Numerical approximation of Backward Stochastic Differential Equations with Jumps

Antoine Lejay, Ernesto Mordecki, Soledad Torres

Numerical approximation of Backward Stochastic Differential Equations with Jumps

Antoine Lejay* Mordecki† Soledad Torres‡

Abstract. In this note we propose a numerical method to approximate the solution of a Backward Stochastic Differential Equations with Jumps (BSDEJ). This method is based on the construction of a discrete BSDEJ driven by a complete system of three orthogonal discrete time-space martingales, the first a random walk converging to a Brownian motion; the second, another random walk, independent of the first one, converging to a Poisson process. The solution of this discrete BSDEJ is shown to weakly converge to the solution of the continuous time BSDEJ. An application to partial integro-differential equations is given.

Keywords: Backward SDEs with jumps, Skorokhod topology, Poisson Process, Monte Carlo method.

2000 AMS MSC. Primary: 60H10, Secondary: 34F05, 93E03

---

*Universit de Lorraine, IECN, UMR 7502, Vandœuvre-lès-Nancy, F-54500, France; CNRS, IECL, UMR 7502, Vandœuvre-lès-Nancy, F-54500, France; Inria, Villers-ls-Nancy, F-54600, France email: `Antoine.Lejay@univ-lorraine.fr`. Partially supported by the MathAmSud Workshop Stochastic Analysis Research Network.

†Centro de Matemática, Facultad de Ciencias, Universidad de la República. Iguá 4225, 11400, Montevideo, Uruguay; email: `mordecki@cmat.edu.uy`. This author acknowledges support of Proyecto Fondo Clemente Estable 10101.

‡DEUV-CIMFAV, Universidad de Valparaíso, Casilla 5030 Valparaíso, Chile; email: `soledad.torres@uv.cl`. Partially supported by Laboratorio de Análisis Estocástico, PBCT-ACT13 and Fondecyt Grant N.1070919.

1 Introduction

Consider a stochastic process $\{Y_t, Z_t, U_t\colon 0 \leq t \leq T\}$ that is the solution of a Backward Stochastic Differential Equation with Jumps (in short BSDEJ) of the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dB_s - \int_{(t,T]\times\mathbb{R}} U_s(x)\tilde{N}(ds, dx), \qquad 0 \leq t \leq T. \quad (1.1)$$

Here $\{B_t\colon 0 \leq t \leq T\}$ is a one dimensional standard Brownian motion; $\tilde{N}(dt, dx) = N(dt, dx) - \nu(dt, dx)$ is a compensated Poisson random measure defined in $[0, T] \times (\mathbb{R} \setminus \{0\})$. Both processes are defined on a stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}_T, \mathbb{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$, and, as usual in this framework, we assume that they are independent. The *terminal condition* ξ is a $\mathcal{F}_T$-measurable random variable in $\mathrm{L}^q(\mathbb{P})$, $q > 2$ (see condition (**B**) in Section 2); the *coefficient* f is a non-anticipative (w.r.t. $(\mathcal{F}_t)_{t\geq 0}$), continuous, bounded and Lipschitz function $f\colon \Omega \times [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ (see condition (**A**) in Section 2). The problem of solving a BSDEJ given the terminal condition ξ, the coefficient f, and the driving processes B and $\tilde{N}$, defined on a stochastic basis $\mathcal{B}$, is to find three adapted processes $\{Y_t, Z_t, U_t\colon 0 \leq t \leq T\}$ such that (1.1) holds.

In this note — once the existence and uniqueness of the solution of a BSDEJ as described above is known — we propose a numerical method that approximates the solution of the equation (1.1) in the case when $\tilde{N}$ is a compensated Poisson process.

Nonlinear backward stochastic differential equations of the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \qquad 0 \leq t \leq T,$$

were first introduced by E. Pardoux and S. Peng [15] in 1990. Under some Lipschitz conditions on the generator f, the authors stated the first existence and uniqueness result. Later on, the same authors in 1992 developed the BSDE theory relaxing the hypotheses that ensure the existence and uniqueness on this type of equations in [16]. Many subsequent efforts have been made in order to relax further the assumptions on the coefficient $f(s, y, z)$, and many applications in mathematical finance have been proposed.

The backward stochastic differential equations with jumps theory begins with an existence result obtained by S. Tang and X. Li [12]. The authors stated such a theorem when the generator satisfies some Lipschitz condition. Another relevant contribution on BSDEJ is the paper by R. Situ [21]. In [2], G. Barles, R. Buckdahn and E. Pardoux consider a BSDEJ when the driving noises are a Brownian motion and an independent Poisson random measure. They show the existence and uniqueness of the solution, and in addition, they establish a link with a partial integro-differential equation (in short PIDE).

A relevant problem in the theory of BSDEs is to propose implementable numerical methods to approximate the solution of such equations. Several efforts have been made in this direction as well. For example, in the Markovian case, J. Douglas, J. Ma and P. Protter [9] proposed a numerical method for a class of forward-backward SDEs, based on a four step scheme developed by J. Ma, P. Protter and J. Yong [14]. On the other hand, D. Chevance [6] proposed a numerical method for BSDEs. In [23], J. Zhang proposed a numerical scheme for a class of backward stochastic differential equations with possible path-dependent terminal values. See also [4, 11], among others, where numerical methods for decoupled forward-backward differential equations are proposed, and [1, 3, 13] for backward differential equations.

In the present work we propose to approximate the solution of a BSDEJ driven by a Brownian Motion and an independent compensated Poisson process, through the solution of a discrete backward equation, following the approach proposed for BSDE by P. Briand, B. Delyon and J. Mémin in [5]. The algorithm to compute this approximation is simple.

In the case without jumps, numerical examples of implementations of this scheme may be found in [17] for example. Note that whatever the method, the computation of a conditional expectation is numerically costly.

However, the rate of convergence is difficult to establish. The difficulty comes from the representation theorem. Not surprisingly, studying closely this term requires sophisticated tools such as Mallavian calculus, on which the work of B. Bouchard and R. Élie relies [3]. Here, we prefer to use relatively simple tools at the price of not studying the rate of convergence.

The rest of the note is organized as follows. In section 2 we present the problem, propose an approximation process and a simple algorithm to compute it, and present the main convergence result of the note. Section 3 is devoted to the proof of the previous main result, in the adequate topology. In section 4 we present an application of the main result to a decoupled system of a stochastic differential equation and a backward stochastic differential equation, resulting in a numerical approximation of the solution of an associated partial integro-differential equation.

2 Main result

In order to propose an implementable numerical scheme we consider that the Poisson random measure is simply generated by the jumps of a Poisson process For the sake of simplicity of notation we work in the time interval $[0,1]$. We then consider a Poisson process $\{N_t\colon 0 \le t \le 1\}$ with intensity λ and jump epochs $\{\tau_k\colon k = 0, 1, \dots\}$. The random measure is then

$$\tilde{N}(dt, dx) = \sum_{k=1}^{N_1} \delta_{(\tau_k,1)}(dt, dx) - \lambda dt \delta_1(dx),$$

where δ_a denotes the Dirac delta at the point a. We also denote $\tilde{N}_t = N_t - \lambda t$. As a consequence, the unknown function $U_t(x)$ that in principle depends on the jump magnitude x becomes $U_t = U_t(1)$, and our BSDEJ in (1.1) becomes

$$\begin{aligned} Y_t &= \xi + \int_t^1 f(s, Y_s, Z_s, U_s) ds - \int_t^1 Z_s dB_s - \sum_{i=N_t+1}^{N_1} U_{T_i} + \lambda \int_t^1 U_s ds, \\ &= \xi + \int_t^1 f(s, Y_s, Z_s, U_s) ds - \int_t^1 Z_s dB_s - \int_{(t,1]} U_s d\tilde{N}_s, \end{aligned} \tag{2.2}$$

for $0 \le t \le 1$. Here, we restrict ourselves without loss of generalities to a time horizon $T = 1$. In order to ensure existence and uniqueness of the solution of this equation we consider the following conditions on the coefficient:

(**A**) The function $f\colon \Omega \times [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is non-anticipative with respect to $(\mathcal{F}_t)_{t \ge 0}$ and there exists $K \ge 0$ and a bounded, non-decreasing continuous function Λ with $\Lambda(0) = 0$ such that

$$\begin{aligned} &|f(\omega, s_1, y_1, z_1, u_1) - f(\omega, s_2, y_2, z_2, u_2)| \\ &\qquad \le \Lambda(s_2 - s_1) + K\big(|y_1 - y_2| + |z_1 - z_2| + |u_1 - u_2|\big), \text{ a.s.} \end{aligned} \tag{2.3}$$

In what respects the terminal condition, we assume:

(**B**) The random variable ξ is $\mathcal{F}_1$-measurable and $\mathbb{E}[|\xi|^q] < \infty$ for some $q > 2$.

2.1 Discrete time BSDE with jumps

We propose to approximate the solution of the BSDEJ in (2.2) by the solution of a *discrete* backward stochastic differential equation with jumps in a discrete stochastic basis with a filtration generated by two independent, centered random walks. In order to obtain the representation property a third martingale is considered. The convergence of this approximation relies on the fact that the first random walk converges to the driving Brownian motion, and the second to the driving compensated Poisson process. Let us define these two random walks.

For $n \in \mathbb{N}$ we introduce the first random walk $\{W_k^n \colon k = 0, \dots, n\}$ by

$$W_0^n = 0, \qquad W_k^n = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} \epsilon_i^n \qquad (k = 1, \dots, n), \tag{2.4}$$

where $\epsilon_1^n, \dots, \epsilon_n^n$ are independent symmetric Bernoulli random variables:

$$\mathbb{P}\big(\epsilon_k^n = 1\big) = \mathbb{P}\big(\epsilon_k^n = -1\big) = 1/2, \qquad (k = 1, \dots, n).$$

The second random walk $\{\tilde{N}_k^n \colon k = 0, \dots, n\}$ is non symmetric, and given by

$$\tilde{N}_0^n = 0, \qquad \tilde{N}_k^n = \sum_{i=1}^{k} \eta_i^n \qquad (k = 1, \dots, n), \tag{2.5}$$

where $\eta_1^n, \dots, \eta_n^n$ are independent and identically distributed random variables with probabilities, for each k, given by

$$\mathbb{P}(\eta_k^n = \kappa_n - 1) = 1 - \mathbb{P}(\eta_k^n = \kappa_n) = \kappa_n \qquad (k = 1, \dots, n), \tag{2.6}$$

where $\kappa_n = e^{-\lambda/n}$. We assume that both sequences $\epsilon_1^n, \dots, \epsilon_n^n$ and $\eta_1^n, \dots, \eta_n^n$ are defined on the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (that can be enlarged if necessary), and that they are mutually independent. The (discrete) filtration in the probability space is $\mathbb{F}^n = \{\mathcal{F}_k^n \colon k = 0, \dots, n\}$ with $\mathcal{F}_0^n = \{\Omega, \emptyset\}$ and $\mathcal{F}_k^n = \sigma\{\epsilon_1^n, \dots, \epsilon_k^n, \eta_1^n, \dots, \eta_k^n\}$ for $k = 1, \dots, n$.

In this discrete stochastic basis, given an $\mathcal{F}_{k+1}^n$-measurable random variable y_{k+1}, to represent the martingale difference $m_{k+1} := y_{k+1} - \mathbb{E}\big(y_{k+1} \mid \mathcal{F}_k^n\big)$ we need a set of three strongly orthogonal martingales. This is a motivation to introduce a third martingale increments sequence $\{\mu_k^n = \epsilon_k^n \eta_k^n \colon k =$

$0, \dots, n\}$. In this context there exist unique $\mathcal{F}_k^n$-measurable random variables z_k, u_k, v_k such that

$$m_{k+1} = y_{k+1} - \mathbb{E}\big(y_{k+1} \mid \mathcal{F}_k^n\big) = \frac{1}{\sqrt{n}} z_k \epsilon_{k+1}^n + u_k \eta_{k+1}^n + v_k \mu_{k+1}^n, \tag{2.7}$$

that can be computed as

$$\begin{aligned} z_k &= \mathbb{E}\big(\sqrt{n} y_{k+1} \epsilon_{k+1}^n \mid \mathcal{F}_k^n\big) = \sqrt{n}\, \mathbb{E}\big(y_{k+1} \epsilon_{k+1}^n \mid \mathcal{F}_k^n\big), &(2.8)\\ u_k &= \frac{\mathbb{E}\big(y_{k+1} \eta_{k+1}^n \mid \mathcal{F}_k^n\big)}{\mathbb{E}\big((\eta_{k+1}^n)^2 \mid \mathcal{F}_k^n\big)} = \frac{1}{\kappa_n(1-\kappa_n)}\, \mathbb{E}\big(y_{k+1} \eta_{k+1}^n \mid \mathcal{F}_k^n\big), &(2.9)\\ v_k &= \frac{\mathbb{E}\big(y_{k+1} \mu_{k+1}^n \mid \mathcal{F}_k^n\big)}{\mathbb{E}\big((\mu_{k+1}^n)^2 \mid \mathcal{F}_k^n\big)} = \frac{1}{\kappa_n(1-\kappa_n)}\, \mathbb{E}\big(y_{k+1} \mu_{k+1}^n \mid \mathcal{F}_k^n\big). \end{aligned}$$

(Observe that the martingales are orthogonal but not orthonormal, hence the normalization.) With this information we proceed to formulate the discrete BSDEJ.

Let us introduce a supplementary condition on f:

(**A'**) There exists a sequence of functions $f_n : \Omega \times [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ non-anticipative w.r.t. $(\mathcal{F}_k^n)_{k=0,\dots,n}$, satisfying (2.3) and such that $f_n(\omega, \cdot, \cdot, \cdot)$ converges uniformly to $f(\omega, \cdot, \cdot, \cdot)$ almost surely.

Remark 1. The prototypal example of such a sequence f_n is when there is an underlying stochastic process X driven by B and N and $f(\omega, s, y, z) = g(X_s(\omega), s, y, z)$. If g is uniformly continuous with respect to its first variable and for $k = 0, \dots, n$, X_k^n is an approximation of $X_{k/n}$ constructed from $\{\epsilon_i^n\}_{i=0,\dots,k}$ and $\{\eta_i^n\}_{i=0,\dots,k}$, then set $f_n(\omega, s, y, z) = g(X_k^n(\omega), k/n, y, z)$ for $s \in [k/n, (k+1)/n)$. This allows one to consider a system of decoupled Forward-Backward Stochastic Differential Equations (See Section 4).

From now, we drop the reference to the randomness ω in f and f_n.

Consider then a square integrable and $\mathcal{F}_n^n$-measurable terminal condition ξ^n and denote $h = 1/n$. Consider the discrete time backward stochastic differential equation with terminal condition $y_n^n := y_{t_n^n}^n = \xi^n$, and for $k = n-1, \dots, 0$ given by

$$y_k^n = \xi^n + \sum_{i=k}^{n-1} h f_n(t_i^n, y_i^n, z_i^n, u_i^n) - \sum_{i=k}^{n-1} \left(\sqrt{h} z_i^n \epsilon_{i+1}^n + u_i^n \eta_{i+1}^n + v_i^n \mu_{i+1}^n \right), \tag{2.10}$$

where $y_i^n := y_{t_i^n}^n$, and $t_i^n = i/n$ $(i = 0, \dots, n)$. A solution to (2.10) is an $\mathbb{F}^n$-adapted sequence $\{y_k^n, z_k^n, u_k^n, v_k^n \colon k = 0, \dots, n\}$ such that $y_n^n = \xi^n$ and (2.10) holds.

Equation (2.10) is equivalent to a backward recursive system, beginning by $y_n^n = \xi^n$, followed by

$$\begin{aligned} y_k^n &= y_{k+1}^n + h f_n(t_k^n, y_k^n, z_k^n, u_k^n) - \sqrt{h} z_k^n \epsilon_{k+1}^n - u_k^n \eta_{k+1}^n - v_k^n \mu_{k+1}^n \\ &= y_{k+1}^n + h f_n(t_k^n, y_k^n, z_k^n, u_k^n) - m_{k+1}^n, \end{aligned} \tag{2.11}$$

for $k = n-1, \dots, 0$. In view of the representation property (2.7), this last equation (2.11) is equivalent to

$$y_k^n = \mathbb{E}(y_{k+1}^n \mid \mathcal{F}_k^n) + h f_n(t_k^n, y_k^n, z_k^n, u_k^n). \tag{2.12}$$

The solution can be computed by the following simple algorithm:

(I) Set $(y_n^n, z_n^n, u_n^n, v_n^n) = (\xi^n, 0, 0, 0)$.

(II) For k from $n-1$ down to 0, compute $\mathbb{E}(y_{k+1}^n \mid \mathcal{F}_k^n)$. Compute z_k^n and u_k^n as in (2.8) and (2.9) and solve (2.12) to find y_k^n, using a fixed point principle.

This algorithm shows the existence and uniqueness of the solution of the discrete BSDEJ defined in (2.10), as it is always possible to solve it using the fixed point principle for n large enough, such that $Kh = K/n < 1$. Bounds on the solution are given below in Section 3.1.

Remark 2 (Computing conditional expectations). Our scheme is very simple to implement. As for any scheme that solves BSDEs numerically, it requires to compute conditional expectations. Various methods have then been proposed: trees [5], quantization [1], regression [11]. Any method faces the explosion of its computational cost as the dimension increases.

In our case, for a function $\Phi \colon \mathbb{R}^{2k+2} \to \mathbb{R}$ we use the formula

$$\begin{aligned} &\mathbb{E}[\Phi(\epsilon_1^n, \dots, \epsilon_{k+1}^n, \eta_1^n, \dots, \eta_{k+1}^n) \,|\, \mathcal{F}_k^n] \\ &\qquad = \frac{\kappa_n}{2} \Phi(\epsilon_1^n, \dots, \epsilon_k^n, 1, \eta_1^n, \dots, \eta_k^n, \kappa_n - 1) \\ &\qquad + \frac{\kappa_n}{2} \Phi(\epsilon_1^n, \dots, \epsilon_k^n, -1, \eta_1^n, \dots, \eta_k^n, \kappa_n - 1) \\ &\qquad + \frac{1-\kappa_n}{2} \Phi(\epsilon_1^n, \dots, \epsilon_k^n, 1, \eta_1^n, \dots, \eta_k^n, \kappa_n) \\ &\qquad + \frac{1-\kappa_n}{2} \Phi(\epsilon_1^n, \dots, \epsilon_k^n, -1, \eta_1^n, \dots, \eta_k^n, \kappa_n). \end{aligned} \tag{2.13}$$

Remark 3. The simplicity of this method is also is drawback. Theoretically, this work for Brownian motion in space dimension and any Poisson process of the form $\Pi(dz) = \sum_{i=1,\dots,\ell} \pi_i \mu_{x_i}(dz)$.

However, in the path-dependant cases, we should consider computing some values for all the possible outcomes of the $\{\epsilon_k^n\}_{k=0}^n$ and the $\{\eta_k^n\}_{k=0}^n$ for each of the Brownian component and each of the Poisson measure μ_{x_i}. This leads to a very high computational cost as soon as the dimension increases.

In the case where f does not depend on a forward part, or only on the Brownian motion, and ξ depends only on the terminal value of W and N, then the computational cost may be reduced since one need only to compute some values at the node of a multi-dimensional tree representing the possibles values of the martingales $\left\{\sum_{i=0}^k \epsilon_i^n\right\}_{k=0}^n$ and $\left\{\sum_{i=0}^k \eta_i^n\right\}_{k=0}^n$. Anyway, the computation cost remains high as soon as the Poisson random measures involves more that one Dirac mass or the Brownian motion have several dimensions.

2.2 The main convergence result

Consider the continuous time version $\{Y_t^n, Z_t^n, U_t^n, V_t^n : 0 \leq t \leq 1\}$ of the solution $\{y_i^n, z_i^n, u_i^n, v_i^n : i = 0, \dots, n\}$ of the discrete equation (2.11) defined by

$$Y_t^n = y_{\lfloor tn \rfloor}^n, \quad Z_t^n = z_{\lfloor tn \rfloor}^n, \quad U_t^n = u_{\lfloor tn \rfloor}^n, \quad V_t^n = v_{\lfloor tn \rfloor}^n, \tag{2.14}$$

for $t \in [0,1]$. Note that Y_t^n, Z_t^n, U_t^n and V_t^n are measurable w.r.t. the σ-algebra $\mathcal{F}_{\lfloor nt \rfloor}^n$ when $t \in [0,1]$. The discrete BSDEJ in (2.10), denoting $c_n(t) = \lfloor nt \rfloor / n$ $(0 \leq t \leq 1)$, can be written as $Y_1^n = \xi^n$ (the square integrable and $\mathcal{F}_n^n$-measurable terminal condition), and

$$\begin{aligned} Y_t^n = \xi^n + \int_{(t,1]} f_n(c_n(s), Y_{s-}^n, Z_{s-}^n, U_{s-}^n) dc_n(s) - \int_{(t,1]} Z_{s-}^n dW_s^n \\ - \int_{(t,1]} U_{s-}^n d\tilde{N}_s^n - \int_{(t,1]} V_{s-}^n d\tilde{M}_s^n \end{aligned}$$

for $t \in [0,1]$, and $\tilde{M}_s^n = W_s^n \times \tilde{N}_s^n$. With our notations (2.11) becomes

$$Y_{t_i^n}^n = Y_{t_{i+1}^n}^n + \frac{1}{n} f\left(t_i^n, Y_{t_i^n}^n, Z_{t_i^n}^n, U_{t_i^n}^n\right) - \frac{1}{\sqrt{n}} Z_{t_i^n}^n \epsilon_{i+1}^n - U_{t_i^n}^n \eta_{i+1}^n - V_{t_i^n}^n \mu_{i+1}^n. \tag{2.15}$$

In Section 2.1 and Remark 2, we gave a numerical scheme to compute the solution $\{Y_{t_i^n}^n, Z_{t_i^n}^n, U_{t_i^n}^n : 0 \leq t \leq 1\}$.

We finally state the convergence assumption on the terminal conditions of the approximating equations:

(**B'**) For some $q > 2$, $\sup_{n\in\mathbb{N}} \mathbb{E}[|\xi^n|^q] + \mathbb{E}[|\xi|^q] < +\infty$ and $\mathbb{E}[|\xi^n - \xi|^2] \xrightarrow[n\to\infty]{} 0$.

Theorem 1. *Under the assumptions* (**A**), (**A'**), (**B**) *and* (**B'**), *the set of processes* $(\xi^n, Y^n, \int_0^\cdot Z_s^n ds, \int_0^\cdot U_s^n ds)$ *converges in the* J_1*-Skorokhod topology, and in probability, towards the solution* $(\xi, Y, \int_0^\cdot Z_s ds, \int_0^\cdot U_s ds)$ *of the BS-DEJ* (1.1).

3 Proofs

Our theorem is inspired in the main result of [5]. The proof follows, when possible, the main steps of the proof of this result. The main difference appears due to the fact that the underlying representation theorem for the simple symmetric Bernoulli random walk does not take place in our case, being necessary to consider a complete system of orthogonal martingales in order to have this representation property, as we have seen in equation (2.11). The idea is then to consider for both the discrete and the continuous time equations the approximations provided by the Picard's method.

In the continuous case denote $Y^{\infty,0} = Z^{\infty,0} = U^{\infty,0} = 0$ and define $\{Y^{\infty,p+1}, Z^{\infty,p+1}, U^{\infty,p+1} \colon 0 \le t \le 1\}$ inductively as the solution of the backward differential equation

$$Y_t^{\infty,p+1} = \xi + \int_t^1 f(s, Y_s^{\infty,p}, Z_s^{\infty,p}, U_s^{\infty,p}) ds - \int_t^1 Z_s^{\infty,p+1} dB_s - \int_{(t,1]} U_s^{\infty,p+1} d\tilde{N}_s.$$

In the discrete case, given n denote $y_k^{n,0} = z_k^{n,0} = u_k^{n,0} = v_k^{n,0} = 0$ for $k = 0, \ldots, n$ and define $\{y_k^{n,p+1}, z_k^{n,p+1}, u_k^{n,p+1}, v_k^{n,p+1} \colon k = 0, \ldots, n\}$ inductively as the solution of the backward difference equation with terminal condition $y_n^{n,p+1} = \xi^n$ and backwards recursion defined by

$$\begin{aligned} y_k^{n,p+1} &= y_{k+1}^{n,p+1} + h f_n(t_k^n, y_k^{n,p}, z_k^{n,p}, u_k^{n,p}) - \sqrt{h} z_k^{n,p+1} \epsilon_{k+1}^n - u_k^{n,p+1} \eta_{k+1}^n \\ &\quad - v_k^{n,p+1} \mu_{k+1}^n \\ &= y_{k+1}^{n,p+1} + h f_n(t_k^n, y_k^{n,p}, z_k^{n,p}, u_k^{n,p}) - m_{k+1}^{n,p+1}. \end{aligned} \tag{3.16}$$

If we consider the continuous time versions of the discrete Picard approximations as defined in (2.14), our method of proof relies on the decompositions

$$\begin{aligned} Y^n - Y &= (Y^n - Y^{n,p}) + (Y^{n,p} - Y^{\infty,p}) + (Y^{\infty,p} - Y), \\ Z^n - Z &= (Z^n - Z^{n,p}) + (Z^{n,p} - Z^{\infty,p}) + (Z^{\infty,p} - Z), \\ U^n - U &= (U^n - U^{n,p}) + (U^{n,p} - U^{\infty,p}) + (U^{\infty,p} - U), \end{aligned}$$

where (Y^n, Z^n, U^n) (resp. $(Y^{n,p}, Z^{n,p}, U^{n,p})$) are the càdlàg processes on $[0,1]$ associated to (y^n, z^n, u^n) (resp. $(y^{n,p}, z^{n,p}, u^{n,p})$) as in (2.14).

We prove the convergence of the discrete solution as $n \to \infty$ to the solution of (1.1), by proving the uniform convergence in the the Picard iteration principle, as well as the convergence of the approximation of this solution given by this iteration principle at each step when the time step is refined.

3.1 Convergence of the Picard's iteration procedure in the discrete case

With standard computations, we have that

$$\sup_{p>0} \mathbb{E}\left[\sup_{t\in[0,1]} |Y_t^{\infty,p}|^2 + \int_0^1 |Z_s^{\infty,p}|^2 ds + \lambda \int_0^1 |U_s^{\infty,p}|^2 ds\right] < +\infty \tag{3.17}$$

and

$$\mathbb{E}\left[\sup_{t\in[0,1]} |Y_t^{\infty,p} - Y_t|^2 + \int_0^1 |Z_s^{\infty,p} - Z_s|^2 ds + \lambda \int_0^1 |U_s^{\infty,p} - U_s|^2 ds\right] \xrightarrow[p\to\infty]{} 0.$$

We now present similar results for the discrete approximations.

The next lemma evaluates the convergence rate of the Picard approximation sequence $(y^{n,p}, z^{n,p}, u^{n,p})$ to (y^n, z^n, u^n) in the discrete scheme, and shows this rate is uniform in the time step $1/n$. It also provides the convergence of the discrete auxiliary sequence $(v_k^{n,p})$ to zero.

We denote $(y, z, u) = \{(y_k, z_k, u_k)\colon k = 0, \ldots, n\}$ (with or without superscript n) and, for $\gamma > 1$ introduce the norms

$$\|(y^n, z^n, u^n)\|_\gamma^2 := \mathbb{E}\left(\sup_{0\le k\le n} \gamma^{k/n} |y_k^n|^2\right) + \frac{1}{n}\mathbb{E}\left(\sum_{k=0}^{n-1} \gamma^{k/n} |z_k^n|^2\right)$$
$$+ \mathbb{E}\left(\kappa_n(1-\kappa_n)\sum_{k=0}^{n-1} \gamma^{k/n} |u_k^n|^2\right),$$
$$\|v^n\|_\gamma^2 := \mathbb{E}\left(\kappa_n(1-\kappa_n)\sum_{k=0}^{n-1} \gamma^{k/n} |v_k^n|^2\right).$$

Lemma 1. *There exists $\gamma > 1$ and $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ and $p \in \mathbb{N}^*$,*

$$\left\|\left(y^{n,p+1} - y^{n,p}, z^{n,p+1} - z^{n,p}, u^{n,p+1} - u^{n,p}\right)\right\|_\gamma^2 + \left\|v^{n,p+1} - v_p^n\right\|_\gamma^2$$
$$\le \frac{1}{4}\left\|\left(y^{n,p} - y^{n,p-1}, z^{n,p} - z^{n,p-1}, u^{n,p} - u^{n,p-1}\right)\right\|_\gamma^2.$$

Proof. We introduce some notations. As n remains fixed during the main part of the proof, it will be omitted in the notations whenever possible. We denote

$$\delta x_k^{p+1} := x_k^{n,p+1} - x_k^{n,p}$$

for a quantity $x = y, z, u, v, m$. With this notation observe that

$$\delta y_k^{p+1} = \delta y_{k+1}^{p+1} + \frac{1}{n}\delta f_k^p - \delta m_{k+1}^{p+1}, \tag{3.18}$$

where m is given in (2.11). We set

$$\delta f_k^p = f_n(t_k^n, y_k^{n,p}, z_k^{n,p}, u_k^{n,p}) - f_n(t_k^n, y_k^{n,p-1}, z_k^{n,p-1}, u_k^{n,p-1}).$$

Now, for some $\beta > 1$ we develop the quantity

$$\beta^n(\delta y_n^{p+1})^2 - \beta^k(\delta y_k^{p+1})^2 = -\beta^k(\delta y_k^{p+1})^2$$

through a discrete (time dependent) Itô Formula (compare with [20, VII §9]):

$$
\begin{aligned}
-\beta^k(\delta y_k^{p+1})^2 &= \sum_{i=k}^{n-1} \left(\beta^{i+1}(\delta y_{i+1}^{p+1})^2 - \beta^i(\delta y_i^{p+1})^2\right) \\
&= (\beta-1)\sum_{i=k}^{n-1}\beta^i(\delta y_i^{p+1})^2 + \sum_{i=k}^{n-1}\beta^{i+1}\left((\delta y_{i+1}^{p+1})^2 - (\delta y_i^{p+1})^2\right) \\
&= (\beta-1)\sum_{i=k}^{n-1}\beta^i(\delta y_i^{p+1})^2 + 2\beta\sum_{i=k}^{n-1}\beta^i\delta y_i^{p+1}(\delta y_{i+1}^{p+1} - \delta y_i^{p+1}) \\
&\quad + \beta\sum_{i=k}^{n-1}\beta^i(\delta y_{i+1}^{p+1} - \delta y_i^{p+1})^2. \qquad (3.19)
\end{aligned}
$$

From (3.18) follows, that

$$
(\delta y_{i+1}^{p+1} - \delta y_i^{p+1})^2 \geq \frac{1}{2}\left(\delta m_{i+1}^{p+1}\right)^2 - \frac{1}{n^2}\left(\delta f_i^p\right)^2.
$$

Changing signs, using the previous inequality and (3.18) again, from (3.19) we obtain

$$
\begin{aligned}
&\beta^k(\delta y_k^{p+1})^2 + \frac{\beta}{2}\sum_{i=k}^{n-1}\beta^i\left(\delta m_{i+1}^{p+1}\right)^2 \\
&\qquad \leq (1-\beta)\sum_{i=k}^{n-1}\beta^i(\delta y_i^{p+1})^2 + 2\beta\sum_{i=k}^{n-1}\beta^i\delta y_i^{p+1}\left(\frac{1}{n}\delta f_{i+1}^p - \delta m_{i+1}^{p+1}\right) \\
&\qquad\quad + \frac{\beta}{n^2}\sum_{i=k}^{n-1}\beta^i\left(\delta f_{i+1}^p\right)^2. \qquad (3.20)
\end{aligned}
$$

We now use the inequality, for $\lambda > 0$,

$$
\left(\delta y_i^{p+1}\right)\left(\frac{2\beta}{n}\delta f_{i+1}^p\right) \leq \lambda\left(\delta y_i^{p+1}\right)^2 + \frac{2\beta^2}{\lambda n^2}\left(\delta f_{i+1}^p\right)^2
$$

in the second term of (3.20) to obtain

$$\beta^k(\delta y_k^{p+1})^2 + \frac{\beta}{2}\sum_{i=k}^{n-1}\beta^i\left(\delta m_{i+1}^{p+1}\right)^2$$
$$\leq (1+\lambda-\beta)\sum_{i=k}^{n-1}\beta^i(\delta y_i^{p+1})^2 + \frac{\beta+4\lambda^{-1}\beta^2}{n^2}\sum_{i=k}^{n-1}\beta^i\left(\delta f_{i+1}^p\right)^2$$
$$- 2\beta\sum_{i=k}^{n-1}\beta^i\delta y_i^{p+1}\delta m_{i+1}^{p+1}.$$

We now assume that $1+\lambda-\beta<0$ and denote $B := \beta+4\lambda^{-1}\beta^2$ to obtain

$$\beta^k(\delta y_k^{p+1})^2 + \frac{\beta}{2}\sum_{i=k}^{n-1}\beta^i\left(\delta m_{i+1}^{p+1}\right)^2$$
$$\leq \frac{B}{n^2}\sum_{i=k}^{n-1}\beta^i\left(\delta f_{i+1}^p\right)^2 - 2\beta\sum_{i=k}^{n-1}\beta^i\delta y_i^{p+1}\delta m_{i+1}^{p+1}. \quad (3.21)$$

Formula (3.21) is our first main inequality. From it we obtain the following two results. First, as the last summand is a martingale, taking expectations with $k=0$, we obtain

$$\frac{\beta}{2}\mathbb{E}\sum_{i=k}^{n-1}\beta^i\left(\delta m_{i+1}^{p+1}\right)^2 \leq \frac{B}{n^2}\mathbb{E}\sum_{i=k}^{n-1}\beta^i\left(\delta f_{i+1}^p\right)^2. \quad (3.22)$$

Second, taking supremum over $k=0,\ldots,n$ we have

$$\sup_{0\leq k\leq n}\beta^k(\delta y_k^{p+1})^2 \leq \frac{B}{n^2}\sum_{i=0}^{n-1}\beta^i\left(\delta f_{i+1}^p\right)^2 + 4\beta\sup_{0\leq k\leq n}\left|\sum_{i=0}^{k}\beta^i\delta y_i^{p+1}\delta m_{i+1}^{p+1}\right|. \quad (3.23)$$

To obtain a convenient bound in the last term of (3.22), we use Davis (see [20, VII §3]) and afterwards Hölder inequalities. With F an universal constant, we obtain

$$
\begin{aligned}
4\beta\mathbb{E}\sup_{0\le k\le n}\left|\sum_{i=0}^{k}\beta^i\delta y_i^{p+1}\delta m_{i+1}^{p+1}\right| &\le F\,\mathbb{E}\sqrt{\sum_{i=0}^{n}\beta^{2i}\left(\delta y_i^{p+1}\right)^2\left(\delta m_{i+1}^{p+1}\right)^2} \\
&\le 4\beta F\,\mathbb{E}\sqrt{\sup_{0\le k\le n}\beta^k(\delta y_k^{p+1})^2\sum_{i=0}^{n}\beta^i\left(\delta m_{i+1}^{p+1}\right)^2} \\
&\le \frac{\beta}{2}\mathbb{E}\sup_{0\le k\le n}\beta^k(\delta y_k^{p+1})^2+32\beta F^2\mathbb{E}\sum_{i=0}^{n}\beta^i\left(\delta m_{i+1}^{p+1}\right)^2.
\end{aligned}
$$

Taking expectation in (3.23), using the previous result, and finally (3.22), we obtain that

$$
(1-\beta/2)\mathbb{E}\sup_{0\le k\le n}\beta^k(\delta y_k^{p+1})^2\le\frac{B}{n^2}\left(1+64\beta F^2\right)\sum_{i=0}^{n-1}\beta^i\left(\delta f_{i+1}^p\right)^2. \tag{3.24}
$$

Combining (3.22) and (3.24) we arrive to

$$
\mathbb{E}\sup_{0\le k\le n}\beta^k(\delta y_k^{p+1})^2+\mathbb{E}\sum_{i=k}^{n-1}\beta^i\left(\delta m_{i+1}^{p+1}\right)^2\le\frac{C}{n^2}\mathbb{E}\sum_{i=k}^{n-1}\beta^i\left(\delta f_{i+1}^p\right)^2, \tag{3.25}
$$

with $C=2B(1+1/\beta+64\beta F^2)/(1-\beta/2)$.

Using now the Lipschitz property (**A**) we see that there exists a constant $\tilde{K}$ such that

$$
|\delta f_i^p|^2\le\tilde{K}\left((\delta y_i^p)^2+(\delta z_i^p)^2+n\kappa_n(1-\kappa_n)(\delta u_i^p)^2\right). \tag{3.26}
$$

Equations (3.25) and (3.26) give

$$
\begin{aligned}
&\mathbb{E}\sup_{0\le k\le n}\beta^k(\delta y_k^{p+1})^2+\mathbb{E}\sum_{i=k}^{n-1}\beta^i\left(\delta m_{i+1}^{p+1}\right)^2 \\
&\quad\le\frac{C\tilde{K}}{n}\Bigg(\mathbb{E}\sup_{0\le k\le n}\beta^k(\delta y_k^p)^2+\frac{1}{n}\mathbb{E}\sum_{k=0}^{n-1}\beta^k\left(\delta z_k^p\right)^2 \\
&\qquad\qquad+\kappa_n(1-\kappa_n)\mathbb{E}\sum_{k=0}^{n-1}\beta^k\left(\delta u_k^p\right)^2\Bigg).
\end{aligned}
$$

It remains to choose properly β and λ as a function of n. For some $\gamma > 1$, set $\beta = \gamma^{1/n}$ and $\lambda = \beta/n$. The condition $1+\lambda < \beta$ is then equivalent to $\gamma > (1-1/n)^{-n}$. Thus, if $\gamma > e$, for n large enough this choice is suitable with our assumptions on β and λ, and C/n remains bounded as $n \to \infty$.

Computing $\mathbb{E}\left(\delta m_{i+1}^{p+1}\right)^2$, we conclude the proof. $\square$

Just like in the continuous case, we can use the Cauchy criterion and the preceding lemma to get the following result.

Proposition 1. *Following the notations of* (2.15) *and* (3.16)*,*

$$\mathbb{E}\left(\sup_{0\le t\le 1}\left|Y_t^{n,p}-Y_t^n\right|^2+\int_0^1\left|Z_t^{n,p}-Z_t^n\right|^2dt+\lambda\int_0^1\left|U_t^{n,p}-U_t^n\right|^2dt\right)$$

converges to 0 *uniformly in* n *as* $p\to\infty$.

We may now state a global bound which will be used to prove $L^2(\mathbb{P})$ convergence in Lemma 4 below.

Lemma 2. *Under Hypotheses* (**A**)*,* (**A'**)*,* (**B**) *and* (**B'**)*,*

$$\sup_{n\in\mathbb{N}}\sup_{p\in\mathbb{N}}\mathbb{E}\left[\left|\frac{1}{n}\sum_{k=0}^{n-1}f_n(t_k^n,y_k^{n,p},z_k^{n,p},u_k^{n,p})\right|^q\right]<+\infty.$$

In addition,

$$\sup_{p\in\mathbb{N}}\mathbb{E}\left[\left|\int_0^1 f(s,Y_s^{\infty,p},Z_s^{\infty,p},U_s^{\infty,p})ds\right|^q\right]<+\infty.$$

Proof. Here, we deal only with the discrete case, which is rather similar to the continuous case whose proof is a variation of the one given in [3, 10].

Again, we drop the superscript n. Here, we assume in a first time that the time horizon is T and $h = T/n$.

We set

$$\|(z^p,u^p)\| := \frac{T}{n}\sum_{k=1}^{n}\left(|z_k^p|^2+|u_k^p|^2\right).$$

Since $n\eta_k^2$ is bounded in n and k, we get from the Burkholder-Davies-Gundy inequality that for some constants C_1 and C_2,

$$\mathbb{E}[\|(z^p,u^p)\|^{q/2}] \le C_1\mathbb{E}\left[\left(\sum_{k=1}^{n}|m_k^p|^2\right)^{q/2}\right] \le C_2\mathbb{E}\left[\sup_{k=1,\dots,n}|m_k^p|^q\right].$$

On the other hand,

$$\mathbb{E}\left[\sup_{\ell=1,\dots,n}|m_\ell^{p+1}|^q\right]\leq\mathbb{E}\left[\sup_{\ell=1,\dots,n}\left|\mathbb{E}\left[\xi+\frac{T}{n}\sum_{k=0}^{n-1}f_n(t_k,y_k^p,z_k^p,u_k^p)\middle|\mathcal{F}_\ell^n\right]\right|^q\right].$$

With Hypothesis (**A**), for some constant C_3 depending only on $\Lambda(T)$ and K,

$$\begin{aligned}\left|\frac{T}{n}\sum_{k=0}^{n-1}f_n(t_k,y_k^p,z_k^p,u_k^p)\right|&\leq\frac{T}{\sqrt{n}}\left(\sum_{k=0}^{n-1}f_n(t_k,y_k^p,z_k^p,u_k^p)^2\right)^{1/2}\\&\leq C_3\frac{T}{\sqrt{n}}\left(n+\sum_{k=0}^{n-1}(|y_k^p|^2+|z_k^p|^2+|u_k^p|^2)\right)^{1/2}\\&\leq C_3T+C_3T\sup_{k=0,\dots,n-1}|y_k^p|+C_3\sqrt{T}\|(z^p,u^p)\|^{1/2}.\end{aligned}$$

With the Jensen inequality for the conditional expectation and the Doob inequality,

$$\mathbb{E}\left[\sup_{k=1,\dots,n}|m_k^{p+1}|^q\right]\leq C_4\mathbb{E}[|\xi|^q]+C_4T^q\|y^p\|_{q,\star}+C_4T+C_4T^q\mathbb{E}[\|(z^p,u^p)\|^{q/2}]$$

with

$$\|y^p\|_{q,\star}:=\mathbb{E}\left[\sup_{k=0,\dots,n-1}|y_k^p|^q\right].$$

On the other hand, we have with similar computations, for some constant C_4 depending only on C_3 and q,

$$\begin{aligned}\|y^{p+1}\|_\star&\leq\mathbb{E}\left[\sup_{\ell=0,\dots,n-1}\left|\mathbb{E}\left[\xi+\frac{T}{n}\sum_{k=\ell}^{n-1}f_n(t_k,y_k^p,z_k^p,u_k^p)\middle|\mathcal{F}_\ell^n\right]\right|^q\right]\\&\leq C_4\mathbb{E}[|\xi|^q]+C_4T+C_4T^q\|y^p\|_{q,\star}+C_4T^q\mathbb{E}[\|(z^p,u^p)\|^{q/2}].\end{aligned}$$

This proves that for $C_5=2C_4$,

$$\begin{aligned}&\|y^{p+1}\|_{q,\star}+\mathbb{E}[\|(z^{p+1},u^{p+1})\|^{q/2}]\\&\qquad\leq C_5\mathbb{E}[|\xi|^q]+C_5T+C_5T^q\left(\|y^p\|_{q,\star}+\mathbb{E}[\|(z^p,u^p)\|^{q/2}]\right).\end{aligned}$$

If T is small enough so that $C_5T<1$, then this proves that

$$\|y^p\|_{q,\star}+\mathbb{E}[\|(z^p,u^p)\|^{q/2}]\leq\frac{1}{1-C_5T}(1+C_5\mathbb{E}[|\xi|^q]).$$

Here, we have obtained a bound when T is small enough. Now, in order to consider the Picard scheme on the time interval $[0,1]$, we may find for each n a time T_n such that $C_5 T_n \le k < 1$ and $T_n = \ell(n)/n$ for some ℓ for some fixed k, and solve recursively the Picard scheme on $[T - T_n, T]$, $[T - 2Tn, T - T_n]$, ... using the terminal condition ξ^n and then $y^{n,p}_{\ell(n)}$, ...

We then obtain the desired bound. $\square$

3.2 Approximation of Brownian motion and Poisson process

In order to establish convergence in probability, we consider that all the processes are defined on the same probability space.

Lemma 3. *(I) Let N be a Poisson process of intensity λ and set $\widetilde{N}_t = N_t - \lambda t$. Then there exists a family of independent random variables $(\eta^n_k)_{k=1,\dots,n}$ whose distribution is given by (2.6) and the process defined by $\widetilde{N}^n_t = \sum_{k=1}^{\lfloor t/h \rfloor - 1} \eta^n_k$, is a martingale which converges in probability to $\widetilde{N}$ in the J_1-Skorokhod topology.*

(II) Let W be a Brownian motion. Then there exists a family of realizations independent random variables ϵ^n_k such that $\mathbb{P}(\epsilon^n_k = 1) = \mathbb{P}(\epsilon^n_k = -1) = 1/2$ and the process defined by $W^n_t = h \sum_{i=1}^{\lfloor t/h \rfloor - 1} \epsilon^n_i$ converges uniformly in probability to W.

(III) The couple $(W^n, \widetilde{N}^n)$ converges in the J_1-Skorokhod topology in probability to $(W, \widetilde{N})$.

Proof. (I) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space on which N is defined. Denote by $\{\tau_i\}_{i=1,\dots,\ell-1}$ the time jumps de N. To simplify the notations, we set $\tau_0 = 0$ and $\tau_\ell = T$. Let A^n the the event

$$A^n = \left\{ \begin{array}{l} \text{there is at most one jump of } N \\ \text{on any interval } [kh, h(k+1)) \text{ for } k = 0, \dots, n \end{array} \right\}.$$

We denote by $\overline{A}^n$ the complementary event of A^n. We set $\eta^n_k = \kappa_n - 1$ with $\kappa_n = e^{-\lambda/n}$ if one jumps occurs on $[kh, h(k+1))$ and $\eta^n_k = \kappa_n$ otherwise. The distribution of η^n_k is given by (2.6).

Let $\phi^n(t,\omega)$ be the random piecewise linear function defined so that for $i = 0, \dots, \ell$, $\phi^n(\tau_i) = c_n(\tau_i)$ on A^n and by $\phi^n(t) = t$ on $\overline{A}^n$. It is easily checked that $0 \le t - \phi^n(t) \le h$ for $t \in [0,1]$.

Then for $t \in [\tau_i, \tau_{i+1}]$,

$$\widetilde{N}^n_{\phi^n(t)} - \widetilde{N}^n_{c_n(\tau_i)} = (e^{-\lambda/n} - 1)(c_n(\phi_n(t)) - c_n(\tau_i)).$$

On the other hand, $\widetilde{N}_t - \widetilde{N}_{\tau_i} = \lambda(t-\tau_i)$ and then there exists some constant K such that

$$|\widetilde{N}^n_{\phi^n(t)} - \widetilde{N}^n_{c_n(\tau_i)} - \widetilde{N}_t + \widetilde{N}_{\tau_i}| \leq Kh \tag{3.27}$$

for $t \in [\tau_i, \tau_{i+1}]$. Besides, on A^n, for h small enough,

$$|\widetilde{N}^n_{c_n(\tau_i)} - \widetilde{N}^n_{c_n(\tau_i)-} - \widetilde{N}_{\tau_i} - \widetilde{N}_{\tau_i-}| = e^{-\lambda/n} - 1 \leq 2\lambda h. \tag{3.28}$$

Combining (3.27) and (3.28), one gets that on A_n,

$$\sup_{t\in[0,1]} |\widetilde{N}^n_{\phi^n(t)} - \widetilde{N}_t| \leq (2+K)\ell h.$$

In addition, $\mathbb{E}[\ell] < +\infty$ so that

$$\mathbb{E}\left[\sup_{t\in[0,1]} |\widetilde{N}^n_{\phi^n(t)} - \widetilde{N}_t|; A_n\right] \xrightarrow[n\to\infty]{} 0.$$

On $\overline{A}^n$, then $\phi^n(t) = t$ and then

$$\widetilde{N}_{ih} - \widetilde{N}^n_{ih} = -\lambda ih - h(\kappa-1)i + \sum_{j=0}^{i-1} (N_{(j+1)h} - N_j - 1)\mathbf{1}_{\{N_{(j+1)h} - N_j \geq 2\}}.$$

From the very definition of A^n and since $N_{(i+1)h} - N_{ih}$ has the distribution of a Poisson random variable with intensity λh,

$$\mathbb{P}\left[\overline{A}^n\right] \leq \sum_{i=0}^{n-1} \mathbb{P}\left[|N_{(i+1)h} - N_{ih}| \geq 2\right] \leq n(1 - e^{-\lambda h} - \lambda h e^{-\lambda h}) \leq \frac{\lambda^2 T^2}{n}.$$

For any $C > 0$,

$$\mathbb{P}\left[\sup_{t\in[0,1]} |\widetilde{N}_t - \widetilde{N}^n_{\phi^n(t)}| > C\right] \leq \mathbb{P}\left[\overline{A}^n\right] + \frac{1}{C}\mathbb{E}\left[\sup_{t\in[0,1]} |\widetilde{N}_t - \widetilde{N}^n_{\phi^n(t)}|; A^n\right]$$

and this quantity converges to 0 as $n \to \infty$. Thus there exists a family $(\phi^n)_{n\in\mathbb{N}}$ of one-to-one random time changes from $[0,1]$ to $[0,1]$ such that

$\sup_{t\in[0,1]} |\phi^n(t) - t| \xrightarrow[n\to\infty]{} 0$ almost surely and $\sup_{t\in[0,1]} |\widetilde{N}_t - \widetilde{N}^n_{\phi^n(t)}| \xrightarrow[n\to\infty]{} 0$ in probability, which means that $\widetilde{N}^n$ converge in the J_1-Skorokhod topology to N.

Point (II) follows from the Donsker theorem, when one uses for example the Skorokhod embedding theorem to construct the ϵ_k's from the Brownian path [18] and (III) holds since W is continuous so that the 2-dimensional path $(W^n, \widetilde{N}^n)$ converges in the J_1-topology to $(W, \widetilde{N})$. $\square$

3.3 Convergence of martingales

Let $H = (W, N)$ be such that W is a Brownian motion and N is an independent Poisson process of intensity λ. Let W^n and $\widetilde{N}^n$ be the one defined in Lemma 3 and set $H^n = (W^n, \widetilde{N}^n)$. Let $(\mathcal{F}_t)_{t\in[0,1]}$ (resp. $(\mathcal{F}^n_t)_{t\in[0,1]}$) be the filtration generated by H (resp. H^n).

Let X (resp. X^n) be a of $\mathcal{F}_1$ (resp. $\mathcal{F}^n_1$)-measurable random variable such that

(H) $\mathbb{E}[X^2] + \sup_{n\in\mathbb{N}} \mathbb{E}[(X^n)^2] < +\infty$ and $\mathbb{E}[|X^n - X|] \xrightarrow[n\to\infty]{} 0.$

Let M (resp. M^n) be the cdlg martingales

$$M^n_t = \mathbb{E}\left(X^n \middle| \mathcal{F}^n_t\right) \text{ and } M_t = \mathbb{E}\left(X \middle| \mathcal{F}_t\right). \tag{3.29}$$

We denote by $[M^n, M^n]$ (resp. $[M, M]$) the quadratic variation of M^n (resp. M) and by $[M^n, W^n], [M^n, \widetilde{N}^n]$ (resp. $[M, W], [M, \widetilde{N}]$) the cross variation of M^n and W^n (resp. $\widetilde{N}^n$).

The following proposition is an adaptation of Theorem 3.1 in [5], and [7] for the convergence of filtrations. Hypothesis (H) ensures the uniform square integrability of M^n and then the convergence of the brackets.

Proposition 2. *Under the above conditions,*

$$(H^n, M^n, [M^n, M^n], [M^n, W^n], [M^n, \widetilde{N}^n]) \xrightarrow[n\to\infty]{} (H, M, [M, M], [M, W], [M, \widetilde{N}])$$

in probability for the J_1-Skorokhod topology.

Corollary 1. *Set $\widetilde{M^n} = \sum_{k=1}^{\lfloor t/h \rfloor - 1} \eta^n_k \epsilon^n_k$, which is a martingale orthogonal to W^n and $\widetilde{N}^n$. Assume in addition to* (H) *that*

(H') $\mathbb{E}[|X^n - X|^2] \xrightarrow[n\to\infty]{} 0.$

Then there exist three sequences $(Z^n_t)_{0\le t\le 1}$, $(V^n_t)_{0\le t\le 1}$ and $(U^n_t)_{0\le t\le 1}$ of $\mathcal{F}^n_\cdot$-predictable processes, and two independent $(Z_t)_{0\le t\le 1}$ and $(U_t)_{0\le t\le 1}$ $\mathcal{F}_\cdot$-predictable processes such that

$$\forall t\in[0,1], \begin{cases} M^n_t = \mathbb{E}[X^n] + \int_0^t Z^n_{s-}dW^n_s + \int_0^t U^n_{s-}d\widetilde{N}^n_s + \int_0^t V^n_{s-}d\widetilde{M}^n_s \\ M_t = \mathbb{E}[X] + \int_0^t Z_{s-}dW_s + \int_0^t U_{s-}d\widetilde{N}_s \end{cases}$$

with

$$\mathbb{E}\left[\int_0^1 (Z^n_t - Z_t)^2 dt + \lambda\int_0^1 (U^n_t - U_t)^2 dt\right] \xrightarrow[n\to\infty]{} 0.$$

Proof. The first part is related to the predictable representation of $\mathcal{F}^n$-martingales in terms of stochastic integrals with respect to three independent random walks, W^n, $\widetilde{N}^n$ and $\widetilde{M}^n$. The increments of $\widetilde{M}^n$ may take up to four different values, which means that we need three orthogonal martingales to represent it [8]. It is then easily obtained that $\widetilde{M}^n$ is a martingale which is orthogonal to both $\widetilde{N}^n$ and W^n. This is why we introduce it. The predictable representation of M with respect to W and $\widetilde{N}$ is classical.

From the Doob inequality,

$$\mathbb{E}[[M^n, M^n]_1] \le \mathbb{E}[|M^n_1|^2] \le 2\mathbb{E}[|X^n|^2].$$

With (H),

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Bigg[\frac{1}{n}\int_0^1 (Z^n_{s-})^2 dc_n(s) + \int_0^1 (\eta^n_{c_n(s)})^2 (U^n_{s-})^2 dc_n(s) + \frac{1}{n}\int_0^1 (\eta^n_{c_n(s)})^2 (V^n_{s-})^2 dc_n(s)\Bigg] < +\infty. \quad (3.30)$$

Since $\mathbb{E}[n(\eta^n_k)^2] \sim_{n\to\infty} \lambda$ and U^n is predictable with respect to $\mathcal{F}^n$, one easily get that

$$\limsup_{n\to\infty} \mathbb{E}\left[\int_0^1 (Z^n_s)^2 ds + \int_0^1 \lambda (U^n_s)^2 ds\right] < +\infty. \quad (3.31)$$

Let $(\mathcal{F}_t^W)_{t\geq 0}$ be the filtration generated by the Brownian motion. Since W and N are independent, for $\overline{X} = X - \mathbb{E}[X]$,

$$\mathbb{E}[\overline{X} \mid \mathcal{F}_1^W] = \int_0^1 Z_s dW_s. \tag{3.32}$$

Let also $\mathcal{G}$ be the σ-algebra generated by $(\epsilon_1, \ldots, \epsilon_n)$. Hence, for $\overline{X}^n = X^n - \mathbb{E}[X^n]$,

$$\mathbb{E}\left[\overline{X}^n \mid \mathcal{G}\right] = \int_0^1 Z_{s-}^n dW_s^n. \tag{3.33}$$

It follows that

$$\mathbb{E}\left[\mathbb{E}\left[\overline{X}^n \mid \mathcal{G}\right]^2\right] = \mathbb{E}\left[\int_0^1 (Z_{s-}^n)^2 dc_n(s)\right].$$

Since the ϵ_k's are constructed from the trajectories of W, one has $\mathcal{F} \subset \mathcal{F}_1^W$. Hence

$$\mathbb{E}\left[\mathbb{E}\left[\overline{X}^n \mid \mathcal{G}\right]^2\right] \leq 2\mathbb{E}\left[\mathbb{E}\left[\overline{X}^n - \overline{X} \mid \mathcal{G}\right]^2\right] + 2\mathbb{E}\left[\mathbb{E}\left[\overline{X} \mid \mathcal{G}\right]^2\right].$$

With the Jensen inequality on conditional expectation and (H'), one gets that

$$\mathbb{E}\left[\mathbb{E}\left[\overline{X}^n - \overline{X} \mid \mathcal{G}\right]^2\right] \xrightarrow[n\to\infty]{} 0$$

and

$$\mathbb{E}\left[\mathbb{E}\left[\overline{X} \mid \mathcal{G}\right]^2\right] \leq \mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[\overline{X} \mid \mathcal{F}_1^W\right] \mid \mathcal{G}\right]^2\right] \leq \mathbb{E}\left[\mathbb{E}\left[\overline{X} \mid \mathcal{F}_1^W\right]^2\right].$$

From (3.32) and (3.33), one gets that

$$\limsup_{n\to\infty} \mathbb{E}\left[\int_0^1 (Z_s^n)^2 ds\right] \leq \mathbb{E}\left[\int_0^1 Z_s^2 ds\right]. \tag{3.34}$$

Similar arguments prove that

$$\limsup_{n\to\infty} \mathbb{E}\left[\int_0^1 (U_s^n)^2 ds\right] \leq \mathbb{E}\left[\int_0^1 U_s^2 ds\right].$$

From Proposition 2, $[M^n, W^n] \to [M, W]$ in probability for the J_1-Skorokhod topology, as well as $[M^n, M^n]$. Then

$$\sup_{0\leq t\leq 1} \left| \int_0^{\psi^n(t)} Z_{s-}^n dc_n(s) - \int_0^t Z_s ds \right| \xrightarrow[n\to\infty]{} 0 \tag{3.35}$$

in probability, where $\psi^n(t) \uparrow t$. Then we get easily that

$$\sup_{0\le t\le 1}\left|\int_0^t Z_s^n ds - \int_0^t Z_s ds\right| \xrightarrow[n\to\infty]{} 0 \tag{3.36}$$

in probability and with (3.31), in $L^1(\mathbb{P})$.

On the other hand, $[M^n, \widetilde{N}^n] \to [M, \widetilde{N}]$ in probability for the J_1-Skorokhod topology. This implies that

$$\sup_{0\le t\le 1}\left|\int_0^{\psi^n(t)} \eta^2_{c_n(s)} U^n_{s-} dc_n(s) - \lambda\int_0^t U_s ds\right| \xrightarrow[n\to\infty]{} 0. \tag{3.37}$$

We can apply the same arguments used for (3.36), Burkholder-Davis-Gundy inequality to control the distance between η_k^2 and $\kappa_n(1-\kappa_n)$, and the fact that $1-\kappa_n \sim_{n\to\infty} \lambda/n$ to get

$$\sup_{0\le t\le 1}\lambda\left|\int_0^t U_s^n ds - \int_0^t U_s ds\right| \xrightarrow[n\to\infty]{} 0 \tag{3.38}$$

in probability and in $L^1(\mathbb{P})$.

The second part relies on the following argument: let $(g_n)_{n\in\mathbb{N}\cup\{\infty\}}$ be a sequence of functions on $[0,1]\times\Omega$ such that

$$\limsup_{n\to\infty}\mathbb{E}\left[\int_0^1 (g_n(s,\omega))^2 ds\right] \le \mathbb{E}\left[\int_0^1 (g_\infty(s,\omega))^2 ds\right] < +\infty, \tag{3.39}$$

$$\text{and } \mathbb{E}\left[\sup_{t\in[0,1]}\left|\int_0^t (g_n(s,\cdot) - g_\infty(s,\cdot))ds\right|\right] \xrightarrow[n\to\infty]{} 0. \tag{3.40}$$

For any given function h_∞ in $L^2([0,1]\times\Omega)$, there exists a sequence of functions $(h_n)_{n\in\mathbb{N}}$ in $L^2([0,1]\times\Omega)$ such that $h_n(s,\omega)$ is of form $\sum_{i=1}^p c_i(\omega)\mathbf{1}_{[t_i,t_{i+1}]}(s)$ and h_n converges to h in $L^2([0,1]\times\Omega)$.

With (3.39),

$$\left|\mathbb{E}\left[\int_0^1 g_n(s,\omega)h_\infty(s,\omega)ds\right] - \mathbb{E}\left[\int_0^1 g_n(s,\omega)h_m(s,\omega)ds\right]\right| \le \sup_{n\in\mathbb{N}}\|g_n\|_{L^2([0,1]\times\Omega}\|h_\infty - h_m\|_{L^2([0,1]\times\Omega}$$

and with (3.39)-(3.40),

$$\mathbb{E}\left[\int_0^1 g_n(s,\omega)h_m(s,\omega)ds\right] \xrightarrow[n\to\infty]{} \mathbb{E}\left[\int_0^1 g_\infty(s,\omega)h_m(s,\omega)ds\right].$$

It follows that g_n converges weakly in $L^2([0,1]\times\Omega)$ to g_∞. In addition, (3.39) implies indeed the strong convergence of g_n to g_∞, which means that $\mathbb{E}\left[\int_0^1 |g_n(s,\omega)-g_\infty(s,\omega)|^2 ds\right]$ converges to 0.

It is now possible to apply this argument to both Z^n and U^n. □

3.4 Convergence of the solution of the BSDE

The idea is now to prove that if $(Y^{n,p}, Z^{n,p}, U^{n,p})$ converges to $(Y^{\infty,p}, Z^{\infty,p}, U^{\infty,p})$ in a given sense, then this is also true for the $(p+1)$-th Picard iteration.

Here, the notion of convergence is

$$\sup_{0\le t\le 1}\left|Y^{n,p}_{\psi^n(t)} - Y^{\infty,p}_t\right|^2 + \int_0^1 |Z^{n,p}_{s-} - Z^{\infty,p}_{s-}|^2\, ds \\ + \lambda\int_0^1 |U^{n,p}_{s-} - U^{\infty,p}_{s-}|^2\, ds \xrightarrow[n\to\infty]{} 0 \quad (3.41)$$

in $L^1(\mathbb{P})$, where ψ^n is a random one-to-one continuous mapping from $[0,1]$ to $[0,1]$ that converges uniformly to $t\mapsto t$ almost surely.

Let us set

$$A^{n,p}_t = \int_0^t f_n(c_n(s), Y^{n,p}_{s-}, Z^{n,p}_{s-}, U^{n,p}_{s-})dc_n(s)$$

and

$$A^{\infty,p}_t = \int_0^t f(s, Y^{\infty,p}_s, Z^{\infty,p}_s, U^{\infty,p}_s)ds.$$

Lemma 4. *If for some integer p, $(Y^{n,p}, Z^{n,p}, U^{n,p})$ converges to $(Y^{\infty,p}, Z^{\infty,p}, U^{\infty,p})$ in the sense of* (3.41)*, then $A^{n,p}_{\psi^n(t)}$ converges uniformly in t to $A^{\infty,p}_t$ in $L^2(\mathbb{P})$.*

Proof. Let us note first that $A^{n,p}$ is piecewise constant on the intervals $[k/n,(k+1)/n)$. Let $\xi^n(t)$ be the inverse of $\psi^n(t)$. Then

$$\sup_{t\in[0,1]}|A^{n,p}_{\psi^n(t)} - A^{\infty,p}_t| = \sup_{t\in[0,1]}|A^{n,p}_t - A^{\infty,p}_{\xi^n(t)}| \\ = \sup_{k=0,\dots,n-1}|A^{n,p}_{k/n} - A^{\infty,p}_{k/n}| + \sup_{k=0,\dots,n-1}\sup_{t\in[k/n,(k+1)/n]}|A^{\infty,p}_{k/n} - A^{\infty,p}_{\xi^n(t)}|.$$

Because of (**A**), (**A'**) and (3.17), we easily get that the last term above converges to 0 in $L^2(\mathbb{P})$ uniformly in p.

On the other hand, with (**A**) and (**A'**),

$$|A^{n,p}_{k/n} - A^{\infty,p}_{k/n}| \leq K \int_0^1 (|Y^{n,p}_s - Y^{\infty,p}_s| + |Z^{n,p}_s - Z^{\infty,p}_s| + |U^{n,p}_s - U^{\infty,p}_s|)ds.$$

Yet let us note that

$$\int_0^1 |Y^{n,p}_s - Y^{\infty,p}_s| ds \leq \int_0^1 |Y^{n,p}_s - Y^{\infty,p}_{\xi^n(s)}| ds + \int_0^1 |Y^{\infty,p}_{\xi^n(s)} - Y^{\infty,p}_s| ds.$$

The first term in the right-hand side of the previous inequality converges to 0 since $Y^{n,p} - Y^{\infty,p} \circ \xi^n$ converges uniformly to 0 in $L^2(\mathbb{P})$.

Regarding the second term, $s \mapsto Y^{\infty,p}_s$ is continuous except at the times at which the Poisson process jumps. Hence, $Y^{\infty,p}_{\xi^n(s)}$ converges to $Y^{\infty,p}_s$ for almost every $s \in [0,1]$ and then $\int_0^1 |Y^{\infty,p}_{\xi^n(s)} - Y^{\infty,p}_s| ds$ converges to 0 almost surely.

With Lemma 2, $\sup_{t\in[0,1]} |A^{n,p}_{\psi^n(t)} - A^{\infty,p}_t|$ converges converges to 0 in $L^2(\mathbb{P})$. □

Proposition 3. *Under Hypotheses* (**A**), (**A'**), (**B**) *and* (**B'**), *for any* $p \in \mathbb{N}$, $(Y^{n,p}, Z^{n,p}, U^{n,p})$ *converges to* $(Y^{\infty,p}, Z^{\infty,p}, U^{\infty,p})$ *in the sense of* (3.41).

Proof. This will be done by induction on p. We rewrite (3.16) as

$$\begin{aligned} Y^{n,p+1}_t = \xi^n + A^{n,p}_1 - A^{n,p}_t \\ - \int_t^1 Z^{n,p+1}_{s-} dW^n_s - \int_t^1 U^{n,p+1}_{s-} d\tilde{N}^n_s - \int_t^1 V^n_s d\tilde{M}^n_s. \end{aligned} \quad (3.42)$$

The induction hypothesis is that $(Y^{n,p}, Z^{n,p}, U^{n,p})$ converges to $(Y^{\infty,p}, Z^{\infty,p}, U^{\infty,p})$ in the sense of (3.41) so that our aim is to prove that the triplet $(Y^{n,p+1}, Z^{n,p+1}, U^{n,p+1})$ converges to $(Y^{\infty,p+1}, Z^{\infty,p+1}, U^{\infty,p+1})$ in the same sense.

As $(Y^{n,0}, Z^{n,0}, U^{n,0}) = (0,0,0)$ and $s \mapsto f(s,0,0,0)$ is continuous, the first step of the induction is immediate from Corollary 1 using (**B'**).

Taking conditional expectations w.r.t. $\mathcal{F}^n_k$ in (3.16) and using the fact that $Y^{n,p+1}_{t_k}$ is $\mathcal{F}^n_k$-measurable, we find that for $t_k \leq c_n(t) < t_{k+1}$,

$$Y^{n,p+1}_t = \mathbb{E}\left[\xi^n + \int_t^1 f_n\left(c_n(s), Y^{n,p}_{s-}, Z^{n,p}_{s-}, U^{n,p}_{s-}\right) dc_n(s) \,|\mathcal{F}^n_k\right].$$

So that

$$\begin{aligned}
M_t^{n,p+1} :=& Y_t^{n,p+1} + \int_0^t f_n\left(c_n(s), Y_{s-}^{n,p}, Z_{s-}^{n,p}, U_{s-}^{n,p}\right) dc_n(s) = Y_t^{n,p+1} + A_t^{n,p} \\
=& \mathbb{E}\left[\xi^n + \int_0^1 f_n\left(s-, Y_{s-}^{n,p}, Z_{s-}^{n,p}, U_{s-}^{n,p}\right) dc_n(s) \big| \mathcal{F}_k^n\right] \\
=& \mathbb{E}\left[M_1^{n,p+1} \big| \mathcal{F}_k^n\right] \quad \text{is a } \mathcal{F}_\cdot^n \text{ martingale.}
\end{aligned}$$

Moreover, we have the representation

$$\begin{aligned}
M_t^{n,p+1} &= \mathbb{E}\left[\int_0^1 Z_{s-}^{n,p+1} dW_s^n + \int_0^1 U_{s-}^{n,p+1} d\tilde{N}_s^n + \int_0^1 V_{s-}^{n,p+1} d\tilde{M}_s^n \big| \mathcal{F}_k^n\right] \\
&= \int_0^{t_k} Z_{s-}^{n,p+1} dW_s^n + \int_0^{t_k} U_{s-}^{n,p+1} d\tilde{N}_s^n + \int_0^{t_k} V_{s-}^{n,p+1} d\tilde{M}_s^n \\
&= \int_0^{t} Z_{s-}^{n,p+1} dW_s^n + \int_0^{t} U_{s-}^{n,p+1} d\tilde{N}_s^n + \int_0^{t} V_{s-}^{n,p+1} d\tilde{M}_s^n.
\end{aligned}$$

The last decomposition corresponds to the martingale representation theorem given in Corollary 1. In order to apply this corollary to the sequence of martingales $\left\{\left(M_t^{n,p+1}\right)_{0\le t\le 1}; n \in \mathbb{N}\right\}$, we have to prove the $L^2(\mathbb{P})$ convergence of $M_1^{n,p+1}$ (the terminal value).

Using the fact that $Y^{n,p}$, $Z^{n,p}$ and $U^{n,p}$ are piecewise constant, we have that

$$|M_1^{n,p+1} - \xi - A_1^{\infty,p}| \le |\xi^n - \xi| + |A_1^{n,p} - A_1^{\infty,p}|.$$

With Lemma 2, Lemma 4 and (**H**), this last quantity tends to zero in $L^2(\mathbb{P})$.

Applying Corollary 1, we see that $M^{n,p+1}$ converges to

$$M_t^{\infty,p+1} := \mathbb{E}\left(\xi + \int_0^1 f\left(s, Y_s^p, Z_s^p, U_s^p\right) ds \big| \mathcal{F}_t\right) = Y_t^{\infty,p+1} + A_t^{\infty,p}, \tag{3.43}$$

in the sense that

$$\begin{aligned}
\sup_{0\le t\le 1} \left|M_{\psi^n(t)}^{n,p+1} - M_t^{\infty,p+1}\right|^2 &+ \int_0^1 \left|Z_s^{n,p+1} - Z_s^{\infty,p+1}\right|^2 ds \\
&+ \lambda \int_0^1 \left|U_s^{n,p+1} - U_s^{\infty,p+1}\right|^2 ds \to 0
\end{aligned}$$

in $L^2(\mathbb{P})$, where ψ^n is a random one-to-one continuous mapping from $[0,1]$ to $[0,1]$ that converges uniformly to $t \mapsto t$ almost surely.

With (3.43) and Lemma 4, then we get the convergence of $(Y^{n,p+1}, Z^{n,p+1}, U^{n,p+1})$ to $(Y^{\infty,p+1}, Z^{\infty,p+1}, U^{\infty,p+1})$ in the sense of (3.41). □

4 Applications to decoupled system of SDE and BSDEJ and to the numerical computations of the solutions of PIDE

Let X be the solution of the d-dimensional SDE with jumps

$$X_t = x + \int_0^t \sigma(s-, X_{s-})dW_s + \int_0^t b(s-, X_{s-})ds + \int_0^t c(s-, X_{s-})d\widetilde{N}_s, \quad (4.44)$$

where we have assumed that

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| + |c(t,x) - c(t,y)| \leq K'|x-y|, \quad (4.45)$$
$$\sup_{t\in[0,1]} \left(|b(t,0)| + |\sigma(t,0)| + |c(t,0)|\right) \leq K'' \quad (4.46)$$

for all $t \in [0,1]$ and for alll $x, y \in \mathbb{R}$. Of course, the BSDEJ

$$Y_t = \xi + \int_t^1 f(s, X_{s-}, Y_s, Z_s, U_s)ds - \int_t^1 Z_s\sigma(s-, X_{s-})dW_s - \int_t^1 U_s d\widetilde{N}_s \quad (4.47)$$

is linked to the non-linear PIDE (with $a = \sigma \cdot \sigma^{\mathrm{T}}$) by

$$\begin{aligned} &\frac{\partial u(t,x)}{\partial t} + \sum_{i,j=1}^d \frac{1}{2} a_{i,j}(t,x) \frac{\partial^2 u(t,x)}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(t,x) \frac{\partial u(t,x)}{\partial x_i} \\ &\quad + \int_{\mathbb{R}} \left(u(t, x + c(t,x,z)) - u(t,x) - c_i(t,x,z) \frac{\partial u(t,x)}{\partial x_i} \right) \Pi(dz) \\ &\quad = f(t, x, u(t,x), \nabla u(t,x)\sigma(t,x), u(t, x + c(t,x,\cdot)) - u(t,x)) \end{aligned} \quad (4.48)$$

with the terminal condition $u(1,x) = g(x)$. It is standard in the theory of BSDE that $u(t, X_t) = Y_t$ and thus $u(0,x) = Y_0$. One can compute similarly $u(s,x)$ for any $s \in [0,t)$ by using the solution to (4.44) starting from (s,x) instead of $(0,x)$.

We use now for $\Pi(dz)$ the measure $\Pi(dz) = \sum_{x_i} \pi_i \delta_{x_i}$, where x_i belongs to the interval I_i.

Following Remark 3, we assume for the sake of simplicity that indeed $\Pi(dz) = \lambda\delta_0$ and that the dimension of the Brownian motion W is 1. Hence, we rewrite $c(t,x,z)$ as $c(t,x)$ since only $c(t,x,0)$ is used.

The SDE (4.44) may be discretized the following way for an integer n: we set $\chi_0^n = x$ and for $i = 0, \dots, n$,

$$\chi_{i+1}^n = \chi_i^n + hb((i+1)h, \chi_i^n) \\ + \sqrt{h}\sigma(((i+1)h, \chi_i^n))\epsilon_{i+1}^n + c((i+1)h, \chi_i^n)\eta_{i+1}^n \quad (4.49)$$

where η^n is a Bernoulli approximation of the compensated Poisson process with intensity λ and ϵ^n is a Bernoulli approximation of the Brownian motion W. Of course, the χ_i^n's are easily simulated. This discrete equation (4.49) may be rewritten in continuous time as

$$X_t^n = x + \int_0^t \sigma(s-, X_{s-}^n) dW_s^n + \int_0^t b(s-, X_{s-}^n) dc^n(s) + \int_0^t c(s-, X_{s-}^n) d\widetilde{N}_s^n \quad (4.50)$$

Thanks to the results in [22], X^n converges in probability in the J_1-Skorokhod topology to the solution X to (4.44).

Using our algorithm, it is then possible to find $(y_i^n, z_i^n, u_i^n)_{i=1,\dots,n}$ adapted to $(\mathcal{F}_i^n)_{i=0,\dots,n}$ that solves the discrete BSDE

$$y_i^n = y_{i+1}^n + hf((i+1)h, \chi_i^n, y_i^n, z_i^n, u_{\cdot,i}^n) - z_i^n \epsilon_{i+1}^n - u_i^n \eta_{i+1}^n - v_i^n \epsilon_{i+1}^n \eta_{i+1}^n \quad (4.51)$$

for $i = 0, \dots, n-1$ with the terminal condition $y_n^n = g(\chi_n^n)$.

We are looking for a function $v^n(i, z)$ such that $y_k^n = v^n(i, \chi_i^n)$ and v^n solves a discrete PDE.

For a function v on $\{0, \dots, n\} \times \mathbb{R}$, we define using (2.13) the discrete operators

$$\begin{aligned} D_0^n v(k, x) &= \mathbb{E}[v(k, x) \,|\, \mathcal{F}_k^n] \\ &= \frac{1-\kappa}{2}\Big(v(k, x + hb(kh, x) + \sqrt{h}\sigma(kh, x) + \kappa c(kh, x)) \\ &\qquad + v(k, x + hb(kh, x) - \sqrt{h}\sigma(kh, x) + \kappa c(kh, x))\Big) \\ &\quad + \frac{\kappa}{2}\Big(v(k, x + hb(kh, x) + \sqrt{h}\sigma(kh, x) + (\kappa - 1)c(kh, x)) \\ &\qquad + v(k, x + hb(kh, x) - \sqrt{h}\sigma(kh, x) + (\kappa - 1)c(kh, x))\Big) \end{aligned}$$

and

$$D_1^n v(k, x) = \mathbb{E}[v(k, x)\epsilon_{k+1}^n \,|\, \mathcal{F}_k^n] \text{ and } D_2^n v(k, x) = \mathbb{E}[v(k, x)\eta_{k+1}^n \,|\, \mathcal{F}_k^n],$$

for which formulae similar to the one for $D_0^n v(k,x)$ can be given.

From these results, one can deduce the representation of the solution of a discrete PDE with the help of the χ_i^n. This representation is similar to the representation of the solution of the BSDEJ in term of $Y_t = u(t, X_t)$, where u is the solution to the PIDE (4.48).

Proposition 4. *Let v^n be the solution to the discrete PDE*

$$\begin{aligned} v_i^n(i,x) &= D_0^n v^n(i+1,x) \\ &\quad + hf((i+1)h, x, v_i^n(i,x), h^{-1/2} D_1^n v^n(i+1,\chi_i^n), D_2^n v^n(i+1,x)) \\ &\qquad\qquad \text{for } i = 0,\dots,n-1,\ x \in \mathbb{R}, \end{aligned} \tag{4.52}$$

with the terminal condition $v_i^n(1,x) = g(x)$. If $hK < 1$, then this solution exists and is unique. In addition, the solution (y^n, z^n, u^n) to the discrete BSDE (2.11) with the terminal condition $\xi = g(\chi_n^n)$ which we construct using our algorithm satisfies $y_i^n = v^n(i,\chi_i^n)$, $z_i^n = h^{-1/2} D_1^n v^n(i+1,\chi_i^n)$ and $u_i^n = D_2^n v^n(i+1,\chi_i^n)$.

Proof. As $hK < 1$ the existence and uniqueness of $v^n(i,\cdot)$ follows from the existence of the solution $\rho(x)$ to

$$\begin{aligned} \rho(x) = D_0^n v^n(i+1,x) + hf((i+1)h, x, \rho(x), h^{-1/2} D_1^n v^n(i+1,x), \\ (1-\kappa)^{-1} D_2^n v^n(i+1,x)) \end{aligned}$$

for any $x \in \mathbb{R}$, once $v^n(i+1,\cdot)$ is known. Thus, one can proceed recursively with $i = n-1$ down to 0.

Let (y^n, z^n, u^n) be given by our algorithm. We assume $v^n(i+1,\chi_{i+1}^n) = y_{i+1}^n$, which is true for $i+1 = n$. Using (2.8), (2.9), and the definitions of D_1^n and D_2^n,

$$\begin{aligned} z_i^n &= h^{-1/2}\mathbb{E}[v^n(i+1,\chi_{i+1}^n)\epsilon_{i+1}^n | \mathcal{F}_i^n] = h^{-1/2} D_1^n v^n(i+1,\chi_i^n), \\ u_i^n &= \mathbb{E}[v^n(i+1,\chi_{i+1}^n)\eta_{i+1}^n | \mathcal{F}_i^n] = D_2^n v^n(i+1,\chi_i^n). \end{aligned}$$

Taking conditional expectation with respect to $\mathcal{F}_i^n$ in (4.52), since $D_0^n v^n(i+1,\chi_i^n) = \mathbb{E}[v^n(i+1,\chi_{i+1}^n)|\mathcal{F}_i^n]$, we get

$$\begin{aligned} &v^n(i,\chi_i^n) = \mathbb{E}[y_{i+1}^n | \mathcal{F}_i^n] \\ &+ hf((i+1)h, \chi_i^n, v^n(i,\chi_i^n), h^{-1/2} D_1^n v^n(i+1,\chi_i^n), D_2^n v^n(i+1,\chi_i^n)), \end{aligned} \tag{4.53}$$

while taking the conditional expectation with respect to $\mathcal{F}_i^n$ in (2.11), we get

$$y_i^n = \mathbb{E}[y_{i+1}^n|\mathcal{F}_i^n] + hf((i+1)h, \chi_i^n, y_i^n, h^{-1/2}D_1^n v^n(i+1,\chi_i^n), D_2^n v^n(i+1,\chi_i^n)). \quad (4.54)$$

As $hf(\cdot,\cdot,\cdot,\cdot,\cdot)$ is Kh-Lipschitz in its third argument with $Kh < 1$, we obtain that y_i^n and $v^n(i,\chi_i^n)$ are equal. □

5 A numerical example

In this section, we deal with a numerical example. We consider N a Poisson process with $\lambda = 1$ and $c < 1$, and the following BSDEJ:

$$dY_t = -cU_t dt + Z_t dW_t + U_t(dN_t - dt), \quad (5.55)$$

with $\xi = N_T$.

The explicit solution of (5.55) is given by

$$(Y_t, Z_t, U_t) = (N_t + (1+c)(T-t), 0, 1).$$

Furthermore if $\xi = 0$ then the solution is equal to $(Y_t, Z_t, U_t) = (0,0,0)$. This example is borrowed from [2].

We have implemented this method on a standard personal computing platform (PC), and have observed that it performs very well using simulated data, as can be seen from the simulated data in the Table 1 and Figure 1. Despite the apparent algebraic complexity of the equations (2.8), (2.9) and (2.12) one needs to solve at each step the conditional expectation to obtain $y_{t_i}^n$, the problem poses no difficulty. Using MATLAB's simulations and algebra capabilities (Version 7.0 running on the University of Valparaíso CIMFAV cluster) yielded best computing times.

In our implementation, and for computational conveniences we consider the case when $T = 1$. The iteration of the algorithm begins from $y_{t_n}^n = \xi^n = N_1^n$ at time $t_n = T = 1$ and proceeds backward to solve $(y_{t_j}^n, z_{t_j}^n, u_{t_j}^n)$, where $t_j = j/n$, at time $j = 0$. Values are given with 4 significant digits.

In the following table and picture we summarize the results.

n	$c = 0.3$	$c = 0.9$	$c = 0.1$	$c = 0.5$
10	1.27	1.81	1.09	1.45
100	1.2970	1.8910	1.0990	1.495
1000	1.2997	1.8991	1.0999	1.4995
2000	1.2999	1.8996	1.1	1.4998
3000	1.2999	1.8997	1.1	1.4998
4000	1.2999	1.8998	1.1	1.4999
4500	1.2999	1.8998	1.1	1.4999
4900	1.2999	1.8998	1.1	1.4999
5000	1.2999	1.8998	1.1	1.4999
Real Value Y_0	**1.3**	**1.9**	**1.1**	**1.5**

Table 1: Numerical Scheme for $dY_t = -cU_t dt + Z_t dB_t - U_t(dN_t - dt)$ with n from 10 until $n = 5000$ steps, $\lambda = 1$, $T = 1$ and different values of c.

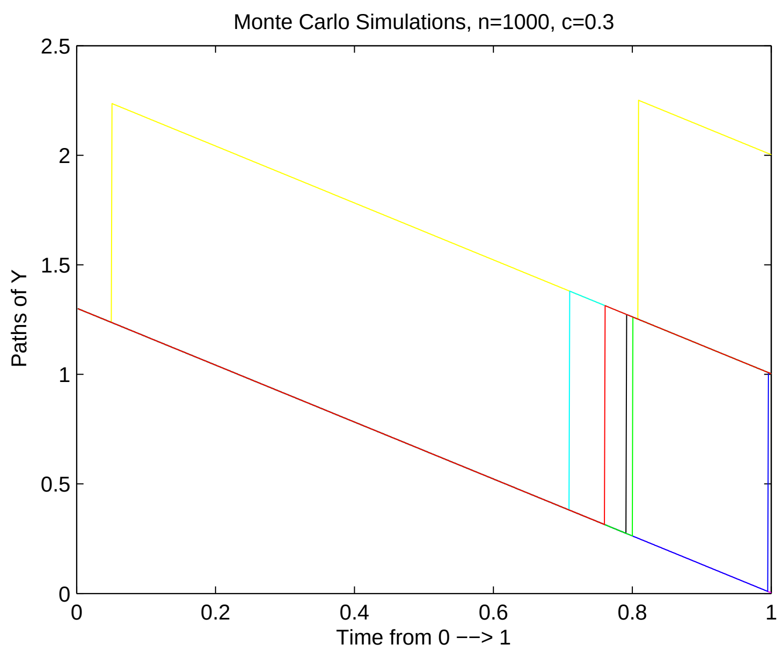

Figure 1: Monte Carlo Simulation; $c = 0.3$, $\lambda = 1$, $n = 1000$ and $T = 1$.

References

[1] Bally, V. and Pagès, G., *A quantization algorithm for solving multi-dimensional discrete-time optimal stopping problems.* Bernoulli **9**:6 (2003) 1003–1049.

[2] Barles, G., Buckdahn, R. and Pardoux, E., *BSDEs and integral-partial differential equations.* Stochastics Stochastics Rep. **60**:1–2 (1997) 57–83.

[3] Bouchard B. and Elie, R., *Discrete time approximation of decoupled Forward-Backward SDE with jumps*, Stochastic Process. Appli. **118** (2005) 53–75.

[4] Bouchard, B. and Touzi, N., *Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations.* Stochastic Process. Appl. **111**:2 (2004) 175–206.

[5] Briand, P., Delyon, B. and Mémin, J., *Donsker-Type Theorem for BS-DEs.* Electron. Comm. Probab., **6** (2001) 1–14.

[6] Chevance, D., *Numerical methods for backward stochastic differential equations.* In *Numerical methods in finance*, 232–244. Publ. Newton Inst., Cambridge Univ. Press, Cambridge, 1997.

[7] Coquet, F., Mémin, J. and Słomiński, L., *On weak convergence of filtrations.* Séminaire de Probabilités, XXXV, 306–328. Lecture Notes in Math. **1755** Springer, Berlin, 2001.

[8] Dothan, M. U., *Prices in financial markets.* The Clarendon Press, Oxford University Press, New York, 1990.

[9] Douglas, J., Ma, J. and Protter, P., *Numerical methods for forward-backward stochastic differential equations.* Ann. Appl. Probab. **6**:3 (1996) 940–968.

[10] Elie, R., *Contrôle Stochastique et Méthodes Numériques*, Ph.D. thesis, Université Paris-Dauphine (2006).

[11] Gobet, E., Lemor, J-P. and Warin, X., *A regression-based Monte Carlo method to solve Backward Stochastic Differential equations.* Ann. Appl. Probab. **15**:3 (2005) 2172–2202.

[12] Li, X. and Tang, S., *Necessary condition for optimal control of stochastic systems with random jumps* SIAM J. Control Optim. **332**:5 (1994), 1447–1475.

[13] Ma J., Protter P., San Martín J. and Torres S., *Numerical method for Backward Stochastic Differential Equations.* Ann. Appl. Probab. **12**:1 (2002) 302–316.

[14] Ma, J., Protter, P. and Yong, J., *Solving forward-backward stochastic differential equations explicitly a four step scheme.* Probab. Theory Related Fields **98**:3 (1994) 339–359.

[15] Pardoux, E. and Peng, S., *Adapted solution of backward stochastic differential equation.* Systems Control Lett. **14**:1 (1990) 55–61.

[16] Pardoux, E. and Peng, S., *Backward stochastic differential equation and quasilinear parabolic partial differential equations.* In *Stochastic partial differential equations and their applications (Charlotte, NC, 1991)*, 200–217. Lecture Notes in Control and Inform. Sci. **176** Springer, Berlin, 1992.

[17] Peng, S. and Xu, M., *Numerical algorithms for backward stochastic differential equations with 1-d Brownian motion: Convergence and simulations.* Math. Model. Numer. Anal. **45** (2011) 335-360.

[18] Rogers, L.C.G. and Williams, D., *Diffusions, Markov Processes, and Martingales: Foundations*, 2^{nd} ed., Cambridge University Press (2000).

[19] San Martin, J. and Torres, S., *Backward stochastic differential equations: numerical methods*, Encyclopedia of Quantitative Finance (2010).

[20] Shiryaev, A. N., *Probability*, 2^{nd} ed., Springer, New York, 1996.

[21] Situ, R., *On solution of backward stochastic differential equations with jumps.* Stochastic Process. Appl. **66**:2 (1997) 209–236.

[22] Słomiński, L. *Stability of strong solutions of stochastic differential equations.* Stochastic Process. Appl. **31**:2 (1989) 173–202.

[23] Zhang, J., *A numerical scheme for BSDEs* Ann. Appl. Probab. **14**:1 (2004) 459–488.