Simulation of exit times and positions for Brownian motions and Diffusions

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Abstract

We present in this note some variations of the Monte Carlo method for the random walk on spheres which allow to solve many elliptic and parabolic problems involving the Laplace operator or second-order differential operators. In these methods, the spheres are replaced by rectangles or parallelepipeds. Our first method constructs the exit time and the exit position of a rectangle for a Brownian motion. The second method exhibits a variance reduction technique. The main point is to reduce the problem only to the use of some distributions related to the standard one-dimensional Brownian motion.

1 Introduction

In this short article, we present the methods introduced in the works [1, 2]. The aim is to construct an algorithm that simulates the first exit time and exit position from a rectangle (or more generally an parallelepiped) for a Brownian particle, with possibly a constant drift. As for the method of random walk on spheres [3] and squares [4, 5], this method allows one to simulate the first exit time and exit position from a general domain D. The idea is to successively sample exit times and exit positions for simple domains (spheres, squares, ...) included in D and containing the current position of the particle. The procedure stops when the calculated position is close enough to the boundary. If the domain is a polygon without sharp angles, our simulation is exact. The advantage we take with this method is that we are not restricted to consider only rectangles centered on the particle's position as is the case for the methods on spheres or on squares. Thus, the rectangles can be chosen before any Monte Carlo simulation.

This technique could be applied to solve many linear problems arising in mathematical modelling, such as solving Poisson equations or bi-harmonic equations,

evaluating effective coefficients in geophysics [6], computing the first eigenvalue of the Laplace operator [7, 8], computing barrier options in finance etc.

Moreover, this method can be easily adapted to perform *importance sampling* which allows to estimate the probability of rare events or to reduce the variance of the Monte Carlo estimator.

2 The method: exact simulation of the first exit time and position for a rectangle

Although the random variables giving the first exit time and the corresponding exit position from the rectangle, with a starting point inside the rectangle, are not independent, the algorithm we propose here relies on the simulation of one-dimensional random variables by performing proper conditioning. The method is adapted from the computations of [5] where the first exit time and exit position from a square are computed by assuming that the particle starts from the center of the square.

Let us present our method for a rectangle $[-L, L] \times [-\ell, \ell]$ and a Brownian particle initially at (x^1, x^2). The algorithm returns a 3-uple giving the first exit time from $[-L, L] \times [-\ell, \ell]$ and the corresponding exit position.

1. Simulate a realization $(\underline{\tau}^1, \underline{y}^1)$ of the first exit time τ^1 from $[-L, L]$ for a 1-dimensional Brownian motion B^1 starting from x^1 and its exit position $B^1_{\tau^1}$.

2. Use a Bernoulli random variable of parameter $\mathbb{P}_{x^2}[\tau < \underline{\tau}^1]$ to deduce whether or not the first exit time from $[-\ell, \ell]$ of a Brownian motion B^2 starting at x^2 is greater or smaller than $\underline{\tau}^1$.

3. If $\tau < \underline{\tau}^1$, then simulate a realization $(\underline{\tau}^2, \underline{y}^2)$ of (τ, B^2_τ) given that $\{\tau < \underline{\tau}^1\}$. Simulate also a realization $\underline{z}^1$ of $B^1_{\underline{\tau}^2}$ given $\{\tau^1 = \underline{\tau}^1, B^1_{\tau^1} = \underline{y}^1\}$ and return $(\underline{\tau}^2, \underline{z}^1, \underline{y}^2)$.

4. If $\tau > \underline{\tau}^1$ then simulate a realization $\underline{z}^2$ of $B^2_{\underline{\tau}^1}$ given $\{\tau > \underline{\tau}^1\}$ and return $(\underline{\tau}^1, \underline{y}^1, \underline{z}^2)$.

Basically, all the quantities we need to simulate are related to the position of the exit time τ from an interval J for a one-dimensional Brownian motion, and to the position at a given time t of the killed Brownian motion. By using the scaling property of the Brownian motion $(B, \mathbb{P}_x)_{x \in J}$, we may assume that $J = [-1, 1]$. The distribution function of B^2_t given $\{t < \underline{\tau}^1\}$, of B^1_t given $\{t = \underline{\tau}^1\}$, of τ^2 given $\{\tau^2 < \underline{\tau}^1\}$ are easily computed by using h-transform and Bayes' formula. In addition, if $p(t, x, y)$ is the transition density function of B killed when it exits

from $[-1,1]$ and $q(t,x)$ is the density of the first exit time from $[-1,1]$, then

$$q(t,x) = -\int_{-1}^{1} \frac{\partial}{\partial t} p(t,x,y)\,\mathrm{d}y.$$

The density transition function $p(t,x,y)$ can be computed either by a spectral decomposition

$$p(t,x,y) = \sum_{n=1}^{+\infty} \exp\left(-\frac{n^2\pi^2}{8}t\right) \sin\left(\frac{n\pi}{2}(x+1)\right) \sin\left(\frac{n\pi}{2}(y+1)\right), \quad (1)$$

or by the method of images

$$p(t,x,y) = \frac{1}{\sqrt{2\pi t}} \sum_{n=-\infty}^{+\infty} \left(\exp\left(-\frac{(x-y-4n)^2}{2t}\right) - \exp\left(-\frac{(x+y-2-4n)^2}{2t}\right)\right). \quad (2)$$

These series converge very quickly, and (1) is suitable for large values of t, while (2) is suitable for small values of t.

Similar formulas can be obtained for a Brownian motion with a constant drift.

3 How to choose the rectangles?

In order to use this method for the simulation of a Brownian motion on a general domain, the idea is to perform successive simulations of the exit time and exit position from rectangles. The rectangles have to be included in the domain and be as large as possible. The algorithm is stopped when the particle is close enough to the boundary, or really on the boundary. When the domain has a polygonal boundary, a good method to choose the rectangles consists in selecting, when possible, rectangles that have at least one side on the boundary. In this situation, the algorithm stops when the particle leave such a rectangle by the side contained on the boundary. We obtain thus an exact value of the exit time and exit position.

Let us note also that our algorithm can be adapted to deal with Brownian motions with constant drift, and also to Neumann boundary conditions, by possibly changing $p(t,x,y)$ with the transition density function of the Brownian motion on $[-1,1]$, killed on 1 and reflected at -1.

4 Importance sampling

The idea behind our importance sampling method is that instead of considering the first exit time and exit position (τ, B_τ) from a rectangle R, we consider the simulation of a realization $(\underline{\theta}, \underline{Z})$ of a couple of random variables (θ, Z) with values on $\mathbb{R}_+^* \times \partial R$. Thus, at each simulation, we compute a *weight* given by the ratio of the densities of (τ, B_τ) and (θ, Z) at the point $(\underline{\theta}, \underline{Z})$. By computing successively the exit time and position from rectangles, the global weight w associated

to the particle can be expressed as the product of all previously calculated weights. Let σ be the first exit time from a domain D or a given time. Then an expression like $\mathbb{E}[\Phi(\sigma, B_\sigma)]$, can then be replaced by $\mathbb{E}[w\Phi(\theta_{n^*}, Z_{n^*})]$. Here (θ_{n^*}, Z_{n^*}) marks the position when the algorithm that computes the intermediate positions (θ_i, Z_i) stops. Analytical expressions of the weight w can be given and, as for the exact simulation, they involve only $p(t, x, y)$ and $q(t, x)$.

With this approach one can "force" the particle to go where it does not naturally go, and thus one can perform variance reduction or estimate rare events.

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