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Counting Quadrics and Delaunay Triangulations and a new Convex Hull Theorem

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Abstract: Given a set $\mathcal{S}$ of n points in three dimensions, we study the maximum numbers of quadrics spanned by subsets of points in $\mathcal{S}$ in various ways. We prove that the set of empty or enclosing ellipsoids has $\Theta(n^4)$ complexity. The same bound applies to empty general cylinders, while for empty circular cylinders a gap remains between the $\Omega(n^3)$ lower bound and the $O(n^4)$ upper bound.

We also take interest in pairs of empty homothetic ellipsoids, with complexity $\Theta(n^6)$, while the specialized versions yield $\Theta(n^5)$ for pairs of general homothetic cylinders, and $\Omega(n^4)$ and $O(n^5)$ for pairs of parallel circular cylinders, respectively. This implies that the number of combinatorially distinct Delaunay triangulations obtained by orthogonal projections of $\mathcal{S}$ on a two-dimensional plane is $\Omega(n^4)$ and $O(n^5)$.

Our lower bounds are derived from a generic geometric construction and its variants. The upper bounds result from tailored linearization schemes, in conjunction with a new result on convex polytopes which is of independent interest: In even dimensions d, the convex hull of a set of n points, where one half lies in a subspace of odd dimension $\delta > \frac{d}{2}$, and the second half is the (multi-dimensional) projection of the first half on another subspace of dimension δ, has complexity only $O\left(n^{\frac{d}{2}-1}\right)$.

Key-words: Complexity, Convex hull, Quadrics, Cylinders, Projection

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Compter quadriques et triangulations de Delaunay, et un nouveau théorème sur les enveloppes convexes

Résumé : Étant donné un ensemble $\mathcal{S}$ de n points en trois dimensions, nous étudions le nombre maximum de quadriques définies de différentes manières par un sous-ensemble de points de $\mathcal{S}$. Nous démontrons que l'ensemble des ellipsoïdes vides ou englobants a une complexité $\Theta(n^4)$. La même borne s'applique au cas des cylindres en général alors que pour les cylindres circulaires un écart demeure entre la borne inférieure $\Omega(n^3)$ et la borne supérieure $O(n^4)$.

Nous nous intéressons également aux paires de d'ellipsoïdes vides et homothétiques dont la complexité est $\Theta(n^6)$, alors que les versions spécialisées donnent $\Theta(n^5)$ pour les paires de cylindres généraux vides et homothétiques et de $\Omega(n^4)$ à $O(n^5)$ pour les paires de cylindres circulaires vides et parallèles. Cela implique que le nombre de triangulation de Delaunay combinatoirement distinctes obtenues par projection orthogonale de $\mathcal{S}$ sur un plan est entre $\Omega(n^4)$ et $O(n^5)$.

Nos bornes inférieures sont démontrées par des variations autour d'une construction géométrique. Les bornes supérieures sont obtenues à partir de schémas de linéarisation et d'un résultat nouveau sur la taille des enveloppes convexes, présentant un intérêt en soi : en dimension paire d l'enveloppe convexe d'un ensemble de n points dont une moitié appartient à un sous-espace de dimension $\delta > \frac{d}{2}$, et l'autre moitié est la projection multidimensionnelle de la première moité sur un autre sous-espace de dimension δ, a une complexité de seulement $O\left(n^{\frac{d}{2}-1}\right)$.

Mots-clés : Complexité, Enveloppe convexe, Quadriques, Cylindres, Projection

1 Introduction

Counting incidences between geometric objects, and analyzing the number of objects spanned by a finite set of points, are topics of frequent interest in computational and combinatorial geometry. Prominent examples are counting the number of faces of many cells in an arrangement of curves [17], determining the number of straight lines touching a given set of spheres [18], bounding the number of empty spheres defined by a finite set of points (i.e., the number of vertices of the Voronoi diagram), and counting the number of plane graphs spanned by a planar set of points [4].

In this paper, we provide combinatorial bounds on the maximum number of 'combinatorially different' quadrics of various types in $\mathbb{R}^3$. Given a set $\mathcal{S}$ of n points in $\mathbb{R}^3$, we consider quadrics having all points of $\mathcal{S}$ on the same side. For ellipsoids or cylinders, inside and outside are clearly defined, and such a quadric will be called *enclosing* or *empty*. We also prove bounds concerning pairs of *homothetic* quadrics. Our original motivation for studying such pairs was the case of circular cylinders, in order to count the number of combinatorially different 2D Delaunay triangulations obtainable by orthogonal projection of $\mathcal{S}$. Such projections are commonly used in 3D surface reconstruction, e.g., for recovering terrains.

Naturally, our bounds depend on the number of degrees of freedom of the considered object class, which is the dimension of a manifold describing the class in any representation, or, in other words, the minimum number of independent real parameters needed to (locally) specify an object of the class. Before describing our results in more detail, let us briefly recall some facts about quadrics.

Types of quadrics

A general quadric is defined by its equation that contains 9 coefficients plus a constant factor, that is, a quadric has 9 degrees of freedom. Given a subset of k points from $\mathcal{S}$ and a family of quadrics having d degrees of freedom, there exists a family of quadrics with $d - k$ degrees of freedom that are passing through these points. Consequently, a quadric (and also an ellipsoid) can be defined by constraining it to pass through 9 given points. We will call two quadrics *combinatorially different* if they have different subsets of $\mathcal{S}$ on their boundaries. Combinatorially different quadrics spanned by $\mathcal{S}$ are the objects of interest in our complexity bounds.

Circular cylinders, which have only 5 degrees of freedom, can be locally specified by 5 points. (In fact, up to six cylinders through these points may exist [14, 9]). It turns out that finding bounds for empty circular cylinders is more involved than for general quadrics. We also study the somehow intermediate case of general (elliptic) cylinders, which have 7 degrees of freedom. A complete classification of quadrics can be found in Dupont et al. [12]. Observe that two quadrics are homothetic if their equations share the same quadratic part. Thus, a second quadric homothetic to a given one has 4 degrees of freedom (3 for translation and one for the homothety factor). For the particular case of cylinders, translations along the axis are irrelevant, and thus the second cylinder has 3 degrees of freedom. For circular cylinders, being homothetic is the same as being parallel. From this discussion we deduce that

— the number of circular cylinders (not necessarily empty) defined by a set of n points is $\Theta(n^5)$,
— the number of pairs of parallel circular cylinders is $\Theta(n^8)$,
— the number of cylinders is $\Theta(n^7)$,
— the number of pairs of homothetic cylinders is $\Theta(n^{10})$,
— the number of quadrics is $\Theta(n^9)$, and
— the number of pairs of homothetic quadrics is $\Theta(n^{13})$.

Results and related work

This paper provides improved asymptotic upper and lower bounds on the number of combinatorially different quadrics defined by a set of n points in $\mathbb{R}^3$. These bounds match in the case of general quadrics, $\Theta(n^4)$, and pairs of them, $\Theta(n^6)$. Surprisingly, both bounds also are valid for general cylinders. Thus, despite the fact that general cylinders have 2 degrees of freedom less than quadrics, the intrinsic complexity does not decrease. Gaps remain in our results for circular cylinders, $\Omega(n^3)$ and $O(n^4)$, and for pairs of parallel circular cylinders, $\Omega(n^4)$ and $O(n^5)$.

The last mentioned result carries over to the number of combinatorially different 2D Delaunay triangulations obtainable by projecting a given point set in $\mathbb{R}^3$. To our knowledge, no nontrivial bounds have been known on this problem, which can be viewed as a three-dimensional generalization of the problem of finding the different orderings of the projection of a set of points. The latter problem is well known as finding circular sequences [13] and has a complexity of $\Theta(n^2)$.

Lower bounds for maximizing the number of certain types of quadrics are derived from a generic construction and its variants. Upper bounds are obtained by applying tailored linearization schemes, introducing a space of quadrics in 9 dimensions, and a space of homothetic quadrics in 13 dimensions (that can be reduced to dimension 12 for cylinders). These spaces are similar to well-known notions such as the spaces of circles, spheres, or conics [8, 10, 13, 15].

Our second linearization scheme becomes powerful in combination with a new theorem (of separate interest) that reduces the complexity of the convex hull of a set of n points in d dimensions in certain special configurations. Namely, if $\frac{n}{2}$ of the points lie in a subspace of dimension $\delta > \frac{d}{2}$, and the second half is the (multi-dimensional) projection of the first half on another subspace of dimension δ, then the complexity is $O\left(n^{\lfloor\frac{\delta}{2}\rfloor+\lfloor\frac{d-\delta}{2}\rfloor}\right)$, which reduces the general upper bound of $O\left(n^{\lfloor\frac{d}{2}\rfloor}\right)$ to $O\left(n^{\frac{d}{2}-1}\right)$ if d is even and δ is odd. The particular case $\delta = d-1$ was already proven [7].

Work related to ours mostly concerns circular cylinders, a kind of quadric useful, for example, in metrology [2, 11] for quality measure of mechanical devices, or in 3D modeling in order to fit special surfaces in a given data set (e.g., trying to model pipes in a factory). Agarwal et al. [3] compute the set of empty unit radius cylinders in $O(n^{3+\varepsilon})$ time and show an $\Omega(n^2)$ complexity bound. Devillers [7] gives $O(n^4)$ and $\Omega(n^3)$ bounds for the number of coaxial circular cylinders defined by 6 points with all points between the two cylinders. There is special interest about smallest enclosing cylinders. Agarwal et al. [3] propose a construction algorithm with near cubic time complexity. Chan [5] proposes a faster approximation algorithm, and Schömer et al. [16] give an algorithm with bit-complexity analysis.

2 Upper bounds

2.1 Empty and enclosing ellipsoids or cylinders

Lemma 1 *The number of combinatorially different* empty *ellipsoids or cylinders defined by a set $\mathcal{S}$ of n points in $\mathbb{R}^3$ is $O(n^4)$.*

Proof: We use a classical linearization scheme: Given a point $p = (x_p, y_p, z_p) \in \mathbb{R}^3$ we define the point $p^\star = (x_p^2, y_p^2, z_p^2, x_p y_p, x_p z_p, y_p z_p, x_p, y_p, z_p) \in \mathbb{R}^9$. Further, given the quadric

$$Q_\phi = \phi_1 x^2 + \phi_2 y^2 + \phi_3 z^2 + \phi_4 xy + \phi_5 xz + \phi_6 yz + \phi_7 x + \phi_8 y + \phi_9 z = \phi_0$$

we define a hyperplane in dimension 9 (where χ_i are the coordinates in $\mathbb{R}^9$):

$$H_\phi : \phi_1\chi_1 + \phi_2\chi_2 + \phi_3\chi_3 + \phi_4\chi_4 + \phi_5\chi_5 + \phi_6\chi_6 + \phi_7\chi_7 + \phi_8\chi_8 + \phi_9\chi_9 = \phi_0$$

Now we have

$$p \in Q_\phi \iff (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7, \phi_8, \phi_9) \cdot p^\star = \phi_0 \iff p^\star \in H_\phi,$$

and if $\phi_1, \phi_2, \phi_3 \geq 0$ and the quadric is an ellipsoid or a cylinder we get

$$p \text{ inside } Q_\phi \iff (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7, \phi_8, \phi_9) \cdot p^\star < \phi_0 \iff p^\star \text{ below } H_\phi.$$

An ellipsoid (cylinder) Q_ϕ passes through points $p_1, p_2 \dots p_k \in \mathcal{S}$ and is empty of points of $\mathcal{S}$ if and only if H_ϕ passes through $p_1^\star, p_2^\star \dots p_k^\star \in \mathcal{S}^\star$ (where $\mathcal{S}^\star = \{p^\star \mid p \in \mathcal{S}\}$) and all other points of $\mathcal{S}^\star$ are above H_ϕ. Thus empty ellipsoids or cylinders correspond to supporting hyperplanes of the lower convex hull of $\mathcal{S}^\star$, which has a complexity of $O\left(n^{\lfloor \frac{d}{2} \rfloor}\right)$, where $d = 9$ is the dimension of the underlying space, that is, a complexity of $O(n^4)$. ∎

Remark 2 The same $O(n^4)$ bound applies to enclosing ellipsoids or cylinders (or quadrics in general), by arguments very similar to the previous proof.

2.2 Pairs of empty homothetic ellipsoids

Lemma 3 *The number of combinatorially different pairs of* empty homothetic *ellipsoids or cylinders defined by a set $\mathcal{S}$ of n points in $\mathbb{R}^3$ is $O(n^6)$.*

Proof: The linearization scheme here is slightly different from the one above and similar to the one used in [1, 7]. To a given point $p = (x_p, y_p, z_p) \in \mathbb{R}^3$ we associate not one but two points in $\mathbb{R}^{13}$:

$$\begin{aligned} p^\star &= (x_p^2, y_p^2, z_p^2, x_p y_p, x_p z_p, y_p z_p, x_p, y_p, z_p, 0, 0, 0, 0) \quad \in \mathbb{R}^{13} \\ \text{and} \quad p^\dagger &= (x_p^2, y_p^2, z_p^2, x_p y_p, x_p z_p, y_p z_p, 0, 0, 0, x_p, y_p, z_p, 1) \quad \in \mathbb{R}^{13}. \end{aligned}$$

Given two homothetic quadrics Q_ϕ and Q'_ϕ with equations

$$\begin{aligned} &\phi_1 x^2 + \phi_2 y^2 + \phi_3 z^2 + \phi_4 xy + \phi_5 xz + \phi_6 yz + \phi_7 x + \phi_8 y + \phi_9 z &= \phi_0 \\ \text{and}\quad &\phi_1 x^2 + \phi_2 y^2 + \phi_3 z^2 + \phi_4 xy + \phi_5 xz + \phi_6 yz + \phi'_7 x + \phi'_8 y + \phi'_9 z &= \phi'_0, \end{aligned}$$

we define a hyperplane

$$\begin{aligned} H_{\phi,\phi'} \quad : \quad & \phi_1\chi_1 + \phi_2\chi_2 + \phi_3\chi_3 + \phi_4\chi_4 + \phi_5\chi_5 + \phi_6\chi_6 \\ & +\phi_7\chi_7 + \phi_8\chi_8 + \phi_9\chi_9 + \phi'_7\chi_{10} + \phi'_8\chi_{11} + \phi'_9\chi_{12} + (\phi_0 - \phi'_0)\chi_{13} = \phi_0 \end{aligned}$$

in dimension 13 (where χ_i are the coordinates in $\mathbb{R}^{13}$). Now we have

$$\begin{aligned} p \in Q_\phi &\iff p^\star \in H_{\phi,\phi'} \\ \text{and}\quad p \in Q_{\phi'} &\iff p^\dagger \in H_{\phi,\phi'}. \end{aligned}$$

The following two statements are equivalent: (1) An ellipsoid Q_ϕ passes through points $p_1, p_2, \ldots, p_j \in \mathcal{S}$, and a ellipsoid $Q_{\phi'}$ homothetic to Q_ϕ passes through $p_{j+1}, p_{j+2}, \ldots, p_k \in \mathcal{S}$, and both Q_ϕ and $Q_{\phi'}$ are empty of other point of $\mathcal{S}$. (2) The corresponding hyperplane $H_{\phi,\phi'}$ passes through $p_1^\star, p_2^\star, \ldots, p_j^\star, p_{j+1}^\dagger, \ldots, p_k^\dagger \in \mathcal{S}^{\star\dagger} = \{p^\star, p^\dagger \mid p \in \mathcal{S}\}$ and all other points of $\mathcal{S}^{\star\dagger}$ are above $H_{\phi,\phi'}$.

Thus pairs of empty ellipsoids correspond to supporting hyperplanes of the convex hull of $\mathcal{S}^{\star\dagger}$, which has a complexity of $O\left(n^{\lfloor \frac{d}{2} \rfloor}\right)$, where $d = 13$ is the dimension of the underlying space, that is, a complexity of $O(n^6)$. ∎

2.3 Pairs of empty homothetic cylinders

Lemma 4 *The number of combinatorially different pairs of* empty homothetic *cylinders defined by a set $\mathcal{S}$ of n points in $\mathbb{R}^3$ is $O(n^5)$.*

Proof: Assume, without loss of generality, that there is no horizontal cylinder. The second cylinder can be defined from the first one by a homothety and a horizontal translation. This leads to a linearization scheme of the following form:

$$\begin{aligned} p^\star &= (x_p^2, y_p^2, z_p^2, x_p y_p, x_p z_p, y_p z_p, x_p, y_p, z_p, 0, 0, 0) &\in \mathbb{R}^{12} \\ \text{and}\quad p^\dagger &= (x_p^2, y_p^2, z_p^2, x_p y_p, x_p z_p, y_p z_p, 0, 0, z_p, x_p, y_p, 1) &\in \mathbb{R}^{12}. \end{aligned}$$

Given two homothetic cylinders Q_ϕ and Q'_ϕ with equations

$$\begin{aligned} &\phi_1 x^2 + \phi_2 y^2 + \phi_3 z^2 + \phi_4 xy + \phi_5 xz + \phi_6 yz + \phi_7 x + \phi_8 y + \phi_9 z &= \phi_0 \\ \text{and}\quad &\phi_1 x^2 + \phi_2 y^2 + \phi_3 z^2 + \phi_4 xy + \phi_5 xz + \phi_6 yz + \phi'_7 x + \phi'_8 y + \phi_9 z &= \phi'_0, \end{aligned}$$

we define a hyperplane

$$\begin{aligned} H_{\phi,\phi'} \quad : \quad & \phi_1\chi_1 + \phi_2\chi_2 + \phi_3\chi_3 + \phi_4\chi_4 + \phi_5\chi_5 + \phi_6\chi_6 \\ & +\phi_7\chi_7 + \phi_8\chi_8 + \phi_9\chi_9 + \phi'_7\chi_{10} + \phi'_8\chi_{11} + (\phi_0 - \phi'_0)\chi_{12} = \phi_0 \end{aligned}$$

in dimension 12. With arguments similar to the ones used in Lemma 3, the number of empty homothetic cylinders can be bounded by the size of the convex hull of the point set $\mathcal{S}^\star \cup \mathcal{S}^\dagger$ in dimension 12, where $\mathcal{S}^\star = \{p^\star \mid p \in \mathcal{S}\}$

and $\mathcal{S}^\dagger = \{p^\dagger \mid p \in \mathcal{S}\}$. Note that $\mathcal{S}^\star$ lies in the 9-dimensional subspace $\chi_{10} = \chi_{11} = \chi_{12} = 0$, and $\mathcal{S}^\dagger$ is its projection on the subspace $\chi_7 = \chi_8 = 1 - \chi_{12} = 0$ parallelly to the 3-dimensional vector space generated by $(0, \ldots, -1, 0, 0, 1, 0, 0)$, $(0, \ldots, 0, -1, 0, 0, 1, 0)$, and $(0, \ldots, 0, 1)$. Thus we can apply our general projection theorem (Theorem 6 in the following section) and get an upper bound of $O\left(n^{\lfloor \frac{9}{2} \rfloor + \lfloor \frac{3}{2} \rfloor}\right) = O(n^5)$. ■

Remark 5 A similar analysis for the simpler case of number of pairs of empty homothetic ellipses in 2D would give a $\Theta(n^3)$ bound. The linearization space has dimension $5 + 3$, thus applying the general projection theorem results in a complexity of $O\left(n^{\lfloor \frac{5}{2} \rfloor + \lfloor \frac{3}{2} \rfloor}\right) = O(n^3)$. A lower bound example is easy to construct; see, for example the proofs of Lemma 9 and 11.

3 General projection theorem

For special configurations of points in d dimensions, the complexity of the convex hull cannot reach the worst case of $\Theta\left(n^{\lfloor \frac{d}{2} \rfloor}\right)$. For example, the projection theorem in [7] proves that for a point set $\mathcal{V}$ of size $2n$, constructed by taking n points in a hyperplane and their parallel projections on another hyperplane, the complexity of its convex hull $CH(\mathcal{V})$ reduces to $O\left(n^{\lfloor \frac{d-1}{2} \rfloor}\right)$. This is of interest if d is even.

A multi-dimensional projection is defined by a vector space $\vec{V}$ of dimension $d - \delta$. Given a point p, its projection p' on Π' is the intersection of Π' and the affine subspace parallel to $\vec{V}$ passing through p; this projection is a unique point if $\vec{V}$ is supplementary to the direction of Π' (no $\vec{v} \in \vec{V}$ is parallel to Π').

We generalize the projection theorem to the case where the points in $\mathbb{R}^d$ live in two δ-dimensional affine subspaces, with $\frac{d}{2} < \delta < d - 1$. If d is even and δ is odd, we save a linear factor like in the original version of the projection theorem. More precisely, we have:

Theorem 6 (General projection theorem) *For any constant dimension d, let Π, Π' be two δ-dimensional affine subspaces in $\mathbb{R}^d$, with $\delta > \frac{d}{2}$, and let $\vec{V}$ be a $(d-\delta)$-dimensional vector space of $\mathbb{R}^d$ supplementary to the directions of both Π and Π'. Let $\mathcal{U} \subset \Pi$ be a set of n points, such that $CH(\mathcal{U}) \cap \Pi' = \emptyset$, and let $\mathcal{U}' \subset \Pi'$ be the projection parallel to $\vec{V}$ of $\mathcal{U}$ on Π'. Let $\mathcal{V} = \mathcal{U} \cup \mathcal{U}'$. The convex hull $CH(\mathcal{V})$ of $\mathcal{V}$ has asymptotic complexity $O\left(n^{\lfloor \frac{\delta}{2} \rfloor + \lfloor \frac{d-\delta}{2} \rfloor}\right)$. If d is even and δ is odd, the complexity becomes $O\left(n^{\frac{d}{2} - 1}\right)$.*

We remark that the case $\delta \leq \frac{d}{2}$ can be solved easily, giving a bound of $O\left(n^{2\lfloor \frac{\delta}{2} \rfloor}\right)$. To prove the theorem we need two technical lemmata. Given two faces G and F of a convex polytope, call G a *superface* of F if F is a face of G.

Lemma 7 *Let F be a face of dimension k of a convex polytope P in d dimensions. The number of superfaces of F is $O\left(n^{\lfloor \frac{d-k-1}{2} \rfloor}\right)$.*

Proof: We project P parallel to its face F to some suplementary affine subspace A of dimension $d-k$. This gives a $(d-k)$-polytope P'. The face F projects to a point p, and each superface of F of dimension $j>k$ projects to a superface of p of dimension $j-k>0$. It therefore suffices to bound the number of the latter faces.

To this end, consider the intersection of P' with a hyperplane in A that cuts off the vertex p (i.e., the vertex figure of p). This is a convex polytope P'' of dimension $d-k-1$, and the faces of P'' correspond in a bijective way to the superfaces of p to be counted. Thus their number is $O\left(n^{\left\lfloor\frac{d-k-1}{2}\right\rfloor}\right)$. ■

Lemma 8 *Let F be a $(d-1)$-face of $CH(\mathcal{V})$. The projection F^Π of F on Π contains a $(\delta-1)$-face $f_\mathcal{U}$ of $CH(\mathcal{U})$.*

Proof:

Projecting F along the $(d-\delta)$-dimensional vector space $\vec{V}$ results in a polyhedron in Π of dimension at least $\delta-1$, because the dimension of F was $d-1$. Therefore F^Π contains a $(\delta-1)$-dimensional face of $CH(\mathcal{U})$, if there exists no pair of points $x,y\in CH(\mathcal{U})\setminus F^\Pi$ such that $\overline{xy}\cap F^\Pi\neq\emptyset$ where $\overline{xy}$ denotes the line segment xy.

To get a contradiction, assume that such a pair of points x,y does exists. Assume that there exist two parallel hyperplanes H_0 and H_1 such that $\Pi\subset H_0$ and $\Pi'\subset H_1$. Without loss of generality, we call H_0 the above hyperplane and H_1 the below hyperplane. Since H_0 and H_1 are parallel, $\mathcal{U}'$ is below H_0 and $\mathcal{U}$ above H_1, thus $CH(\mathcal{V})$ is between these two hyperplanes. Let $H_\lambda, \lambda\in[0,1]$, be a hyperplane parallel to H_0 sweeping the slice of $\mathbb{R}^d$ between H_0 and H_1 containing $CH(\mathcal{V})$. Let z_0 and z_1^Π be the intersections of $\overline{xy}$ with the boundary of F^Π, z_1 the projection of z_1^Π on Π'. Let z_λ the projection of $(1-\lambda)z_0+\lambda z_1$ on H_λ and z_λ^Π its projection on Π. Let also F_λ be $F\cap H_\lambda$ and F_λ^Π its projection on Π. During the sweep, z_λ describes a 1-dimensional line-segment, z_λ^Π is moving from z_0 to z_1^Π inside F^Π, and F_λ^Π is sweeping F^Π from F_0 to F_1^Π. Thus for some $\lambda\in]0,1[$, $z_\lambda^\Pi\in F_\lambda^\Pi$ which gives a contradiction since z_λ must be in $CH(x,y,x',y')$ (where x',y' are the projections of x,y in Π') which is stricly inside $CH(\mathcal{V})$ and at the same time in F_λ which is on the boundary of $CH(\mathcal{V})$.

The two hyperplanes H_0 and H_1 containing Π and Π' can be easily constructed, but they are not necessarily parallel. This can be solved using a projective transformation moving $\Pi\cap\Pi'\subset H_0\cap H_1$ to infinity which makes H_0 and H_1 parallel and does not change the convex hull of $\mathcal{V}$ because of the hypothesis that $CH(\mathcal{U})$ does not intesect $\Pi\cap\Pi'$.

■

Proof: (of Theorem 6) Using Lemma 8 we charge each $(d-1)$-face F of $CH(\mathcal{V})$ to $F_\mathcal{U}$, a $(\delta-1)$-face of $CH(\mathcal{U})$ contained in F^Π. Let G be the face of $CH(\mathcal{V})$ formed by the δ vertices of F that project to the vertices of $F_\mathcal{U}$.

Given $F_\mathcal{U}$, there are at most 2^δ such faces G (choosing points from $\mathcal{U}$ or $\mathcal{U}'$). All the superfaces of G can be charged on $F_\mathcal{U}$. By applying Lemma 7 to G, we get that $F_\mathcal{U}$ is charged at most $O\left(2^\delta n^{\left\lfloor\frac{d-\delta}{2}\right\rfloor}\right)=O\left(n^{\left\lfloor\frac{d-\delta}{2}\right\rfloor}\right)$ times. The number of $(\delta-1)$-faces $F_\mathcal{U}$ is bounded by the size of $CH(\mathcal{U})$, which is $O\left(n^{\left\lfloor\frac{\delta}{2}\right\rfloor}\right)$.

It remains to prove that the bound applies also to lower dimensional faces of $CH(\mathcal{V})$. Because of the special configuration of the points, by construction some $(d-1)$-faces of $CH(\mathcal{V})$ are not simplices and have more than d points. If we assume general position of the point set $\mathcal{U}$ in Π, any k-face of $CH(\mathcal{U})$ has exactly $k+1 \leq \delta$ vertices. Thus a $(d-1)$-face of $CH(\mathcal{V})$, which links a face of $CH(\mathcal{U})$ to a face of $CH(\mathcal{U}')$, has less than 2δ vertices. Since $2\delta = O(1)$ the number of subfaces of such a $(d-1)$-face is constant and the theorem follows. The general position hypothesis in $\mathcal{U}$ can be removed as for the classical bound on convex hull by perturbation scheme and by observing that degeneracies in $\mathcal{U}$ only decrease the convex hull size. ■

4 Lower bounds

We now turn our attention to lower bound constructions. We exhibit families of points lying on a few straight lines that allow for a large number of empty (or enclosing) quadrics of several types. By slightly perturbing these sets we can achieve general position without altering the number of objects counted.

4.1 Empty circular cylinders

Lemma 9 *There exists a set $\mathcal{S}$ of n points in $\mathbb{R}^3$ such that the number of combinatorially different empty circular cylinders defined by $\mathcal{S}$ is $\Omega(n^3)$.*

Proof: Consider the points $p_0 = (1,0,0)$ and $q_0 = (0,1,0)$. Then the ellipses in the plane $z = 0$ and tangent to the x axis at p_0 and to the y axis at q_0 form a continuous family that can be parameterized by $\lambda \in [1,\infty[$, the ratio of the long axis to the small axis (Figure 1-right). Denote with E_λ one of these ellipses and consider the ellipses $E_{1+\frac{k}{n}}$ for $0 \leq k < N$ for $N = \frac{n}{3}$.

Through each of these ellipses, we can fit one circular cylinder such that the vector of the direction of the cylinder axis is positive in all three coordinates (there is another one with only x and y values negative).

Now, since the slopes of those cylinders are different, we get disjoint ellipses if we consider a cross section of the cylinders by a plane $z = h$ for large enough h. Thus we can add points r_k on the line $x - y = z - h = 0$ that separate these ellipses as indicated in Figure 1-left. By slightly tilting the cylinders until for every k the cylinder corresponding to $E_{1+\frac{k}{n}}$ touches r_k, we create a linear size family of cylinders through p_0, q_0 and r_k which are tangent to the x and y axes.

Considering points $p_i = (1 + \frac{i\varepsilon}{n}, 0, 0)$ and $q_j = (0, 1 + \frac{j\varepsilon}{n}, 0)$ $(0 \leq i, j < N)$ for small enough ε performs a small perturbation on the ellipses and so does neither change the existence of the cylinders nor the fact that they are empty, since all cylinders are tangent to x and y axes which contain the p_i and q_j points (Figure 2). This construction produces empty circular cylinders through p_i, q_j, r_k for all i, j, k. ■

Remark 10 To construct circular cylinders passing through 5 points, one can proceed in the following way: Considering one of the cylinders constructed above passing through 3 points of $\mathcal{S}$, one can relax the tangency conditions with respect to the x-axis and y-axis. Then the cylinder can be moved in a two degrees of freedom family incident to the three points. Playing with these two degrees of freedom we can obtain circular cylinders through five points $p_i, p_{i+1}, q_j, q_{j+1}, r_k$ for all i, j, k.

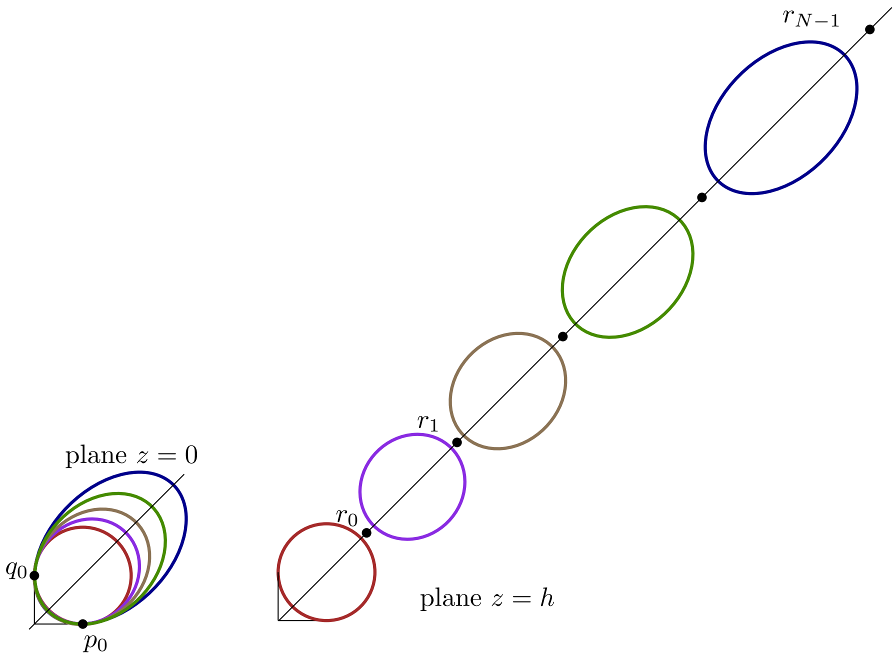

Figure 1: Cross sections of cylinders through p_0 and q_0 and tangent to x and y axes.

4.2 Empty pairs of parallel circular cylinders

Lemma 11 *There exists a set $\mathcal{S}$ of n points in $\mathbb{R}^3$ such that the number of combinatorially different pairs of empty parallel circular cylinders defined by $\mathcal{S}$ is $\Omega(n^4)$.*

Proof: Let now $N = \frac{n}{4}$. We add a family of points to the construction in the proof of Lemma 9, namely $s_l = (L + \frac{l\varepsilon}{n}, 0, 0)$, $0 \leq l < N$, and a single point $u = (L-1, 1, 0)$, for L large enough. For each of the $\Omega(n^3)$ cylinders through p_i, q_j, r_k for all $0 \leq i, j, k < N$ we can place a disjoint parallel circular cylinder through u and any of the points s_l and tangent to the x axis, avoiding all the points p_i, q_j, r_k. ■

Remark 12 Getting more than a linear number of empty circular cylinders parallel to a given cylinder is not possible. If one fixes a cylinder, the projection in the direction of the cylinder's axis associates empty parallel cylinders to empty circles which are well known to be in linear size (Delaunay triangulation has linear complexity). Thus extending the example of the proof of Lemma 9 for getting more than an extra linear factor seems to be a difficult task.

4.3 Empty general cylinders (and pairs of them)

Lemma 13 *There exists a set $\mathcal{S}$ of n points in $\mathbb{R}^3$ such that the number of combinatorially different empty general cylinders defined by $\mathcal{S}$ is $\Omega(n^4)$. More-*

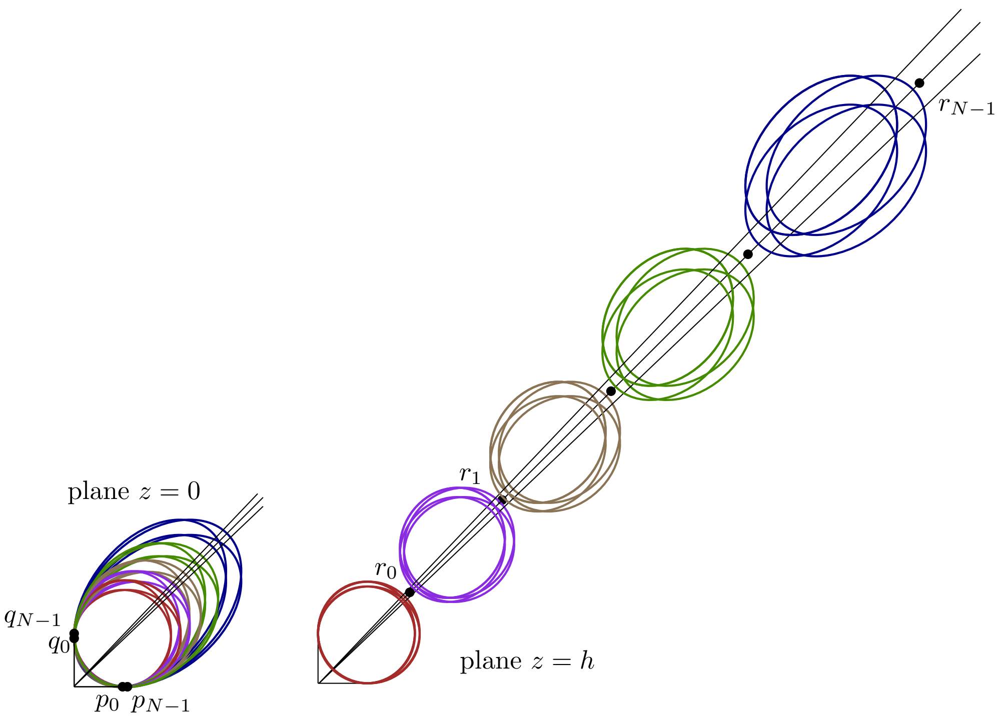

Figure 2: Cross sections of cylinders through p_i and q_j and tangent to x and y axes.

over, the number of combinatorially different pairs of empty homothetic general cylinders defined by $\mathcal{S}$ is $\Omega(n^5)$.

Proof: As in the proof of Lemma 9, we again take the points p_0 and q_0 and the ellipses $E_{1+\frac{k}{n}}$ for $0 \leq k < N$ with $N = \frac{n}{4}$. In addition we consider the point $t_0 = (0, 0, -1)$. Then we construct the general cylinders that pass through the ellipses $E_{1+\frac{k}{n}}$ and the point t_0, and that are tangent to the line $x+y = z+1 = 0$. Now the situation is very similar to the one in Lemma 9 and an intersection of the cylinders with a plane $z = h$ for large enough h leads to disjoint ellipses as shown in Figure 1-right. Thus adding the points p_i and q_j as in Lemma 9, and points $t_m = (\frac{m\varepsilon}{n}, -\frac{m\varepsilon}{n}, -1)$, $1 \leq i, j, m < N$, gives $\Omega(n^4)$ cylinders through points p_i, q_j, r_k and t_m for any i, j, k, m.

Similar to the proof of Lemma 11, taking $N = \frac{n}{5}$ and adding points $s_l = (L+\frac{l\varepsilon}{n}, 0, 0)$, $0 \leq l < N$ completes the construction for $\Omega(n^5)$ pairs of homothetic cylinders. ∎

Remark 14 The $\Omega(n^4)$ lower bound on general empty cylinders matches the $O(n^4)$ upper bound for empty quadrics in Lemma 1. Thus we have reached a tight situation, $\Theta(n^4)$, for both cases.

4.4 Empty ellipsoids (and pairs of them)

Lemma 15 *There exists a set $\mathcal{S}$ of n points in $\mathbb{R}^3$ such that the number of combinatorially different empty ellipsoids defined by $\mathcal{S}$ is $\Omega(n^4)$. Moreover, the number of combinatorially different pairs of empty homothetic ellipsoids defined by $\mathcal{S}$ is $\Omega(n^6)$.*

Proof: We need 9 constraints to define a quadric, e.g. if we consider the following families of points: $p_i = (1, \frac{i\varepsilon}{n}, 0)$, $q_j = (-1, 0, \frac{j\varepsilon}{n})$, $r_k = (\frac{k\varepsilon}{n}, 1, 0)$, and $s_l = (0, \frac{l\varepsilon}{n}, 1)$, $0 \leq i, j, k, l < N = \frac{n}{5}$ and $t = (0, 0, -1)$, there are ellipsoids through p_i, q_j, r_k, s_l, t and tangent to the lines containing our four families of points at p_i, q_j, r_k, s_l for any i, j, k, l. This construction gives an example of an $\Omega(n^4)$ size set of empty ellipsoids. For $i = j = k = l = 0$ the ellipsoid is a sphere, and thus a homothet is nothing else than another sphere. Thus using a set of points providing a quadratic example for the 3D Delaunay triangulation such as $t_m = (L + m, m^2, m^3)$, $0 \leq m \leq N$, for L large enough, we get a quadratic number of pairs of spheres: the sphere through p_0, q_0, r_0, and s_0 and the $\Theta(n^2)$ spheres corresponding to the edges of the Delaunay triangulation of the points t_m.

For ε small enough, the picture remains valid for each of the $\Theta(n^4)$ almost spherical ellipsoids defined by p_i, q_j, r_k, and s_l, providing the claimed bound. ∎

4.5 Enclosing ellipsoids or cylinders

Lemma 16 *There exists a set $\mathcal{S}$ of n points in $\mathbb{R}^3$ such that the number of combinatorially different enclosing ellipsoids defined by $\mathcal{S}$ is $\Omega(n^4)$.*

Proof: We can slightly modify the lower bound example of Lemma 15. Given an ellipsoid E and five points p_0, q_0, r_0, s_0, and t on it, we add families p_i, q_j,

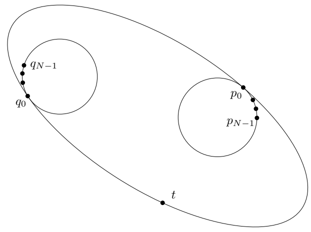

Figure 3: Construction of a set of $\Omega(n^2)$ enclosing ellipses (see the proof of Lemma 16)

r_k, and s_l lying on circles tangent to E at points p_0, q_0, r_0, and s_0 respectively and inside E (see Figure 3-left for a 2D illustration). There is a unique quadric passing through p_i, q_j, r_k, s_l, and t and tangent to the circles. If each family is close to its reference point, and the circles radii are sufficiently large, then the quadric is close to E and is an ellipsoid with all points inside. We have defined $\Omega(n^4)$ perturbed copies of E that are combinatorially different. ■

Lemma 17 *There exists a set $\mathcal{S}$ of n points in $\mathbb{R}^3$ such that the number of combinatorially different enclosing cylinders defined by $\mathcal{S}$ is $\Omega(n^3)$.*

Proof: The proof is very similar to the previous one, with only three families of points p_i, q_j, r_k, and a fourth point t since we have only 7 degrees of freedom. ■

Remark 18 The $\Omega(n^2)$ lower bound for enclosing circular cylinders of a given radius shown by Agarwal et al. [3] applies also to enclosing circular cylinders in general.

5 2D Delaunay triangulations from projection

Corollary 19 *Given a set of n points in $\mathbb{R}^3$, the number of combinatorially different 2D Delaunay triangulations obtainable by projecting these points on a plane is $O(n^5)$. Moreover, there exist sets of n points in $\mathbb{R}^3$ such that this number is $\Omega(n^4)$.*

Proof: Given a set $\mathcal{S}$ of n points we consider the possible directions of projection in $\mathbb{R}^3$, i.e. elements of the sphere $\mathbb{S}_2$. When that direction moves, the 2D Delaunay triangulation of the projection remains the same except at some places where the diagonal of some quadrilateral flips in the triangulation. The directions where a given quadrilateral defines a flip, or in other words, where the four points look cocircular, are exactly the directions of the empty circular cylinders passing through the four corresponding points in $\mathbb{R}^3$. These directions define a cubic curve in $\mathbb{S}_2$ [9]. On the sphere $\mathbb{S}_2$ a diagram can be drawn such that, if the direction of projection remains within one cell, the Delaunay triangulation of the projection remains combinatorially the same. Edges of that diagram correspond to a single flip and are pieces of the cubic curves described

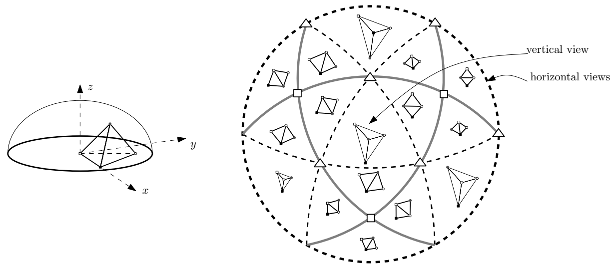

Figure 4: Diagram of the different Delaunay triangulations of the 4 vertices of a regular tetrahedron. (Figure taken from [9]).

above. The vertices of the diagram correspond to the intersections of these curves. In this way, we obtain a planar graph on $\mathbb{S}_2$. Its complexity is linear in the number of its vertices, which correspond to the directions of projection where two (or more) flips occur simultaneously. Such a direction is the one of (at least) two empty parallel circular cylinders, each with four points on its boundary. The claimed upper bound now follows immediately from Lemma 4. See Figure 4 where a simple example of the diagram on $\mathbb{S}_2$ is shown.

For the lower bound we have to modify the construction used in the proof of Lemma 11 so that both cylinders of a pair have 4 points on their boundary. In the cross section plane $z = h$ we fit the points r_k (see the proof of Lemma 9) between the disjoint ellipses of the cylinders through the points s_l (see proof of Lemma 11), instead of putting them between the ellipses corresponding to the cylinders built on $E_{1+\frac{k}{n}}$. This construction further ensures that the $\Omega(n^4)$ pairs of empty parallel circular cylinders have $\Omega(n^4)$ different cylinder axis directions. Thus we obtain $\Omega(n^4)$ different projected 2D Delaunay triangulations. ■

Remark 20 We consider Voronoi diagrams for convex metrics associated to ellipses, an ellipse being defined (up to translation and homothety) by two parameters such as orientation and excentricity. We can draw a diagram in parameter space whose cells describe ellipses producing the same Voronoi diagram. The vertices of such a diagram correspond to pairs of homothetic empty ellipses and have complexity $\Theta(n^3)$ by Remark 5.

6 Conclusion and open problems

Let us summarize the obtained complexity results for quadrics in the table below.

Using random sampling theory [6], our bounds can be extended to the number of quadrics that contain at most k points, resulting in bounds of $O(kn^4)$ for circular cylinders, $O(k^3n^4)$ for general cylinders, and $O(k^5n^4)$ for ellipsoids.

One obvious open problem is to reduce the gaps between upper and lower bounds for enclosing cylinders, empty circular cylinders, and pairs of parallel empty circular cylinders.

We conjecture that the lower bound $\Omega(n^3)$ for empty circular cylinders is tight: When considering empty circles and circles containing at most k points in 2D, the complexities are respectively $O(n)$ and $O(k^2 n)$ i.e. quadratic in k, so the linear increase in k for circular cylinders in 3D shown by the above $O(kn^4)$ bound would in fact be surprising. An $O(n^3)$ bound on empty circular cylinders would imply, again using random sampling theory, an $O(k^2 n^3)$ bound on circular cylinders containing at most k points. This seems to be more reasonable than the above $O(kn^4)$ bound. Similarly, the lower bound for pairs of empty circular cylinders is probably tight.

Object	Property	Degrees of Freedom	Ref. or Lemmas	Complexity
Circular Cylinders	Empty	5	1, 9	$O(n^4)$, $\Omega(n^3)$
	Fixed radius and empty	4	[1]	$O(n^{3+\varepsilon})$, $\Omega(n^2)$
	Enclosing	5	2, [1]	$O(n^4)$, $\Omega(n^2)$
Pairs of	Coaxial and "sandwiching"⋆	6	[7]	$O(n^4)$, $\Omega(n^3)$
	Parallel and empty	8	4, 11	$O(n^5)$, $\Omega(n^4)$
General Cylinders	Empty	7	1, 13	$\Theta(n^4)$
	Enclosing	7	2, 17	$O(n^4)$, $\Omega(n^3)$
Pairs of	Homothetic and empty	10	4, 13	$\Theta(n^5)$
Ellipsoids	Empty	9	1, 13	$\Theta(n^4)$
	Enclosing	9	2, 16	$\Theta(n^4)$
Pairs of	Empty	13	3, 15	$\Theta(n^6)$

From an algorithmic point of view, it is possible to use our approach to actually find the (pairs of) empty quadrics or cylinders in $O(n^4)$ time ($O(n^6)$ or $O(n^5)$ time for pairs). For circular cylinders, we compute the convex hull in dimension 9 giving the empty quadrics. One 4-dimensional face of the convex hull describes all the empty quadrics passing through 5 points. We then have to check if one of the at most 6 circular cylinders, passing through the 5 points, belongs to this set of quadrics. The method is similar for other cases, using the relevant convex hull and dimension for a face.

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⋆ a pair of coaxial circular cylinder sandwiches a set of points if all points are in between the inner and outer cylinders.

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