Multihomogeneous Resultant Formulae for Systems with Scaled Support

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Abstract

Constructive methods for matrices of multihomogeneous (or multigraded) resultants for unmixed systems have been studied by Weyman, Zelevinsky, Sturmfels, Dickenstein and Emiris. We generalize these constructions to *mixed* systems, whose Newton polytopes are scaled copies of one polytope, thus taking a step towards systems with arbitrary supports. First, we specify matrices whose determinant equals the resultant and characterize the systems that admit such formulae. Bézout-type determinantal formulae do not exist, but we describe all possible Sylvester-type and hybrid formulae. We establish tight bounds for all corresponding degree vectors, and specify domains that will surely contain such vectors; the latter are new even for the unmixed case. Second, we make use of multiplication tables and strong duality theory to specify resultant matrices explicitly, for a general scaled system, thus including unmixed systems. The encountered matrices are classified; these include a new type of Sylvester-type matrix as well as Bézout-type matrices, known as partial Bezoutians. Our public-domain Maple implementation includes efficient storage of complexes in memory, and construction of resultant matrices.

Key words: multihomogeneous system, resultant matrix, Sylvester, Bézout, determinantal formula, Maple implementation

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1 Introduction

Resultants provide efficient ways for studying and solving polynomial systems by means of their matrices. They are most efficiently expressed by a generically nonsingular matrix, whose determinant is a multiple of the resultant, so that the determinant degree with respect to the coefficients of one polynomial equals that of the resultant. For two univariate polynomials there are matrix formulae named after Sylvester and Bézout, whose determinant equals the resultant; we refer to them as detererminantal formulae. Unfortunately, such determinantal formulae do not generally exist for more variables, except for specific cases; this is the topic of our paper.

We consider the sparse (or toric) resultant, which exploits *a priori* knowledge on the support of the equations. Matrix formulae have been studied for systems where the variables can be partitioned into groups so that every polynomial is homogeneous in each group, i.e. *mixed multihomogeneous*, or multigraded, systems. This study is an intermediate stage from the theory of homogeneous and unmixed multihomogeneous systems, towards fully exploiting arbitrary sparse structure. Multihomogeneous systems are encountered in several areas, e.g. [3,12,8]. Few foundational works exist, such as [13], where bigraded systems are analyzed. Our work continues that of [7,14,16], where the unmixed case has been treated, and generalizes their results to systems whose Newton polytopes are scaled copies of one polytope. These are known as generalized unmixed systems, and allow us to take a step towards systems with arbitrary supports. This is the first work that treats *mixed* multihomogeneous equations, and provides explicit resultant matrices.

Sparse resultant matrices are of different types. On the one end of the spectrum are the *pure Sylvester-type* matrices, filled in by polynomial coefficients; such are Sylvester's and Macaulay's matrices. On the other end are the *pure Bézout-type* matrices, filled in by coefficients of the *Bezoutian* polynomial. Hybrid matrices contain blocks of both pure types.

We examine Weyman complexes (defined below), which generalize the Cayley-Koszul complex and yield the multihomogeneous resultant as the determinant of a complex. These complexes are parameterized by a *degree vector* $\boldsymbol{m}$. When the complex has two terms, its determinant is that of a matrix expressing the map between these terms, and equals the resultant. In this case, there is a *determinantal* formula, and the corresponding vector $\boldsymbol{m}$ is *determinantal*. The resultant matrix is then said to be *exact*, or *optimal*, in the sense that there is no extraneous factor in the determinant. As is typical in all such approaches, including this paper, the polynomial coefficients are assumed to be sufficiently generic for the resultant, as well as any extraneous factor, to be nonzero.

In [16], the unmixed multihomogeneous systems for which a determinantal formula exists were classified, but no formula was given; see also [10, Sect.13.2]. Identifying explicitly the corresponding morphisms and the vectors $\boldsymbol{m}$ was the focus of [7]. The main result of [14] was to establish that a determinantal formula of Sylvester type exists (for unmixed systems) precisely when the condition of [16] holds on the cardinalities of the groups of variables and their degrees. In [14, Thm.2] all such formulae are characterized by showing a bijection with the permutations of the variable groups and by defining the corresponding vector $\boldsymbol{m}$. This includes all known Sylvester-type formulae, in particular, of linear systems, systems of two univariate polynomials, and bihomogeneous systems of 3 polynomials whose resultant is, respectively, the coefficient determinant, the Sylvester resultant and the classic Dixon formula.

In [14], they characterized all determinantal Cayley-Koszul complexes, which are instances of *Weyman complexes* when all the higher cohomologies vanish. In [7], this characterization is extended to the whole class of unmixed Weyman complexes. It is also shown that there exists a determinantal pure Bézout-type resultant formula if and only if there exists such a Sylvester-type formula. Explicit choices of determinantal vectors are given for any matrix type, as well as a choice yielding pure Bézout type formulae, if one exists. The same work provides tight bounds for the coordinates of all possible determinantal vectors and, furthermore, constructs a family of (rectangular) pure Sylvester-type formulae among which lies the smallest such formula. This paper shall extend these results to unmixed systems with scaled supports.

Studies exist, e.g. [3], for computing hybrid formulae for the resultant in specific cases. In [1], the Koszul and Cech cohomologies are studied in the mixed multihomogeneous case so as to define the resultant in an analogous way to the one used in Section 2. In [5], hybrid resultant formulae were proposed in the mixed homogeneous case; this work is generalized here to multihomogeneous systems. Similar approaches are applied to Tate complexes [4] to handle mixed systems.

The main contributions of this paper are as follows: Firstly, we establish the analog of the bounds given in [7, Sect.3]; in so doing, we simplify their proof in the unmixed case. We characterize the scaled systems that admit a determinantal formula, either pure or hybrid. If pure determinantal formulae exist, we explicitly provide the $\boldsymbol{m}$-vectors that correspond to them. In the search for determinantal formulae we discover box domains that consist of determinantal vectors thus improving the wide search for these vectors adopted in [7]. We conjecture that a formula of minimum dimension can be recovered from the centers of such boxes, analogous to the homogeneous case.

Second, we make the differentials in the Weyman complex explicit and provide details of the computation. Note that the actual construction of the matrix, given the terms of the complex, is nontrivial. Our study has been motivated by [7], where similar ideas are used in the (unmixed) examples of their Section 7, with some constructions which we specify in Example 4.3. Finally, we deliver a complete, publicly available Maple package for the computation of multihomogeneous resultant matrices. Based on the software of [7], it has been enhanced with new functions, including some even for the unmixed case, such as the construction of resultant matrices and the efficient storage of complexes.

The rest of the paper is organized as follows. We start with sparse multihomogeneous resultants and Weyman complexes in Section 2 below. Section 3 presents bounds on the coordinates of all determinantal vectors and classifies the systems that admit hybrid and pure determinantal formulae; explicit vectors are provided for pure formulae and minimum dimension choices are conjectured. In Section 4 we construct the actual matrices; we present Sylvester- and Bézout-type constructions that also lead to hybrid matrices. We conclude with the presentation of our MAPLE implementation along with examples of its usage.

Some of these results have appeared in preliminary form in [9].

2 Resultants via complexes

We define the resultant, and connect it to complexes by homological constructions. Take the product $X := \mathbb{P}^{l_1} \times \cdots \times \mathbb{P}^{l_r}$ of projective spaces over an algebraically closed field $\mathbb{F}$ of characteristic zero, for $r \in \mathbb{N}$. Its dimension equals the number of affine variables $n = \sum_{k=1}^{r} l_k$. We consider polynomials over X of scaled degree: their multidegree is a multiple of a base degree $\boldsymbol{d} = (d_1, \ldots, d_r) \in \mathbb{N}^r$, say $\deg f_i = s_i \boldsymbol{d}$. We assume $s_0 \leq \cdots \leq s_n$ and $\gcd(s_0, \ldots, s_n) = 1$, so that the data $\boldsymbol{l}, \boldsymbol{d}, \boldsymbol{s} = (s_0, \ldots, s_n) \in \mathbb{N}^{n+1}$ fully characterize the system. We denote by $S(\boldsymbol{d})$ the vector space of multihomogeneous forms of degree $\boldsymbol{d}$ defined over X. These are homogeneous of degree d_k in the variables $\boldsymbol{x}_k$ for $k = 1, \ldots, r$. By a slight abuse of notation, we also write $S(d_k) \subset \mathbb{P}^{l_k}$ for the subspace of homogeneous polynomials in l_k variables, of degree d_k. A system of type $(\boldsymbol{l}, \boldsymbol{d}, \boldsymbol{s})$ belongs to $V = S(s_0\boldsymbol{d}) \oplus \cdots \oplus S(s_n\boldsymbol{d})$.

Definition 2.1. Consider a generic scaled multihomogeneous system $\boldsymbol{f} = (f_0, \ldots, f_n)$ defined by the cardinalities $\boldsymbol{l} \in \mathbb{N}^r$, base degree $\boldsymbol{d} \in \mathbb{N}^r$ and $\boldsymbol{s} \in \mathbb{N}^{n+1}$. The *multihomogeneous* resultant $\mathcal{R}(f_0, \ldots, f_n) = \mathcal{R}_{\boldsymbol{l},\boldsymbol{d},\boldsymbol{s}}(f_0, \ldots, f_n)$ is the unique up to sign, irreducible polynomial of $\mathbb{Z}[V]$, which vanishes if and only if there exists a common root of $f_0, \ldots, f_n$ in X.

This polynomial exists for any data $\boldsymbol{l}$, $\boldsymbol{d}$, $\boldsymbol{s}$, since it is an instance of the sparse resultant. It is itself multihomogeneous in the coefficients of each f_i, with degree given by the multihomogeneous Bézout bound:

Lemma 2.2. The resultant polynomial is homogeneous in the coefficients of each f_i, $i = 0, \ldots, n$, with degree

$$\deg_{f_i} \mathcal{R} = \binom{n}{l_1, \ldots, l_r} \frac{d_1^{l_1} \cdots d_r^{l_r} s_0 \cdots s_n}{s_i}.$$

Proof. The degree $\deg_{f_i} \mathcal{R}$ of $\mathcal{R}(\boldsymbol{f})$ with respect to f_i is the coefficient of $y_1^{l_1} \cdots y_r^{l_r}$ in the new polynomial:

$$\prod_{j \neq i}(s_j d_1 y_1 + \cdots + s_j d_r y_r) = \prod_{j \neq i} s_j (d_1 y_1 + \cdots + d_r y_r) = \frac{s_0 s_1 \cdots s_n}{s_i}(d_1 y_1 + \cdots + d_r y_r)^n.$$

In [14, Sect.4] the coefficient of $y_1^{l_1}\cdots y_r^{l_r}$ in $(d_1y_1+\cdots+d_ry_r)^n$ is shown to be equal to

$$\binom{n}{l_1,\ldots,l_r} d_1^{l_1}\cdots d_r^{l_r},$$

thus proving the formula in the unmixed case. Hence the coefficient of $y_1^{l_1}\cdots y_r^{l_r}$ in our case is this number multiplied by $\dfrac{s_0s_1\cdots s_n}{s_i}$. □

This yields the total degree of the resultant, that is, $\sum_{i=0}^{n}\deg_{f_i}\mathcal{R}$.

The rest of the section gives details on the underlying theory. The vanishing of the multihomogeneous resultant can be expressed as the failure of a complex of sheaves to be exact. This allows to construct a class of complexes of finite-dimensional vector spaces whose determinant is the resultant polynomial. This definition of the resultant was introduced by Cayley [10, App. A], [15].

For $\boldsymbol{u}\in\mathbb{Z}^r$, $H^q(X,\mathcal{O}_X(\boldsymbol{u}))$ denotes the q-th cohomology of X with coefficients in the sheaf $\mathcal{O}_X(\boldsymbol{u})$. Throughout this paper we write for simplicity $H^q(\boldsymbol{u})$, even though we also keep the reference to the space whenever it is different than X, for example $H^0(\mathbb{P}^{l_k},u_k)$. To a polynomial system $\boldsymbol{f}=(f_0,\ldots,f_n)$ over V, we associate a finite complex of sheaves $K_\bullet$ on X :

$$0\to K_{n+1}\to\cdots\xrightarrow{\delta_2}K_1\xrightarrow{\delta_1}K_0\xrightarrow{\delta_0}\cdots\to K_{-n}\to 0 \tag{1}$$

This complex (whose terms are defined in Definition 2.3 below) is known to be exact if and only if $f_0,\ldots,f_n$ share no zeros in X; it is hence generically exact. When passing from the complex of sheaves to a complex of vector spaces there exists a degree of freedom, expressed by a vector $\boldsymbol{m}=(m_1,\ldots,m_r)\in\mathbb{Z}^r$. For every given $\boldsymbol{f}$ we specialize the differentials $\delta_i:K_i\to K_{i-1}$, $i=1-n,\ldots,n+1$ by evaluating at $\boldsymbol{f}$ to get a complex of finite-dimensional vector spaces. The main property is that the complex is exact if and only if $\mathcal{R}(f_0,\ldots,f_n)\neq 0$ [15, Prop.1.2].

The main construction that we study is this complex, which we define in our setting. It extends the unmixed case, where for given p the direct sum collapses to $\binom{n+1}{p}$ copies of a single cohomology group.

Definition 2.3. For $\boldsymbol{m}\in\mathbb{Z}^r$, $\nu=-n,\ldots,n+1$ and $p=0,\ldots,n+1$ set

$$K_{\nu,p}=\bigoplus_{0\le i_1<\cdots<i_p\le n} H^{p-\nu}\left(\boldsymbol{m}-\sum_{\theta=1}^{p}s_{i_\theta}\boldsymbol{d}\right)$$

where the direct sum is over all possible indices $i_1<\cdots<i_p$. The *Weyman complex* $K_\bullet=K_\bullet(\boldsymbol{l},\boldsymbol{d},\boldsymbol{s},\boldsymbol{m})$ is generically exact and has terms $K_\nu=\bigoplus_{p=0}^{n+1}K_{\nu,p}$.

This generalizes the classic *Cayley-Koszul* complex. The determinant of the complex can be expressed as a quotient of products of minors from the δ_i. It is invariant under different choices of $\boldsymbol{m}\in\mathbb{Z}^r$ and equals the multihomogeneous resultant $\mathcal{R}(f_0,\ldots,f_n)$.

We present a combinatorial description of the terms in our complex, applicable to the unmixed case as well. For details on the co-homological tools that we use, see [10].

By the Künneth formula, we have the decomposition

$$H^q(\boldsymbol{\alpha}) = \bigoplus_{j_1+\cdots+j_r=q}^{j_k\in\{0,l_k\}} \bigotimes_{k=1}^{r} H^{j_k}\left(\mathbb{P}^{l_k}, \alpha_k\right), \tag{2}$$

where $q = p - \nu$ and the direct sum runs over all integer sums $j_1 + \cdots + j_r = q$, $j_k \in \{0, l_k\}$. In particular, $H^0(\mathbb{P}^{l_k}, \alpha_k)$ is isomorphic to $S(\alpha_k)$, the graded piece of $\mathbb{P}^{l_k}$ in degree α_k or, equivalently, the space of all homogeneous polynomials in $l_k + 1$ variables with total degree α_k, where $\boldsymbol{\alpha} = \boldsymbol{m} - z\boldsymbol{d} \in \mathbb{Z}^r$ for $z \in \mathbb{Z}$.

By Serre duality, for any $\boldsymbol{\alpha} \in \mathbb{Z}^r$, we know that

$$H^q(\boldsymbol{\alpha}) \simeq H^{n-q}(-\boldsymbol{l} - \mathbf{1} - \boldsymbol{\alpha})^*, \tag{3}$$

where * denotes dual, and $\mathbf{1} \in \mathbb{N}^r$ a vector full of ones. Therefore $H^j(\alpha_k)^* \simeq H^{l_k-j}(-\alpha_k - 1 - l_k)$.

Furthermore, we identify $H^{l_k}(\mathbb{P}^{l_k}, \alpha_k)$ as the dual space $S(-\alpha_k - l_k - 1)^*$. This is the space of linear functions $\Lambda : S(\alpha_k) \to \mathbb{F}$. Sometimes we use the negative symmetric powers to interpret dual spaces, see also [16, p.576]. This notion of duality is naturally extended to the direct sum of cohomologies: the dual of a direct sum is the direct sum of the duals of the summands. The next proposition (Bott's formula) implies that this dual space is nontrivial if and only if $-\alpha_k - l_k - 1 \geq 0$.

Proposition 2.4. [2] For any $\boldsymbol{\alpha} \in \mathbb{Z}^r$ and $k \in \{1, \ldots, r\}$,
(a) $H^j(\mathbb{P}^{l_k}, \alpha_k) = 0, \ \forall j \neq 0, l_k$,
(b) $H^{l_k}(\mathbb{P}^{l_k}, \alpha_k) \neq 0 \Leftrightarrow \alpha_k < -l_k$, $\dim H^{l_k}(\mathbb{P}^{l_k}, \alpha_k) = \binom{-\alpha_k-1}{l_k}$.
(c) $H^0(\mathbb{P}^{l_k}, \alpha_k) \neq 0 \Leftrightarrow \alpha_k \geq 0$, $\dim H^0(\mathbb{P}^{l_k}, \alpha_k) = \binom{\alpha_k+l_k}{l_k}$.

Definition 2.5. Given $\boldsymbol{l}, \boldsymbol{d} \in \mathbb{N}^r$ and $\boldsymbol{s} \in \mathbb{N}^{n+1}$, define the *critical degree* vector $\rho \in \mathbb{N}^r$ by $\boldsymbol{\rho}_k := d_k \sum_{\theta=0}^{n} s_\theta - l_k - 1$, for all $k = 1, \ldots r$.

The Künneth formula (2) states that $H^q(\boldsymbol{\alpha})$ is a sum of products. We can give a better description:

Lemma 2.6. If $H^q(\boldsymbol{\alpha})$ is nonzero, then it is equal to a product $H^{j_1}(\mathbb{P}^{l_k}, \alpha_k) \otimes \cdots \otimes H^{j_r}(\mathbb{P}^{l_k}, \alpha_k)$ for some integers $j_1, \ldots, j_r$ with $j_k \in \{0, l_k\}$, $\sum_{k=1}^{r} j_k = q$.

Proof. By Proposition 2.4(a), only $H^0(\mathbb{P}^{l_k}, \alpha_k)$ or $H^{l_k}(\mathbb{P}^{l_k}, \alpha_k)$ may be nonzero. By Proposition 2.4(b,c) at most one of them appears. □

Combining Lemma 2.6 with Definition 2.3 and (2) we get

$$K_{\nu,p} = \bigoplus_{0\le i_1<\cdots<i_p\le n} \bigotimes_{k=1}^{r} H^{j_k}\left(\mathbb{P}^{l_k}, m_k - \sum_{\theta=1}^{p} s_{i_\theta} d_k\right) \tag{4}$$

for some integer sums $j_1+\cdots+j_r = p-\nu,\ j_k \in \{0, l_k\}$ such that all the terms in the product do not vanish. Consequently, $\dim H^q(\boldsymbol{\alpha}) = \prod_{k=1}^{r} \dim H^{j_k}\left(\mathbb{P}^{l_k}, \alpha_k\right)$. The dimension of $K_{\nu,p}$ follows by taking the sum over all $\boldsymbol{\alpha} = \boldsymbol{m} - \sum_{\theta=1}^{p} s_{i_\theta}\boldsymbol{d}$, for all combinations $\{i_1 < \cdots < i_p\} \subseteq \{0,\ldots,n\}$.

Throughout this paper we denote $[u,v] := \{u, u+1, \ldots, v\}$; given $p \in [0, n+1]$, the set of possible sums of p coordinates out of vector $\boldsymbol{s}$ is

$$S_p := \left\{ \sum_{\theta=1}^{p} s_{i_\theta} \ : \ 0 \le i_1 < \cdots < i_p \le n \right\}$$

and by convention $S_0 = \{0\}$. By Proposition 2.4, the set of integers z such that both $H^0(\mathbb{P}^{l_k}, m_k - zd_k)$ and $H^{l_k}(\mathbb{P}^{l_k}, m_k - zd_k)$ vanish is:

$$P_k := \left(\frac{m_k}{d_k}, \frac{m_k + l_k}{d_k} \right] \cap \mathbb{Z}.$$

We adopt notation from [16]: for $u \in \mathbb{Z}$, $P_k < u \iff u > \frac{m_k+l_k}{d_k}$ and $P_k > u \iff u \le \frac{m_k}{d_k}$. Note that we use this notation even if $P_k = \varnothing$. As a result, the $z \in \mathbb{Z}$ that lead to a nonzero $H^{j_k}(\mathbb{P}^{l_k}, m_k - zd_k)$, for $j_k = l_k$ or $j_k = 0$, and $p \in [0, n+1]$, lie in:

$$Q_p = S_p \setminus \cup_1^r P_k, \text{ and } Q = \cup_{p=0}^{n+1} Q_p. \tag{5}$$

Now $\#P_k \le l_k$ implies $\#(\cup_k P_k) \le n$. So $\#(\cup_p S_p) \ge n+2$ implies $\#Q \ge 2$. We define a function $q : Q \to [0, n]$ by

$$q(z) := \sum_{P_k < z} l_k. \tag{6}$$

Observe that $H^j(X, \boldsymbol{m} - z\boldsymbol{d}) \ne 0 \iff z \in Q$ and $j = q(z)$; also the system is unmixed if and only if $S_p = \{p\}$. Clearly $1 \le \#S_p \le \binom{n+1}{p}$, the former inequality being strict for $\boldsymbol{s} \ne \boldsymbol{1} \in \mathbb{N}^{n+1}$ and $p \ne 0, n+1$.

The following lemma generalizes [16, Prop.2.4].

Lemma 2.7. Let $\nu \in \mathbb{Z}$, $p \in \{0, \ldots, n+1\}$ and $K_{\nu,p}$ given by Definition 2.3; then $K_{\nu,p} \ne 0 \iff \nu \in \{p - q(z) \ : \ z \in Q_p\}$.

Proof. Assuming $K_{\nu,p} \ne 0$, there exists a nonzero summand $H^{p-\nu}(\boldsymbol{m} - z\boldsymbol{d}) \ne 0$. By Lemma 2.6 it is equal to $H^{j_1}(\mathbb{P}^{l_k}, m_k - zd_k) \otimes \cdots \otimes H^{j_r}(\mathbb{P}^{l_k}, m_k - zd_k) \ne 0,\ j_k \in \{0, l_k\}$ and

$$p - \nu = \sum_{k=1}^{r} j_k = \sum_{P_k < z} l_k \Rightarrow \nu = p - \sum_{P_k < z} l_k.$$

Conversely, if $\nu \in \{p - q(z) : z \in Q_p\}$ then $Q_p \neq \varnothing$. Now $z \in Q_p$ implies $z \notin P$, which means $H^{q(z)}(\boldsymbol{m} - z\boldsymbol{d}) \neq 0$, the latter being a summand of $K_{\nu,p}$. □

One instance of the complexity of the mixed case is that in the unmixed case, given $p \in [0, n+1]$, there exists at most one integer ν such that $K_{\nu,p} \neq 0$.

All formulae (including determinantal ones) come in dual pairs, thus generalizing [7, Prop.4.4].

Lemma 2.8. Assume $\boldsymbol{m}, \boldsymbol{m}' \in \mathbb{Z}^r$ satisfy $\boldsymbol{m} + \boldsymbol{m}' = \boldsymbol{\rho}$, where $\boldsymbol{\rho}$ is the critical degree vector. Then, $K_\nu(\boldsymbol{m})$ is dual to $K_{1-\nu}(\boldsymbol{m}')$ for all $\nu \in \mathbb{Z}$. In particular, $\boldsymbol{m}$ is determinantal if and only if $\boldsymbol{m}'$ is determinantal, yielding matrices of the same size, namely $\dim(K_0(\boldsymbol{m})) = \dim(K_1(\boldsymbol{m}'))$.

Proof. Based on the equality $\boldsymbol{m} + \boldsymbol{m}' = \boldsymbol{\rho}$ we deduce that for all $J \subseteq [0, n+1]$, it holds that $\boldsymbol{m}' - \sum_{i \in J} s_i \boldsymbol{d} = -\boldsymbol{l} - \boldsymbol{1} - (\boldsymbol{m} - \sum_{i \notin J} s_i \boldsymbol{d})$. Therefore, for all $q = 0, \ldots, n$, Serre's duality (3) implies that $H^q(X, \boldsymbol{m}' - \sum_{i \in J} s_i \boldsymbol{d})$ and $H^{n-q}(X, \boldsymbol{m} - \sum_{i \notin J} s_i \boldsymbol{d})$ are dual.

Let $\#J = p$ and $\nu = p - q$; since $(n + 1 - p) - (n - q) = 1 - (p - q) = 1 - \nu$, we deduce that $K_{\nu,p}(\boldsymbol{m})$ is dual to $K_{1-\nu,n+1-p}(\boldsymbol{m}')$ for all $p \in [0, n+1]$ which leads to $K_\nu(\boldsymbol{m})^* \simeq K_{1-\nu}(\boldsymbol{m}')$ for all $\nu \in \mathbb{Z}$, as desired. In particular, $K_{-1}(\boldsymbol{m}) \simeq K_2^*(\boldsymbol{m}')$ and $K_0(\boldsymbol{m}) \simeq K_1^*(\boldsymbol{m}')$, the latter giving the matrix dimension in the case of determinantal formulae. □

3 Determinantal formulae

This section focuses on formulae that yield square matrices expressing the resultant without extraneous factors and prescribes the corresponding determinantal $\boldsymbol{m}$-vectors.

Determinantal formulae occur only if there is exactly one nonzero differential, so the complex consists of two consecutive nonzero terms. The determinant of the complex is the determinant of this differential. We now specify this differential; for the unmixed case see [16, Lem.3.3].

Lemma 3.1. If $\boldsymbol{m} \in \mathbb{Z}^r$ is determinantal then the nonzero part of the complex is $\delta_1 : K_1 \to K_0$.

Proof. The condition that $\boldsymbol{m}$ is determinantal is equivalent to the fact that $N := \{p - q(z) : \ z \in Q_p, p \in [0, n+1]\}$ consists of two consecutive integers.

Let $z_1 = \min Q < z_2 = \max Q$, since $\#Q \geq 2$. There exist p_1, p_2 with $p_1 < p_2$ such that $z_1 = \sum_{\theta=1}^{p_1} s_{i_\theta}$ and $z_2 = \sum_{\lambda=1}^{p_2} s_{j_\lambda}$ where the indices are sub-sequences of $[0, n]$, of length p_1 and p_2 resp. The p_1 integers

$$0, s_0, s_0 + s_1, \ldots, s_0 + \cdots + s_{p_1-2} \in \mathbb{Z}$$

are distinct, smaller than z_1, hence belong to $\cup_{P_k<z_1} P_k$. Also, it is clear that, for all $k \in [1, r]$, $\#P_k \leq \lceil l_k/d_k \rceil \leq l_k$ thus

$$p_1 \leq \# \bigcup_{P_k<z_1} P_k \leq \sum_{P_k<z_1} \#P_k \leq q(z_1). \tag{7}$$

This means $p_1 - q(z_1) \leq 0$. Similarly, the $n + 1 - p_2$ integers $s_n + \cdots + s_0, \ldots, s_n + \cdots + s_{n-p_2}$ are distinct, larger than z_2, hence belong to $\cup_{P_k>z_2} P_k$, so:

$$n - p_2 + 1 \leq \# \bigcup_{P_k>z_2} P_k \leq \sum_{P_k>z_2} \#P_k \leq \sum_{P_k>z_2} l_k. \tag{8}$$

This means $n+1-p_2 \leq n-q(z_2)$, thus $p_2 - q(z_2) \geq 1$. Hence there exists a positive integer in N; from (7) we must have a non-positive integer in N. Since $\#N = 2$ and the integers of N are consecutive we deduce that $N = \{0, 1\}$. □

Corollary 3.2. If $\boldsymbol{m} \in \mathbb{Z}^r$ is determinantal, then equality holds in (7). In particular, for $k \in [1, r]$ s.t. either $P_k < z_1$ or $P_k > z_2$, we have $\#P_k = l_k$ and any two such P_k are disjoint.

Proof. Lemma 3.1 combined with Lemma 2.7 imply $p_1 - q(z_1) \geq 0$, and (7) implies $p_1 - q(z_1) \leq 0$, hence we deduce $p_1 - q(z_1) = 0$. Now equality in (7) gives $\sum_{P_k<z_1} \#P_k = q(z_1) = \sum_{P_k<z_1} l_k$; combining with $\#P_k \leq l_k$ we deduce $\#P_k = l_k$ for all k in this sum. Similarly for $n + 1 - p_2 = \sum_{P_k>z_2} l_k = n - q(z_2)$. □

3.1 Bounds for determinantal vectors

We generalize the bounds in [7, Sect.3] to the mixed case, for the coordinates of all determinantal $\boldsymbol{m}-$vectors. We follow a simpler and more direct approach based on a global view of determinantal complexes.

Lemma 3.3. If a vector $\boldsymbol{m} \in \mathbb{Z}^r$ is determinantal then the corresponding $\bigcup_1^r P_k$ is contained in $[0, \sum_0^n s_i]$.

Proof. It is enough to establish that $P_k > 0$ and $P_k < \sum_0^n s_i$ for all $k \in [1, r]$. Proof by contradiction: Let $\boldsymbol{m}$ be a determinantal vector, and $P = \cup_0^k P_k$. Let z_1, z_2 as in the proof of Lemma 3.1. If $z_1 < P_k < z_2$ it is clear that $0 \leq z_1 < P_k < z_2 \leq \sum_0^n s_i \Rightarrow P_k \subseteq [0, \sum_0^n s_i]$.

If $z_2 < P_k$, Corollary 3.2 implies $\# \bigcup_{P_k>z_2} P_k = \#R$, where $R := \{s_n + \cdots + s_0, \ldots, s_n + \cdots + s_{n-p_2}\}$. By the definition of z_2, $R \subseteq \bigcup_{P_k>z_2} P_k$, thus $\bigcup_{P_k>z_2} P_k = R \subseteq [0, \sum_0^n s_i]$. Similarly, $\bigcup_{P_k<z_1} P_k = \{0, s_0, s_0 + s_1, \ldots, s_0 + \cdots + s_{p_1-2}\} \subseteq [0, \sum_0^n s_i]$, which proves the lemma for $P_k < z_1$. □

The bound below is proved in [7, Cor.3.9] for the unmixed case. They also show with an example that this bound is tight with respect to individual coordinates. We give an independent, significantly simplified proof, which extends that result to the scaled case.

Theorem 3.4. For determinantal $\boldsymbol{m} \in \mathbb{Z}^r$, for all k we have

$$\max\{-d_k, -l_k\} \leq m_k \leq d_k \sum_0^n s_i - 1 + \min\{d_k - l_k, 0\}.$$

Proof. Observe that by Lemma 3.1 there are no $k \in [1, r]$ such that $P_k < 0$ or $P_k > \sum_0^n s_i$. Combining this fact with Lemma 3.3, we get

$$m_k/d_k \geq -1 \quad \text{and} \quad (m_k + l_k)/d_k < 1 + \sum_0^n s_i \tag{9}$$

for all $k \in [1, r]$. Furthermore, the sets P_k, $k \in [1, r]$ can be partitioned into two (not necessarily non-empty) classes, by considering the integers z_1, z_2 of Lemma 3.1:

- $P_k < z_1$ or $P_k > z_2$, with cardinalities $\#P_k = l_k$.
- $z_1 < P_k < z_2$, without cardinality restrictions (possibly empty).

Taking into account that $P_k = \left(\frac{m_k}{d_k}, \frac{m_k + l_k}{d_k}\right] \cap \mathbb{Z}$ we get

$$(m_k + l_k)/d_k \geq 0 \quad \text{and} \quad m_k/d_k < \sum_0^n s_i \tag{10}$$

for all $k \in [1, r]$. □

Our implementation in Section 5 conducts a search in the box defined by the above bounds. For each $\boldsymbol{m}$ in the box, the dimension of K_2 and K_{-1} is calculated; if both are zero the vector is determinantal. Finding these dimensions is time consuming; the following lemma provides a cheap necessary condition to check before calculating them.

Lemma 3.5. If $\boldsymbol{m} \in \mathbb{Z}^r$ is determinantal then there exist indices $k, k' \in [1, r]$ such that $m_k < d_k(s_{n-1} + s_n)$ and $m_{k'} \geq d_{k'} \sum_0^{n-2} s_i - l_{k'}$.

Proof. If for all k, $m_k/d_k \geq s_0 + s_1$ then $q(s_{n-1} + s_n) = 0$ by (5), so for $p = 2$ we have $p - q(s_{n-1} + s_n) = 2 - 0 = 2$ which contradicts the fact that m is determinantal. Similarly, if for all k, $(m_k + l_k)/d_k < \sum_0^{n-2} s_i \Rightarrow q\left(\sum_0^{n-2} s_i\right) = n$ and for $p = n - 1$ we have $p - q\left(\sum_2^n s_i\right) = (n-1) - n = -1$, which is again infeasible. □

A formula is determinantal if and only if $K_2 = K_{-1} = 0$. In this section we provide necessary and sufficient conditions for the data $\boldsymbol{l}, \boldsymbol{d}, \boldsymbol{s}$ to admit a determinantal formula; we call this data *determinantal*. Also, we derive multidimensional integer intervals (boxes) that yield determinantal formulae and conjecture that minimum dimension formulae appear near the center of these intervals.

Lemma 3.6. If $\boldsymbol{m} \in \mathbb{Z}^r$ is a determinantal vector for the data $\boldsymbol{l}, \boldsymbol{d}, \boldsymbol{s}$, then this data admits a determinantal vector $\boldsymbol{m}' \in \mathbb{Z}^r$ with $P_k \cap P_{k'} = \varnothing$ for all $k, k' \in [1, r]$.

Proof. Suppose $m_i/d_i \leq m_j/d_j$. Let $P_i(\boldsymbol{m}) \cap P_j(\boldsymbol{m}) = [u, v] \subset \mathbb{Z}$. Set $m'_j = m_j + td_j$ where $t \in \mathbb{Z}$ is the minimum shift so that $P_i(\boldsymbol{m}') \cap P_j(\boldsymbol{m}') = \varnothing$ and $P_j(\boldsymbol{m}')$ satisfies Theorem 3.4. For all $k \neq j$, let $m'_k = m_k$.

Any vector in $\mathbb{Z}^r$ defines a nontrivial complex, since $Q \neq \varnothing$. In particular, $\boldsymbol{m}'$ is determinantal because $P(\boldsymbol{m}) \subseteq P(\boldsymbol{m}')$, i.e. no new terms are introduced, but possibly some terms vanish. Repeat until all $P_k \cap P_{k'} = \varnothing$. $\square$

Let $\sigma : [1, r] \to [1, r]$ be any permutation. One can identify at most $r!$ classes of determinantal complexes, indexed by the permutations of $\{1, \ldots, r\}$. This classification arises if we look at the nonzero terms that can occur in the complex, provided that the sets P_k satisfy

$$P_{\sigma(1)} \leq P_{\sigma(2)} \leq \cdots \leq P_{\sigma(r)}$$

where we set $P_i \leq P_j \iff m_i/d_i \leq m_j/d_j$. Any given $\boldsymbol{m}$ defines these sets, as well as an ordering between them. This fact allows us to classify determinantal $\boldsymbol{m}$-vectors and the underlying complexes.

For this configuration, expressed by σ, the only nonzero summands of K_ν can be $K_{\nu,\nu+q}$ where q takes values in the set $\{0, l_{\sigma(1)}, l_{\sigma(1)} + l_{\sigma(2)}, \ldots, n\}$. To see this, observe that $q(z) = \sum_{P_k < z} l_k$, $z \in \mathbb{N}$ cannot attain more than $r + 1$ distinct values; so if the relative ordering of the P_k is fixed as above, then these are the only possible values of q. This leads us to the following description of K_2 and K_{-1}:

$$K_2^\sigma = \bigoplus_{k=1}^{r} K_{2,2+\sum_{i=1}^{k-1} l_{\sigma(i)}} \quad , \quad K_{-1}^\sigma = \bigoplus_{k=1}^{r} K_{-1,-1+\sum_{i=1}^{k} l_{\sigma(i)}} \tag{11}$$

As a side remark, note that the proof of Lemma 2.8 implies that the dual of $K_\nu^\sigma(\boldsymbol{m})$ is $K_{1-\nu}^\tau(\boldsymbol{\rho} - \boldsymbol{m})$ where τ is the permutation s.t. $\tau(i) := r + 1 - \sigma(i)$.

Let $\pi[k] := \sum_{\pi(i) \leq \pi(k)} l_i$. If $\pi = \mathrm{Id}$ this is $\mathrm{Id}[k] = l_1 + \cdots + l_k$. We now characterize determinantal data:

Theorem 3.7. The data $\boldsymbol{l}, \boldsymbol{d}, \boldsymbol{s}$ admit a determinantal formula if and only if there exists $\pi : [1,r] \to [1,r]$ s.t.

$$d_k \sum_{n-\pi[k]+2}^{n} s_i - l_k < d_k \sum_{0}^{\pi[k-1]+1} s_i, \ \forall k.$$

Proof. We assume without loss of generality that $\pi = \mathrm{Id}$. This is not restrictive, since if $\pi \neq \mathrm{Id}$ then we can re-number the variable groups such that $k' := \pi^{-1}(k)$. Hence if we set $L_k := \sum_0^{\mathrm{Id}[k-1]+1} s_i = \min S_{\mathrm{Id}[k-1]+2}$ and $R_k := \sum_{n-\mathrm{Id}[k]+2}^{n} s_i = \max S_{\mathrm{Id}[k]-1}$ then the relations become:

$$d_k R_k - l_k < d_k L_k, \ \forall k.$$

Throughout this proof, whenever we use non-positive indices $j \leq 0$ for l_j or $\mathrm{Id}[j]$, these quantities will be zero, and the results in this case are straightforward to verify. Also, note that the dual complex is given by the "reversed" permutation, and in particular, $K_{-1}^* \simeq K_2$, therefore any results on the nullity of K_{-1} can be directly used to prove the nullity of K_2.

($\Leftarrow$) Assume that the inequalities hold. Then for all k there exists an integer m_k such that

$$d_k R_k - l_k \leq m_k \leq d_k L_k - 1. \tag{12}$$

Let $\boldsymbol{m} = (m_1, \ldots, m_r)$. We shall prove that this vector gives a determinantal formula; it suffices to show that for all $k \in [1,r]$, $K_{2,2+\mathrm{Id}[k-1]} = K_{-1,-1+\mathrm{Id}[k]} = 0$, since in (11) we have $\sigma = \pi^{-1} = \mathrm{Id}$.

• If $l_k \geq 3$, we have

$$\mathrm{Id}[k] - 1 = \mathrm{Id}[k-1] + l_k - 1 \geq \mathrm{Id}[k-1] + 2. \tag{13}$$

Thus $L_k \leq R_k$, since $s_i \geq 1$; also our hypothesis (12) translates into the inclusion

$$\big[L_k, R_k\big] \subseteq \left(\frac{m_k}{d_k}, \frac{m_k + l_k}{d_k}\right] = P_k. \tag{14}$$

Now, by (13) we derive

$$\min S_{\mathrm{Id}[k-1]+2} = L_k \leq \min S_{\mathrm{Id}[k]-1} \quad \text{and} \quad \max S_{\mathrm{Id}[k]-1} = R_k \geq \max S_{\mathrm{Id}[k-1]+2}$$

so (14) implies $S_{\mathrm{Id}[k-1]+2} \subseteq P_k$ as well as $S_{\mathrm{Id}[k]-1} \subseteq P_k$ and thus $K_{2,2+\mathrm{Id}[k-1]} = 0$, $K_{-1,-1+\mathrm{Id}[k]} = 0$ by Proposition 2.4.

• If $l_k \leq 2$, it is $\mathrm{Id}[k] - 1 = \mathrm{Id}[k-1] + l_k - 1 \leq \mathrm{Id}[k-1] + 1$. In this case we will prove $K_{2,2+\mathrm{Id}[k-1]} = 0$, $K_{-1,-1+\mathrm{Id}[k]} = 0$ using Lemma 2.7.

Let $z \in Q_p$ for $p = -1 + \mathrm{Id}[k]$. From $R_k \leq (m_k + l_k)/d_k$ it is clear that $P_k \not< z$, thus $q(z) \leq \mathrm{Id}[k-1]$. Also,

$$\mathrm{Id}[k-2] + 2 = \mathrm{Id}[k] - l_{k-1} - l_k + 2 \leq \mathrm{Id}[k] - 1.$$

where the last inequality is taken under the assumption $\max\{l_k, l_{k-1}\} \geq 2$. We treat the case $l_k = l_{k-1} = 1$ seperately. Hence

$$z \geq \min S_{-1+\mathrm{Id}[k]} \geq \min S_{2+\mathrm{Id}[k-2]} = L_{k-1} > m_{k-1}/d_{k-1}.$$

This implies $P_{k-1} < z \Rightarrow q(z) \geq \mathrm{Id}[k-1]$. We conclude that $q(z) = \mathrm{Id}[k-1]$ and thus

$$p - q(z) = (-1 + \mathrm{Id}[k]) - \mathrm{Id}[k-1] = -1 + l_k \in [0, 1]. \tag{15}$$

By Lemma 2.7 we see that $K_{-1,-1+\mathrm{Id}[k]} = 0$, since $p - q(z) \neq -1$.

To complete the proof, suppose $l_k = l_{k-1} = 1$ and $p = \mathrm{Id}[k] - 1$. We get $\mathrm{Id}[k] - 1 = \mathrm{Id}[k-2] + 1$ and therefore $z > P_{k-2} \Rightarrow q(z) \geq \mathrm{Id}[k-2]$. Recall that $q(z)$ is also upper bounded by $\mathrm{Id}[k-1]$. We derive that for $z \in Q_p$ it holds $p - 1 \leq q(z) \leq p$, therefore $p - q(z) \in [0, 1]$, and again by Lemma 2.7, $K_{-1,-1+\mathrm{Id}[k]} = 0$.

As already pointed out, by using duality one can see that, for $z' \in Q_{2+\mathrm{Id}[k-1]}$, it holds $p - q(z') \neq 2$, therefore $K_{2,2+\mathrm{Id}[k-1]} = 0$.

($\Rightarrow$) Suppose that $\boldsymbol{m} \in \mathbb{Z}^r$ is determinantal, namely $K_2(\boldsymbol{m}) = K_{-1}(\boldsymbol{m}) = 0$. Lemma 3.6 implies that we may assume the sets P_j are pairwise disjoint. By a permutation of the variable groups we also assume that the P_j sets induced by $\boldsymbol{m}$ satisfy

$$P_1 \leq P_2 \leq \cdots \leq P_r.$$

The sets P_j have to be distributed along $I := [0, \sum_0^n s_i]$ (Lemma 3.3) and the connected components of $I \setminus \cup P_j$ are subsets of $S_p \cup S_{p+1}$, $p \in [0, n+1]$ since they define a determinantal complex. In particular, for $p = \mathrm{Id}[k] - 1$ we get $P_{k-1} \leq S_p \cup S_{p+1} \leq P_{k+1}$. Now the definition of R_k as an element of $S_{\mathrm{Id}[k]-1}$ implies $P_{k-1} < R_k < P_{k+1}$, i.e. we have the implications (similarly for L_k):

$$R_k \in \cup_1^r P_j \implies R_k \in P_k \quad \text{and} \quad L_k \in \cup_1^r P_j \implies L_k \in P_k. \tag{16}$$

Suppose $m_k < d_k R_k - l_k$, or equivalently $\dfrac{m_k + l_k}{d_k} < R_k$. Then $R_k \notin P_k$, hence by (16) we must have $R_k \notin \cup_j P_j$ which leads to $z = R_k \in Q_p$, $p = \mathrm{Id}[k] - 1$. This implies

$$q(z) \geq \mathrm{Id}[k] \quad \Rightarrow \quad p - q(z) \leq \mathrm{Id}[k] - 1 - \mathrm{Id}[k] = -1 \;\Rightarrow\; K_{-1} \neq 0,$$

which is a contradiction. In the same spirit, if $m_k \geq d_k L_k$, we are led to $z' = L_k \in Q_p$, $p = \mathrm{Id}[k-1] + 2$, then

$$q(z') \leq \mathrm{Id}[k-1] \quad \Rightarrow \quad p - q(z') \geq p - \mathrm{Id}[k-1] = 2 \;\Rightarrow\; K_2 \neq 0,$$

which again contradicts our hypothesis on $\boldsymbol{m}$.

We conclude that any coordinate m_k of $\boldsymbol{m}$ must satisfy $d_k R_k - l_k \leq m_k < d_k L_k$, hence the existence of $\boldsymbol{m}$ implies the inequality relations we had to prove. $\square$

Corollary 3.8. For any permutation $\pi : [1,r] \to [1,r]$, the vectors $\boldsymbol{m} \in \mathbb{Z}^r$ contained in the box

$$d_k \sum_{n-\pi[k]+2}^{n} s_i - l_k \leq m_k \leq d_k \sum_{0}^{\pi[k-1]+1} s_i - 1$$

for $k = 1, \ldots, r$ are determinantal.

It would be good to have a characterization that does not depend on the permutations of $[1,r]$; this would further reduce the time needed to check if some given data is determinantal. One can see that if $r \leq 2$ an equivalent condition is $d_k \sum_{n-l_k+2}^{n} s_i - l_k < d_k(s_0 + s_1)$ for all $k \in [1,r]$; see [5, Lem.5.3] for the case $r = 1$. It turns out that for any $r \in \mathbb{N}$ this condition is *necessary* for the existence of determinantal vectors, but not always sufficient: the smallest counterexample is $\boldsymbol{l} = (1,2,2)$, $\boldsymbol{d} = (1,1,1)$, $\boldsymbol{s} = (1,1,1,1,2,3)$: this data is not determinantal, although the condition holds. In our implementation this condition is used as a filter when checking if some data is determinantal. Also, [5, Cor.5.5] applies coordinate-wise: if for some k, $l_k \geq 7$ then a determinantal formula cannot possibly exist unless $d_k = 1$ and all the s_i's equal 1, or at most, $s_{n-1} = s_n = 2$, or all of them equal 1 except $s_n = 3$.

We deduce that there exist at most $r!$ boxes, defined by the above inequalities that consist of determinantal vectors, or at most $r!/2$ matrices up to transpose. One can find examples of data with any even number of nonempty boxes, but by Theorem 3.7 there exists at least one that is nonempty.

If $r = 1$ then a minimum dimension formula lies in the center of an interval [5]. We conjecture that a similar explicit choice also exists for $r > 1$. Experimental results indicate that minimum dimension formulae tend to appear near the *center* of the nonempty boxes:

Conjecture 3.9. If the data $\boldsymbol{l}, \boldsymbol{d}, \boldsymbol{s}$ is determinantal then determinantal degree vectors of minimum matrix dimension lie *close* to the center of the nonempty boxes of Corollary 3.8.

We conclude this section by treating the homogeneous case, as an example.

Example 3.10. The case $r = 1$, arbitrary degree, has been studied in [5]. We shall formulate the problem in our setting and provide independent proofs. Let $n, d \in \mathbb{Z}$, $\boldsymbol{s} \in \mathbb{Z}_{>0}^{n+1}$. This data define a scaled homogeneous system in $\mathbb{P}^n$; given $m \in \mathbb{Z}$, we obtain $P = \left(\frac{m}{d}, \frac{m+n}{d}\right] \cap \mathbb{Z}$. In this case there exist only zero and nth cohomologies; zero cohomologies can exist only for $\nu \geq 0$ and nth cohomologies can exist only for

$\nu \leq 1$. Thus in principle both of them exist for $\nu \in \{0, 1\}$. Hence,

$$K_\nu = \begin{cases} K_{\nu,\nu}, & 1 < \nu \leq n+1 \\ K_{\nu,\nu} \oplus K_{\nu,n+\nu}, & 0 \leq \nu \leq 1 \\ K_{\nu,n+\nu}, & -n \leq \nu < 0 \end{cases},$$

i.e. the complex is of the form:

$$0 \to K_{n+1,n+1} \to \cdots \to K_{1,1} \oplus K_{1,n+1} \to K_{0,0} \oplus K_{0,n} \to K_{-1,n-1} \to \cdots \to K_{-n,0} \to 0$$

We can explicitly give all determinantal integers in this case:

$$K_2 = 0 \iff K_{2,2} = 0 \iff Q_2 = \varnothing \iff S_2 \subseteq P,$$

thus

$$\min S_2 > \frac{m}{d} \iff s_0 + s_1 > \frac{m}{d} \iff m < (s_0 + s_1)d.$$

Similarly $K_{-1} = 0 \iff Q_{n-1} = \varnothing \iff S_{n-1} \subseteq P$ and thus

$$\max S_{n-1} \leq \frac{m+n}{d} \iff m \geq d\sum_{i=2}^{n} s_i - n.$$

Consequently, a determinantal formula exists iff $d\sum_2^n s_i - n < (s_0+s_1)d$, also verified by Theorem 3.7. In this case the integers contained in the interval

$$\left(d\sum_{i=2}^{n} s_i - n - 1\ ,\ d(s_0 + s_1)\right)$$

are the only determinantal vectors, also verifying Corollary 3.8. Notice that the sum of the two endpoints is exactly the critical degree ρ.

In [5, Cor.4.2,Prop.5.6] it is proved that the minimum-dimension determinantal formula is attained at $m = \lfloor \rho/2 \rfloor$ and $m = \lceil \rho/2 \rceil$, ie. the center(s) of this interval. For an illustration see Ex. 4.5. □

3.3 Pure formulae

A determinantal formula is pure if it is of the form $K_{1,a} \to K_{0,b}$ for $a, b \in [0, n+1]$ with $a > b$. These formulae are either Sylvester- or Bézout-type, named after the matrices for the resultant of two univariate polynomials.

In the unmixed case both kinds of pure formulae exist exactly when for all $k \in [1, r]$ it holds that $\min\{l_k, d_k\} = 1$ [14,7]. The following theorem extends this characterization to the scaled case, by showing that only pure Sylvester formulae are possible and the only data that admit such formulae are univariate and bivariate-bihomogeneous systems.

Theorem 3.11. If $\boldsymbol{s} \neq \mathbf{1}$ a pure Sylvester formula exists if and only if $r \leq 2$ and $\boldsymbol{l} = (1)$ or $\boldsymbol{l} = (1,1)$. If $l_1 = n = 1$ the degree vectors are given by

$$m = d_1 \sum_0^1 s_i - 1 \quad \text{and} \quad m' = -1,$$

whereas if $\boldsymbol{l} = (1,1)$ the vectors are given by

$$\boldsymbol{m} = \left(-1, d_2 \sum_0^2 s_i - 1\right) \quad \text{and} \quad \boldsymbol{m}' = \left(d_1 \sum_0^2 s_i - 1, -1\right).$$

Pure Bézout determinantal formulae *cannot* exist.

Notice the duality $\boldsymbol{m} + \boldsymbol{m}' = \rho$.

Proof. It is enough to see that if a pure formula is determinantal the following inequalities hold

$$n \leq \# \bigcup_{p \neq a,b} S_p \leq \# \cup_1^r P_k \leq n$$

which implies that equalities hold. The inequality on the left follows from the fact that every S_p, $p \in [0, n+1]$ contains at least one distinct integer since the sequence $0, s_0, s_0 + s_1, \ldots, \sum_0^n s_i$ is strictly increasing. For the right inequality, note that the vanishing of all $K_{\nu,p}$ with $p \neq a, b$ implies $Q_p = \varnothing$ (see Lemma 2.7). Thus $\cup_{p \neq a,b} S_p \subseteq \cup_{k=1}^r P_k$ so the cardinality is bounded by $\# \cup_1^r P_k \leq \sum_1^r \# P_k \leq \sum_1^r l_k = n$. Consequently $\# \cup_{p \neq a,b} S_p = n$. Suppose $n > 2$; the fact $\#(S_i \cup S_j) > 2$ for $\{i,j\} \neq \{0,1\}$ implies $\cup_{p \neq a,b} S_p = S_i \cup S_j$ for some i, j, i.e. $\#\{a,b\} = n$, contradiction. Thus $n \leq 2$.

Take $n = 2$. Since $\#(S_0 \cup S_3) = 2$, the above condition is satisfied for $a = 2, b = 1$: it is enough to set $\cup_1^r P_k = S_0 \cup S_3 = \{0, \sum_0^2 s_i\}$, thus the integers of $\cup_1^r P_k$ are not consecutive, so $r > 1$ and $l = (1,1)$. Similarly, if $n = l = 1$ two formulae are possible; for $\cup_1^r P_k = S_0 = \{0\}$ $(a = 2, b = 1)$ or $\cup_1^r P_k = S_2 = \{s_0 + s_1\}$ $(a = 1, b = 0)$.

All stated $\boldsymbol{m}$-vectors follow easily in both cases from $(m_k + l_k)/d_k = 0$ and $(m_k + l_k)/d_k = \sum_0^n s_i$. A pure Bézout determinantal formula comes from $K_{1,n+1} \to K_{0,0}$. Now $\cup_k P_k$ contains $S_1 \cup \cdots \cup S_n$ hence $\# \cup_k P_k > n$. Thus it cannot exist for $\boldsymbol{s} \neq \mathbf{1}$. □

All pure formulae above are of Sylvester-type, made explicit in Section 4. If $n = 1$, both formulae correspond to the classical Sylvester matrix.

If $\boldsymbol{s} = \mathbf{1}$ pure determinantal formulae are possible for arbitrary n, r and a pure formula exists if and only if for all k, $l_k = 1$ or $d_k = 1$ [7, Thm.4.5]; if a pure Sylvester formula exists for $a, b = a - 1$ then another exists for $a = 1, b = 0$ [7, p. 15]. Observe in the proof above that this is not the case if $\boldsymbol{s} \neq \mathbf{1}, n = 2$, thus the construction of the corresponding matrices for $a \neq 1$ now becomes important and highly nontrivial, in contrast to [7].

4 Explicit matrix construction

In this section we provide algorithms for the construction of the resultant matrix expressed as the matrix of the differential δ_1 in the natural monomial basis and we clarify all the different morphisms that may be encountered.

Before we continue, let us justify the necessity of our matrices, using $\boldsymbol{l} = \boldsymbol{d} = (1, 1)$ and $\boldsymbol{s} = (1, 1, 2)$, that is, the system of two bilinear and one biquadratic equation to be examined in Example 5.1. It turns out that a (hybrid) resultant matrix of minimum dimension is of size 4×4. The standard Bézout-Dixon construction has size 6×6 but its determinant is identically zero, hence it does not express the resultant of the system.

The matrices constructed are unique up to row and column operations, reflecting the fact that monomial bases may be considered with a variety of different orderings. The cases of pure Sylvester or pure Bézout matrix can be seen as a special case of the (generally hybrid, consisting of several blocks) matrix we construct in this section.

In order to construct a resultant matrix we must find the matrix of the linear map $\delta_1 : K_1 \to K_0$ in some basis, typically the natural monomial basis, provided that $K_{-1} = 0$. In this case we have a generically surjective map with a maximal minor divisible by the sparse resultant. If additionally $K_2 = 0$ then $\dim K_1 = \dim K_0$ and the determinant of the square matrix is equal to the resultant, i.e. the formula is determinantal. We consider restrictions $\delta_{a,b} : K_{1,a} \to K_{0,b}$ for any direct summand $K_{1,a}$, $K_{0,b}$ of K_1, K_0 respectively. Every such restriction yields a block of the final matrix of size defined by the corresponding dimensions. Throughout this section the symbols a and b will refer to these indices.

4.1 Sylvester blocks

The Sylvester-type formulae we consider generalize the classical univariate Sylvester matrix and the multigraded Sylvester matrices of [14] by introducing multiplication matrices with block structure. Even though these Koszul morphisms are known to correspond to some Sylvester blocks since [16] (see Proposition 4.1 below), the exact interpretation of the morphisms into matrix formulae had not been made explicit until now. We also rectify the Sylvester-type matrix presented in [7, Sect.7.1].

By [16, Prop.2.5,Prop.2.6] we have the following

Proposition 4.1. [16] If $a - 1 < b$ then $\delta_{a,b} = 0$. Moreover, if $a - 1 = b$ then $\delta_{a,b}$ is a Sylvester map.

If $a = 1$ and $b = 0$ then every coordinate of $\boldsymbol{m}$ is non-negative and there are only zero cohomologies involved in $K_{1,1} = \bigoplus_i H^0(\boldsymbol{m} - s_i\boldsymbol{d})$ and $K_{0,0} = H^0(\boldsymbol{m})$. This map is a well known Sylvester map expressing the multiplication $(g_0, \ldots, g_n) \longmapsto \sum_{i=0}^n g_i f_i$.

The entries of the matrix are indexed by the exponents of the basis monomials of $\bigoplus_i S(\boldsymbol{m} - s_i\boldsymbol{d})$ and $S(\boldsymbol{m})$ as well as the chosen polynomial f_i. The entry indexed $(i, \boldsymbol{\alpha}), \boldsymbol{\beta}$ can be computed as:

$$\operatorname{coef}\left(f_i, \boldsymbol{x}^{\boldsymbol{\beta}-\boldsymbol{\alpha}}\right) \quad , \quad i = 0, 1, \ldots, n$$

where $\boldsymbol{x}^{\boldsymbol{\alpha}}$ and $\boldsymbol{x}^{\boldsymbol{\beta}}$ run through the corresponding monomial bases. The entry $(i, \boldsymbol{\alpha}), \boldsymbol{\beta}$ is zero if the support of f_k does not contain $\boldsymbol{\beta} - \boldsymbol{\alpha}$. Also, by Serre duality a block $K_{1,n+1} \to K_{0,n}$ corresponds to the dual of $K_{1,1} \to K_{0,0}$, i.e. to the degree vector $\boldsymbol{\rho} - \boldsymbol{m}$, and yields the same matrix transposed.

The following theorem constructs corresponding Sylvester-type matrix in the general case.

Theorem 4.2. The entry of the transposed matrix of $\delta_{a,b} : K_{1,a} \to K_{0,a-1}$ in row $(I, \boldsymbol{\alpha})$ and column $(J, \boldsymbol{\beta})$ is

$$\begin{cases} 0, \text{ if } J \not\subset I, \\ (-1)^{k+1}\operatorname{coef}\left(f_{i_k}, \boldsymbol{x}^{\boldsymbol{u}}\right), \text{ if } I \setminus J = \{i_k\}, \end{cases}$$

where $I = \{i_1 < i_2 < \cdots < i_a\}$ and $J = \{j_1 < j_2 < \cdots < j_{a-1}\}$, $I, J \subseteq \{0, \ldots, n\}$. Moreover, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^n$ run through the exponents of monomial bases of $H^{a-1}(\boldsymbol{m} - \boldsymbol{d}\sum_{\theta=1}^{a} s_{i_\theta})$, $H^{a-1}(\boldsymbol{m} - \boldsymbol{d}\sum_{\theta=1}^{a-1} s_{j_\theta})$, and $\boldsymbol{u} \in \mathbb{N}^n$, with $u_t = |\beta_t - \alpha_t|$.

Proof. Consider a basis of $\bigwedge^a V$, $\{e_{i_1,i_2,\ldots,i_a} : 0 \le i_1 < i_2 < \cdots < i_a \le n\}$ and similarly for $\bigwedge^{a-1} V$, where $e_0, \ldots, e_n$ is a basis for V. This differential expresses a classic Koszul map

$$\partial_a(e_{i_1}, \ldots, e_{i_a}) = \sum_{k=0}^{n} (-1)^{k+1} f_{i_k} e_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_a}$$

and by [16, Prop.2.6], this is identified as multiplication by f_{i_k}, when passing to the complex of modules.

Now fix two sets $I \subseteq J$ with $I \setminus J = \{i_k\}$, corresponding to a choice of basis elements e_I, e_J of the exterior algebra; then the part of the Koszul map from e_I to e_J gives

$$(-1)^{k+1} M(f_{i_k}) : H^{a-1}(\boldsymbol{m} - \boldsymbol{d}\sum_{\theta \in I} s_\theta) \to H^{a-1}(\boldsymbol{m} - \boldsymbol{d}\sum_{\theta \in J} s_\theta)$$

This multiplication map is a product of homogeneous multiplication operators in the symmetric power basis. This includes operators between negative symmetric powers, where multiplication is expressed by applying the element of the dual space to f_{i_k}.

To see this, consider basis elements $\boldsymbol{w}^{\boldsymbol{\alpha}}$, $\boldsymbol{w}^{\boldsymbol{\beta}}$ that index a row and column resp. of the matrix of $M(f_{i_k})$. Here the part $\boldsymbol{w}_k$ of $\boldsymbol{w}$ associated with the k-th variable group is either $\boldsymbol{x}_k^{\boldsymbol{\alpha}_k}$ or a dual element indexed by $\boldsymbol{\alpha}_k$. We identify dual elements with the negative symmetric powers, thus this can be thought as $\boldsymbol{x}_k^{-\boldsymbol{\alpha}_k}$. This defines

$\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}} \in \mathbb{Z}^n$; the generalized multihomogeneous multiplication by f_{i_k} as in [16, p.577] is, in terms of multidegrees, incrementing $|\tilde{\boldsymbol{\alpha}}_k|$ by $s_i d_k$ to obtain $|\tilde{\boldsymbol{\beta}}_k|$, and hence the corresponding matrix has entry $\operatorname{coef}(f_{i_k}, \boldsymbol{x}^{\boldsymbol{u}})$, where $u_t = |\beta_t - \alpha_t|$, $t \in [1, n]$. The absolute value is needed because for multiplication in dual spaces, the degrees satisfy $-|\boldsymbol{\alpha}_k| + s_i d_k = -|\boldsymbol{\beta}_k| \Rightarrow s_i d_k = |\boldsymbol{\alpha}_k| - |\boldsymbol{\beta}_k| = -(|\boldsymbol{\beta}_k| - |\boldsymbol{\alpha}_k|)$. □

In [7, Sect.7.1], an example is studied that admits a Sylvester formula with $a = 2, b = 1$. The matrix derived by such a complex is described by Theorem 4.2 above and does not coincide with the matrix given there. The following example is taken from there and presents the correct formula.

Example 4.3. Consider the unmixed case $\boldsymbol{l} = (1, 1)$, $\boldsymbol{d} = (1, 1)$, as in [7, Sect.7.1]. This is a system of three bi-linear forms in two affine variables. The vector $\boldsymbol{m} = (2, -1)$ gives $K_1 = K_{1,2} = H^1(0, -3)^{\binom{3}{2}}$ and $K_0 = K_{0,1} = H^1(1, -2)^{\binom{3}{1}}$. The Sylvester map represented here is

$$\delta_1 \ : \ (g_0, g_1, g_2) \mapsto (-g_0 f_1 - g_1 f_2, g_0 f_0 - g_2 f_2, g_1 f_0 + g_2 f_1)$$

and is similar to the one in [6]. By Theorem 4.2, it yields the following (transposed) matrix, given in 2×2 block format:

$$\begin{bmatrix} -M(f_1) & M(f_0) & \mathbf{0} \\ -M(f_2) & \mathbf{0} & M(f_0) \\ \mathbf{0} & -M(f_2) & M(f_1) \end{bmatrix}$$

If $g = c_0 + c_1 x_1 + c_2 x_2 + c_3 x_1 x_2$ the matrix of the multiplication map

$$M(g) \ : \quad S(0) \otimes S(1)^* \ni w \ \longmapsto \ wg \in S(1) \otimes S(0)^*$$

in the natural monomial basis is $\begin{bmatrix} c_2 & c_3 \\ c_0 & c_1 \end{bmatrix}$ as one can easily verify by hand calculations or using procedure `multmap` of our Maple package presented in Section 5. □

4.2 Bézout blocks

A Bézout-type block comes from a map of the form $\delta_{a,b} : K_{1,a} \to K_{0,b}$ with $a - 1 > b$. In the case $a = n + 1$, $b = 0$ this is a map corresponding to the Bezoutian of the system, whereas in other cases some Bézout-like matrices occur, from square subsystems obtained by hiding certain variables.

Consider the Bézoutian, or Morley form [11], of $f_0, \ldots, f_n$. This is a polynomial of multidegree $(\boldsymbol{\rho}, \boldsymbol{\rho})$ in $\mathbb{F}[\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}]$ and can be decomposed as

$$\Delta := \sum_{u_1=0}^{\rho_1} \cdots \sum_{u_r=0}^{\rho_r} \Delta_u(\bar{x}) \cdot \bar{y}^u$$

where $\Delta_u(\bar{x}) \in S$ has $\deg \Delta_u(\bar{x}) = \boldsymbol{\rho} - \boldsymbol{u}$. Here $\bar{\boldsymbol{x}} = (\bar{\boldsymbol{x}}_1, \ldots, \bar{\boldsymbol{x}}_r)$ is the set of homogeneous variable groups and $\bar{\boldsymbol{y}} = (\bar{\boldsymbol{y}}_1, \ldots, \bar{\boldsymbol{y}}_r)$ a set of new variables with the same cardinalities.

The Bezoutian gives a linear map

$$\bigwedge^{n+1} V \to \bigoplus_{m_k \leq \rho_k} S(\boldsymbol{\rho} - \boldsymbol{m}) \otimes S(\boldsymbol{m}).$$

where the space on the left is the $(n+1)$-th exterior algebra of $V = S(s_0 \boldsymbol{d}) \oplus \cdots \oplus S(s_n \boldsymbol{d})$ and the direct sum runs over all vectors $\boldsymbol{m} \in \mathbb{Z}^r$ with $m_k \leq \rho_k$ for all $k \in [1, r]$. In particular, the graded piece of Δ in degree $(\boldsymbol{\rho} - \boldsymbol{m}, \boldsymbol{m})$ in $(\bar{x}, \bar{y})$ is

$$\Delta_{\boldsymbol{\rho}-\boldsymbol{m},\boldsymbol{m}} := \sum_{u_k = m_k} \Delta_u(\bar{\boldsymbol{x}}) \cdot \bar{\boldsymbol{y}}^{\boldsymbol{u}}$$

for all monomials $\bar{\boldsymbol{y}}^{\boldsymbol{u}}$ of degree $\boldsymbol{m}$ and coefficients in $\mathbb{F}[\bar{x}]$ of degree $\boldsymbol{\rho} - \boldsymbol{m}$. It yields a map

$$S(\boldsymbol{\rho} - \boldsymbol{m})^* \longrightarrow S(\boldsymbol{m})$$

known as the Bézoutian in degree m of $f_0, \ldots, f_n$. The differential of $K_{1,n+1} \to K_{0,0}$ can be chosen to be exactly this map, since evidently $K_{0,0} = H^0(\boldsymbol{m}) \simeq S(\boldsymbol{m})$ and

$$K_{1,n+1} = H^n \left(\boldsymbol{m} - \sum_0^n s_i \boldsymbol{d} \right) \simeq S \left(-\boldsymbol{m} + \sum_0^n s_i \boldsymbol{d} + \boldsymbol{l} + \boldsymbol{1} \right)^*$$

according to Serre duality (see Section 2.1). Thus, substituting the critical degree vector, we get $K_{1,n+1} = S(\boldsymbol{\rho} - \boldsymbol{m})^*$.

The polynomial Δ defined above has $n + r$ homogeneous variables and its homogeneous parts can be computed using a determinant construction in [1], which we adopt here. We recursively consider, for $k = 1, \ldots, r$ the uniquely defined polynomials $f_{i,j}^{(1)}$, where $0 \leq j \leq l_k$, as follows:

$$f_i = x_{1,0} f_{i,0}^{(1)} + \ldots + x_{1,l_1} f_{i,l_1}^{(1)}, \quad f_{i,j}^{(1)} \in \mathbb{F}\left[x_{1,j}, \ldots, x_{1,l_1}\right]\left[\overline{x}_2, \ldots, \overline{x}_r\right], \tag{17}$$

for all $i = 1, \ldots, n$. To define $f_{i,j}^{(k)}$, for $2 \leq k \leq r$ and $0 \leq j \leq l_j$, we decompose $f_{i,l_{k-1}}^{(k-1)}$ as in (17) with respect to the group $\boldsymbol{x}_j$:

$$\begin{aligned} f_{i,l_{k-1}}^{(k-1)} &= x_{k,1} f_{i,1}^{(k)} + \ldots + x_{k,l_k} f_{i,l_k}^{(k)} \\ f_{i,j}^{(k)} &\in \mathbb{F}\left[x_{1,l_1}, \ldots, x_{k-1,l_{k-1}}\right]\left[x_{k,j}, \ldots, x_{k,l_k}\right]\left[\overline{x}_{k+1}, \ldots, \overline{x}_r\right]. \end{aligned}$$

Overall we obtain a decomposition

$$\begin{aligned} f_i = \sum_{j=0}^{l_1-1} x_{1,j} f_{i,j}^{(1)} + x_{1,l_1} \sum_{j=0}^{l_2-1} f_{i,j}^{(2)} + \cdots + \prod_{t=1}^{k-1} x_{t,l_t} \sum_{j=1}^{l_k-1} x_{k,j} f_{i,j}^{(k)} + \cdots \\ \cdots + \prod_{t=1}^{r-1} x_{t,l_t} \sum_{j=1}^{l_r-1} x_{r,j} f_{i,j}^{(r)} + \prod_{t=1}^{r} x_{t,l_t} f_{i,l_r}^{(r)} f_{i,l_r} \in \mathbb{F}[x_{1,l_1}, \ldots, x_{1,l_r}] \end{aligned}$$

of the polynomial f_i, for all $i = 1, \ldots, n$. The order of the variable groups, from left to right, corresponds to choosing the permutation $\pi = \mathrm{Id}$. The determinant of size $(n+1) \times (n+1)$ given by

$$\mathcal{D} = \begin{vmatrix} f^{(1)}_{0,0} & \cdots & f^{(1)}_{0,l_1-1} & \cdots & f^{(k)}_{0,0} & \cdots & f^{(k)}_{0,l_k-1} & \cdots & f^{(r)}_{0,0} & \cdots & f^{(r)}_{0,l_r} \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ f^{(1)}_{i,0} & \cdots & f^{(1)}_{i,l_1-1} & \cdots & f^{(k)}_{i,0} & \cdots & f^{(k)}_{i,l_k-1} & \cdots & f^{(r)}_{i,0} & \cdots & f^{(r)}_{i,l_r} \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ f^{(1)}_{n,0} & \cdots & f^{(1)}_{n,l_1-1} & \cdots & f^{(k)}_{n,0} & \cdots & f^{(k)}_{n,l_k-1} & \cdots & f^{(r)}_{n,0} & \cdots & f^{(r)}_{n,l_r} \end{vmatrix},$$

is equal to $\Delta_{\boldsymbol{\rho}-\boldsymbol{m},\boldsymbol{m}}$, in our setting, as we have the following:

Theorem 4.4. [1] The determinant $\mathcal{D}$ is an inertia form of degree $\rho_k - m_k$ with respect to the variable group $\boldsymbol{x}_k$, $k = 1, \ldots, r$.

Let us show a more simple construction of some part $\Delta_{\boldsymbol{\rho}-\boldsymbol{m},\boldsymbol{m}}$ using an affine Bézoutian. Let $\boldsymbol{x}_k = (x_{k,1}, \ldots, x_{k,l_k})$ the (dehomogenized) k-th variable group, and $\boldsymbol{y}_k = (y_{k,1}, \ldots, y_{k,l_k})$. As a result the totality of variables is $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r)$ and $\boldsymbol{y} = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_r)$.

We set $\boldsymbol{w}_t,\ t = 1, \ldots n-1$ the conjunction of the first t variables of $\boldsymbol{y}$ and the last $n - t$ variables of $\boldsymbol{x}$.

If $a = n+1, b = 0$ the affine Bézoutian construction follows from the expansion of

$$\begin{vmatrix} f_0(\boldsymbol{x}) & f_0(\boldsymbol{w}_1) & \cdots & f_0(\boldsymbol{w}_{n-1}) & f_0(\boldsymbol{y}) \\ \vdots & \vdots & & \vdots & \vdots \\ f_n(\boldsymbol{x}) & f_n(\boldsymbol{w}_1) & \cdots & f_n(\boldsymbol{w}_{n-1}) & f_n(\boldsymbol{y}) \end{vmatrix} / \prod_{k=1}^{r} \prod_{j=1}^{l_k} (x_{kj} - y_{kj})$$

as a polynomial in $\mathbb{F}[y]$ with coefficients in $\mathbb{F}[x]$. Hence the entry indexed $\boldsymbol{\alpha}, \boldsymbol{\beta}$ of the Bézoutian in some degree can be computed as the coefficient of $\boldsymbol{x}^{\alpha}\boldsymbol{y}^{\boldsymbol{\beta}}$ of this polynomial.

We propose generalizations of this construction for arbitrary a, b that are called *partial Bezoutians*, as in [7]. It is clear that $a - 1 = q(z_1)$ and $b = q(z_2)$, for $z_1 \in Q_a$ and $z_2 \in Q_b$. The difference $a - b - 1 = \sum_{\theta=1}^{t} l_{k_\theta}$ where $k_1, \ldots, k_t$ is a subsequence of $[1, r]$, since if $P_k < b$ then $P_k < a$ thus

$$q(a) - q(b) = \sum_{P_k < a} l_k - \sum_{P_k < b} l_k = \sum_{b < P_k < a} l_k.$$

These indices suggest the variable groups that we should substitute in the partial Bezoutian. Note that in the case of Bezoutian blocks, it holds $a - b - 1 > 0$ thus some substitutions will actually take place. Let $i_1, \ldots, i_{a-b}$ be a subsequence of

$[0,n]$. We can define a partial Bezoutian polynomial with respect to $f_{i_1},\ldots,f_{i_{a-b}}$ and $\boldsymbol{y}_{k_1},\ldots,\boldsymbol{y}_{k_t}$ as

$$\begin{vmatrix} f_{i_1}(\boldsymbol{x}) & \cdots & f_{i_1}(\boldsymbol{w}) \\ \vdots & & \vdots \\ f_{i_{a-b}}(\boldsymbol{x}) & \cdots & f_{i_{a-b}}(\boldsymbol{w}) \end{vmatrix} / \prod_{\theta=1}^{t} \prod_{j=1}^{l_{i_\theta}} (x_{i_\theta,j} - y_{i_\theta,j}). \tag{18}$$

In this Bezoutian, only the indicated $\boldsymbol{y}$-variable substitutions take place, in successive columns: The variable vector $\boldsymbol{w}$ differs from $\boldsymbol{x}$ at $\boldsymbol{x}_{k_1},\ldots,\boldsymbol{x}_{k_t}$, these have been substituted gradually with $\boldsymbol{y}_{k_1},\ldots,\boldsymbol{y}_{k_t}$. Note that $\boldsymbol{w}$ generalizes the vectors $\boldsymbol{w}_t$ defined earlier, in the sense that the variables of only specific groups are substituted. The total number of substituted variables is $a-b-1$, so this is indeed a Bézout type determinant.

For given a and b, there exist $\binom{n+1}{a-b}$ partial Bezoutian polynomials. The columns of the final matrix are indexed by the $\boldsymbol{x}$-part of their support, and the rows are indexed by the $\boldsymbol{y}$-part as well as the chosen polynomials $f_{i_1},\ldots,f_{i_{a-b}}$.

Example 4.5. Consider the unmixed data $l=2,\ d=2,\ \mathbf{s}=(1,1,1)$. Determinantal formulae are $m\in[0,3]$, which is just $m=0$, $m=1$ and their transposes. Notice how these formulae correspond to the decompositions of $\rho=3=3+0=2+1$. In both cases the complex is of block type $K_{1,3}\to K_{0,0}\oplus K_{0,2}$. The Sylvester part $K_{1,3}\to K_{0,2}$ can be retrieved as in Ex. 4.3. For $m=0$ the Bézout part is $H^2(-6)\simeq S(3)^*\to H^0(0)\simeq S(0)$, whose 5×1 matrix is in terms of brackets

$$\begin{bmatrix} [142]\ [234]+[152]\ [235]\ [042]\ [052] \end{bmatrix}^T.$$

A bracket $[ijk]$ is defined as

$$[ijk] := \det \begin{bmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix},$$

where a_i,b_i,c_i denote coefficients of f_0,f_1,f_2 respectively, for instance $f_2=c_0+c_1x_2+c_2{x_2}^2+c_3x_1+c_4x_1x_2+c_5{x_1}^2$. Now, for $m=1$ we have Bézout part $H^2(-5)\simeq S(2)^*\to H^0(1)\simeq S(1)$, which yields the 5×3 matrix

$$\begin{bmatrix} [142] & [152]+[234] & [235] & [042] & [052] \\ [152] & [154]+[235] & [354] & [052] & [054] \\ [132]+[042] & [052]+[134] & [135] & [041]+[032] & [051] \end{bmatrix}^T.$$

$\square$

routine	function
`Makesystem`	output polynomials of type $(\boldsymbol{l}, \boldsymbol{d}, \boldsymbol{s})$
`mBezout`	compute the m-Bézout bound
`allDetVecs`	enumerate all determinantal $\boldsymbol{m}-$vectors
`detboxes`	output the vector boxes of Cor. 3.8
`findSyl`	output Sylvester type vectors (unmixed case)
`findBez`	find all pure Bézout-type vectors.
`MakeComplex`	compute the complex of an $\boldsymbol{m}-$vector
`printBlocks`	print complex as $\oplus_a K_{1,a} \to \oplus_b K_{0,b}$
`printCohs`	print complex as $\oplus H^q(\boldsymbol{u}) \to \oplus H^q(\boldsymbol{v})$
`multmap`	construct matrix $M(f_i) : S(\boldsymbol{u}) \to S(\boldsymbol{v})$
`Sylvmat`	construct Sylv. matrix $K_{1,p} \to K_{0,p-1}$
`Bezoutmat`	construct Bézout matrix $K_{1,a} \to K_{0,b}$
`makeMatrix`	construct matrix $K_1 \to K_0$

Table 1
The main routines of our software.

5 Implementation

We have implemented the search for formulae and construction of the corresponding resultant matrices in Maple. Our code is based on that of [7, Sect.8] and extends it to the scaled case. We also introduce new features, including construction of the matrices of Section 4; hence we deliver a full package for multihomogeneous resultants, publicly available at `www-sop.inria.fr/galaad/amantzaf/soft.html`.

Our implementation has three main parts; given data $(\boldsymbol{l}, \boldsymbol{d}, \boldsymbol{s})$ it discovers all possible determinantal formula; this part had been implemented for the unmixed case in [7]. Moreover, for a specific $\boldsymbol{m}-$vector the corresponding resultant complex is computed and saved in memory in an efficient representation. As a final step the results of Section 4 are being used to output the resultant matrix coming from this complex. The main routines of our software are illustrated in Table 1.

The computation of all the $\boldsymbol{m}-$vectors can be done by searching the box defined in Theorem 3.4 and using the filter in Lemma 3.5. For every candidate, we check whether the terms K_2 and K_{-1} vanish to decide if it is determinantal.

For a vector $\boldsymbol{m}$, the resultant complex can be computed in an efficient data structure that captures its combinatorial information and allows us to compute the corresponding matrix. More specifically, a nonzero cohomology summand $K_{\nu,p}$ is represented as a list of pairs (c_q, e_p) where $c_q = \{k_1, \dots, k_t\} \subseteq [1, r]$ such that $q = \sum_{i=1}^{t} l_{k_i} = p - \nu$ and $e_p \subseteq [0, n]$ with $\#e_p = p$ denotes a collection of polynomials (or a basis element in the exterior algebra). Furthermore, a term K_ν is a list of $K_{\nu,p}$'s and a complex a list of terms K_ν.

The construction takes place block by block. We iterate over all morphisms $\delta_{a,b}$ and after identifying each of them the corresponding routine constructs a Sylvester or Bézout block. Note that these morphisms are not contained in the representation of the complex, since they can be retrieved from the terms $K_{1,a}$ and $K_{0,b}$.

Example 5.1. We show how our results apply to a concrete example and demonstrate the use of the MAPLE package on it. The system we consider admits a standard Bézout-Dixon construction of size 6×6. But its determinant is identically zero, due to the sparsity of the supports, hence it neither expresses the multihomogeneous resultant, nor provides any information on the roots. Instead our method constructs a non-singular 4×4 hybrid matrix.

Let $\boldsymbol{l} = \boldsymbol{d} = (1, 1)$ and $\boldsymbol{s} = (1, 1, 2)$.

```
> read mhomo-scaled.mpl:
> l:=vector([1,1]): d:=l: s:= vector([1,1,2]):

> f:= Makesystem(l,d,s);
```

$$\begin{aligned}
f_0 &= a_0 + a_1 x_1 + a_2 x_2 + a_3 x_1 x_2\\
f_1 &= b_0 + b_1 x_1 + b_2 x_2 + b_3 x_1 x_2\\
f_2 &= c_0 + c_1 x_1 + c_2 x_2 + c_3 x_1 x_2 + c_4 {x_1}^2 + c_5 {x_1}^2 x_2 +\\
&\quad + c_6 {x_2}^2 + c_7 x_1 {x_2}^2 + c_8 {x_1}^2 {x_2}^2
\end{aligned}$$

We check that this data is determinantal, using Theorem 3.7:

```
> has_deter( l, d, s);
```

$$true$$

Below we apply a search for all possible determinantal vectors, by examining all vectors in the boxes of Corollary 3.8. The condition used here is that the dimension of K_2 and K_{-1} is zero, which is both necessary and sufficient.

```
> allDetVecs( l, d, s) ;
```

$$\begin{gathered}
[[2,0,4],[0,2,4],[3,0,6],[2,1,6],[2,-1,6],[1,2,6],[1,1,6],\\
[1,0,6],[0,3,6],[0,1,6],[-1,2,6],[3,1,8],[1,3,8],[1,-1,8],\\
[-1,1,8],[3,-1,10],[-1,3,10]]
\end{gathered}$$

The vectors are listed with matrix dimension as third coordinate. The search returned 17 vectors; the fact that the number of vectors is odd reveals that there exists a self-dual vector. The critical degree is $\boldsymbol{\rho} = (2, 2)$, thus $\boldsymbol{m} = (1, 1)$ yields the self-dual formula. Since the remaining 16 vectors come in dual pairs, we only mention one formula for each pair; finally, the first 3 formulae listed have a symmetric formula, due to the symmetries present to our data, so it suffices to list 6 distinct formulae.

Using Theorem 3.7 we can compute directly determinantal boxes:

```
> detboxes( l, d, s) ;
```

$$[[-1,1],[1,3]],[[1,3],[-1,1]]$$

Note that the determinantal vectors are exactly the vectors in these boxes. These intersect at $\boldsymbol{m} = (1,1)$ which yields the self-dual formula. In this example minimum dimension formulae correspond to the centers of the intervals, at $\boldsymbol{m} = (2,0)$ and $\boldsymbol{m} = (0,2)$ as noted in Conj. 3.9.

A pure Sylvester matrix comes from the vector

```
> m:= vector([d[1]*convert(op(s),`+`)-1, -1]);
```

$$\boldsymbol{m} = (3,-1)$$

We compute the complex:

```
> K:= makeComplex(l,d,s,m):
> printBlocks(K); printCohs(K);
```

$$K_{1,2} \to K_{0,1}$$

$$H^1(1,-3) \oplus H^1(0,-4)^2 \to H^1(2,-2)^2 \oplus H^1(1,-3)$$

The dual vector $(-1,3)$ yields the same matrix transposed. The block type of the matrix is deduced by the first command, whereas `printCohs` returns the full description of the complex. The dimension is given by the multihomogeneous Bézout bound, see Lemma 2.2, which is equal to:

```
> mbezout( l, d, s) ;
```

$$10$$

It corresponds to a "twisted" Sylvester matrix:

```
> makematrix(l,d,s,m);
```

$$\begin{bmatrix} -b_1 & -b_3 & 0 & a_1 & a_3 & 0 & 0 & 0 & 0 & 0 \\ -b_0 & -b_2 & 0 & a_0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -b_1 & -b_3 & 0 & a_1 & a_3 & 0 & 0 & 0 & 0 \\ 0 & -b_0 & -b_2 & 0 & a_0 & a_2 & 0 & 0 & 0 & 0 \\ -c_4 & -c_5 & -c_8 & 0 & 0 & 0 & a_1 & 0 & a_3 & 0 \\ -c_1 & -c_3 & -c_7 & 0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 \\ -c_0 & -c_2 & -c_6 & 0 & 0 & 0 & 0 & a_0 & 0 & a_2 \\ 0 & 0 & 0 & -c_4 & -c_5 & -c_8 & b_1 & 0 & b_3 & 0 \\ 0 & 0 & 0 & -c_1 & -c_3 & -c_7 & b_0 & b_1 & b_2 & b_3 \\ 0 & 0 & 0 & -c_0 & -c_2 & -c_6 & 0 & b_0 & 0 & b_2 \end{bmatrix}$$

The rest of the matrices are presented in block format; the same notation is used for both the map and its matrix. The dimension of these maps depend on $\boldsymbol{m}$, which we omit to write. Also, $B(x_k)$ stands for the partial Bézoutian with respect to variables $\boldsymbol{x}_k$.

For $\boldsymbol{m} = (3,1)$ we get $K_{1,1} \oplus K_{1,2} \to K_{0,0}$, or

$$H^0(2,0)^2 \oplus H^1(0,-2)^2 \to H^0(3,1)$$

$$\begin{bmatrix} M(f_0) \\ M(f_1) \\ B(x_2) \end{bmatrix}$$

Symmetric is $\boldsymbol{m} = (1,3)$.

For $\boldsymbol{m} = (3,0)$, $K_{1,2} \to K_{0,0} \oplus K_{0,1}$:

$$H^1(1,-2) \oplus H^1(0,-3)^2 \to H^0(3,0) \oplus H^1(1,-2)^2$$

$$\left[\begin{array}{c|c} 0 & \\ M(f_0) & B(x_2) \\ -M(f_1) & \end{array}\right]$$

Symmetric is $\boldsymbol{m} = (0,3)$.

For $\boldsymbol{m} = (2,1)$, we compute $K_{1,1} \oplus K_{1,3} \to K_{0,0}$, or

$$H^1(1,0)^2 \oplus H^2(-2,-3) \to H^0(2,1)$$

$$\begin{bmatrix} \begin{matrix} M(f_1) \\ M(f_2) \end{matrix} & \Bigg| \Delta_{(0,1),(2,1)} \end{bmatrix}$$

Symmetric is $\boldsymbol{m} = (1, 2)$.

If $\boldsymbol{m} = (1, 1)$, $K_{1,1} \oplus K_{1,3} \to K_{0,0} \oplus K_{0,2}$, yielding

$$H^0(0,0)^2 \oplus H^2(-3,-3) \to H^0(1,1) \oplus H^2(-2,-2)^2$$

$$\begin{bmatrix} f_0 & & \\ & 0 & \\ f_1 & & \\ \Delta_{(1,1),(1,1)} & M(f_0) & -M(f_1) \end{bmatrix}$$

We write here f_i instead of $M(f_i)$, since this matrix is just the 1×4 vector of coefficients of f_i.

For $\boldsymbol{m} = (2, 0)$, we get $K_{1,2} \oplus K_{1,3} \to K_{0,0} \oplus K_{0,1}$, or

$$H^1(0,-2) \oplus H^2(-2,-4) \to H^0(2,0) \oplus H^1(0,-2)$$

$$\begin{bmatrix} B(x_2) & 0 \\ \Delta_{(2,0),(0,2)} & B(x_1) \end{bmatrix}$$

This is the minimum dimension determinantal complex, yielding a 4×4 matrix. $\square$

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