement rule is presented below. Since multiple levels of hanging nodes are avoided, the set of marked calls has to be increased. The control of the number of elements of the set of additionally marked cells is an important step in order to achieve complexity estimates. To the authors knowledge, such estimates have not been established in the literature before.

Following the idea of (5), we use an adaptive markings strategy which either performs the refinement according to a solution-dependent estimator introduced below, or according to a data-dependent oscillation term. In this paper, we prove convergence and optimal complexity of the adaptive algorithm. This seems to be the first theoretical result concerning convergence of AFEM on quadrilateral meshes.

The paper is organized as follows: In Section 2 we describe the family of meshes and the local refinement algorithm. For readability reasons, the proof of the complexity estimate which is used later on is postponed to Section 7. The adaptive algorithm which is the subject of our study is defined in Section 3. Here we introduce an a posteriori error estimator which consists of a standard edge residual term augmented by a term which measures the deviation from a parallelogram. We prove geometric convergence of the algorithm in Section 4. Here we make use of a generalization of the Carstensen quasi-interpolation operator to quadrilateral meshes with hanging nodes. For readability, we postpone the proofs of the employed properties of this operator to Section 8. The optimal complexity of the sequence of generated meshes is proved in Section 5. Numerical experiments are the subject of Section 6, and some conclusions are drawn in Section 9.

Throughout the paper we use the following notation. For the norm of the standard Sobolev space $H_0^1(\Omega)$ we write $|u|_1 := \left(\int_\Omega |\nabla u|^2\,dx\right)^{1/2}$. The $L^2(A)$-scalar product and norm are denoted by $\langle\cdot,\cdot\rangle_A$ and $|\cdot|_A$, respectively, omitting the subscript in case $A=\Omega$, for either a measurable subset $A \subset \Omega$ or for a side A of a finite element mesh (with obvious modification of the measure).

In order to deviate as less as possible from standard notation, we denote by h a mesh in a family $\mathcal{H}$, and by u_h the corresponding finite element solution. The set of cells of mesh h is denoted by $\mathcal{K}_h$, the set of sides by $\mathcal{S}_h$, and the subset of interior sides by $\mathcal{S}_h^{int}$. In addition, the set of nodes is $\mathcal{N}_h$. The diameter of $K \in \mathcal{K}_h$ is denoted by d_K and in addition we define $d_{max}(h) := \max\limits_{K\in\mathcal{K}_h} d_K$. As compared to standard notation in finite element literature, h denotes a mesh in a family of meshes $\mathcal{H}$ and *not* a global maximal cell width.

2 Local mesh refinement algorithm

The purpose of this section is the definition of the family of locally refined admissible meshes $\mathcal{H}$ and a local refinement algorithm $\mathcal{R}ef(H,\mathcal{M})$ which has as input $H \in \mathcal{H}$ and a subset of marked quadrilaterals $\mathcal{M} \subset \mathcal{K}_H$. It produces as output an admissible mesh $h = \mathcal{R}ef(H,\mathcal{M})$ such that at least each $K \in \mathcal{M}$ is refined. The refinement of a quadrilateral is always done by regular subdivision joining the midpoints of opposite edges.

Let h_0 be a given quadrilateral mesh without hanging nodes. We associated to each cell K of $\mathcal{K}_{h_0}$ the integer zero as its level of refinement. Then we introduce the graph $\mathcal{G}(h_0)$ which corresponds to all possible meshes obtained by a number n_K of local bisections of the cells of $K \in \mathcal{K}_{h_0}$. The level $\mathrm{lev}(L)$ of a leave L of K in the graph is n_K. For different reasons, we wish to avoid multiple hanging nodes. We therefore impose the following regularity condition. For $g \in \mathcal{G}(h_0)$ we denote by $\mathcal{N}_g$ the set of nodes, defined as the nodes of any cell $K \in \mathcal{K}_g$. We denote by $\mathcal{N}_g^*$ the set of regular nodes which are not hanging nodes, i.e. they are not located in the interior of a side of any cell $K \in \mathcal{K}_g$, see Figure 1. To a hanging node N, there corresponds a hanging side S, i.e. a side that properly contains sides of other cells of the mesh. We denote by $\mathcal{S}_h^*$ the set of sides without the hanging sides and the subset of interior sides by $\mathcal{S}_h^{int*}$. For a given node $N \in \mathcal{N}_g$, let us denote by $\mathcal{K}(N)$ the set of neighboring quadrilaterals, $\mathcal{K}(N) := \{K \in \mathcal{K}_g : \overline{K} \cap N \neq \emptyset\}$. We then define the following

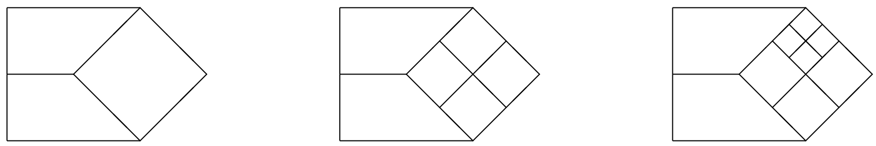

Figure 1: Refinement creating hanging nodes (middle) and not admissible mesh (right).

regularity condition

$$\max_{K\in\mathcal{K}(N)} \operatorname{lev}(K) - \min_{K\in\mathcal{K}(N)} \operatorname{lev}(K) \leq 1 \qquad \forall N \in \mathcal{N}_g. \tag{3}$$

The set of admissible meshes is now defined as

$$\mathcal{H} := \{g \in \mathcal{G}(h_0) :\ \text{g satisfies (3)}\}. \tag{4}$$

We call a family of meshes $\mathcal{H}$ shape regular, if the minimum angle of all cells $K \in \mathcal{K}$ is bounded below by a strictly positive number uniformly for all $h \in \mathcal{H}$.

Next we define the local mesh refinement algorithm $\mathcal{R}ef$. Let $H \in \mathcal{H}$ and $\mathcal{M} \subset \mathcal{K}_H$ be given. Let us denote by $\mathcal{B}ref(H,\mathcal{L}) \in \mathcal{G}(h_0)$ the mesh obtained by bisection of all K in $\mathcal{L} \subset \mathcal{K}_H$. The refinement algorithm is defined in the following way. Let

$$\mathbb{M}(\mathcal{M}) := \{\mathcal{L} \subset \mathcal{K}_H :\ \mathcal{L} \supset \mathcal{M} \text{ and } \mathcal{B}ref(H,\mathcal{L}) \in \mathcal{H}\} \tag{5}$$

be the sets of cells leading to an admissible mesh and

$$\mathcal{R}ef(H,\mathcal{M}) = \mathcal{B}ref(H,\widetilde{\mathcal{M}}), \quad \text{if} \quad \#\widetilde{\mathcal{M}} = \inf_{\mathcal{L}\in\mathbb{M}(\mathcal{M})} \#\mathcal{L}. \tag{6}$$

The set $\widetilde{\mathcal{M}}$ is well-defined as the solution of a finite-dimensional optimization problem. We will give a recursive algorithm for its construction later on in Section 7.

For the complexity estimate of the adaptive algorithm, it is important to control $\#\widetilde{\mathcal{M}} - \#\mathcal{M}$. This is done in the following result.

Lemma 1. *Let $h_k, k = 0,\dots n$ be a sequence of locally refined meshes, $h_{k+1} = \mathcal{R}ef(h_k,\mathcal{M}_k)$ with $\mathcal{M}_k \subset \mathcal{K}_{h_k}$. Then $\{h_k\}$ is a shape regular family and we have with $N_k := \#\mathcal{K}_{h_k}$*

$$N_{h_n} \leq N_{h_0} + C_0 \sum_{k=0}^{n-1} \#\mathcal{M}_k. \tag{7}$$

The analogue of Lemma 1 in the case of triangular meshes is known to be true for the newest vertex bisection algorithm, see Theorem 2.4 of (6). We give a proof of Lemma 1 in Section 7.

3 Adaptive algorithm

Let $h \in \mathcal{H}$. We denote for given $K \in \mathcal{K}_h$ by T_K the bi-linear transformation from the reference element $\hat{K}$ to K mapping nodes into nodes. We define the finite element space

$$Q_h^1 := \left\{ v_h \in C(\bar{\Omega}) \cap H_0^1(\Omega) : \ v_h|_K \circ T_K \in Q^1 \ \forall K \in \mathcal{K}_h \right\}, \tag{8}$$

where $Q^1 = \mathrm{Vect}(1, x, y, xy)$. The continuity requirement implies that the value of $v_h \in Q_h^1$ at a hanging node is determined as the average of its values at the extrema of the corresponding hanging side, see for example (8). A local Lagrangian basis can then be constructed, such that $Q_h^1 = \mathrm{Vect}(\phi_N : \ N \in \mathcal{N}_h^*)$.

The discretization of (1) reads : Find $u_h \in Q_h^1$ such that for all $v_h \in Q_h^1$

$$\langle \nabla u_h, \nabla v_h \rangle = \langle f, v_h \rangle. \tag{9}$$

It follows from the conformity that for $v_h \in Q_h^1$

$$\langle \nabla(u - u_h), \nabla v_h \rangle = 0, \tag{10}$$

and for $v \in H_0^1(\Omega)$

$$\langle \nabla(u - u_h), \nabla v \rangle = \langle f + \tilde{\Delta} u_h, v \rangle - \frac{1}{2} \sum_{K \in \mathcal{K}_h} \langle [\frac{\partial u_h}{\partial n}], v \rangle_{\partial K}, \tag{11}$$

where $\tilde{\Delta} u_h \in L^2(\Omega)$ denotes the piecewise defined function such that $\tilde{\Delta} u_h|_K = \Delta(u_h|_K)$ for $K \in \mathcal{K}_h$. We note that $\Delta u_h|_K = 0$ if the transformation T_K is affine-linear, i.e. K is a parallelogram.

The a posteriori error estimator is derived from (11) by choosing $v = (u - u_h) - C_h(u - u_h)$, where $C_h : L^1(\Omega) \to Q_h^1$ is the following interpolation operator, see (9). Let $\omega_N = \mathrm{supp}(\phi)_N$, that is the set of all cells joining a node $N \in \mathcal{N}_h^*$. Recall that $\{\phi_N\}_{N \in \mathcal{N}_h^*}$ is the Lagrangian basis and define $\tilde{\phi}_N := \phi_N / \sum_{M \in \mathcal{N}_h^*} \phi_M$. We define for $v \in L^1(\Omega)$

$$C_h v := \sum_{N \in \mathcal{N}_h^*} c_N(v) \phi_N, \quad c_N(v) = \int_{\omega_N} v \tilde{\phi}_N \, dx \Big/ \int_{\omega_N} \phi_N \, dx. \tag{12}$$

This operator is a generalization of the operator analyzed in (9) in the case of regular triangular meshes. Its definition is motivated by the fact that, using the partition of unity $\{\tilde{\phi}_N\}_{N \in \mathcal{N}_h^*}$, the interpolation error has vanishing weighted average. In addition, we will use below the analogues of the interpolation properties of C_h proven in (9) for triangular meshes. The proofs for the considered case of quadrilateral meshes with hanging nodes will be provided below in Section 8.

We define for given $N \in \mathcal{N}_h^*$ the mean-value operator π_N

$$\pi_N(f) := \int_{\omega_N} f \, dx / |\omega_N|$$

and for $\mathcal{M} \subset \mathcal{K}_h$ the oscillation term

$$\mathrm{osc}_N(f) := |\omega_N|^{1/2} \, |f - \pi_N f|_{\omega_N}, \quad \mathrm{osc}_h(f, \mathcal{M}) := \left(\sum_{K \in \mathcal{M}} \sum_{N \in \mathcal{N}^*(K)} \mathrm{osc}_N^2(f) \right)^{1/2} \tag{13}$$

Let $v_h \in Q_h^1$. We define

$$J_K(v_h) := \frac{|K|}{\sqrt{2}} \|[\frac{\partial v_h}{\partial n}]\|_{\partial K \setminus \partial\Omega}, \quad D_K(v_h) := |K|^{1/2} |\tilde{\Delta} v_h|_K, \tag{14}$$

and the error estimator

$$\eta_h(v_h) := \left(\sum_{K \in \mathcal{K}_h} J_K^2(v_h) + D_K^2(v_h) \right)^{1/2}.$$

and for given $\mathcal{M} \subset \mathcal{K}_h$

$$\eta(v_h, \mathcal{M}) := \left(\sum_{K \in \mathcal{M}} J_K^2(v_h) + D_K^2(v_h) \right)^{1/2}. \tag{15}$$

Finally let $J_h(v_h)$ and be $J_h(v_h, \mathcal{M})$ the terms,

$$J_h(v_h) = \left(\sum_{K \in \mathcal{K}_h} J_K^2(v_h) \right)^{1/2},$$

and

$$J_h(v_h, \mathcal{M}) = \left(\sum_{K \in \mathcal{M}} J_K^2(v_h) \right)^{1/2}.$$

We expect the jump terms to dominate the estimator since the term involving D_K depends on the deviation of the cells from parallelograms which tends to zero faster under the considered mesh refinement algorithm.

Remark 2. *Instead of D_K one could alternatively employ the term*

$$\tilde{D}_K(v_h) := |\omega_N| |\tilde{\Delta} v_h - \pi_N \tilde{\Delta} v_h|^2_{\omega_N}.$$

However, since $\tilde{\Delta} u_h$ is a discontinuous function on general meshes, $\tilde{D}_K(u_h)$ cannot be expected significantly smaller, but is more involved to compute than $D_K(u_h)$.

The purpose of this article is to analyze the following adaptive finite element algorithm:

Remark 3. *The refinement is determined by the oscillation term, only if it is big compared to the estimator, following the idea of (3). Therefore, in most practical cases, the side residuals alone dominate the error estimation.*

The choice of parameters can be guided by our theoretical results. The parameters θ, σ, and γ are arbitrary for convergence. The fact that γ is arbitrary indicates that the side residuals play the dominant role in the overall refinement.

In order to achieve optimal complexity, the marking parameter θ has to be small enough, as known from other complexity estimates (6; 15), and γ has to satisfy a condition, whereas σ is free.

$\mathcal{AFEM}$

(0) Choose parameters $0 < \theta, \sigma < 1$, $\gamma > 0$, and an initial mesh h_0, and set $k = 0$.

(1) Solve the discrete problem (9) with h replaced by h_k in order to obtain the finite element solution u_{h_k}.

(2) Compute $\eta_{h_k}(u_{h_k})$, and oscillation term $\mathrm{osc}_{h_k}(f)$.

(3) – If $\mathrm{osc}^2_{h_k}(f) \leq \gamma\, \eta^2_{h_k}(u_{h_k})$ then mark a set $\mathcal{M} \subset \mathcal{K}_{h_k}$ with minimal cardinality such that

$$\eta^2_{h_k}(u_{h_k}, \mathcal{M}) \geq \theta\, \eta^2_{h_k}(u_{h_k}). \tag{16}$$

– else find a set $\mathcal{P} \subset \mathcal{N}^*_{h_k}$ with minimal cardinality such that

$$\mathrm{osc}^2_{h_k}(f, \mathcal{P}) \geq \sigma\, \mathrm{osc}^2_{h_k}(f). \tag{17}$$

and define $\mathcal{M}$ to be the set of cells containing at least one node in $\mathcal{P}$.

(4) Adapt the mesh : $h_{k+1} := \mathcal{R}ef(h_k, \mathcal{M})$.

(5) Set $k := k + 1$ and go to step (1).

4 Convergence

Our convergence proof relies on the following global upper bound and the decrease of the estimator under refinement.

Lemma 4. ***(global upper bound)*** *Let u be the solution of the Poisson equation (1), $h \in \mathcal{H}$, and $u_h \in Q^1_h$ the solution of its discrete analogue (9). There exists a constant $C_1 > 0$ independent of $\mathcal{H}$ such that*

$$|u - u_h|^2_1 \leq C_1 \left(\eta^2_h(u_h) + osc^2_h(f)\right) \tag{18}$$

Proof. The proof follows from (11) and the Galerkin orthogonality by setting $v = u - u_h$. This yields with (62)

$$|u - u_h|^2_1 = \langle f + \tilde{\Delta} u_h, v - C_h v\rangle - \frac{1}{2} \sum_{K \in \mathcal{K}_h} \langle [\frac{\partial u_h}{\partial n}], v - C_h v\rangle_{\partial K}$$

$$\leq C_{i0} C_{i1} |v|_1 \left(\left(\sum_{N \in \mathcal{N}^*_h} |\omega_N|\, |f - \pi_N f|^2_{\omega_N} \right)^{1/2} + \left(\sum_{N \in \mathcal{N}^*_h} |\tilde{\Delta} u_h|^2_{\omega_N} |\omega_N| \right)^{1/2} \right)$$

$$+ C_{i2} \sum_{S \in \mathcal{S}^{int*}_h} |[\frac{\partial u_h}{\partial n}]|_S |S|^{1/2} |v|_{1,\omega_S},$$

where C_{i1} and C_{i1} are the constants from Lemma 14 and ω_S is the set of cells having S as side. The result follows with $C_1 = C_0 \max(C_{i1}, C_{i2})^2$, where $C_0 = \max_{K \in \mathcal{K}_h, N \in \mathcal{N}_h(K)} |K|/|\omega_N|$, which is bounded by our refinement rule. $\square$

The next result serves as a replacement for Verfürth's lower bound, which does not hold on quadrilateral meshes. It is basically shown that the error estimator decreases locally under mesh refinement, following the idea of (10) in the case of triangular meshes, where only the reduction of the jumps of the normal derivatives has to be shown.

Lemma 5. ***(decrease of estimator)*** *There exist constants $C_2 > 0$ and $0 < \kappa < 1$ independent of $\mathcal{H}$ such that, if $h = \mathcal{R}ef(H, \mathcal{M})$, for any $\delta > 0$*

$$\eta_h^2(u_h) \leq (1+\delta)\eta_H^2(u_H) - \kappa(1+\delta)\eta_H^2(u_H, \mathcal{M}) + C_2(1+1/\delta)|u_h - u_H|_1^2. \quad (19)$$

Proof. The proof of (19) is separated into two parts corresponding to the two contributions to the estimator. First, for the jump terms we have with a strictly positive constant absolute κ_1 the inequality

$$J_h^2(u_h) \leq (1+\delta)J_H^2(u_H) - \kappa_1(1+\delta)J_H^2(u_H, \mathcal{F}) + C_4(1+1/\delta)|u_h - u_H|_1^2. \quad (20)$$

Its prove relies on the decrease in mesh-size and is identical to the one for the case of the triangular meshes, see (3).

It remains to prove that there exists a strictly positive absolute constant κ_2

$$\sum_{K\in\mathcal{K}_h} |K|\,|\tilde{\Delta} u_h|_K^2 \leq (1+\delta) \sum_{K\in\mathcal{K}_K} |K|\,|\tilde{\Delta} u_H|_K^2 -$$
$$\kappa_2(1+\delta) \sum_{K\in\mathcal{M}} |\tilde{\Delta} u_H|_K^2 + C(1+1/\delta)|u_h - u_H|_1^2, \quad (21)$$

and we conclude by choosing $\kappa = \min(\kappa_1, \kappa_2)$.

In order to prove (21), we first remark that there exists $p < 1$ such that for any $K \in \mathcal{M}$ and $K' \in \mathcal{K}_h$ such that $K' \subset K$ we have $|K'| \leq p|K|$ ($p = 1/4$ in the case of parallelograms). We now have by an inverse estimate

$$\begin{aligned}
&\sum_{K\in\mathcal{K}_h} |K|\,|\tilde{\Delta} u_h|_K^2 \\
&\leq (1+1/\delta) \sum_{K\in\mathcal{K}_h} |K|\,|\tilde{\Delta}(u_h - u_H)|_K^2 + (1+\delta) \sum_{K\in\mathcal{K}_h} |K|\,|\tilde{\Delta} u_H|_K^2 \\
&\leq C(1+1/\delta)|\nabla(u_h - u_H)|_1^2 + (1+\delta) \sum_{K\in\mathcal{K}_h} |K|\,|\tilde{\Delta} u_H|_K^2
\end{aligned}$$

For the last term we have

$$\begin{aligned}
\sum_{K\in\mathcal{K}_h} |K|\,|\tilde{\Delta} u_H|_K^2 &= \sum_{K\in\mathcal{K}_H\setminus\mathcal{M}} |K|\,|\tilde{\Delta} u_H|_K^2 + \sum_{K\in\mathcal{M}} \sum_{K'\subset K, K'\in\mathcal{K}_h} |K'|\,|\tilde{\Delta} u_H|_{K'}^2 \\
&\leq \sum_{K\in\mathcal{K}_H\setminus\mathcal{M}} |K|\,|\tilde{\Delta} u_H|_K^2 + p \sum_{K\in\mathcal{M}} \sum_{K'\subset K, K'\in\mathcal{K}_h} |K|\,|\tilde{\Delta} u_H|_{K'}^2 \\
&= \sum_{K\in\mathcal{K}_H} |K|\,|\tilde{\Delta} u_H|_K^2 - (1-p) \sum_{K\in\mathcal{M}} |\tilde{\Delta} u_H|_K^2.
\end{aligned}$$

This concludes the proof. □

We next state our convergence result with respect to the following error measure:

$$e_h := |u - u_h|_1^2 + \beta_1 \eta_h^2(u_h) + \beta_2 \operatorname{osc}_h^2(f) \tag{22}$$

for some constants $\beta_1 > 0$ and $\beta_2 > 0$.

Theorem 6. *Let $\{h_k\}_{k\geq 0}$ be a sequence of meshes generated by algorithm $\mathcal{AFEM}$and let $\{u_{h_k}\}_{k\geq 0}$ be the corresponding sequence of finite element solutions. There exist constants $\beta_1 > 0$, $\beta_2 > 0$, and $\rho < 1$ such that for all $k = 1, 2, \ldots$*

$$e_{h_{k+1}} \leq \rho \, e_{h_k}. \tag{23}$$

Remark 7. *For the convergence result of Theorem 6, γ, $\theta < 1$, and $\sigma < 1$ can be chosen arbitrarily.*

Proof. We use the Galerkin orthogonality and Lemma 5 in order to obtain

$$\begin{aligned} |u - u_h|_1^2 + \beta_1 \eta_h^2(u_h) + \beta_2 \operatorname{osc}_h^2(f) \leq |u - u_H|_1^2 - (1 - \beta_1 C_2(1 + 1/\delta)) \, |u_h - u_H|_1^2 \\ + \beta_1 (1+\delta) \eta_H^2(u_H) - \beta_1 \kappa (1+\delta) \eta_H^2(u_H, \mathcal{M}) + \beta_2 \operatorname{osc}_h^2(f). \end{aligned} \tag{24}$$

We now split the proof into two parts depending on the two cases of the algorithm.

In the first case we have $\operatorname{osc}_H^2(f) \leq \gamma \, \eta_H^2(u_H)$ and the refinement is made such that $\eta_H^2(u_H, \mathcal{M}) \geq \theta \, \eta_H^2(u_H)$. Using in addition the monotonicity of the oscillation term, (24) becomes

$$\begin{aligned} |u - u_h|_1^2 + \beta_1 \eta_h^2(u_h) + \beta_2 \operatorname{osc}_h^2(f) \quad &\leq |u - u_H|_1^2 - (1 - \beta_1 C_2(1 + 1/\delta)) \, |u_h - u_H|_1^2 \\ &\quad + \beta_1 \left(1 + \delta - \theta \kappa (1+\delta)\right) \eta_H^2(u_H) + \beta_2 \operatorname{osc}_H^2(f). \end{aligned}$$

Under the condition

$$1 - \beta_1 C_2 (1 + 1/\delta) \geq 0, \tag{25}$$

we find

$$\begin{aligned} &|u - u_h|_1^2 + \beta_1 \eta_h^2(u_h) + \beta_2 \operatorname{osc}_h^2(f) \leq \\ &(1 - \rho_1)|u - u_H|_1^2 + (1 - \rho_2)\beta_1 \eta_H^2(u_H) + (1 - \rho_3)\beta_2 \operatorname{osc}_H^2(f) \\ &+ \rho_1 |u - u_H|_1^2 + \left(\rho_2 + \delta - \theta\kappa(1+\delta)\right) \beta_1 \eta_H^2(u_H) + \rho_3 \beta_2 \operatorname{osc}_H^2(f), \end{aligned} \tag{26}$$

with positive ρ_1, ρ_2, and ρ_3 to be determined below. First we set

$$\rho_2 = \theta \kappa \frac{1+\delta}{2} - \delta, \tag{27}$$

which has to be made positive by appropriate choice of $\delta > 0$ below. Let us denote the sum in the last line of (26) by A. Using the upper bound of the error and the condition of case one, we get

$$\begin{aligned} A &= \rho_1 |u - u_H|_1^2 - \theta\kappa \frac{1+\delta}{2} \beta_1 \eta_H^2(u_H) + \rho_3 \beta_2 \operatorname{osc}_H^2(f) \\ &\leq \left(\rho_1 C_1 - \beta_1 \theta \kappa \frac{1+\delta}{2}\right) \eta_H^2(u_H) + \left(\rho_1 C_1 + \rho_3 \beta_2\right) \operatorname{osc}_H^2(f) \\ &\leq \left((\rho_1 C_1 - \beta_1 \theta \kappa \frac{1+\delta}{2}) + \gamma(\rho_1 C_1 + \rho_3 \beta_2)\right) \eta_H^2(u_H). \end{aligned} \tag{28}$$

It remains to impose the following four conditions: the term in brackets in the last line of (28) has to be negative, inequality (25) has to be satisfied, and $\rho_i > 0$.

Condition (25) implies that

$$\delta \geq \frac{\beta_1 C_2}{1-\beta_1 C_2}, \tag{29}$$

and the condition $\rho_2 > 0$ implies

$$\delta < \frac{\theta\kappa}{2-\theta\kappa}. \tag{30}$$

We can chose $\delta > 0$ such that (29) and (30) are satisfied under the condition

$$\frac{\beta_1 C_2}{1-\beta_1 C_2} < \frac{\theta\kappa}{2-\theta\kappa},$$

which is veryfied if

$$\beta_1 < \frac{\theta\kappa}{2} C_2^{-1}. \tag{31}$$

In order to make the term in the last line of (28) negative we choose ρ_1 and ρ_3 such that

$$\rho_1 \leq (1+\gamma)^{-1} C_1^{-1} \frac{\beta_1\theta\kappa}{4} \quad \text{and} \quad \rho_3 \leq \beta_2^{-1} \frac{\beta_1\theta\kappa}{4}. \tag{32}$$

The fact that β_2 is arbitrary up to now will be used in the second case of the proof. This concludes the convergence proof in the first case.

Now we consider the second case. We have the following property concerning the oscillation term involving a constant $0 < \mu < 1$:

$$\mathrm{osc}_H^2(f) - \mathrm{osc}_h^2(f) \geq \mu\, \mathrm{osc}_H^2(f,\mathcal{P}). \tag{33}$$

This implies

$$\mathrm{osc}_h^2(f) \leq (1-\mu\sigma)\mathrm{osc}_H^2(f). \tag{34}$$

We therefore obtain from (24)

$$\begin{aligned} |u-u_h|_1^2 + \beta_1\eta_h^2(u_h) + \beta_2\, \mathrm{osc}_h^2(f) \leq |u-u_H|_1^2 - (1-\beta_1 C_2(1+1/\delta))\, |u_h-u_H|_1^2 \\ + \beta_1(1+\delta)\eta_H^2(u_H) + \beta_2(1-\mu\sigma)\mathrm{osc}_H^2(f). \end{aligned} \tag{35}$$

Under the condition

$$1-\beta_1 C_2(1+1/\delta) \geq 0, \tag{36}$$

and introducing positive constants ρ_1 and ρ_3 we have

$$\begin{aligned} &|u-u_h|_1^2 + \beta_1\eta_h^2(u_h) + \beta_2\, \mathrm{osc}_h^2(f) \leq \\ &(1-\rho_1)|u-u_h|_1^2 + \beta_1(1-\rho_2)\eta_h^2(u_h) + \beta_2(1-\frac{1}{2}\mu\sigma)\mathrm{osc}_H^2(f) \\ &+ \rho_1|u-u_h|_1^2 + \beta_1(\delta+\rho_2)\eta_h^2(u_h) - \frac{1}{2}\beta_2\mu\sigma\mathrm{osc}_H^2(f). \end{aligned} \tag{37}$$

Denote the last line of (37) by A. Using the global upper bound and $\eta_h^2(u_h) \leq \gamma^{-1}\text{osc}_H^2(f)$ yields

$$\begin{aligned} A &= \rho_1|u-u_h|_1^2 + \beta_1(\delta+\rho_2)\eta_h^2(u_h) - \frac{1}{2}\beta_2\mu\sigma\text{osc}_H^2(f) \\ &\leq (\rho_1 C_1 + \beta_1(\delta+\rho_2))\,\eta_h^2(u_h) + \left(\rho_1 C_1 - \frac{1}{2}\beta_2\mu\sigma\right)\text{osc}_H^2(f) \\ &\leq \left(\gamma^{-1}\left(\rho_1 C_1 + \beta_1(\delta+\rho_2)\right) + \rho_1 C_1 - \frac{1}{2}\beta_2\mu\sigma\right)\text{osc}_H^2(f). \end{aligned}$$

In order to obtain convergence we have to choose the different parameters in such a way that (36) as well as the following two inequalities are satisfied:

$$\rho_1 > 0, \tag{38}$$

$$\frac{1}{2}\beta_2\mu\sigma \geq (1+\gamma^{-1})\rho_1 C_1 + \gamma^{-1}\beta_1(\delta+\rho_2). \tag{39}$$

With the same choice of δ and β_1 as in the first case, condition (36) is verified in connection with (38) as before. It remains to choose β_2 sufficiently large in order to ensure (39). This is possible since β_2 was arbitrary in the first part of the proof. □

5 Complexity estimate

In order to estimate the complexity of the generated meshes, we use the following local upper and global lower bounds.

Lemma 8. ***(local upper bound)*** *Let u be the solution of the Poisson equation (1) and $u_H \in Q_H^1$ be the solution of its discrete analogue (9), and h a refinement of H. Let $\mathcal{M}$ be the set of refined cells and their neighbors. There exists a constant $C_3 > 0$ independent of $\mathcal{H}$ such that*

$$|u_h - u_H|_1^2 \leq C_3\left(\eta_H^2(u_H,\mathcal{M}) + osc_H^2(f,\mathcal{M})\right) \tag{40}$$

and

$$\#\mathcal{M} \leq C_4(\#N_h - \#N_H). \tag{41}$$

Proof. Since the Carstensen interpolation operator is not idempotent, we make additional use of the Clément operator I_H, and set $e_h = u_h - u_H$, $w_h = e_h - I_H e_h$.We have

$$|u_h - u_H|_1^2 = <\nabla e_h, \nabla(e_h - I_H e_h)> = <\nabla e_h, \nabla w_h> = <\nabla e_h, \nabla(w_h - C_H w_h)>$$

Then similarly to the proof of Lemma 4 we obtain

$$|u_h - u_H|_1^2 = \langle f + \tilde{\Delta}u_H, w_h - C_H w_h\rangle - \frac{1}{2}\sum_{K\in\mathcal{K}_h}\langle[\frac{\partial u_H}{\partial n}], w_h - C_H w_h\rangle_{\partial K}$$

We remark that $I_h e_h|_{\omega_N} = e_h|_{\omega_N}$ if for all $K \subset \omega_N$ we have $K \notin \mathcal{M}$, which implies $w_h|_{\omega_N} = 0$. Therefore only the terms involving $N \in \mathcal{N}_H^*$ such that $\omega_N \cap K \neq \emptyset$ for some $K \in \mathcal{M}$ have to be taken into account. We conclude with the H^1-stability of the Clément operator. □

Lemma 9. ***(global lower bound)*** *There exists a constant $C_5 > 0$ independent of $\mathcal{H}$ such that*

$$\eta_h^2(u_h) \le C_5 \left(|u - u_h|_1^2 + osc_h^2(f) \right). \tag{42}$$

Proof. The proof follows Verfürth's technique for the lower bound on triangular meshes. The difference with that case is that the jump of the normal derivative over a given edge is a linear function. We therefore use two linearly independent bubble functions on each side in order to obtain the bound with no further modification with respect to (16). □

In order to express the optimal complexity, we introduce some notation from nonlinear approximation theory, developed in (6; 11). Let $\mathcal{H}_N$ be the set of all meshes h which satisfy $N_h \le N$.

Next we define the approximation class

$$\mathcal{W}^s := \Big\{ (u, f) \in (H_0^1(\Omega), L^2(\Omega)) : \|(u, f)\|_{\mathcal{W}^s} < +\infty \Big\}. \tag{43}$$

with

$$\|(u, f)\|_{\mathcal{W}^s}^2 := \sup_{N \ge N_0} N^s \inf_{h \in \mathcal{H}_N} \Big(|u - u_h|_1^2 + \operatorname{osc}_h^2(f) \Big).$$

We say that an adaptive finite element method realizes the optimal convergence rate if, whenever $(u, f) \in \mathcal{W}^s$, there exists an absolute constant C such that the generated sequence of triangulations $\{h_k\}$ with dimensions N_k and corresponding approximations u_{h_k} satisfies

$$|u - u_{h_k}|_1^2 + \operatorname{osc}_k^2(f) \le C\, N_k^{-s}. \tag{44}$$

Alternatively, setting $\varepsilon_k = |u - u_{h_k}|_1^2 + \operatorname{osc}_k^2(f)$ we may ask for

$$N_k \le C\, \varepsilon_k^{-1/s}. \tag{45}$$

Theorem 10. *The adaptive algorithm $\mathcal{AFEM}$ realizes the optimal convergence rate, if the parameters γ and θ satisfy*

$$\gamma \le \frac{1}{4} C_5^{-1} (1 + C_3)^{-1}, \quad \theta < \frac{1}{4} C_3^{-1} C_5^{-1}. \tag{46}$$

Proof. Let k be given.

From the regularity assumption $(u, f) \in \mathcal{W}^s$ we have existence of a mesh $h^* \in \mathcal{H}$ such that with $\varepsilon_{h^*} = |u - u_{h^*}|_1^2 + \operatorname{osc}_{h^*}^2(f)$ and for $\lambda > 0$ to be chosen below there holds

$$\varepsilon_{h^*} \le \lambda\, \varepsilon_k, \tag{47}$$

and

$$N_{h^*} \le C \varepsilon_k^{-1/s}. \tag{48}$$

Following the proof of Stevenson (15) (proof of Lemma 5.2), we can suppose that h^* is a refinement of h_k, if we replace (48) by:

$$N_{h^*} - N_{h_k} \le C\, \varepsilon_{h_k}^{-1/s}. \tag{49}$$

Let $\mathcal{M}^* \subset \mathcal{K}_{h_k}$ be the set of refined cells and $\mathcal{M}_k$ be the set of marked cells in iteration k.

We will prove below the estimate

$$\#\mathcal{M}_k \leq C\,\varepsilon_k^{-1/s}. \tag{50}$$

This implies the complexity estimate (45) as follows. Let $e_l := |u - u_{h_l}|_1^2 + \beta_1 \eta_{h_l}^2(u_{h_l}) + \beta_2 \operatorname{osc}_{h_l}^2(f)$. From Theorem 6.1 we know that for some constant $\rho < 1$

$$e_k \leq \rho^{k-l}\, e_l, \quad 0 \leq l \leq k.$$

We obviously have $\varepsilon_l \leq \max(1, \beta_2) e_l$. By the global lower bound (42) we also have $e_l \leq C\,\varepsilon_l$ with an absolute constant C. This implies

$$\varepsilon_k \leq C\,\rho^{k-l}\,\varepsilon_l, \quad 0 \leq l \leq k. \tag{51}$$

The bound (51) and Lemma 1 imply

$$\begin{aligned}
N_{k+1} - N_0 \quad &\leq \quad C \sum_{l=0}^{k} \#\mathcal{M}_l \leq C \sum_{l=0}^{k} \varepsilon_l^{-1/s} \\
&\leq \quad C \left(\sum_{l=0}^{k} \rho_l^{(k-l)/s} \right) \varepsilon_k^{-1/s} \leq \frac{C}{1 - \rho^{1/s}}\, \varepsilon_k^{-1/s}.
\end{aligned}$$

yielding (45).

We now turn to the proof of (50). As before, we consider the two cases of the algorithm separately.

In the first case we have

$$\operatorname{osc}_{h_k}^2(f) \leq \gamma\, \eta_{h_k}^2(u_{h_k}). \tag{52}$$

We will prove below that

$$\eta_{h_k}^2(u_{h_k}, \mathcal{M}^*) \geq \theta \eta_{h_k}^2(u_{h_k}). \tag{53}$$

This implies the estimate (50): Since $\mathcal{M}$ is chosen to be the set with minimal cardinality satisfying the bound (53), we find using (41) that

$$\#\mathcal{M}_k \leq \#\mathcal{M}^* \leq C_4 (N_{h^*} - N_k) \leq C\,\varepsilon_k^{-1/s}. \tag{54}$$

The proof of (53) is obtained as follows. First we note that

$$\begin{aligned}
|u - u_{h_k}|_1^2 \quad &= \quad |u - u_{h^*}|_1^2 + |u_{h^*} - u_{h_k}|_1^2 \\
&\leq \quad \lambda \left(|u - u_{h_k}|_1^2 + \operatorname{osc}_{h_k}^2(f) \right) + |u_{h^*} - u_{h_k}|_1^2,
\end{aligned}$$

which implies

$$|u - u_{h_k}|_1^2 \leq (1-\lambda)^{-1} \left(|u_{h^*} - u_{h_k}|_1^2 + \lambda \operatorname{osc}_{h_k}^2(f) \right).$$

We now successively use (42), the Galerkin orthogonality, (47) and (52), introducing a parameter $a = (1-\lambda)^{-1}$, in order to obtain

$$\begin{aligned}
C_5^{-1} \eta_{h_k}^2(u_{h_k}) \quad &\leq \quad |u - u_{h_k}|_1^2 + \operatorname{osc}_{h_k}^2(f) \\
&\leq \quad (1-\lambda)^{-1} \left(|u_{h^*} - u_{h_k}|_1^2 + \operatorname{osc}_{h_k}^2(f) \right) \\
&\leq \quad (1-\lambda)^{-1} \left(C_3 \left(\eta_{h_k}^2(u_{h_k}, \mathcal{M}^*) + \operatorname{osc}_{h_k}^2(f, \mathcal{M}^*) \right) + \operatorname{osc}_{h_k}^2(f) \right) \\
&\leq \quad (1-\lambda)^{-1} \left(C_3 \eta_{h_k}^2(u_{h_k}, \mathcal{M}^*) + (1 + C_3) \operatorname{osc}_{h_k}^2(f) \right) \\
&\leq \quad (1-\lambda)^{-1} \left(C_3 \eta_{h_k}^2(u_{h_k}, \mathcal{M}^*) + \gamma (1 + C_3) \eta_{h_k}^2(u_{h_k}) \right)
\end{aligned}$$

It follows with $\lambda \leq \frac{1}{2}$ that

$$\left(C_5^{-1} - 2\gamma(1 + C_3))\right) \eta_{h_k}^2(u_{h_k}) \leq 2C_3\, \eta_{h_k}^2(u_{h_k}, \mathcal{M}^*). \tag{55}$$

The assumption on γ $(46)_1$ implies that

$$C_5^{-1} - 2\gamma(1 + C_3) \geq \frac{1}{2} C_5^{-1}$$

and it follows that

$$\eta_{h_k}^2(u_{h_k}, \mathcal{M}^*) \quad \geq \quad \frac{1}{4} C_3^{-1} C_5^{-1} \eta_{h_k}^2(u_{h_k}).$$

The assumption on θ $(46)_2$ completes the proof in the first case.

Now we consider the second case. We thus have

$$\eta_{h_k}^2(u_{h_k}) \leq \gamma^{-1} \mathrm{osc}_{h_k}^2(f). \tag{56}$$

We will prove that

$$\mathrm{osc}_{h_k}^2(f, \mathcal{M}^*) \geq \sigma \mathrm{osc}_{h_k}^2(f). \tag{57}$$

This implies (50) as before by the optimality of the choice of $\mathcal{P}$. First we note that by (42) and (57) we have

$$\begin{aligned} \left(|u - u_{h_k}|_1^2\right) &\leq C_3 \eta_{h_k}^2(u_{h_k}) \\ &\leq C_3(1 + \gamma^{-1}) \mathrm{osc}_{h_k}^2(f). \end{aligned}$$

This implies together with (47) that

$$\begin{aligned} \mathrm{osc}_{h_k}^2(f) - \mathrm{osc}_{h_k}^2(f, \mathcal{M}^*) &\leq \mathrm{osc}_{h^*}^2(f) \\ &\leq \lambda |u - u_{h_k}|_1^2 + \mathrm{osc}_{h_k}^2(f) \\ &\leq \lambda \left(1 + C_3(1 + \gamma^{-1})\right) \mathrm{osc}_{h_k}^2(f), \end{aligned}$$

and therefore with λ small enough

$$\sigma\, \mathrm{osc}_{h_k}^2(f) \leq \left(1 - \lambda \left(1 + C_3(1 + \gamma^{-1})\right)\right) \mathrm{osc}_{h_k}^2(f) \leq \mathrm{osc}_{h_k}^2(f, \mathcal{M}^*)$$

This concludes the proof. □

Corollary 11. *The algorithm $\mathcal{AFEM}$, combined with multigrid iteration (7; 17), has optimal work count in the sense that for a given accuracy $\varepsilon > 0$, the algorithm provides a discrete solution u_h satisfying $|u - u_h|_1 \leq \varepsilon$ with a number of operations proportional to $\varepsilon^{-1/s}$. The combination of the adaptive algorithm with multigrid requires the introduction of a stopping criterion leading to an additional iteration error. Such an algorithm has been proposed and analyzed in (4).*

We finally remark that the regularity assumption $(f, u) \in \mathcal{W}^s$ is difficult to verify in practice. However, the a priori error analysis on meshes adapted to corner singularities suggests that $s = 1$ if $f \in L^2(\Omega)$ in the two-dimensional case under mild restrictions on the domain.

6 Numerical experiments

We illustrate the behavior of the adaptive algorithm at hand of two classical boundary value problems with singular solution,

$$\begin{aligned} -\Delta u &= 1 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

In the first case, we consider the L-shaped domain $\Omega = (-1,1)\times(0,1)\cup(-1,0)\times(-1,0]$, and in the second case the slit domain $\Omega = (-1,1)\times(-1,1)\setminus\{0\}\times[-1,0]$.

In Figure 2 and 3, we show a comparison between the asymptotic complexity of the algorithm for different values of θ (note that $\theta = 1$ corresponds to uniform refinement) for the first case. The results indicate that the optimal behavior, $\varepsilon \approx N^{-1/2}$ is recovered by the adaptive algorithm. As is well-known, the sequence of uniformly refined meshes leads to $\varepsilon \approx N^{-1/2}$.

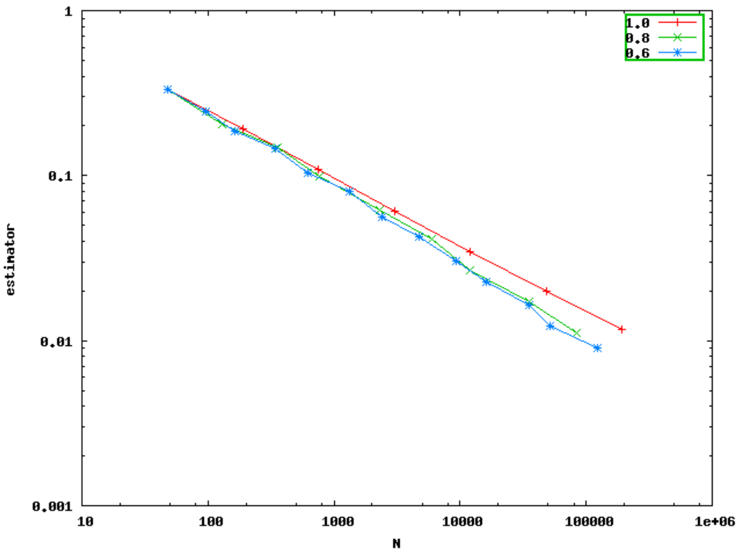

Figure 2: L-shaped domain: Estimator error versus N (log-log-scale) for $\theta = 1.0, 0.8, 0.6$.

In Figures 4 and 5, the same comparison is made for the slit domain. The results indicate that the optimal beahvior, $\varepsilon \approx N^{-1/2}$ is recovered by the adaptive algorithm. In this case, the sequence of uniformly refined gives $\varepsilon \approx N^{-1/2}$.

7 Complexity estimate of the refinement algorithm

We now turn to the proof of Lemma 1 which we recall here.

Lemma 12. *Let $h_k, k = 0, \dots n$ be a sequence of locally refined meshes, $h_{k+1} = \mathcal{R}ef(h_k, \mathcal{M}_k)$ with $\mathcal{M}_k \subset \mathcal{K}_{h_k}$. Then $\{h_n\}$ is uniformly shape regular and we*

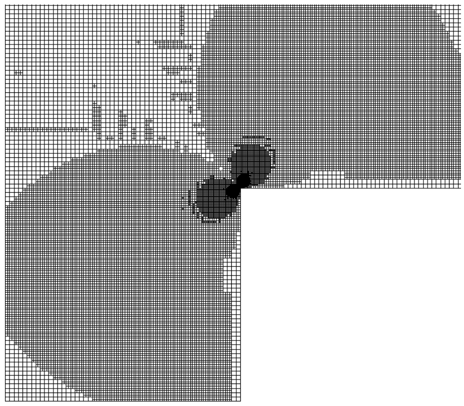

Figure 3: L-shaped domain:typical mesh

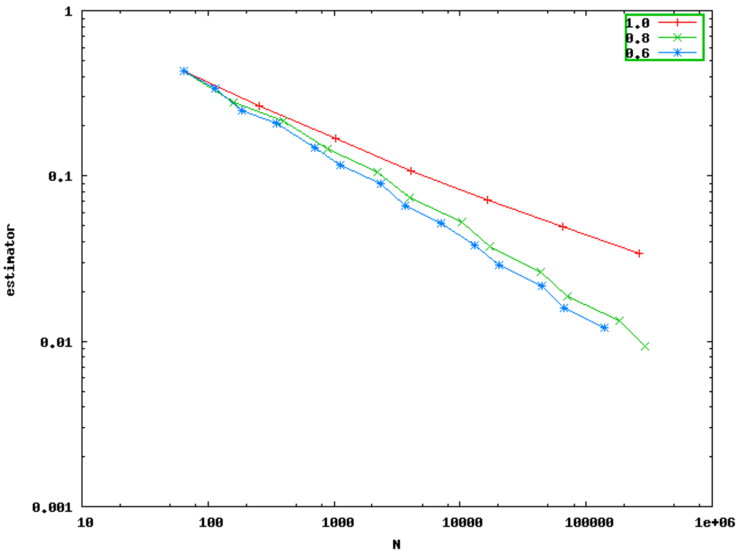

Figure 4: Slit domain: Estimator error versus N (log-log-scale) for $\theta = 1.0, 0.8, 0.6$.

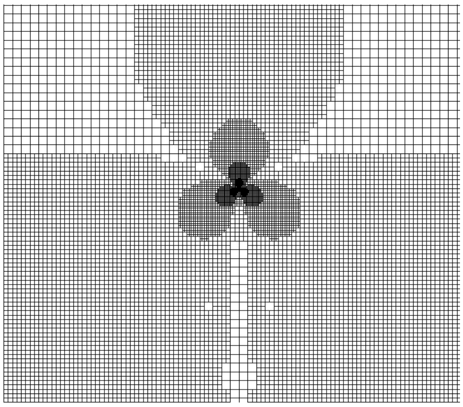

Figure 5: Slit domain: typical mesh.

have with $N_k := \#\mathcal{K}_{h_k}$

$$N_{h_n} \leq N_{h_0} + C_0 \sum_{k=0}^{n-1} \#\mathcal{M}_k. \tag{58}$$

We begin with a short discussion of why the proof of (6) for the new-vertex bisection algorithm does not carry over to the present case. In the following we denote by $\widetilde{\mathcal{M}_k} \supset \mathcal{M}_k$ the set of cells such that $\mathcal{R}ef(h_k, \mathcal{M}_k) = \mathcal{B}ref(h_k, \widetilde{\mathcal{M}_k})$, and by $\Delta\mathcal{M}_k := \widetilde{\mathcal{M}_k} \setminus \mathcal{M}_k$ the set of additionally refined quadrilaterals. The proof of Lemma 12 is based on the estimate

$$\#\Delta\mathcal{M}_n \leq C \sum_{k=0}^{n} \#\mathcal{M}_k, \tag{59}$$

with an absolute constant C. The idea to prove (59) is to relate to a cell $K' \in \Delta\mathcal{M}_{n-1}$ previously marked cells K. This is done in (6) by only considering a local neighborhood of a marked cell K. In our case, however, the marking of a quadrilateral K may lead to additional refinement of a cell K' at any distance from K, as illustrated in Figure 6. We call such a situation, where all the intermediate quadrilaterals are marked additionally, a stairway with r steps ($r = 2$ in Figure 6).

This example also shows that the sum in the left-hand side of (59) cannot be replaced by only the last term of the sum (as is also the case for other local mesh refinement techniques).

Proof. Let h_n be given. We consider the whole tree of cells $\mathcal{G}(h_n)$ which has $\mathcal{G}(h_{n-1})$ as a sub-tree. For a given quadrilateral $K \in \mathcal{G}(h_n)$ with level $l = \text{lev}(K)$ we recursively define circles of surrounding quadrilaterals. To this end, we call a direct neighbor a quadrilateral K' which shares a vertex with K, satisfies $\text{lev}(K') \in \{l, l-1\}$ and is not a parent of K.

We thus set $\mathcal{A}^0(K) := \{K\}$ and for $r \geq 0$ we define

$$\mathcal{A}^{r+1}(K) := \{L \in \mathcal{G}(h) : \ L \text{ is a direct neighbor of some } K' \in \mathcal{A}^r(K)\}\,.$$

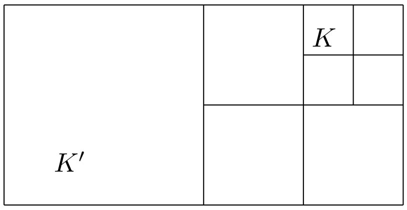

Figure 6: Example of local refinement: marking of K leads to additional refinement of K'.

This definition leads to transitivity in the following sense: if $K' \in \mathcal{A}^r(K)$ and $K'' \in \mathcal{A}^s(K')$, we have $K'' \in \mathcal{A}^{r+s}(K)$.

In order to prove (59), we set $\mathcal{M} := \bigcup_{k=0}^n \mathcal{M}_k$ and define a mapping $\lambda : \mathcal{M} \times \mathcal{G}(h_n) \to \mathbb{R}$ by

$$\lambda(K, K') := \begin{cases} 2^{-r} & \text{if } K' \in \mathcal{A}^r(K), \\ 0 & \text{else.} \end{cases}$$

Thus λ decreases by a factor of two in each step we go away from K. For given $K' \in \mathcal{G}(h_n)$, we also introduce the set $\mathcal{M}(K') := \{K \in \mathcal{M} : \ \lambda(K, K') > 0\}$. It follows by construction, that, if K' is a parent of K'', we have that $\mathcal{M}(K') \cap \mathcal{M}(K'') = \emptyset$.

Since the number of neighbors is growing linearly with r, that is, there exists an absolute constant C_0 such that $\#\mathcal{A}^r(K) \leq C_0 r$, we have for $K \in \mathcal{M}$

$$\sum_{K' \in \mathcal{G}(h_n)} \lambda(K, K') = \sum_{r \geq 0} \sum_{K' \in \mathcal{A}^r(K)} \lambda(K, K') \leq C_0 \sum_{r \geq 0} r\, 2^{-r} \leq C_1, \tag{60}$$

it follows that $\sum\limits_{K \in \mathcal{M}} \sum\limits_{K' \in \mathcal{G}(h_n)} \lambda(K, K') \leq C_1 \,\#\mathcal{M}$.

Next prove by induction on n that for all $K' \in \Delta\mathcal{M}_n$ the following inequality holds:

$$\sum_{K \in \mathcal{M}} \lambda(K, K') \geq 1. \tag{61}$$

The assertion is trivially true for $n = 0$, since the first mesh does not contain hanging nodes and therefore only marked quadrilaterals are refined.

Suppose now that (61) is true for $n - 1$. Let $K \in \Delta\mathcal{M}_n$. Then there exists a refined cell K_1 which has K as a neighbor and which has a marked child K_{11} and K is also a neighbor of K_{11}, see Figure 7. It follows by induction that $\sum_{M \in \mathcal{M}} \lambda(M, K_1) \geq 1$. The cell K is related to a marked cell by a stairway of minimal length $r \geq 1$. Next we prove (61) by induction on r, that is the inequality is true for all $K \in \Delta\mathcal{M}_n$ with minimal stairway length r. For $r = 1$ the assertion is true since $K_{11} \in \mathcal{M}_n$ and therefore

$$\sum_{M \in \mathcal{M}} \lambda(M, K) \geq \frac{1}{2} \sum_{M \in \mathcal{M}} \lambda(M, K_1) + \lambda(K_{11}, K) \geq 1.$$

Let now the assertion be true for $r - 1 \geq 1$. Let $K \in \Delta\mathcal{M}_n$ with minimal stairway length r. Then it follows that $K_1 \Delta\mathcal{M}_n$ with minimal stairway length $r - 1$. By induction we have that $\sum_{M \in \mathcal{M}} \lambda(M, K_1) \geq 1$.

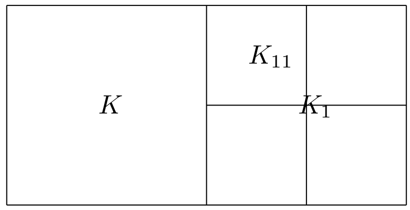

Figure 7: Situation if $K \in \Delta\mathcal{M}$.

Since $\mathcal{M}(K_1) \cap \mathcal{M}(K_{11}) = \emptyset$, we obtain

$$\sum_{M\in\mathcal{M}} \lambda(M,K) \geq \frac{1}{2} \sum_{M\in\mathcal{M}(K_{11})} \lambda(M,K_1) + \frac{1}{2} \sum_{M\in\mathcal{M}(K_1)} \lambda(M,K_1) \geq 1.$$

Finally it follows from (60) and (61) that

$$\#\Delta\mathcal{M} \leq \sum_{K'\in\Delta\mathcal{M}} \sum_{K\in\mathcal{M}} \lambda(K,K') \leq \sum_{K\in\mathcal{M}} \sum_{K'\in\mathcal{G}(h)} \lambda(K,K') \leq C_1 \,\#\mathcal{M}.$$

This concludes the proof. □

8 Carstensen interpolation on quadrilateral meshes

The purpose of this section is to prove the orthogonality and interpolation properties of the Carstensen operator which have been used for the upper bounds.

Lemma 13.

$$\langle v - C_h v, w\rangle \leq C_{i0}\, |v|_1 \left(\sum_{N\in\mathcal{N}_h^*} |\omega_N|\, |w - \pi_N w|^2_{\omega_N} \right)^{1/2} \tag{62}$$

Proof. Follows by construction. □

Lemma 14. *Let $v \in H^1_0(\Omega)$ and C_h be the quasi-interpolation defined in (12). Then there exist constants C_{i1} and C_{i1} both independant from $\mathcal{H}$ such that for $N \in \mathcal{N}_h^*$ and $S \in \mathcal{S}_h^{int^*}$*

$$|v - C_h v|_{\omega_N} \leq C_{i1}\, |\omega_N|^{1/2} |\phi_N \nabla v|_{\omega_N}, \quad |v - C_h v|_S \leq C_{i2}\, |S|^{1/2} |v|_{1,\omega_S}. \tag{63}$$

Proof. The proof follows from the Bramble-Hilbert lemma with a finite number of reference elements and the fact that the topology of ω_N may only have a finite number of configurations. □

9 Conclusions

We have proposed a new adaptive finite element algorithm on quadrilateral meshes. The proposed adaptive algorithm is based on a modification of the standard residual error estimator and uses an adaptive marking strategy.

We have given proofs for geometric convergence of the error and for the asymptotical optimality of the complexity of the resulting meshes. To this end, we have generalized the Carstensen interpolation operator to quadrilateral meshes and established a complexity estimate for the local refinement algorithm using hanging nodes.

We have restricted ourselves to the case of lowest order conforming finite elements in two dimensions. The generalization to the three dimensions and higher-order finite elements is the subject of further work.

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