Design of Stable Feedback Controllers for Second Order Systems with Varying Sampling Rate: LQ and Lie-Algebraic Approaches

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DESIGN OF STABLE FEEDBACK CONTROLLER FOR SECOND ORDER SYSTEMS WITH VARYING SAMPLING TIME RATE: LQ AND LIE-ALGEBRAIC APPROACHES

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Thème – Real Time Control Systems

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Abstract: In this article, we consider the design of a controller family that given a second-order continuous-time linear plant controlled at a varying rate, asymptotically stabilizes the closed loop and provides a good performance. Rate adaptation of control task execution is increasingly used in order to optimize allocation and throughput of shared resources in embedded systems.

The LQ technique, tipically used to adapt the controller parameters to rate variation, is compared with a Lie-algebraic method [7] which guarantees the existence of a common Lyapunov function for the varying-time system. The use of a performance index as a function of the rate, derived from the discrete Lyapunov function and related with closed-loop eigenvalues, simplifies the evaluation of the cost associated with each rate.

Keywords: Varying Rate Control Design, Embedded systems, Cost Functions, Lie-Algebraic Approach.

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1. INTRODUCTION

In embedded systems, an important amount of information is exchanged between the system nodes (sensors, controllers, actuators) through the common and limited resource, causing that the real-time deadlines of control tasks may be violated due to execution delays, which can lead to performance degradation or even instability of a control system.

In order to guarantee that the instances of a control task are correctly executed (completing their execution before its respective deadline), different resource management strategies have been proposed in several papers.

One strategy adjusts tasks periods at run time [5], [4], [9], and another one uses a dropout policy called (*m,k*)-firm [10] [12], which means that if only *m* out of any *k* consecutive task instances can be schedulable, then only *m* instances are executed and the other (*k-m*) instances are rejected.

But, as the control performance under this variable instance's execution time can suffer a very important degradation, an useful solution is the adaptation of control law parameters to these variations.

To address this problem, [5], [4], [9] have proposed the use of optimal design of state-feedback linear quadratic controllers (LQ) to adapt control law parameters. But, as shown in [15], control systems designed with this optimal-LQ technique, may suffer from instability under certain switching sequences.

As a given control system undergoing rate variation can be thought of as a concatenation of systems in time, it can be modeled as a Discrete-Time Switched System (DTSS). A review of available results to study asymptotic stability of switched systems was presented in [11]. In particular, it was established that if a family of systems that constitutes a DTSS has a Common Lyapunov Function (CLF), then, asymptotic stability is guaranteed for any switching sequence

In consequence, in [6] [3], the LQ design have been complemented with "a-posteriori" search of a common Lyapunov function, in order to guarantee stability. The problem with this technique is that it not gives information about how to re-design the LQ controller in the case that a CLF does not exist for the original LQ design.

Another proposition to solve this design problem have been approached in [14] adopting a LMI framework to synthesize optimal controllers that guarantee closed-loop stability under any sampling sequence, once again searching a CLF. In spite of Sala solves the problem with a unique common controller, his result can be extended to find a family of adaptative controllers.

In [7] a Lie-algebraic approach was proposed to solve the problem of adaptation of control laws. Specifically, we propose to choose the controller parameters in order for the Lie algebra generated by the closed-loop matrices (each matrix is associated with a different rate) to be solvable. The solvability of the Lie algebra generated by a family of stable matrices, is a sufficient condition for the asymptotic stability of the discrete-time switching system represented by them, because it guarantees the existence of a CLF for the family. In the case of second order linear systems, we obtained an exact and analytical solution to controller adaptation, used in this article.

Then, the goal of the resource management strategy is to maximizes control performance within the available resources, so it implies to satisfy both, the highest level of the overall control performance and the control task schedulability (solved by rate variation). Therefore, it requires to define the relation between each control task period and control performance. In [4], the LQ cost was defined as the perfomance index (because controllers are designed with LQ technique) which is calculated using the solution to the algebraic Riccati equation S(h) over an horizon of time and in [3] it was extended this idea to the evaluation of an exact Lyapunov matrix. Both matrices takes the intersample behavior into account. But, as we analyse in section 2, the assumption that each cost function is a function monotonically increasing of the control task period is not general, it depends on the selected time-horizon.

In section 3, we propose to use a quadratic cost which considers the common Lyapunov quadratic function, that in the case of Lie-algebraic designed control can be simplified to the evaluation of products of spectral radius of closed-loop matrices involved in the sequences.

2. PROBLEMS WITH LQ DESIGN.

As already pointed-out, a system controlled by the control laws adapted to execution tasks variation, can suffer from instability when the problem design was solved with the LQ approach. But, supposing this stabiliy problem has been solved by the verification of the existence of a CLF for the DTSS, the use of LQ cost to adjust the rate cand depend on the time horizon.

The objective of the LQ control design is to minimize a continuous-time cost function see [1], over a time horizon (H). Disregarding noise, this cost function can be replace by a sampled cost function as follow

$$J\left(N,seq\right)=\underbrace{\sum_{k=0}^{N-1}\left(x_k^T\left(Q_k^{'}+2*M_k^{'}L_k+L_k^TR_k^{'}L_k\right)x_k\right)}_{1}+\underbrace{x_N^TQx_N}_{2} \qquad (1)$$

where N=Tbs/h is assumed to be an integer and the matrices

$$Q_k^{'}=\int_0^{h_k}\left(e^{At}\right)^T(t)\,Q\,e^{At}dt\,,\;M_k^{'}=\int_0^{h_k}\left(e^{At}\right)^T Q\,\Gamma(t)\,dt\quad R_k^{'}=\int_0^{h_k}\left(\Gamma^T(t)Q\Gamma(t)+R\right)dt \qquad (2)$$

take the intersample behavior into account, and being Q and R are the design parameters.

The first term of equation (1) represents the exact energy used to go from initial condition x_0, following a given sequence *seq*, to the final state value x_N (at time H). The second term takes into account this final value.

For any sequence, equation (1) represents a sum as

$$J(N,seq) = x_0^T\left(Q_0^{'} + 2*M_0^{'}L_0 + L_0^T R_0^{'} L_0\right)x_0 + x_1^T\left(Q_1^{'} + 2*M_1^{'}L_1 + L_1^T R_1^{'} L_1\right)x_1 + x_2^T\left(Q_2^{'} + 2*M_2^{'}L_2 + L_2^T R_2^{'} L_2\right)x_2 + .. + x_N^T Q x_N \tag{3}$$

If optimal control is used, the optimal cost would be

$$J(N,seq) = x_0^T\, S_N(seq)\, x_0 \tag{4}$$

where $S_N(seq)$ is the solution to the algebraic Riccati equation (calculated iteratively) for a given sequence.

In the seminal paper [5] an infinite horizon (H=∞) was used to solve the design of LQ controllers (they considered also the noise influence). But, as the management strategy must to adjust the task periods, and consequently the controller parameters, as soon as a change in the resource utilization is detected, then it is more appropiate to solve the optimization problem over a finite time horizon. I.e. to find $\min_{h1,..hn} \sum_{j=1}^{n} J_j\left(x_j, hj, H\right)$ where H is a finite number (5)

In some articles [4] [9] it was proposed that the time horizon H is equal to the sampling time of the management strategy Tbs executed by the "task scheduler". But in this case, there is a trade-off between an adequate control of the ressource utilization and the "introduced overhead" due to the management strategy evaluation. Other approach considers an event-based execution of the management strategy (it only reacts when some utilization measure changes) [6].

The assumption considered in several articles is that the number Tbs/hj into equation (5) is integer for all the plants *j=1.. n*, but, of course, it is not always true. So, if we do not take into account the real relation between control task and scheduler periods, a "sub-optimal" can be obtained.

In [5], [9], [4] and [14] it was considered that cost functions can be described as exponential, linear or quadratic functions of hj. This approximation yields explicit solutions for task frequency assignment, by applying the Kuhn-Tucker conditions. However, as tipically the cost function can not be aproximated by this kind of functions, in [3] it was proposed to calculate off-line the controller parameters and the factor S(hj) for each plant and to store these matrices into a table to be used by the optimization routine. But, the use of this technique requires that each cost function (4) will be monotonically increasing in h. And as we show in next subsection it is not necesarilly true in the case of S(hj). In [3], the cost functions are convex functions of the control period because they are evaluated for a particular common initial condition.

$S_{TBS}(h)$ used in the cost function

Given two sampling periods h1 and h5=5*h1, and the system described by

$$A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \quad B=\begin{pmatrix}0\\ b_0\end{pmatrix}$$

where b_0 =128.

Q=diag(1250, 250) and R=10^6 and the nominal period is 0.1s.

To compare both performances, we chose a time horizon Tbs=5*h1 (equal maximum sampling period).

The exact finite horizon cost is calculated using equation (3)

$$\begin{aligned}J(N,seq) &= x_0^T\left(Q_0^{'}+M_0^{'}L_0+L_0^TR_0^{'}L_0\right)x_0+x_0^T\left(\Phi_0^{CL}\right)^T\left(Q_1^{'}+M_1^{'}L_1+L_1^TR_1^{'}L_1\right)\Phi_0^{CL}x_0+\\ &\ldots+x_0^T\left(\Phi_{N-1}^{CL}\right)^T*\ldots*\left(\Phi_1^{CL}\right)^T*\left(\Phi_0^{CL}\right)^T\left(Q_N^{'}+M_N^{'}L_N+L_N^TR_N^{'}L_N\right)\Phi_0^{CL}*\Phi_1^{CL}*\ldots*\Phi_{N-1}^{CL}x_0\\ &+x_0^T\left(\Phi_{N-1}^{CL}\right)^T*\ldots*\left(\Phi_1^{CL}\right)^T*\left(\Phi_0^{CL}\right)^T Q\,\Phi_0^{CL}*\Phi_1^{CL}*\ldots*\Phi_{N-1}^{CL}x_0\end{aligned} \tag{6}$$

For the period h1 (Tbs=5*h1 then N=5) then

$$\begin{aligned}J(Tbs,h1) &= x_0^T\left(Q_0^{'}+M_0^{'}L_0+L_0^TR_0^{'}L_0\right)x_0+x_0^T\left(\Phi_1^{CL}\right)^T\left(Q_1^{'}+M_1^{'}L_1+L_1^TR_1^{'}L_1\right)\Phi_1^{CL}x_0+..\\ &+x_0^T\left(\left(\Phi_1^{CL}\right)^4\right)^T\left(Q_5^{'}+M_5^{'}L_5+L_5^TR_5^{'}L_5\right)\left(\Phi_1^{CL}\right)^4x_0+x_0^T\left(\left(\Phi_1^{CL}\right)^5\right)^T Q\left(\Phi_1^{CL}\right)^5x_0=x_0^TS_{Tbs}(h1)x_0\end{aligned} \tag{7}$$

and for the period h5

$$J(Tbs,h5)=x_0^T\left(Q_0^{'}+M_0^{'}L_0+L_0^TR_0^{'}L_0\right)x_0+x_0^T\left(\Phi_5^{CL}\right)^T Q\,\Phi_5^{CL}x_0=x_0^TS_{Tbs}(h5)x_0 \tag{8}$$

Then, if we would like to compare the cost for these two different sequences, from the same initial condition, we must calculate the difference between costs (7) and (8)

$$J(Tbs,h1)-J(Tbs,h5)=x_0^T\left(S_{Tbs}(h1)-S_{Tbs}(h5)\right)x_0 \tag{9}$$

In order to find the best sequence (in the sense of lower cost) over the horizon Tbs, we must analyze if the matrix $S_{Tbs}(h1)-S_{Tbs}(h5)$ is positive or negative definite.

In this example this matrix $S_{Tbs}(h1)-S_{Tbs}(h5)=\begin{bmatrix}-1760.1 & -992.7\\ -658.5 & -362.1\end{bmatrix}$ is an indefinite matrix.

Therefore, there are some initial conditions were $J(Tbs,h1)>J(Tbs,h5)$ and another ones were $J(Tbs,h1)<J(Tbs,h5)$.

Let be the initial condition x_0= [x_{01} x_{02}]', if {-.58869 x_{02}< x_{01}< -.349493 x_{02}}, the cost for h5 is lower than the cost for h1 as we can see in the figure 1. I.e., the cost function (4) is not monotonically increasing in h.

But, if we evaluate both cost functions when Tbs=∞, then $S_\infty(h5) = \begin{bmatrix} 1066.4 & 349.2080 \\ 281.516 & 246.20378 \end{bmatrix} > S_\infty(h1) = \begin{bmatrix} 1003.88 & 291.139 \\ 264.047 & 222.10 \end{bmatrix}$

It means that, considering an infinite horizon, the cost function for h5 gives an upper value than the calculated for h1, for all initial conditions. In consequence, this cost is a function of the selected time horizon Tbs.

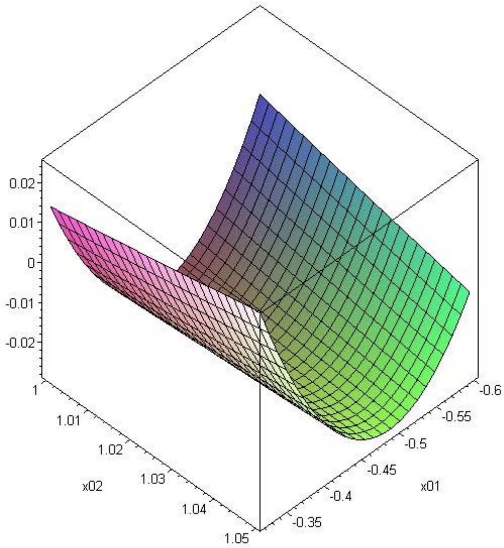

Figure 1: $J(Tbs, h5)$ - $J(Tbs, h1)$ as functions of initial conditions.

In figure 2 we show the evolution of cost functions, starting from the same initial condition, and applying three sequences.

Seq1	h1-h1-h1-h1-h1,	h1-h1-h1-h1-h1,	h1-h1-h1-h1-h1,	h1-h1-h1-h1-h1
Seq2,	h5 ,	h5 ,	h5 ,	h5
Seq3,	h5 ,	h1-h1-h1-h1-h1,	h1-h1-h1-h1-h1,	h1-h1-h1-h1-h1
0	Tbs,	2Tbs,	3Tbs,	4Tbs

Using equation (9), and for the initial condition [-0.57 1] ^{T}, we verify that $J(Tbs, h5) <$ $J(Tbs, h1)$. If the lowest cost selection routine is used, the task period selected will be h5 (best sequence is given by $x_5 = \left(\Phi_5^{CL}\right) x_0$).

From time Tbs to time 2Tbs, the cost is lower for period h1 than for h5, $J\left(2Tbs-Tbs,h1\right)<J\left(2Tbs-Tbs,h5\right)$, then the task period selected by the routine will be h1. A similar result results for the interval of time [2Tbs, 3Tbs]. I.e., until 3Tbs, the sequence that gives lowest cost is $x_{10}=\left(\Phi_1^{CL}\right)^5 x_5$.

The same result remains from 3Tbs to the infinite. Then, the cost associated with the task period h1 corresponds to the sequence $x_{15}=\left(\Phi_1^{CL}\right)^5 x_{10}$.

Using the previous statements, the sequence that gives the lowest cost should be $x_{15}=\left(\Phi_1^{CL}\right)^5\left(\Phi_1^{CL}\right)^5\left(\Phi_5^{CL}\right)x_0$. But, as we can see in figure 2, it is not true, because the minimum cost is obtained for sequence $x_{15}=\left(\Phi_1^{CL}\right)^5\left(\Phi_1^{CL}\right)^5\left(\Phi_1^{CL}\right)^5 x_0$. In consequence, if the cost function proposed in [3] is used by the routine, a suboptimal result, according with an inadequate task period selection, will be obtained.

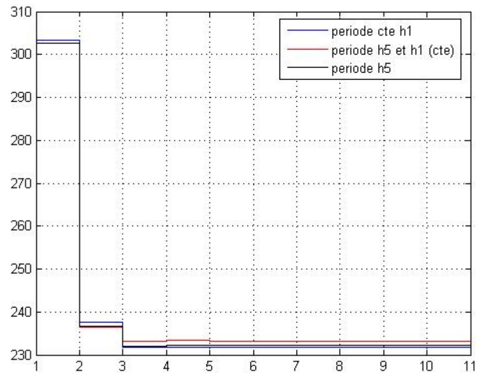

Figure 2: Cost evolution of three sequences as function of time.

Then, we remark that the use of LQ design can give a “non-optimal” performance solution even when it is its main goal, due to the cost criterion depends on the horizon time used. An optimal solution can be obtained if we recalculate the controller parameters associated with each period and for each particular initial condition, i.e. as function of *x*(k*Tbs), but solving this problem on-line is very time consuming.

In section 3, we consider a Lie- Algebra design for second order systems, taken from [7], and we define a discrete cost function that, using the common Lyapunov function, simplifies the performance analysis via the evaluation of an exponential stability term.

3. CO-DESIGN: STABLE AND ADAPTATIVE CONTROL LAWS UNDER RATE VARYING

To support changes in the ressource utilisation (changes in the task computation time, activation/non activation of tasks), two strategies have been proposed: a) the variation of the task period or b) the execution of only m of k consecutives instances (m,k-pattern).

In both cases the control law is updated in a varying time, and then it can be thought of as a sampling period variation. Therefore, the controlled plant can be thought of as a concatenation of systems in time, and it can be modeled as a Discrete-Time Switched System (DTSS).

Here, we resume the results presented in [7] and [8] to solve the second order control law adaptation problem. According with this control design we evaluate the cost function, to measure the control performance, by using the CQLF and the ellipsoid norm.

3.1 DTSS model and Stabilization Problem

The discretization with a sampling time $h_n = t_{i+1} - t_i$ of the open-loop continuous-time plant described by the linear model

$$\begin{aligned} \dot{x} &= A\,x + B\,u \\ y &= C\,x + D\,u \end{aligned} \tag{10}$$

yields the discrete-time linear system

$$\begin{aligned} x_{i+1} &= \Phi(h_n)\,x_i + \Gamma(h_n)\,u_i \\ y_i &= C_d(h_n)\,x_i + D_d(h_n)\,u_i \end{aligned}, \tag{11}$$

where $x \in \Re^n$ is the vector state, $u \in \Re^m$ is the input and $y \in \Re^p$ is the system output, where $u_i = u(t_i)$, $x_i = x(t_i)$, and $y_i = y(t_i)$.

Matrices in (11) are given in (12), if a zero-order hold is considered.

$$\begin{aligned} \Phi_n &= \Phi(h_n) = e^{Ah_n} & C_d(h_n) &= C \\ \Gamma_n &= \Gamma(h_n) = \int_0^{h_n} e^{As} B\, ds & D_d(h_n) &= D \end{aligned} \tag{12}$$

A Closed-Loop Discrete-Time System results from the interconnection of the sampled plant with the linear discrete time dynamic controller. Particularly, for a static state feedback controller $u_i = L_n\, x_i$, the closed-loop matrix of equation is

$$x_{i+1} = \Phi_n\, x_i + \Gamma_n\, u_i = (\Phi_n + \Gamma_n L_n)\, x_i = \Phi_n^{CL} x_i \tag{13}$$

Considering we have a finite number of sampling periods $\{h_n,\ n=1,..,k\}$, then it constitutes a family describing the open-loop systems (21), and the problem to be solved is to find a family of controllers (L_n ; $n=1,..,k$) such that the DTSS closed-loop system (23) is asymptotically stable.

3.2 Adaptative Lie-Algebra Controller Design

As we already pointed-out, we propose to use the results presented in [7] to choose the controller parameters (L_n ; $n=1,..,k$) in order for the Lie algebra generated by the family of closed-loop DTSS -matrices

$$\mathbf{M}=\{\ \Phi_n^{CL}\ ,\ n=1,..,k\} \tag{14}$$

to be solvable.

The solvability of the Lie Algebra generated by the DTSS matrices family guarantees the existence of a Common Quadratic Lyapunov Function CQLF [2] (it was established that if a family of systems that constitutes a switched system has a CQLF, then, asymptotic stability is guaranteed for any switching sequence).

For second-order systems, the following solution has been found:

Let be the family $\mathbf{M} = \{M_1, M_2\}$, where the matrices are

$$M_1=\begin{bmatrix} n_1 & n_2 \\ n_3 & n_4 \end{bmatrix} \text{ and } M_2=\begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix} \tag{15}$$

If we can select the values of e_1 and e_2 in order that

$$e_2=n_2\, e_3\,/n_3 \text{ and } e_1 = -\ n_4\ e_3+e_3\ n_1+e_4\,/n_3 \tag{16}$$

then, the Lie-algebra generated by the family $\mathbf{M}$ is solvable. The matrices are pairwise commutative (abelian case) ($[M_1, M_2]=0$) [6].

If we set the parameters as in (16) for each new matrix M_{i+2} added to this family (each new matrix is designed using (16) respect to M_1 and commutes with M_1 and M_2), then the augmented family $\mathbf{M} = \{M_1, M_2, \boldsymbol{M_3}, \boldsymbol{M_4}..\}$ generates a solvable Lie-algebra.

By using that, for second-order systems with one input, and a state-feedback controller, we can calculate the "exact" explicit solution to adapt the controller parameters of family (14) as follow:

Let be $h_r=r*h$, the adaptative control law parameters are $L_r = \begin{bmatrix} l_{r,1} & l_{r,2} \end{bmatrix}$ where

$$\begin{aligned} l_{r,1} = -\Big(&-l_{1,2}\, a_{2,1}^r b_2^r b_1^1 + l_{1,2} a_{2,1}^r b_1^r b_2^1 - l_{1,1} a_{2,1}^r b_1^r b_1^1 - l_{1,1} a_{2,2}^r b_1^r b_2^1 + l_{1,1} a_{1,2}^r b_2^r b_2^1 \\ &- a_{1,2}^1 a_{2,1}^r b_2^r - a_{1,1}^1 a_{2,1}^r b_1^r - a_{2,1}^1 a_{2,2}^r b_1^r + a_{2,1}^1 a_{1,2}^r b_2^r + l_{1,1} a_{1,1}^r b_1^r b_2^1 + a_{1,1}^r a_{2,1}^1 b_1^r \\ &+ a_{2,1}^r a_{2,2}^1 b_1^r \Big) / \Big(-a_{1,1}^1 b_1^r b_2^r + l_{1,2} b_2^1 b_1^r b_2^r + l_{1,1} b_2^1 b_1^r b_1^r - l_{1,2} b_1^1 b_2^r b_2^r - l_{1,1} b_1^1 b_1^r b_2^r \\ &- a_{1,2}^1 b_2^r b_2^r + a_{2,2}^1 b_1^r b_2^r + a_{2,1}^1 b_1^r b_1^r \Big) \end{aligned}$$

$$\begin{aligned}
l_{r,2} = -\Big(& l_{1,1}\, a^r_{1,2} b^r_1 b^1_2 - l_{1,2} a^r_{2,1} b^r_1 b^1_1 - l_{1,1} a^r_{1,2} b^r_2 b^1_1 - a^1_{1,2} a^r_{2,1} b^r_1 + l_{1,2} a^r_{1,2} b^r_2 b^1_2 - a^1_{1,2} a^r_{2,2} b^r_2 \\
& + a^1_{2,2} a^r_{1,2} b^r_2 - l_{1,2} a^r_{2,2} b^r_2 b^1_1 + a^1_{1,2} a^r_{1,1} b^r_2 + a^1_{2,1} a^r_{1,2} b^r_1 - a^1_{1,1} a^r_{1,2} b^r_2 + l_{1,2} a^r_{1,1} b^r_2 b^1_1 \Big) / \\
\Big(& -a^1_{1,1} b^r_1 b^r_2 + l_{1,2} b^1_2 b^r_1 b^r_2 + l_{1,1} b^1_2 b^r_1 b^r_1 - l_{1,2} b^1_1 b^r_2 b^r_2 - l_{1,1} b^1_1 b^r_1 b^r_2 \\
& - a^1_{1,2} b^r_2 b^r_2 + a^1_{2,2} b^r_1 b^r_2 + a^1_{2,1} b^r_1 b^r_1 \Big)
\end{aligned} \tag{17}$$

and the parameters $l_{1,1}$ and $l_{1,2}$ are designed for the nominal sampling time h, ($L_1 = \begin{bmatrix} l_{1,1} & l_{1,2} \end{bmatrix}$).

3.3 Definition of the sequence

For the task periods adjustement technique, a nominal sampling period h is changed by a period h_i (for a finite number of i), then, over an horizon of time H, we must to satisfy

$$\sum_{j=0}^{N-1} t_j = H \text{ where } N = \lfloor H/h \rfloor \text{, and } t_j = h_i \text{ for all } j=0,1..N-2 \text{ and } t_{N-1} = H - N * h_i \tag{18}$$

If we replace t_j by $f_{j,1} * h$ then

$$\sum_{j=0}^{N-1} f_{j,l}\, h = H \text{, then } f_{0,1} = f_{1,1} = \ldots = f_{N-2,1} = h_i/h \text{ and } f_{N-1,1} = H/h - N \tag{19}$$

The number $f_{j,1}$ is a real and positive number and the index l is equal 1 because there are only one solution, as uniformly as possible (required in this case), which satisfies the constraint (19).

For the (m,k)-firm approach, the number $f_{j,l}$ is defined as the number of consecutive rejected task instances plus one, it means the difference between two consecutive control law updates. Then, inside a H=k*h interval of time

$$\sum_{j=0}^{m-1} f_{j,l}\, h = H \tag{20}$$

is valid.

In this case, the number $f_{j,l}$ is an integer, varying in the set [1,..k]. Different solutions identified by l can be found for a m value.

Without loss of generality, we can set N=m in (19).

Then, for a l solution which belongs to the space solution of problems (19) or (20), there are a sequence $\{f_{0,l}, f_{1,l}, \ldots, f_{m-1,l}\}$ such that the evolution matrix is

$$\Theta_{m,l} = \prod_{j=0}^{m-1} \Phi^{CL}\left(f_{m-1-j,l} \right) = \prod_{j=0}^{m-1} \Phi_{f_{m-1-j,l}}{}^{CL} \tag{21}$$

In (m,k)-pattern case, for each value of m, there are S(m)= (k-1)!/((m-1)!*(k-m)!) different solutions.

In [12], the instance (m,k)-pattern is chosen as uniformly as possible. Then, for each m, only one solution is considered, and also $f_{j,1}$ only can be an integer value. In the example of section 4 we prove that that uniform pattern gives the optimal solution in terms of control system performance.

4. PERFORMANCE EVALUATION: COMMON QUADRATIC LYAPUNOV FUNCTION AND COST FUNCTION

4.1 Common Quadratic Lyapunov Function for Second Order Systems

A matrix Lie algebra g is solvable if and only if there exists a non singular matrix T such that $\tilde{M}_i = T^{-1} M_i T$ is upper triangular for all $M_i \in \mathbf{M}$. In consequence, all the matrices in the set that generates a solvable algebra can be put simultaneously into upper-triangular form and they share at least one eigenvector.

Even more, for the second-order abelian case, all the matrices have the same eigenvectors, and then they can be put simultaneously into diagonal form $\tilde{M}_i$ (a particular case of an upper triangular matrix) by using as the similar transformation the eigenvector common matrix.

Then, each sequence expressed in the transformed space has a diagonal form

$$\tilde{\Theta}_{m,l} = \prod_{j=0}^{m-1} \tilde{M}_{f_{m-1-j,l}} \text{, where } \tilde{\Theta}_{m,i} = \begin{bmatrix} a_{1,m,i} & 0 \\ 0 & a_{2,m,i} \end{bmatrix} \tag{22}$$

Proposing the common quadratic Lyapunov function as $V(x_i) = x_i^T P\, x_i$, then from the initial state $x(t_i)$ to the state $x(t_i+kh)$, $V(x_i) - V(x_{i+k}) = x_i^T \left(P - (\Theta_{m,l})^T P(\Theta_{m,l})\right) x_i$ (23)

, and equivalently for the transformed state vector $\tilde{x}_i = T\, x_i$,

$$V(\tilde{x}_i) - V(\tilde{x}_{i+k}) = \tilde{x}_i^T \left(\tilde{P} - (\tilde{\Theta}_{m,l})^T \tilde{P}(\tilde{\Theta}_{m,l})\right) \tilde{x}_i \tag{24}$$

The ellipsoid norm of each matrix in the set is defined as the smallest scalar value γ for which:

$$M_i^T P M_i \le \gamma P \quad \forall i \in k \quad \text{for some } P \tag{25}$$

If $\tilde{P}$ has a diagonal form, then the ellipsoid norm is

$$\gamma\tilde{P} - \tilde{\Theta}_{m,l}{}^T \tilde{P} \tilde{\Theta}_{m,l} = \begin{bmatrix} \left(\gamma - (a_{1,m,l})^2\right)\tilde{p}_1 & 0 \\ 0 & \left(\gamma - (a_{2,m,l})^2\right)\tilde{p}_2 \end{bmatrix} \tag{26}$$

If the modulus of the eigenvalues is lower than unity, then (26) is always definite positive. Without loss of generality, we can consider the identity I as matrix $\tilde{P}$, then

$$\gamma I - \tilde{\Theta}_{m,l}{}^{T} I \,\tilde{\Theta}_{m,l} = \begin{bmatrix} \left(\gamma - \left(a_{1,m,l}\right)^2\right) & 0 \\ 0 & \left(\gamma - \left(a_{2,m,l}\right)^2\right) \end{bmatrix} \tag{27}$$

Setting $\gamma = \rho^2\left(\tilde{\Theta}_{m,l}\right) = \rho^2\left(\Theta_{m,l}\right) = \max\left(\left|a_{1,m,l}\right|^2, \left|a_{2,m,l}\right|^2\right)$, i.e. equal the spectral radius (largest modulus of matrix eigenvalues), then (27) is semidefinite positive.

Even more, if $\left|a_{1,m,l}\right| = \left|a_{2,m,l}\right|$ (complex conjugate-eigenvalues or double real eigenvalue), equation (27) is reduced to equality. In this case we can verify that if $\gamma < \rho^2\left(\Theta_{m,l}\right)$, matrix (27) is definite negative, if $\gamma = \rho^2\left(\Theta_{m,l}\right)$ is a zero matrix, and if $\gamma > \rho^2\left(\Theta_{m,l}\right)$ it is always positive definite. So, the selection of $\gamma = \rho^2\left(\Theta_{m,l}\right)$ divides the region where the Lyapunov function cost can be positive or negative.

We can derive also the equality $\gamma = \left(\left\|\Theta_{m,l}\right\|_I\right)^2 = \rho^2\left(\Theta_{m,l}\right)$ by following the work in [2] for symmetric matrices (the diagonal matrix is a particular case of the symmetric one).

In the case of complexes conjugate-eigenvalues, the matrix *P*, in the original space, was calculated explicitly in [8]. I.e., it is

$$P = \begin{bmatrix} \dfrac{-n_3\, c}{n_2} & \dfrac{\left(n_1 - n_4\right)c}{2n_2} \\ \dfrac{\left(n_1 - n_4\right)c}{2n_2} & c \end{bmatrix} \tag{29}$$

Equivalently, this matrix can be obtained as $P=T^* \tilde{P}\ T$ (here $\tilde{P}$ =I, and symbol * indicates Hermitian transpose), and *T* is the common transformation which put simultaneously all the matrices into diagonal form $T^{-1} M_i\, T$ and c is a constant.

It means that all the matrices M_i into the set **M** satisfy the equality $\left(M_i\right)^T P M_i = \left(\rho\left(M_i\right)\right)^2 P$,

and from (27) all the sequences $\left(\Theta_{m,i}\right)^T P\, \Theta_{m,i} = \left(\rho\left(\Theta_{m,i}\right)\right)^2 P \qquad (29)$

, too. It means that the ellipsoid norm of each matrix M_i (or $\Theta_{m,i}$), is $\gamma = \left\|M_i\right\|_P = \rho\left(M_i\right)$ (or $\gamma = \left\|\Theta_{m,l}\right\|_P = \rho\left(\Theta_{m,l}\right)$).

From (15) and (28), the spectral radius of each matrix are $\rho(M_1)^2 = n_1 n_4 - n_2 n_3$, $\rho(M_2)^2 = e_4\,(-n_4 e_3 + e_3 n_1 + e_4 n_3)/n_3 - n_2 e_3^2/n_3$.

In the case of real eigenvalues, $\rho\left(\Theta_{m,l}\right)=\rho\left(\tilde{\Theta}_{m,l}\right)=\max\left(\left|a_{1,m,l}\right|,\left|a_{2,m,l}\right|\right)$, the matrix (27) is semidefinite positive. It means that $V\left(\tilde{x}_i\right)-V\left(\tilde{x}_{i+k}\right)\le\left(1-\rho^2\left(\tilde{\Theta}_{m,l}\right)\right)\left\|\tilde{x}_i\right\|^2$ and

$$V\left(x_i\right)-V\left(x_{i+k}\right)\le\left(1-\rho^2\left(\Theta_{m,l}\right)\right)x_i^T P x_i \tag{30}$$

, when $P=T^* T$.

4.2 Cost function

To evaluate the performance we adopt a discrete-time quadratic cost for a finite horizon k^*h. So, it measures the cost to go from $x(t_i)$ to the state $x(t_i+k\,h)$ for a given sequence (21)

$$J\left(k,seq\right)=x\left(t_i\right)^T Q\,x\left(t_i\right)+x\left(t_i+kh\right)^T Q\,x\left(t_i+kh\right) \tag{31}$$

We select $Q=P$ (the cost's weight matrix is the common Lyapunov matrix). And the sequence depends on m.

In the case that all the eigenvalues are complexes conjugate, we can use the equality (29), then equation (31) is simplified as

$$J\left(k,seq\right)=\left(1+\gamma\left(\Theta_{m,l}\right)\right)\left(x\left(t_i\right)^T P\,x\left(t_i\right)\right) \tag{32}$$

where as we defined before

$$\gamma\left(\Theta_{m,l}\right)=\rho^2\left(\Theta_{m,l}\right)=\prod_{j=0}^{m-1}\rho^2\left(\Phi_{f_{m,l}}^{CL}\right)=\rho^2\left(\Phi_{f_{m-1,l}}^{CL}\right)\rho^2\left(\Phi_{f_{m-2,l}}^{CL}\right)*..*\rho^2\left(\Phi_{f_{0,l}}^{CL}\right) \tag{33}$$

according to the sequence Θ_m defined in (21).

It means that, according with equation (33), the cost evaluation is reduced to do the product of the m spectral radius.

Then, if we would like to compare the cost for two different sequences, each one characterized by a value *m1,l1* and *m2,l2* from the same initial condition, we must calculate the difference between costs (33)

$$J\left(k,seq1\right)-J\left(k,seq2\right)=\left(\gamma\left(\Theta_{m1,l1}\right)-\gamma\left(\Theta_{m2,l2}\right)\right)\left(x\left(t_i\right)^T P\,x\left(t_i\right)\right) \tag{34}$$

As P is definite positive, to decide which is the best sequence (in the sense of lowest cost), we must analyze the sign of

$$\gamma\left(\Theta_{m1,l1}\right)-\gamma\left(\Theta_{m2,l2}\right) \tag{35}$$

Therefore, we must compare the product of spectral radius of each closed-loop matrix in the sequences *m1* and *m2* which is independent of the matrix *P* and also of the initial condition value. We remark that the spectral radius of closed-loop systems is typically used to compare the temporal response of time-invariant systems, but here we establish that it can be used to compare the performance of varying systems.

Example 2

Let be m1=k, then $\{f_{0,1}=1, \ldots, f_{k-1,1}=1\}$, and m2=1 then $\{f_{0,1}=\mathrm{k}\}$

Equation (35) becomes $\left(\rho^2\left(\Phi_1^{CL}\right)\right)^k - \rho^2\left(\Phi_k^{CL}\right)$ (36)

If this value is negative it means that, from the same initial condition, the final state will be lower (in terms of *P* norm) for the first sequence than for the second one, or viceversa. So, this cost evaluation takes the response time evolution into account.

Using the values in table 1 and k=5, to compare sequences $\Theta_{5,1} = \left(\Phi_1^{CL}\right)^5$ and $\Theta_{1,1} = \Phi_5^{CL}$, then $\left(\rho^2\left(\Phi_1^{CL}\right)\right)^5 - \rho^2\left(\Phi_5^{CL}\right)$=-0.010142. This leads us to conclude that the former sequence gives the lowest cost, independently of the initial condition value.

Example 3

For the second order system from section 2, we show in Table 1 the values of $\gamma(\Theta_{m,l})$ obtained for different sequences satisfying equation (20), it means only integer solutions of f_j are considered. As the matrices are commuting pair-wise the cost is the same for commutative sequences, for example the final state following the sequence {2,3} is equal to {3,2}, from the same initial condition, i.e. $\left(\Phi_3^{CL}\right)*\left(\Phi_2^{CL}\right) = \left(\Phi_2^{CL}\right)*\left(\Phi_3^{CL}\right)$.

m,l	**$\{f_{0,l},.f_{m-1,l}\}$**	$\gamma(\Theta_{m,l})$
5,1	**{1,1,1,1,1}**	**0.16294**
4,1	**{1,1,1,2}**	**0.16310**
3,1	**{1,2,2}**	**0.16326**
3,2	**{1,1,3}**	**0.16380**
2,1	**{2,3}**	**0.16397**
2,2	**{1,4}**	**0.16593**
1,1	**{5}**	**0.17123**

Table 1. $\gamma(\Theta_m)$ values. To compare cost as function of m.

Un table I, one can see that the values of $\gamma(\Theta_{m,l})$ are arranged in increasing order when m decreases. That is, if *m1>m2* then $\Theta_{m1} < \Theta_{m2}$, and by using (29) *J*(*k*,*m1*) < *J*(*k*,*m2*), i.e. if we can execute more instances of a control task we will obtain a lower cost, and a lower value represents a better control performance. This conclusion is independent of the initial condition value. We can also verify that a m-sequence chosen as uniformly as possible, marked in gray color, gives always a lower cost than the non-uniform sequence. So, the typical assertion that uniform sequence gives lower cost is proved for this example.

Let us to cosider the solutions of equation (19), where $f_{j,l}$ can assume non integer values, $f_{0,1}$= $f_{1,1}$=…= $f_{m-1,1}$=n, Tbs=12*h and h=0.1, then the cost function is a convex function of the period as it is shown in Figure 3. This function can be approximated by a non-linear function $\gamma(\Theta_{Tbs})$=0.01254+.0004709 n-.00021 n^2+.0003901 n^3.

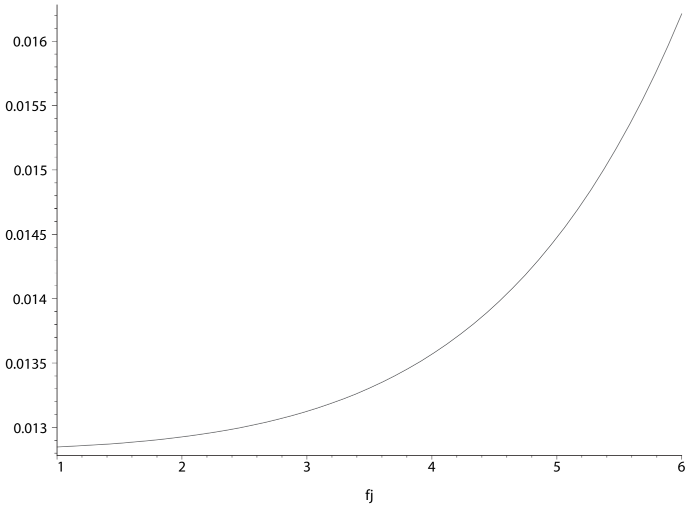

Figure 3: $\gamma(\Theta_{Tbs})$ as function of task period.

In the case of real eigenvalues, approximation (30) is not always adequate for lowest cost sequence determination. In consequence, different cases must be considered.

As we mentioned before, the use of the transformed space (27) allows the computation of the modulus of the eigenvalues of closed-loop systems, and as it has been established in digital control theory, there are a relation between eigenvalues and temporal response.

Consider two diagonal matrices in the transformed space

$$\tilde{\Theta}_r = \begin{bmatrix} a_{1,r} & 0 \\ 0 & a_{2,r} \end{bmatrix}, \quad \tilde{\Theta}_j = \begin{bmatrix} a_{1,j} & 0 \\ 0 & a_{2,j} \end{bmatrix} \tag{37}$$

where the transformed state vector is $\tilde{x}(t_i) = \begin{bmatrix} \tilde{x}_1(t_i) \\ \tilde{x}_2(t_i) \end{bmatrix}$

The difference between the cost function for each sequence is

$$\tilde{x}_i^T\left(\left\|\tilde{\Theta}_r\right\|^2-\left\|\tilde{\Theta}_j\right\|^2\right)\tilde{x}_i=\tilde{x}_i^T\begin{bmatrix}\left(a_{1,r}\right)^2-\left(a_{1,j}\right)^2 & 0\\ 0 & \left(a_{2,r}\right)^2-\left(a_{2,j}\right)^2\end{bmatrix}\tilde{x}_i=\tilde{x}_i^T\,\Delta\,\tilde{x}_i \tag{38}$$

Then, we can distinguish two cases:

a) if $\left|a_{1,r}\right|>\left|a_{1,j}\right|$ and $\left|a_{2,j}\right|>\left|a_{2,r}\right|$ (or $\left|a_{1,j}\right|>\left|a_{1,r}\right|$ and $\left|a_{2,r}\right|>\left|a_{2,r}\right|$) then, matrix Δ is definite positive (or negative), and costs difference is positive (or negative) for all initial conditions. This result is translated to the original space by doing $J\left(k,seq1\right)-J\left(k,seq2\right)=x\left(t_i\right)^T T^T \Delta T\,x\left(t_i\right)$

b) otherwise, the matrix Δ is indefinite causing that equation (38) depends on the initial condition value. We analyse this case in example 4.

Example 4

For the same example in section 2, but using a nominal control law which makes that closed-loop eigenvalues are real ones L1=[-0.025, -0.05], then we can verify that the sequence $\Theta_{5,1}=\left(\Phi_1^{CL}\right)^5$ has the highest and the lowest eigenvalues respect to $\Theta_{1,1}=\Phi_5^{CL}$ (case b) ($\lambda\left(\Theta_{5,1}\right)=\{.76135,0.0098\}$, $\lambda\left(\Theta_{1,1}\right)=\{0.7601,\text{-}0.36699\}$), then the difference between costs $T^T\Delta T$ is illustrated in figure 4 as functions of x_1 and x_2. This difference assumes negatives values for most of the initial conditions, but for a small region it is positive.

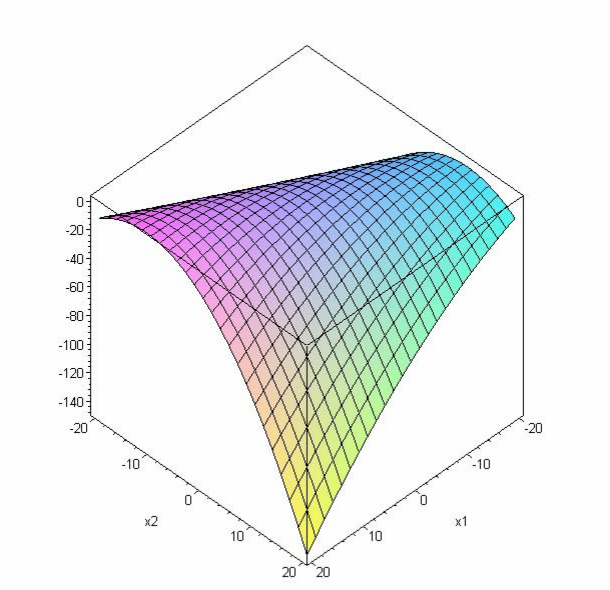

Figure 4: Cost difference as a function of initial condition.

For example, if we calculate this difference from an initial condition that belongs to the positive region, then we verify that the best sequence (lowest cost) if obtained to h5.

It implies that the state vector obtained by applying the best sequence is lower (in the P-norm) than the calculated for the other sequence. In consequence, in the each step k*Tbs we can decide which sequence gives the lowest cost, verifying in which region the initial condition is included.

In figure 5 we show the cost evolution for three sequences (period h1, period h5 and varying period {h1 or h5} selected according with lowest cost condition). We can observe that this minimum cost selection does not fail as the exposed in section 2, because it takes the response time of the controlled system into account, by using the eigenvalues which are invariant for each closed-loop matrix in the set.

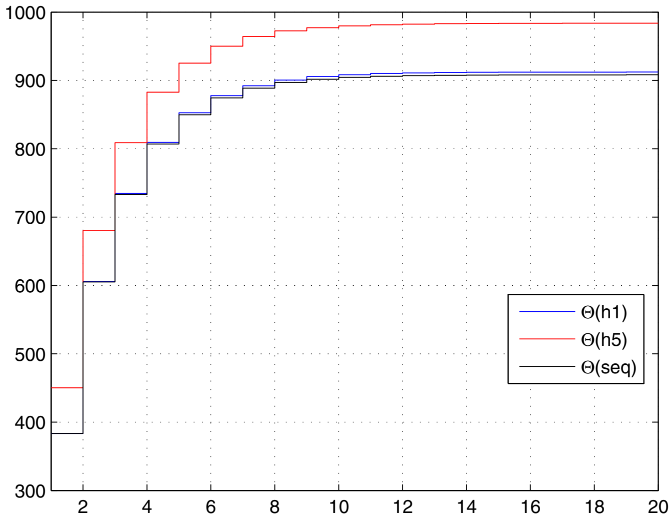

Figure 5: Cost evolution for three sequences.

5 - CONCLUSIONS

Rate adaptation of task execution is typically used in order to optimize allocation and throughput of shared resources in embedded systems.

In order to find the control law parameters adapted to rate variation, a LQ approach has been considered in several papers. But, the use of this method can gives a non-stable solution as it has been noted in [15], or as we illustrated in the example 1, for a stable solution, the LQ cost function may provoke a non-optimal selection of the task period, because it depends on the chosen time horizon.

In this article, an alternative approach to solve the adaptation problem, based on a Lie-algebraic approach, has also been presented based in [7]. It provides, for second order systems, an exact explicit solution to controller parameter adaptation. According with this design, the use of a quadratic Lyapunov function common to all closed-loop matrices (each one corresponds to a task period), allows the simplification of the cost function evaluation.

In the complex-conjugate closed-loop eigenvalues case, it remains to compare the product of spectral radius of each closed-loop matrix, i.e. for each possible sequence, which is independent of the matrix P and also of the initial condition value. Otherwise, the sequence eigenvalues can be used to determinate the cost associated with both, the sequence and the initial condition of the vector state.

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