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Abstract

In this paper we present a tail inequality for the maximum of partial sums of a weakly dependent sequence of random variables that is not necessarily bounded. The class considered includes geometrically and subgeometrically strongly mixing sequences. The result is then used to derive asymptotic moderate deviation results. Applications include classes of Markov chains, functions of linear processes with absolutely regular innovations and ARCH models.

1 Introduction

Let us consider a sequence $X_1, X_2, \dots$ of real valued random variables. The aim of this paper is to present nonasymptotic tail inequalities for $S_n = X_1 + X_2 + \cdots + X_n$ and to use them to derive moderate deviations principles.

For independent and centered random variables $X_1, X_2, \dots$, one of the main tools to get an upper bound for the large and moderate deviations principles is the so-called Bernstein

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inequalities. We first recall the Bernstein inequality for random variables satisfying Condition (1.1) below. Suppose that the random variables $X_1, X_2, \dots$ satisfy

$$\log \mathbb{E} \exp(tX_i) \le \frac{\sigma_i^2 t^2}{2(1-tM)} \quad \text{for positive constants} \quad \sigma_i \quad \text{and } M, \tag{1.1}$$

for any t in $[0, 1/M[$. Set $V_n = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2$. Then

$$\mathbb{P}(S_n \ge \sqrt{2V_n x} + Mx) \le \exp(-x).$$

When the random variables $X_1, X_2, \dots$ are uniformly bounded by M then (1.1) holds with $\sigma_i^2 = \mathrm{Var} X_i$, and the above inequality implies the usual Bernstein inequality

$$\mathbb{P}(S_n \ge y) \le \exp\Big(-y^2(2V_n + 2yM)^{-1}\Big). \tag{1.2}$$

Assume now that the random variables $X_1, X_2, \dots$ satisfy the following weaker tail condition: for some γ in $]0, 1[$ and any positive t, $\sup_i \mathbb{P}(X_i \ge t) \le \exp(1 - (t/M)^\gamma)$. Then, by the proof of Corollary 5.1 in Borovkov (2000-a) we infer that

$$\mathbb{P}(|S_n| \ge y) \le 2 \exp\Big(-c_1 y^2/V_n\Big) + n \exp\Big(-c_2 (y/M)^\gamma\Big), \tag{1.3}$$

where c_1 and c_2 are positive constants (c_2 depends on γ). More precise results for large and moderate deviations of sums of independent random variables with *semiexponential* tails may be found in Borovkov (2000-b).

In our terminology the moderate deviations principle (MDP) stays for the following type of asymptotic behavior:

Definition 1. We say that the MDP holds for a sequence $(T_n)_n$ of random variables with the speed $a_n \to 0$ and rate function $I(t)$ if for each A Borelian,

$$\begin{aligned} -\inf_{t \in A^o} I(t) &\le \liminf_n a_n \log \mathbb{P}(\sqrt{a_n} T_n \in A) \\ &\le \limsup_n a_n \log \mathbb{P}(\sqrt{a_n} T_n \in A) \le -\inf_{t \in \bar{A}} I(t), \end{aligned} \tag{1.4}$$

where $\bar{A}$ denotes the closure of A and A^o the interior of A.

Our interest is to extend the above inequalities to strongly mixing sequences of random variables and to study the MDP for $(S_n/stdev(S_n))_n$. In order to cover a larger class of examples we shall also consider less restrictive coefficients of weak dependence, such as the τ-mixing coefficients defined in Dedecker and Prieur (2004) (see Section 2 for the definition of these coefficients).

Let $X_1, X_2, \ldots$ be a strongly mixing sequence of real-valued and centered random variables. Assume that there exist a positive constant γ_1 and a positive c such that the strong mixing coefficients of the sequence satisfy

$$\alpha(n) \leq \exp(-cn^{\gamma_1}) \text{ for any positive integer } n\,, \tag{1.5}$$

and there is a constant γ_2 in $]0, +\infty]$ such that

$$\sup_{i>0} \mathbb{P}(|X_i| > t) \leq \exp(1 - t^{\gamma_2}) \text{ for any positive } t \tag{1.6}$$

(when $\gamma_2 = +\infty$ (1.6) means that $\|X_i\|_\infty \leq 1$ for any positive i).

Obtaining exponential bounds for this case is a challenging problem. One of the available tools in the literature is Theorem 6.2 in Rio (2000), which is a Fuk-Nagaev type inequality, that provides the inequality below. Let γ be defined by $1/\gamma = (1/\gamma_1) + (1/\gamma_2)$. For any positive λ and any $r \geq 1$,

$$\mathbb{P}(\sup_{k\in[1,n]} |S_k| \geq 4\lambda) \leq 4\Big(1 + \frac{\lambda^2}{rnV}\Big)^{-r/2} + 4Cn\lambda^{-1}\exp\Big(-c(\lambda/r)^\gamma\Big), \tag{1.7}$$

where

$$V = \sup_{i>0}\Big(\mathbb{E}(X_i^2) + 2\sum_{j>i} |\mathbb{E}(X_iX_j)|\Big).$$

Selecting in (1.7) $r = \lambda^2/(nV)$ leads to

$$\mathbb{P}(\sup_{k\in[1,n]} |S_k| \geq 4\lambda) \leq 4\exp\Big(-\frac{\lambda^2 \log 2}{2nV}\Big) + 4Cn\lambda^{-1}\exp\Big(-c(nV/\lambda)^\gamma\Big)$$

for any $\lambda \geq (nV)^{1/2}$. The above inequality gives a subgaussian bound, provided that

$$(nV/\lambda)^\gamma \geq \lambda^2/(nV) + \log(n/\lambda),$$

which holds if $\lambda \ll (nV)^{(\gamma+1)/(\gamma+2)}$ (here and below $\ll$ replaces the symbol o). Hence (1.7) is useful to study the probability of moderate deviation $\mathbb{P}(|S_n| \geq t\sqrt{n/a_n})$ provided $a_n \gg n^{-\gamma/(\gamma+2)}$. For $\gamma = 1$ this leads to $a_n \gg n^{-1/3}$. For bounded random variables and geometric mixing rates (in that case $\gamma = 1$), Proposition 13 in Merlevède and Peligrad (2009) provides the MDP under the improved condition $a_n \gg n^{-1/2}$. We will prove in this paper that this condition is still suboptimal from the point of view of moderate deviation.

For stationary geometrically mixing (absolutely regular) Markov chains, and bounded functions f (here $\gamma = 1$), Theorem 6 in Adamczak (2008) provides a Bernstein's type inequality for

$S_n(f) = f(X_1) + f(X_2) + \cdots + f(X_n)$. Under the centering condition $\mathbb{E}(f(X_1)) = 0$, he proves that

$$\mathbb{P}(|S_n(f)| \geq \lambda) \leq C \exp\Big(-\frac{1}{C} \min\Big(\frac{\lambda^2}{n\sigma^2}, \frac{\lambda}{\log n}\Big)\Big), \tag{1.8}$$

where $\sigma^2 = \lim_n n^{-1}\mathrm{Var} S_n(f)$ (here we take $m = 1$ in his condition (14) on the small set). Inequality (1.8) provides exponential tightness for $S_n(f)/\sqrt{n}$ with rate a_n as soon as $a_n \gg n^{-1}(\log n)^2$, which is weaker than the above conditions. Still in the context of Markov chains, we point out the recent Fuk-Nagaev type inequality obtained by Bertail and Clémençon (2008). However for stationary subgeometrically mixing Markov chains, their inequality does not lead to the optimal rate which can be expected in view of the results obtained by Djellout and Guillin (2001).

To our knowledge, Inequality (1.8) has not been extended yet to the case $\gamma < 1$, even for the case of bounded functions f and absolutely regular Markov chains. In this paper we improve inequality (1.7) in the case $\gamma < 1$ and then derive moderate deviations principles from this new inequality under the minimal condition $a_n n^{\gamma/(2-\gamma)} \to \infty$. The main tool is an extension of inequality (1.3) to dependent sequences. We shall prove that, for α-mixing or τ-mixing sequences satisfying (1.5) and (1.6) for $\gamma < 1$, there exists a positive η such that, for $n \geq 4$ and $\lambda \geq C(\log n)^\eta$

$$\mathbb{P}(\sup_{j \leq n} |S_j| \geq \lambda) \leq (n+1)\exp(-\lambda^\gamma/C_1) + \exp(-\lambda^2/(C_2 + C_2 nV)), \tag{1.9}$$

where C, C_1 and C_2 are positive constants depending on c, γ_1 and γ_2 and V is some constant (which differs from the constant V in (1.7) in the unbounded case), depending on the covariance properties of truncated random variables built from the initial sequence. In order to define precisely V we need to introduce truncation functions φ_M.

Notation 1. For any positive M let the function φ_M be defined by $\varphi_M(x) = (x \wedge M) \vee (-M)$.

With this notation, (1.9) holds with

$$V = \sup_{M \geq 1} \sup_{i > 0} \Big(\mathrm{Var}(\varphi_M(X_i)) + 2\sum_{j > i} |\mathrm{Cov}(\varphi_M(X_i), \varphi_M(X_j))|\Big). \tag{1.10}$$

To prove (1.9) we use a variety of techniques and new ideas, ranging from the big and small blocks argument based on a Cantor-type construction, diadic induction, adaptive truncation along with coupling arguments. In a forthcoming paper, we will study the case $\gamma_1 = 1$ and $\gamma_2 = \infty$. We now give more definitions and precise results.

2 Main results

We first define the dependence coefficients that we consider in this paper.

For any real random variable X in $\mathbb{L}^1$ and any σ-algebra $\mathcal{M}$ of $\mathcal{A}$, let $\mathbb{P}_{X|\mathcal{M}}$ be a conditional distribution of X given $\mathcal{M}$ and let $\mathbb{P}_X$ be the distribution of X. We consider the coefficient $\tau(\mathcal{M}, X)$ of weak dependence (Dedecker and Prieur, 2004) which is defined by

$$\tau(\mathcal{M}, X) = \Big\| \sup_{f \in \Lambda_1(\mathbb{R})} \Big| \int f(x)\mathbb{P}_{X|\mathcal{M}}(dx) - \int f(x)\mathbb{P}_X(dx) \Big| \Big\|_1 , \tag{2.1}$$

where $\Lambda_1(\mathbb{R})$ is the set of 1-Lipschitz functions from $\mathbb{R}$ to $\mathbb{R}$.

The coefficient τ has the following coupling property: If Ω is rich enough then the coefficient $\tau(\mathcal{M}, X)$ is the infimum of $\|X - X^*\|_1$ where X^* is independent of $\mathcal{M}$ and distributed as X (see Lemma 5 in Dedecker and Prieur (2004)). This coupling property allows to relate the coefficient τ to the strong mixing coefficient Rosenblatt (1956) defined by

$$\alpha(\mathcal{M}, \sigma(X)) = \sup_{A \in \mathcal{M}, B \in \sigma(X)} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| ,$$

as shown in Rio (2000, p. 161) for the bounded case, and by Peligrad (2002) for the unbounded case. For equivalent definitions of the strong mixing coefficient we refer for instance to Bradley (2007, Lemma 4.3 and Theorem 4.4).

If Y is a random variable with values in $\mathbb{R}^k$, the coupling coefficient τ is defined as follows: If $Y \in \mathbb{L}^1(\mathbb{R}^k)$,

$$\tau(\mathcal{M}, Y) = \sup\{\tau(\mathcal{M}, f(Y)), f \in \Lambda_1(\mathbb{R}^k)\} , \tag{2.2}$$

where $\Lambda_1(\mathbb{R}^k)$ is the set of 1-Lipschitz functions from $\mathbb{R}^k$ to $\mathbb{R}$.

The τ-mixing coefficients $\tau_X(i) = \tau(i)$ of a sequence $(X_i)_{i\in\mathbb{Z}}$ of real-valued random variables are defined by

$$\tau_k(i) = \max_{1\le \ell \le k} \frac{1}{\ell} \sup \Big\{ \tau(\mathcal{M}_p, (X_{j_1}, \cdots, X_{j_\ell})),\, p + i \le j_1 < \cdots < j_\ell \Big\} \text{ and } \tau(i) = \sup_{k \ge 0} \tau_k(i) , \tag{2.3}$$

where $\mathcal{M}_p = \sigma(X_j, j \le p)$ and the above supremum is taken over p and $(j_1, \ldots j_\ell)$. Recall that the strong mixing coefficients $\alpha(i)$ are defined by:

$$\alpha(i) = \sup_{p \in \mathbb{Z}} \alpha(\mathcal{M}_p, \sigma(X_j, j \ge i + p)) .$$

Define now the function $Q_{|Y|}$ by $Q_{|Y|}(u) = \inf\{t > 0, \mathbb{P}(|Y| > t) \le u\}$ for u in $]0, 1]$. To compare the τ-mixing coefficient with the strong mixing coefficient, let us mention that, by Lemma 7 in Dedecker and Prieur (2004),

$$\tau(i) \le 2 \int_0^{2\alpha(i)} Q(u)du , \text{ where } Q = \sup_{k\in\mathbb{Z}} Q_{|X_k|}. \tag{2.4}$$

Let $(X_j)_{j\in\mathbb{Z}}$ be a sequence of centered real valued random variables and let $\tau(i)$ be defined by (2.3). Let $\tau(x) = \tau([x])$ (square brackets denoting the integer part). Throughout, we assume that there exist positive constants γ_1 and γ_2 such that

$$\tau(x) \leq \exp(-cx^{\gamma_1}) \quad \text{for any } x \geq 1\,, \tag{2.5}$$

where $c > 0$ and for any positive t,

$$\sup_{k>0} \mathbb{P}(|X_k| > t) \leq \exp(1 - t^{\gamma_2}) := H(t)\,. \tag{2.6}$$

Suppose furthermore that

$$\gamma < 1 \text{ where } \gamma \text{ is defined by } 1/\gamma = 1/\gamma_1 + 1/\gamma_2\,. \tag{2.7}$$

Theorem 1. *Let $(X_j)_{j\in\mathbb{Z}}$ be a sequence of centered real valued random variables and let V be defined by (1.10). Assume that (2.5), (2.6) and (2.7) are satisfied. Then V is finite and, for any $n \geq 4$, there exist positive constants C_1, C_2, C_3 and C_4 depending only on c, γ and γ_1 such that, for any positive x,*

$$\mathbb{P}\Big(\sup_{j\leq n} |S_j| \geq x\Big) \leq n\exp\Big(-\frac{x^\gamma}{C_1}\Big) + \exp\Big(-\frac{x^2}{C_2(1+nV)}\Big) + \exp\Big(-\frac{x^2}{C_3 n}\exp\Big(\frac{x^{\gamma(1-\gamma)}}{C_4(\log x)^\gamma}\Big)\Big)\,.$$

Remark 1. *Let us mention that if the sequence $(X_j)_{j\in\mathbb{Z}}$ satisfies (2.6) and is strongly mixing with strong mixing coefficients satisfying (1.5), then, from (2.4), (2.5) is satisfied (with an other constant), and Theorem 1 applies.*

Remark 2. *If $\mathbb{E}\exp(|X_i|^{\gamma_2})) \leq K$ for any positive i, then setting $C = 1 \vee \log K$, we notice that the process $(C^{-1/\gamma_2} X_i)_{i\in\mathbb{Z}}$ satisfies (2.6).*

Remark 3. *If $(X_i)_{i\in\mathbb{Z}}$ satisfies (2.5) and (2.6), then*

$$\begin{aligned} V &\leq \sup_{i>0}\Big(\mathbb{E}(X_i^2) + 4\sum_{k>0}\int_0^{\tau(k)/2} Q_{|X_i|}(G(v))dv\Big) \\ &= \sup_{i>0}\Big(\mathbb{E}(X_i^2) + 4\sum_{k>0}\int_0^{G(\tau(k)/2)} Q_{|X_i|}(u)Q(u)du\Big), \end{aligned}$$

where G is the inverse function of $x \mapsto \int_0^x Q(u)du$ (see Section 3.3 for a proof). Here the random variables do not need to be centered. Note also that, in the strong mixing case, using (2.4), we have $G(\tau(k)/2) \leq 2\alpha(k)$.

This result is the main tool to derive the MDP below.

Theorem 2. *Let $(X_i)_{i\in\mathbb{Z}}$ be a sequence of random variables as in Theorem 1 and let $S_n = \sum_{i=1}^n X_i$ and $\sigma_n^2 = \mathrm{Var} S_n$. Assume in addition that $\liminf_{n\to\infty} \sigma_n^2/n > 0$. Then for all positive sequences a_n with $a_n \to 0$ and $a_n n^{\gamma/(2-\gamma)} \to \infty$, $\{\sigma_n^{-1} S_n\}$ satisfies (1.4) with the good rate function $I(t) = t^2/2$.*

If we impose a stronger degree of stationarity we obtain the following corollary.

Corollary 1. *Let $(X_i)_{i\in\mathbb{Z}}$ be a second order stationary sequence of centered real valued random variables. Assume that (2.5), (2.6) and (2.7) are satisfied. Let $S_n = \sum_{i=1}^n X_i$ and $\sigma_n^2 = \mathrm{Var} S_n$. Assume in addition that $\sigma_n^2 \to \infty$. Then $\lim_{n\to\infty} \sigma_n^2/n = \sigma^2 > 0$, and for all positive sequences a_n with $a_n \to 0$ and $a_n n^{\gamma/(2-\gamma)} \to \infty$, $\{n^{-1/2} S_n\}$ satisfies (1.4) with the good rate function $I(t) = t^2/(2\sigma^2)$.*

2.1 Applications

2.1.1 Instantaneous functions of absolutely regular processes

Let $(Y_j)_{j\in\mathbb{Z}}$ be a strictly stationary sequence of random variables with values in a Polish space E, and let f be a measurable function from E to $\mathbb{R}$. Set $X_j = f(Y_j)$. Consider now the case where the sequence $(Y_k)_{k\in\mathbb{Z}}$ is absolutely regular (or β-mixing) in the sense of Rozanov and Volkonskii (1959). Setting $\mathcal{F}_0 = \sigma(Y_i, i \le 0)$ and $\mathcal{G}_k = \sigma(Y_i, i \ge k)$, this means that

$$\beta(k) = \beta(\mathcal{F}_0, \mathcal{G}_k) \to 0\,, \text{ as } k \to \infty\,,$$

with $\beta(\mathcal{A},\mathcal{B}) = \frac{1}{2}\sup\{\sum_{i\in I}\sum_{j\in J} |\mathbb{P}(A_i \cap B_j) - \mathbb{P}(A_i)\mathbb{P}(B_j)|\}$, the maximum being taken over all finite partitions $(A_i)_{i\in I}$ and $(B_i)_{i\in J}$ of Ω respectively with elements in $\mathcal{A}$ and $\mathcal{B}$. If we assume that

$$\beta(n) \le \exp(-cn^{\gamma_1}) \text{ for any positive } n, \tag{2.8}$$

where $c > 0$ and $\gamma_1 > 0$, and that the random variables X_j are centered and satisfy (2.6) for some positive γ_2 such that $1/\gamma = 1/\gamma_1 + 1/\gamma_2 > 1$, then Theorem 1 and Corollary 1 apply to the sequence $(X_j)_{j\in\mathbb{Z}}$. Furthermore, as shown in Viennet (1997), by Delyon's (1990) covariance inequality,

$$V \le \mathbb{E}(f^2(X_0)) + 4\sum_{k>0} \mathbb{E}(B_k f^2(X_0)),$$

for some sequence $(B_k)_{k>0}$ of random variables with values in $[0,1]$ satisfying $\mathbb{E}(B_k) \le \beta(k)$ (see Rio (2000, Section 1.6) for more details).

We now give an example where $(Y_j)_{j\in\mathbb{Z}}$ satisfies (2.8). Let $(Y_j)_{j\ge 0}$ be an E-valued irreducible ergodic and stationary Markov chain with a transition probability P having a unique invariant probability measure π (by Kolmogorov extension Theorem one can complete $(Y_j)_{j\geq 0}$ to a sequence $(Y_j)_{j\in\mathbb{Z}}$). Assume furthermore that the chain has an atom, that is there exists $A \subset E$ with $\pi(A) > 0$ and ν a probability measure such that $P(x,\cdot) = \nu(\cdot)$ for any x in A. If

$$\text{there exists } \delta > 0 \text{ and } \gamma_1 > 0 \text{ such that } \mathbb{E}_\nu(\exp(\delta\tau^{\gamma_1})) < \infty\,, \tag{2.9}$$

where $\tau = \inf\{n \geq 0;\, Y_n \in A\}$, then the β-mixing coefficients of the sequence $(Y_j)_{j\geq 0}$ satisfy (2.8) with the same γ_1 (see Proposition 9.6 and Corollary 9.1 in Rio (2000) for more details). Suppose that $\pi(f) = 0$. Then the results apply to $(X_j)_{j\geq 0}$ as soon as f satisfies

$$\pi(|f| > t) \leq \exp(1 - t^{\gamma_2}) \ \text{ for any positive } t\,.$$

Compared to the results obtained by de Acosta (1997) and Chen and de Acosta (1998) for geometrically ergodic Markov chains, and by Djellout and Guillin (2001) for subgeometrically ergodic Markov chains, we do not require here the function f to be bounded.

2.1.2 Functions of linear processes with absolutely regular innovations

Let f be a 1-Lipshitz function. We consider here the case where

$$X_n = f\Big(\sum_{j\geq 0} a_j\xi_{n-j}\Big) - \mathbb{E}f\Big(\sum_{j\geq 0} a_j\xi_{n-j}\Big)\,,$$

where $A = \sum_{j\geq 0}|a_j| < \infty$ and $(\xi_i)_{i\in\mathbb{Z}}$ is a strictly stationary sequence of real-valued random variables which is absolutely regular in the sense of Rozanov and Volkonskii.

Let $\mathcal{F}_0 = \sigma(\xi_i, i \leq 0)$ and $\mathcal{G}_k = \sigma(\xi_i, i \geq k)$. According to Section 3.1 in Dedecker and Merlevède (2006), if the innovations $(\xi_i)_{i\in\mathbb{Z}}$ are in $\mathbb{L}^2$, the following bound holds for the τ-mixing coefficient associated to the sequence $(X_i)_{i\in\mathbb{Z}}$:

$$\tau(i) \leq 2\|\xi_0\|_1 \sum_{j\geq i}|a_j| + 4\|\xi_0\|_2 \sum_{j=0}^{i-1}|a_j|\,\beta_\xi^{1/2}(i-j)\,.$$

Assume that there exists $\gamma_1 > 0$ and $c' > 0$ such that, for any positive integer k,

$$a_k \leq \exp(-c'k^{\gamma_1}) \text{ and } \beta_\xi(k) \leq \exp(-c'k^{\gamma_1})\,.$$

Then the τ-mixing coefficients of $(X_j)_{j\in\mathbb{Z}}$ satisfy (2.5). Let us now focus on the tails of the random variables X_i. Assume that $(\xi_i)_{i\in\mathbb{Z}}$ satisfies (2.6). Define the convex functions ψ_η for $\eta > 0$ in the following way: $\psi_\eta(-x) = \psi_\eta(x)$, and for any $x \geq 0$,

$$\psi_\eta(x) = \exp(x^\eta) - 1 \text{ for } \eta \geq 1 \ \text{ and } \ \psi_\eta(x) = \int_0^x \exp(u^\eta)du \text{ for } \eta \in]0,1].$$

Let $\| \,.\, \|_{\psi_\eta}$ be the usual corresponding Orlicz norm. Since the function f is 1-Lipshitz, we get that $\|X_0\|_{\psi_{\gamma_2}} \leq 2A\|\xi_0\|_{\psi_{\gamma_2}}$. Next, if $(\xi_i)_{i\in\mathbb{Z}}$ satisfies (2.6), then $\|\xi_0\|_{\psi_{\gamma_2}} < \infty$. Furthermore, it can easily be proven that, if $\|Y\|_{\psi_\eta} \leq 1$, then $\mathbb{P}(|Y| > t) \leq \exp(1 - t^\eta)$ for any positive t. Hence, setting $C = 2A\|\xi_0\|_{\psi_{\gamma_2}}$, we get that $(X_i/C)_{i\in\mathbb{Z}}$ satisfies (2.6) with the same parameter γ_2, and therefore the conclusions of Theorem 1 and Corollary 1 hold with γ defined by $1/\gamma = 1/\gamma_1 + 1/\gamma_2$, provided that $\gamma < 1$.

This example shows that our results hold for processes that are not necessarily strongly mixing. Recall that, in the case where $a_i = 2^{-i-1}$ and the innovations are iid with law $\mathcal{B}(1/2)$, the process fails to be strongly mixing in the sense of Rosenblatt.

2.1.3 ARCH(∞) models

Let $(\eta_t)_{t\in\mathbb{Z}}$ be an iid sequence of zero mean real random variables such that $\|\eta_0\|_\infty \leq 1$. We consider the following ARCH(∞) model described by Giraitis *et al.* (2000):

$$Y_t = \sigma_t \eta_t \,, \text{ where } \sigma_t^2 = a + \sum_{j\geq 1} a_j Y_{t-j}^2 \,, \tag{2.10}$$

where $a \geq 0$, $a_j \geq 0$ and $\sum_{j\geq 1} a_j < 1$. Such models are encountered, when the volatility $(\sigma_t^2)_{t\in\mathbb{Z}}$ is unobserved. In that case, the process of interest is $(Y_t^2)_{t\in\mathbb{Z}}$. Under the above conditions, there exists a unique stationary solution that satisfies

$$\|Y_0\|_\infty^2 \leq a + a \sum_{\ell \geq 1} \Big(\sum_{j\geq 1} a_j \Big)^\ell = M < \infty \,.$$

Set now $X_j = (2M)^{-1}(Y_j^2 - \mathbb{E}(Y_j^2))$. Then the sequence $(X_j)_{j\in\mathbb{Z}}$ satisfies (2.6) with $\gamma_2 = \infty$. If we assume in addition that $a_j = O(b^j)$ for some $b < 1$, then, according to Proposition 5.1 (and its proof) in Comte *et al.* (2008), the τ-mixing coefficients of $(X_j)_{j\in\mathbb{Z}}$ satisfy (2.5) with $\gamma_1 = 1/2$. Hence in this case, the sequence $(X_j)_{j\in\mathbb{Z}}$ satisfies both the conclusions of Theorem 1 and of Corollary 1 with $\gamma = 1/2$.

3 Proofs

3.1 Some auxiliary results

The aim of this section is essentially to give suitable bounds for the Laplace transform of

$$S(K) = \sum_{i\in K} X_i \,, \tag{3.1}$$

where K is a finite set of integers.

$$c_0 = (2(2^{1/\gamma} - 1))^{-1}(2^{(1-\gamma)/\gamma} - 1)\,,\ c_1 = \min(c^{1/\gamma_1} c_0/4, 2^{-1/\gamma})\,, \tag{3.2}$$

$$c_2 = 2^{-(1+2\gamma_1/\gamma)} c_1^{\gamma_1}\ ,\ c_3 = 2^{-\gamma_1/\gamma}\ ,\ \text{and } \kappa = \min\big(c_2, c_3\big)\,. \tag{3.3}$$

Proposition 1. *Let $(X_j)_{j\geq 1}$ be a sequence of centered and real valued random variables satisfying (2.5), (2.6) and (2.7). Let A and ℓ be two positive integers such that $A2^{-\ell} \geq (1 \vee 2c_0^{-1})$. Let $M = H^{-1}(\tau(c^{-1/\gamma_1}A))$ and for any j, set $\overline{X}_M(j) = \varphi_M(X_j) - \mathbb{E}\varphi_M(X_j)$. Then, there exists a subset $K_A^{(\ell)}$ of $\{1, \dots, A\}$ with $\mathrm{Card}(K_A^{(\ell)}) \geq A/2$, such that for any positive $t \leq \kappa\big(A^{\gamma-1} \wedge (2^\ell/A)\big)^{\gamma_1/\gamma}$, where κ is defined by (3.3),*

$$\log \exp\Big(t \sum_{j \in K_A^{(\ell)}} \overline{X}_M(j)\Big) \leq t^2 v^2 A + t^2\big(\ell(2A)^{1+\frac{\gamma_1}{\gamma}} + 4A^\gamma(2A)^{\frac{2\gamma_1}{\gamma}}\big) \exp\Big(-\frac{1}{2}\Big(\frac{c_1 A}{2^\ell}\Big)^{\gamma_1}\Big)\,, \tag{3.4}$$

with

$$v^2 = \sup_{T\geq 1} \sup_{K \subset \mathbb{N}^*} \frac{1}{\mathrm{Card} K} \mathrm{Var} \sum_{i \in K} \varphi_T(X_i) \tag{3.5}$$

(the maximum being taken over all nonempty finite sets K of integers).

Remark 4. *Notice that $v^2 \leq V$ (the proof is immediate).*

Proof of Proposition 1. The proof is divided in several steps.

Step 1. The construction of $K_A^{(\ell)}$. Let c_0 be defined by (3.2) and $n_0 = A$. $K_A^{(\ell)}$ will be a finite union of 2^ℓ disjoint sets of consecutive integers with same cardinal spaced according to a recursive "Cantor"-like construction. We first define an integer d_0 as follows:

$$d_0 = \begin{cases} \sup\{m \in 2\mathbb{N}\,,\ m \leq c_0 n_0\} & \text{if } n_0 \text{ is even} \\ \sup\{m \in 2\mathbb{N}+1\,,\ m \leq c_0 n_0\} & \text{if } n_0 \text{ is odd.} \end{cases}$$

It follows that $n_0 - d_0$ is even. Let $n_1 = (n_0 - d_0)/2$, and define two sets of integers of cardinal n_1 separated by a gap of d_0 integers as follows

$$\begin{aligned} I_{1,1} &= \{1, \dots, n_1\} \\ I_{1,2} &= \{n_1 + d_0 + 1, \dots, n_0\}\,. \end{aligned}$$

We define now the integer d_1 by

$$d_1 = \begin{cases} \sup\{m \in 2\mathbb{N}\,,\ m \leq c_0 2^{-(\ell \wedge \frac{1}{\gamma})} n_0\} & \text{if } n_1 \text{ is even} \\ \sup\{m \in 2\mathbb{N}+1\,,\ m \leq c_0 2^{-(\ell \wedge \frac{1}{\gamma})} n_0\} & \text{if } n_1 \text{ is odd.} \end{cases}$$

Noticing that $n_1 - d_1$ is even, we set $n_2 = (n_1 - d_1)/2$, and define four sets of integers of cardinal n_2 by

$$\begin{aligned} I_{2,1} &= \{1, \dots, n_2\} \\ I_{2,2} &= \{n_2 + d_1 + 1, \dots, n_1\} \\ I_{2,i+2} &= (n_1 + d_0) + I_{2,i} \text{ for } i = 1, 2\,. \end{aligned}$$

Iterating this procedure j times (for $1 \le j \le \ell$), we then get a finite union of 2^j sets, $(I_{j,k})_{1 \le k \le 2^j}$, of consecutive integers, with same cardinal, constructed by induction from $(I_{j-1,k})_{1 \le k \le 2^{j-1}}$ as follows: First, for $1 \le k \le 2^{j-1}$, we have

$$I_{j-1,k} = \{a_{j-1,k}, \dots, b_{j-1,k}\}\,,$$

where $1 + b_{j-1,k} - a_{j-1,k} = n_{j-1}$ and

$$1 = a_{j-1,1} < b_{j-1,1} < a_{j-1,2} < b_{j-1,2} < \cdots < a_{j-1,2^{j-1}} < b_{j-1,2^{j-1}} = n_0.$$

Let $n_j = 2^{-1}(n_{j-1} - d_{j-1})$ and

$$d_j = \begin{cases} \sup\{m \in 2\mathbb{N}\,,\, m \le c_0 2^{-(\ell \wedge \frac{j}{\gamma})} n_0\} & \text{if } n_j \text{ is even} \\ \sup\{m \in 2\mathbb{N} + 1\,,\, m \le c_0 2^{-(\ell \wedge \frac{j}{\gamma})} n_0\} & \text{if } n_j \text{ is odd.} \end{cases}$$

Then $I_{j,k} = \{a_{j,k}, a_{j,k} + 1, \dots, b_{j,k}\}$, where the double indexed sequences $(a_{j,k})$ and $(b_{j,k})$ are defined as follows:

$$a_{j,2k-1} = a_{j-1,k}\,,\ b_{j,2k} = b_{j-1,k}\,,\ b_{j,2k} - a_{j,2k} + 1 = n_j \text{ and } b_{j,2k-1} - a_{j,2k-1} + 1 = n_j\,.$$

With this selection, we then get that there is exactly d_{j-1} integers between $I_{j,2k-1}$ and $I_{j,2k}$ for any $1 \le k \le 2^{j-1}$.

Finally we get

$$K_A^{(\ell)} = \bigcup_{k=1}^{2^\ell} I_{\ell,k}\,.$$

Since $\mathrm{Card}(I_{\ell,k}) = n_\ell$, for any $1 \le k \le 2^\ell$, we get that $\mathrm{Card}(K_A^{(\ell)}) = 2^\ell n_\ell$. Now notice that

$$A - \mathrm{Card}(K_A^{(\ell)}) = \sum_{j=0}^{\ell-1} 2^j d_j \le A c_0 \Big(\sum_{j \ge 0} 2^{j(1-1/\gamma)} + \sum_{j \ge 1} 2^{-j} \Big) \le A/2\,.$$

Consequently

$$A \ge \mathrm{Card}(K_A^{(\ell)}) \ge A/2 \text{ and } n_\ell \le A 2^{-\ell}\,.$$

The following notation will be useful for the rest of the proof: For any k in $\{0,1,\ldots,\ell\}$ and any j in $\{1,\ldots,2^\ell\}$, we set

$$K^{(\ell)}_{A,k,j} = \bigcup_{i=(j-1)2^{\ell-k}+1}^{j2^{\ell-k}} I_{\ell,i}\,. \tag{3.6}$$

Notice that $K^{(\ell)}_A = K^{(\ell)}_{A,0,1}$ and that for any k in $\{0,1,\ldots,\ell\}$

$$K^{(\ell)}_A = \bigcup_{j=1}^{2^k} K^{(\ell)}_{A,k,j}\,, \tag{3.7}$$

where the union is disjoint.

In what follows we shall also use the following notation: for any integer j in $[0,\ell]$, we set

$$M_j = H^{-1}\big(\tau(c^{-1/\gamma_1}A2^{-(\ell\wedge\frac{j}{\gamma})})\big)\,. \tag{3.8}$$

Since $H^{-1}(y) = \big(\log(e/y)\big)^{1/\gamma_2}$ for any $y \le e$, we get that for any $x \ge 1$,

$$H^{-1}(\tau(c^{-1/\gamma_1}x)) \le \big(1 + x^{\gamma_1}\big)^{1/\gamma_2} \le (2x)^{\gamma_1/\gamma_2}\,. \tag{3.9}$$

Consequently since for any j in $[0,\ell]$, $A2^{-(\ell\wedge\frac{j}{\gamma})} \ge 1$, the following bound is valid:

$$M_j \le \big(2A2^{-(\ell\wedge\frac{j}{\gamma})}\big)^{\gamma_1/\gamma_2}\,. \tag{3.10}$$

For any set of integers K and any positive M we also define

$$\overline{S}_M(K) = \sum_{i\in K}\overline{X}_M(i)\,. \tag{3.11}$$

Step 2. Proof of Inequality (3.4) with $K^{(\ell)}_A$ defined in step 1.

Consider the decomposition (3.7), and notice that for any $i = 1,2$, $\mathrm{Card}(K^{(\ell)}_{A,1,i}) \le A/2$ and

$$\tau\big(\sigma(X_i\,:\,i\in K^{(\ell)}_{A,1,1}), \bar{S}_{M_0}(K^{(\ell)}_{A,1,2})\big) \le A\tau(d_0)/2\,.$$

Since $\overline{X}_{M_0}(j) \le 2M_0$, we get that $|\bar{S}_{M_0}(K^{(\ell)}_{A,1,i})| \le AM_0$. Consequently, by using Lemma 2 from Appendix, we derive that for any positive t,

$$|\mathbb{E}\exp\big(t\bar{S}_{M_0}(K^{(\ell)}_A)\big) - \prod_{i=1}^{2}\mathbb{E}\exp\big(t\bar{S}_{M_0}(K^{(\ell)}_{A,1,i})\big)| \le \frac{At}{2}\tau(d_0)\exp(2tAM_0)\,.$$

Since the random variables $\bar{S}_{M_0}(K^{(\ell)}_A)$ and $\bar{S}_{M_0}(K^{(\ell)}_{A,1,i})$ are centered, their Laplace transform are greater than one. Hence applying the elementary inequality

$$|\log x - \log y| \le |x-y| \text{ for } x \ge 1 \text{ and } y \ge 1, \tag{3.12}$$

we get that, for any positive t,

$$|\log \mathbb{E} \exp \big(t\bar{S}_{M_0}(K_A^{(\ell)})\big) - \sum_{i=1}^{2} \log \mathbb{E} \exp \big(t\bar{S}_{M_0}(K_{A,1,i}^{(\ell)})\big)| \leq \frac{At}{2}\tau(d_0)\exp(2tAM_0)\,.$$

The next step is to compare $\mathbb{E}\exp\big(t\bar{S}_{M_0}(K_{A,1,i}^{(\ell)})\big)$ with $\mathbb{E}\exp\big(t\bar{S}_{M_1}(K_{A,1,i}^{(\ell)})\big)$ for $i=1,2$. The random variables $\bar{S}_{M_0}(K_{A,1,i}^{(\ell)})$ and $\bar{S}_{M_1}(K_{A,1,i}^{(\ell)})$ have values in $[-AM_0, AM_0]$, hence applying the inequality

$$|e^{tx} - e^{ty}| \leq |t||x-y|(e^{|tx|} \vee e^{|ty|})\,, \tag{3.13}$$

we obtain that, for any positive t,

$$\big|\mathbb{E}\exp\big(t\bar{S}_{M_0}(K_{A,1,i}^{(\ell)})\big) - \mathbb{E}\exp\big(t\bar{S}_{M_1}(K_{A,1,i}^{(\ell)})\big)\big| \leq te^{tAM_0}\mathbb{E}\big|\bar{S}_{M_0}(K_{A,1,i}^{(\ell)}) - \bar{S}_{M_1}(K_{A,1,i}^{(\ell)})\big|\,.$$

Notice that

$$\mathbb{E}\big|\bar{S}_{M_0}(K_{A,1,i}^{(\ell)}) - \bar{S}_{M_1}(K_{A,1,i}^{(\ell)})\big| \leq 2\sum_{j\in K_{A,1,i}^{(\ell)}} \mathbb{E}|(\varphi_{M_0} - \varphi_{M_1})(X_j)|\,.$$

Since for all $x \in \mathbb{R}$, $|(\varphi_{M_0} - \varphi_{M_1})(x)| \leq M_0 \ \mathbb{1}_{|x|>M_1}$, we get that

$$\mathbb{E}|(\varphi_{M_0} - \varphi_{M_1})(X_j)| \leq M_0\mathbb{P}(|X_j| > M_1) \leq M_0\tau(c^{-\frac{1}{\gamma_1}}A2^{-(\ell\wedge\frac{1}{\gamma})})\,.$$

Consequently, since $\mathrm{Card}(K_{A,1,i}^{(\ell)}) \leq A/2$, for any $i=1,2$ and any positive t,

$$\big|\mathbb{E}\exp\big(t\bar{S}_{M_0}(K_{A,1,i}^{(\ell)})\big) - \mathbb{E}\exp\big(t\bar{S}_{M_1}(K_{A,1,i}^{(\ell)})\big)\big| \leq tAM_0e^{tAM_0}\tau(c^{-\frac{1}{\gamma_1}}A2^{-(\ell\wedge\frac{1}{\gamma})})\,.$$

Using again the fact that the variables are centered and taking into account the inequality (3.12), we derive that for any $i=1,2$ and any positive t,

$$\big|\log\mathbb{E}\exp\big(t\bar{S}_{M_0}(K_{A,1,i}^{(\ell)})\big) - \log\mathbb{E}\exp\big(t\bar{S}_{M_1}(K_{A,1,i}^{(\ell)})\big)\big| \leq e^{2tAM_0}\tau(c^{-\frac{1}{\gamma_1}}A2^{-(\ell\wedge\frac{1}{\gamma})})\,. \tag{3.14}$$

Now for any $k=1,\dots,\ell$ and any $i=1,\dots,2^k$, $\mathrm{Card}(K_{A,k,i}^{(\ell)}) \leq 2^{-k}A$. By iterating the above procedure, we then get for any $k=1,\dots,\ell$, and any positive t,

$$\begin{aligned} |\sum_{i=1}^{2^{k-1}} \log\mathbb{E}\exp\big(t\bar{S}_{M_{k-1}}(K_{A,k-1,i}^{(\ell)})\big) \quad &- \quad \sum_{i=1}^{2^k}\log\mathbb{E}\exp\big(t\bar{S}_{M_{k-1}}(K_{A,k,i}^{(\ell)})\big)| \\ &\leq \quad 2^{k-1}\frac{tA}{2^k}\tau(d_{k-1})\exp\Big(\frac{2tAM_{k-1}}{2^{k-1}}\Big)\,, \end{aligned}$$

and for any $i=1,\dots,2^k$,

$$|\log\mathbb{E}\exp\big(t\bar{S}_{M_{k-1}}(K_{A,k,i}^{(\ell)})\big) - \log\mathbb{E}\exp\big(t\bar{S}_{M_k}(K_{A,k,i}^{(\ell)})\big)| \leq \tau(c^{-\frac{1}{\gamma_1}}A2^{-(\ell\wedge\frac{k}{\gamma})})\exp\Big(\frac{2tAM_{k-1}}{2^{k-1}}\Big)\,.$$

Hence finally, we get that for any $j = 1, \dots, \ell$, and any positive t,

$$
\begin{aligned}
&| \sum_{i=1}^{2^{j-1}} \log \mathbb{E} \exp \big(t \bar{S}_{M_{j-1}} (K^{(\ell)}_{A,j-1,i}) \big) - \sum_{i=1}^{2^j} \log \mathbb{E} \exp \big(t \bar{S}_{M_j} (K^{(\ell)}_{A,j,i}) \big) | \\
&\qquad \le \frac{tA}{2} \tau(d_{j-1}) \exp(2tAM_{j-1} 2^{1-j}) + 2^j \tau(c^{-\frac{1}{\gamma_1}} A 2^{-(\ell \wedge \frac{j}{\gamma})}) \exp(2tAM_{j-1} 2^{1-j}) \,.
\end{aligned}
$$

Set

$$
k_\ell = \sup\{ j \in \mathbb{N} \,,\, j/\gamma < \ell \} \,,
$$

and notice that $0 \le k_\ell \le \ell - 1$. Since $K^{(\ell)}_A = K^{(\ell)}_{A,0,1}$, we then derive that for any positive t,

$$
\begin{aligned}
&| \log \mathbb{E} \exp \big(t \bar{S}_{M_0} (K^{(\ell)}_A) \big) - \sum_{i=1}^{2^{k_\ell+1}} \log \mathbb{E} \exp \big(t \bar{S}_{M_{k_\ell+1}} (K^{(\ell)}_{A,k_\ell+1,i}) \big) | \\
&\qquad \le \frac{tA}{2} \sum_{j=0}^{k_\ell} \tau(d_j) \exp \big(\frac{2tAM_j}{2^j} \big) + 2 \sum_{j=0}^{k_\ell - 1} 2^j \tau(2^{-1/\gamma} c^{-1/\gamma_1} A 2^{-j/\gamma}) \exp \big(\frac{2tAM_j}{2^j} \big) \\
&\qquad\quad + 2^{k_\ell+1} \tau(c^{-1/\gamma_1} A 2^{-\ell}) \exp(2tAM_{k_\ell} 2^{-k_\ell}) \,. \qquad (3.15)
\end{aligned}
$$

Notice now that for any $i = 1, \dots, 2^{k_\ell+1}$, $S_{M_{k_\ell+1}}(K^{(\ell)}_{A,k_\ell+1,i})$ is a sum of $2^{\ell-k_\ell-1}$ blocks, each of size n_ℓ and bounded by $2M_{k_\ell+1} n_\ell$. In addition the blocks are equidistant and there is a gap of size $d_{k_\ell+1}$ between two blocks. Consequently, by using Lemma 2 along with Inequality (3.12) and the fact that the variables are centered, we get that

$$
\begin{aligned}
&| \log \mathbb{E} \exp \big(t \bar{S}_{M_{k_\ell+1}} (K^{(\ell)}_{A,k_\ell+1,i}) \big) - \sum_{j=(i-1)2^{\ell-k_\ell-1}+1}^{i 2^{\ell-k_\ell-1}} \log \mathbb{E} \exp \big(t \bar{S}_{M_{k_\ell+1}} (I_{\ell,j}) \big) | \\
&\qquad \le t n_\ell 2^\ell 2^{-k_\ell-1} \tau(d_{k_\ell+1}) \exp(2tM_{k_\ell+1} n_\ell 2^{\ell-k_\ell-1}) \,. \qquad (3.16)
\end{aligned}
$$

Starting from (3.15) and using (3.16) together with the fact that $n_\ell \le A2^{-\ell}$, we obtain:

$$
\begin{aligned}
&| \log \mathbb{E} \exp \big(t \bar{S}_{M_0} (K^{(\ell)}_A) \big) - \sum_{j=1}^{2^\ell} \log \mathbb{E} \exp \big(t \bar{S}_{M_{k_\ell+1}} (I_{\ell,j}) \big) | \\
&\qquad \le \frac{tA}{2} \sum_{j=0}^{k_\ell} \tau(d_j) \exp \big(\frac{2tAM_j}{2^j} \big) + 2 \sum_{j=0}^{k_\ell - 1} 2^j \tau(2^{-1/\gamma} c^{-1/\gamma_1} A 2^{-j/\gamma}) \exp \big(\frac{2tAM_j}{2^j} \big) \\
&\qquad\quad + 2^{k_\ell+1} \tau(c^{-1/\gamma_1} A 2^{-\ell}) \exp \big(\frac{2tAM_{k_\ell}}{2^{k_\ell}} \big) + tA \tau(d_{k_\ell+1}) \exp(tM_{k_\ell+1} A 2^{-k_\ell}) \,. \qquad (3.17)
\end{aligned}
$$

Notice that for any $j = 0, \dots, \ell - 1$, we have $d_j + 1 \ge [c_0 A 2^{-(\ell \wedge \frac{j}{\gamma})}]$ and $c_0 A 2^{-(\ell \wedge \frac{j}{\gamma})} \ge 2$. Whence

$$
d_j \ge (d_j + 1)/2 \ge c_0 A 2^{-(\ell \wedge \frac{j}{\gamma}) - 2} \,.
$$

Consequently setting $c_1 = \min(\frac{1}{4}c^{1/\gamma_1}c_0, 2^{-1/\gamma})$ and using (2.5), we derive that for any positive t,

$$
\begin{aligned}
& |\log \mathbb{E}\exp\big(t\bar{S}_{M_0}(K_A^{(\ell)})\big) - \sum_{j=1}^{2^\ell} \log \mathbb{E}\exp\big(t\bar{S}_{M_{k_\ell+1}}(I_{\ell,j})\big)| \\
\leq \; & \frac{tA}{2}\sum_{j=0}^{k_\ell} \exp\Big(-\big(c_1 A 2^{-j/\gamma}\big)^{\gamma_1} + \frac{2tAM_j}{2^j}\Big) + 2\sum_{j=0}^{k_\ell-1} 2^j \exp\Big(-\big(c_1 A 2^{-j/\gamma}\big)^{\gamma_1} + \frac{2tAM_j}{2^j}\Big) \\
& + 2^{k_\ell+1}\exp\Big(-\big(A2^{-\ell}\big)^{\gamma_1} + \frac{2tAM_{k_\ell}}{2^{k_\ell}}\Big) + tA\exp\Big(-\big(c_1 A 2^{-\ell}\big)^{\gamma_1} + tM_{k_\ell+1}A2^{-k_\ell}\Big).
\end{aligned}
$$

By (3.10), we get that for any $0 \leq j \leq k_\ell$,

$$2AM_j 2^{-j} \leq 2^{\gamma_1/\gamma}(2^{-j}A)^{\gamma_1/\gamma}.$$

In addition, since $k_\ell + 1 \geq \gamma\ell$ and $\gamma < 1$, we get that

$$M_{k_\ell+1} \leq (2A2^{-\ell})^{\gamma_1/\gamma_2} \leq (2A2^{-\gamma\ell})^{\gamma_1/\gamma_2}.$$

Whence,

$$M_{k_\ell+1}A2^{-k_\ell} = 2M_{k_\ell+1}A2^{-(k_\ell+1)} \leq 2^{\gamma_1/\gamma}A^{\gamma_1/\gamma}2^{-\gamma_1\ell}.$$

In addition,

$$2AM_{k_\ell}2^{-k_\ell} \leq 2^{2\gamma_1/\gamma}(A2^{-k_\ell-1})^{\gamma_1/\gamma} \leq 2^{2\gamma_1/\gamma}A^{\gamma_1/\gamma}2^{-\gamma_1\ell}.$$

Hence, if $t \leq c_2 A^{\gamma_1(\gamma-1)/\gamma}$ where $c_2 = 2^{-(1+2\gamma_1/\gamma)}c_1^{\gamma_1}$, we derive that

$$
\begin{aligned}
& |\log \mathbb{E}\exp\big(t\bar{S}_{M_0}(K_A^{(\ell)})\big) - \sum_{j=1}^{2^\ell} \log \mathbb{E}\exp\big(t\bar{S}_{M_{k_\ell+1}}(I_{\ell,j})\big)| \\
\leq \; & \frac{tA}{2}\sum_{j=0}^{k_\ell} \exp\Big(-\frac{1}{2}\big(c_1 A 2^{-j/\gamma}\big)^{\gamma_1}\Big) + 2\sum_{j=0}^{k_\ell-1} 2^j \exp\Big(-\frac{1}{2}\big(c_1 A 2^{-j/\gamma}\big)^{\gamma_1}\Big) \\
& + (2^{k_\ell+1} + tA)\exp\big(-(c_1A2^{-\ell})^{\gamma_1}/2\big).
\end{aligned}
$$

Since $2^{k_\ell} \leq 2^{\ell\gamma} \leq A^\gamma$, it follows that for any $t \leq c_2 A^{\gamma_1(\gamma-1)/\gamma}$,

$$|\log \mathbb{E}\exp\big(t\bar{S}_{M_0}(K_A^{(\ell)})\big) - \sum_{j=1}^{2^\ell} \log \mathbb{E}\exp\big(t\bar{S}_{M_{k_\ell+1}}(I_{\ell,j})\big)| \leq (2\ell tA + 4A^\gamma)\exp\Big(-\frac{1}{2}\Big(\frac{c_1A}{2^\ell}\Big)^{\gamma_1}\Big). \tag{3.18}$$

We bound up now the log Laplace transform of each $\bar{S}_{M_{k_\ell+1}}(I_{\ell,j})$ using the following fact: from l'Hospital rule for monotonicity (see Pinelis (2002)), the function $x \mapsto g(x) = x^{-2}(e^x - x - 1)$ is increasing on $\mathbb{R}$. Hence, for any centered random variable U such that $\|U\|_\infty \leq M$, and any positive t,

$$\mathbb{E}\exp(tU) \leq 1 + t^2 g(tM)\mathbb{E}(U^2). \tag{3.19}$$

Notice that

$$\|\bar{S}_{M_{k_\ell+1}}(I_{\ell,j})\|_\infty \le 2M_{k_\ell+1}n_\ell \le 2^{\gamma_1/\gamma}(A2^{-\ell})^{\gamma_1/\gamma}.$$

Since $t \le 2^{-\gamma_1/\gamma}(2^\ell/A)^{\gamma_1/\gamma}$, by using (3.5), we then get that

$$\log \mathbb{E}\exp\big(t\bar{S}_{M_{k_\ell+1}}(I_{\ell,j})\big) \le t^2v^2n_\ell\,.$$

Consequently, for any $t \le \kappa\big(A^{\gamma_1(\gamma-1)/\gamma} \wedge (2^\ell/A)^{\gamma_1/\gamma}\big)$, the following inequality holds:

$$\log \mathbb{E}\exp\big(t\bar{S}_{M_0}(K_A^{(\ell)})\big) \le t^2v^2A + (2\ell tA + 4A^\gamma)\exp\big(-\big(c_1A2^{-\ell}\big)^{\gamma_1}/2\big)\,. \tag{3.20}$$

Notice now that $\|\bar{S}_{M_0}(K_A^{(\ell)})\|_\infty \le 2M_0A \le 2^{\gamma_1/\gamma}A^{\gamma_1/\gamma}$. Hence if $t \le 2^{-\gamma_1/\gamma}A^{-\gamma_1/\gamma}$, by using (3.19) together with (3.5), we derive that

$$\log \mathbb{E}\exp\big(t\bar{S}_{M_0}(K_A^{(\ell)})\big) \le t^2v^2A\,, \tag{3.21}$$

which proves (3.4) in this case.

Now if $2^{-\gamma_1/\gamma}A^{-\gamma_1/\gamma} \le t \le \kappa\big(A^{\gamma_1(\gamma-1)/\gamma} \wedge (2^\ell/A)^{\gamma_1/\gamma}\big)$, by using (3.20), we derive that (3.4) holds, which completes the proof of Proposition 1. $\diamond$

We now bound up the Laplace transform of the sum of truncated random variables on $[1, A]$. Let

$$\mu = \big(2(2 \vee 4c_0^{-1})/(1-\gamma)\big)^{\frac{2}{1-\gamma}} \text{ and } c_4 = 2^{\gamma_1/\gamma}3^{\gamma_1/\gamma_2}c_0^{-\gamma_1/\gamma_2}\,, \tag{3.22}$$

where c_0 is defined in (3.2). Define also

$$\nu = \big(c_4\big(3 - 2^{(\gamma-1)\frac{\gamma_1}{\gamma}}\big) + \kappa^{-1}\big)^{-1}\big(1 - 2^{(\gamma-1)\frac{\gamma_1}{\gamma}}\big)\,, \tag{3.23}$$

where κ is defined by (3.3).

Proposition 2. *Let $(X_j)_{j\ge1}$ be a sequence of centered real valued random variables satisfying (2.5), (2.6) and (2.7). Let A be an integer. Let $M = H^{-1}(\tau(c^{-1/\gamma_1}A))$ and for any j, set $\overline{X}_M(j) = \varphi_M(X_j) - \mathbb{E}\varphi_M(X_j)$. Then, if $A \ge \mu$ with μ defined by (3.22), for any positive $t < \nu A^{\gamma_1(\gamma-1)/\gamma}$, where ν is defined by (3.23), we get that*

$$\log \mathbb{E}\Big(\exp(t\textstyle\sum_{k=1}^A \overline{X}_M(k))\Big) \le \dfrac{AV(A)t^2}{1 - t\nu^{-1}A^{\gamma_1(1-\gamma)/\gamma}}\,, \tag{3.24}$$

where $V(A) = 50v^2 + \nu_1\exp(-\nu_2A^{\gamma_1(1-\gamma)}(\log A)^{-\gamma})$ and ν_1, ν_2 are positive constants depending only on c, γ and γ_1, and v^2 is defined by (3.5).

Proof of Proposition 2. Let $A_0 = A$ and $X^{(0)}(k) = X_k$ for any $k = 1, \dots, A_0$. Let ℓ be a fixed positive integer, to be chosen later, which satisfies

$$A_0 2^{-\ell} \geq (2 \vee 4c_0^{-1}) . \tag{3.25}$$

Let $K_{A_0}^{(\ell)}$ be the discrete Cantor type set as defined from $\{1, \dots, A\}$ in Step 1 of the proof of Proposition 1. Let $A_1 = A_0 - \mathrm{Card} K_{A_0}^{(\ell)}$ and define for any $k = 1, \dots, A_1$,

$$X^{(1)}(k) = X_{i_k} \text{ where } \{i_1, \dots, i_{A_1}\} = \{1, \dots, A\} \setminus K_A .$$

Now for $i \geq 1$, let $K_{A_i}^{(\ell_i)}$ be defined from $\{1, \dots, A_i\}$ exactly as $K_A^{(\ell)}$ is defined from $\{1, \dots, A\}$. Here we impose the following selection of ℓ_i:

$$\ell_i = \inf\{j \in \mathbb{N} \, , \, A_i 2^{-j} \leq A_0 2^{-\ell} \} . \tag{3.26}$$

Set $A_{i+1} = A_i - \mathrm{Card} K_{A_i}^{(\ell_i)}$ and $\{j_1, \dots, j_{A_{i+1}}\} = \{1, \dots, A_{i+1}\} \setminus K_{A_{i+1}}^{(\ell_{i+1})}$. Define now

$$X^{(i+1)}(k) = X^{(i)}(j_k) \text{ for } k = 1, \dots, A_{i+1} .$$

Let

$$m(A) = \inf\{m \in \mathbb{N} \, , \, A_m \leq A 2^{-\ell}\} . \tag{3.27}$$

Note that $m(A) \geq 1$, since $A_0 > A2^{-\ell}$ ($\ell \geq 1$). In addition, $m(A) \leq \ell$ since for all $i \geq 1$, $A_i \leq A 2^{-i}$.

Obviously, for any $i = 0, \dots, m(A) - 1$, the sequences $(X^{(i+1)}(k))$ satisfy (2.5), (2.6) and (3.5) with the same constants. Now we set $T_0 = M = H^{-1}(\tau(c^{-1/\gamma_1} A_0))$, and for any integer $j = 0, \dots, m(A)$,

$$T_j = H^{-1}(\tau(c^{-1/\gamma_1} A_j)) .$$

With this definition, we then define for all integers i and j,

$$X_{T_j}^{(i)}(k) = \varphi_{T_j}\big(X^{(i)}(k)\big) - \mathbb{E}\varphi_{T_j}\big(X^{(i)}(k)\big) .$$

Notice that by (2.5) and (2.6), we have that for any integer $j \geq 0$,

$$T_j \leq (2A_j)^{\gamma_1/\gamma_2} . \tag{3.28}$$

For any $j = 1, \dots, m(A)$ and $i < j$, define

$$Y_i = \sum_{k \in K_{A_i}^{(\ell_i)}} X_{T_i}^{(i)}(k) \, , \; Z_i = \sum_{k=1}^{A_i} (X_{T_{i-1}}^{(i)}(k) - X_{T_i}^{(i)}(k)) \text{ for } i > 0, \text{ and } R_j = \sum_{k=1}^{A_j} X_{T_{j-1}}^{(j)}(k) .$$

The following decomposition holds:

$$\sum_{k=1}^{A_0} X_{T_0}^{(0)}(k) = \sum_{i=0}^{m(A)-1} Y_i + \sum_{i=1}^{m(A)-1} Z_i + R_{m(A)}\,. \tag{3.29}$$

To control the terms in the decomposition (3.29), we need the following elementary lemma.

Lemma 1. *For any $j = 0, \dots, m(A)-1$, $A_{j+1} \geq \frac{1}{3} c_0 A_j$.*

Proof of Lemma 1. Notice that for any i in $[0, m(A)[$, we have $A_{i+1} \geq [c_0 A_i] - 1$. Since $c_0 A_i \geq 2$, we derive that $[c_0 A_i] - 1 \geq ([c_0 A_i] + 1)/3 \geq c_0 A_i / 3$, which completes the proof. $\diamond$

Using (3.28), a useful consequence of Lemma 1 is that for any $j = 1, \dots, m(A)$

$$2 A_j T_{j-1} \leq c_4 A_j^{\gamma_1/\gamma} \tag{3.30}$$

where c_4 is defined by (3.22)

A bound for the Laplace transform of $R_{m(A)}$.

The random variable $|R_{m(A)}|$ is a.s. bounded by $2 A_{m(A)} T_{m(A)-1}$. By using (3.30) and (3.27), we then derive that

$$\|R_{m(A)}\|_\infty \leq c_4 (A_{m(A)})^{\gamma_1/\gamma} \leq c_4 \big(A 2^{-\ell}\big)^{\gamma_1/\gamma}\,. \tag{3.31}$$

Hence, if $t \leq c_4^{-1} (2^\ell / A)^{\gamma_1/\gamma}$, by using (3.19) together with (3.5), we obtain

$$\log \mathbb{E}\big(\exp(t R_{m(A)})\big) \leq t^2 v^2 A 2^{-\ell} \leq t^2 (v\sqrt{A})^2 := t^2 \sigma_1^2\,. \tag{3.32}$$

A bound for the Laplace transform of the Y_i's.

Notice that for any $0 \leq i \leq m(A)-1$, by the definition of ℓ_i and (3.25), we get that

$$2^{-\ell_i} A_i = 2^{1-\ell_i} (A_i/2) > 2^{-\ell} (A/2) \geq (1 \vee 2 c_0^{-1})\,.$$

Now, by Proposition 1, we get that for any $i \in [0, m(A)[$ and any $t \leq \kappa \big(A_i^{\gamma-1} \wedge 2^{-\ell_i} A_i\big)^{\gamma_1/\gamma}$ with κ defined by (3.3),

$$\log \mathbb{E}\big(e^{tY_i}\big) \leq t^2 \Big(v\sqrt{A_i} + \big(\sqrt{\ell_i} (2A_i)^{\frac{1}{2} + \frac{\gamma_1}{2\gamma}} + 2 A_i^{\gamma/2} (2A_i)^{\gamma_1/\gamma}\big) \exp\big(-\frac{1}{4} \big(c_1 A_i 2^{-\ell_i}\big)^{\gamma_1}\big)\Big)^2 .$$

Notice now that $\ell_i \leq \ell \leq A$, $A_i \leq A 2^{-i}$ and $2^{-\ell-1} A \leq 2^{-\ell_i} A_i \leq 2^{-\ell} A$. Taking into account these bounds and the fact that $\gamma < 1$, we then get that for any i in $[0, m(A)[$ and any $t \leq \kappa (2^i / A)^{1-\gamma} \wedge (2^\ell / A)^{\gamma_1/\gamma}$,

$$\log \mathbb{E}\big(e^{tY_i}\big) \leq t^2 \Big(v \frac{A^{1/2}}{2^{i/2}} + \Big(2^{2+\frac{\gamma_1}{\gamma}} \frac{A^{1+\frac{\gamma_1}{\gamma}}}{(2^i)^{\frac{\gamma}{2} + \frac{\gamma_1}{2\gamma}}} \Big) \exp\Big(-\frac{c_1^{\gamma_1}}{2^{2+\gamma_1}} \Big(\frac{A}{2^\ell}\Big)^{\gamma_1}\Big)\Big)^2 := t^2 \sigma_{2,i}^2\,, \tag{3.33}$$

A bound for the Laplace transform of the Z_i's.

Notice first that for any $1 \leq i \leq m(A)-1$, Z_i is a centered random variable, such that

$$|Z_i| \leq \sum_{k=1}^{A_i} \Big(\big|(\varphi_{T_{i-1}} - \varphi_{T_i})X^{(i)}(k)\big| + \mathbb{E}\big|(\varphi_{T_{i-1}} - \varphi_{T_i})X^{(i)}(k)\big| \Big) .$$

Consequently, using (3.30) we get that

$$\|Z_i\|_\infty \leq 2A_i T_{i-1} \leq c_4 A_i^{\gamma_1/\gamma} .$$

In addition, since $|(\varphi_{T_{i-1}} - \varphi_{T_i})(x)| \leq (T_{i-1} - T_i)\ \mathbb{I}_{x>T_i}$, and the random random variables $(X^{(i)}(k))$ satisfy (2.6), by the definition of T_i, we get that

$$\mathbb{E}|Z_i|^2 \leq (2A_i T_{i-1})^2 \tau(c^{-1/\gamma_1} A_i) \leq c_4^2 A_i^{2\gamma_1/\gamma} .$$

Hence applying (3.19) to the random variable Z_i, we get for any positive t,

$$\mathbb{E}\exp(tZ_i) \leq 1 + t^2 g(c_4 t A_i^{\gamma_1/\gamma}) c_4^2 A_i^{2\gamma_1/\gamma} \exp(-A_i^{\gamma_1}) .$$

Hence, since $A_i \leq A2^{-i}$, for any positive t satisfying $t \leq (2c_4)^{-1}(2^i/A)^{\gamma_1(1-\gamma)/\gamma}$, we have that

$$2tA_i T_{i-1} \leq A_i^{\gamma_1}/2 .$$

Since $g(x) \leq e^x$ for $x \geq 0$, we infer that for any positive t with $t \leq (2c_4)^{-1}(2^i/A)^{\gamma_1(1-\gamma)/\gamma}$,

$$\log \mathbb{E}\exp(tZ_i) \leq c_4^2 t^2 (2^{-i}A)^{2\gamma_1/\gamma} \exp(-A_i^{\gamma_1}/2) .$$

By taking into account that for any $1 \leq i \leq m(A)-1$, $A_i \geq A_{m(A)-1} > A2^{-\ell}$ (by definition of $m(A)$), it follows that for any i in $[1, m(A)[$ and any positive t satisfying $t \leq (2c_4)^{-1}(2^i/A)^{\gamma_1(1-\gamma)/\gamma}$,

$$\log \mathbb{E}\exp(tZ_i) \leq t^2 \big(c_4 (2^{-i}A)^{\gamma_1/\gamma} \exp(-(A2^{-\ell})^{\gamma_1}/4)\big)^2 := t^2 \sigma_{3,i}^2 . \tag{3.34}$$

End of the proof. Let

$$C = c_4 \Big(\frac{A}{2^\ell}\Big)^{\gamma_1/\gamma} + \frac{1}{\kappa} \sum_{i=0}^{m(A)-1} \Big(\Big(\frac{A}{2^i}\Big)^{1-\gamma} \vee \frac{A}{2^\ell}\Big)^{\gamma_1/\gamma} + 2c_4 \sum_{i=1}^{m(A)-1} \Big(\frac{A}{2^i}\Big)^{\gamma_1(1-\gamma)/\gamma} ,$$

and

$$\sigma = \sigma_1 + \sum_{i=0}^{m(A)-1} \sigma_{2,i} + \sum_{i=1}^{m(A)-1} \sigma_{3,i} ,$$

where σ_1, $\sigma_{2,i}$ and $\sigma_{3,i}$ are respectively defined in (3.32), (3.33) and (3.34).

Notice that $m(A) \leq \ell$, and $\ell \leq 2\log A/\log 2$. We select now ℓ as follows

$$\ell = \inf\{j \in \mathbb{N}\,,\, 2^j \geq A^\gamma (\log A)^{\gamma/\gamma_1}\}\,.$$

This selection is compatible with (3.25) if

$$(2 \vee 4c_0^{-1})(\log A)^{\gamma/\gamma_1} \leq A^{1-\gamma}\,. \tag{3.35}$$

Now we use the fact that for any positive δ and any positive u, $\delta \log u \leq u^\delta$. Hence if $A \geq 3$,

$$(2 \vee 4c_0^{-1})(\log A)^{\gamma/\gamma_1} \leq (2 \vee 4c_0^{-1}) \log A \leq 2(1-\gamma)^{-1}(2 \vee 4c_0^{-1})A^{(1-\gamma)/2}\,,$$

which implies that (3.35) holds as soon as $A \geq \mu$ where μ is defined by (3.22). It follows that

$$C \leq \nu^{-1} A^{\gamma_1(1-\gamma)/\gamma}\,. \tag{3.36}$$

In addition

$$\sigma \leq 5v\sqrt{A} + 10 \times 2^{2\gamma_1/\gamma} A^{1+\gamma_1/\gamma} \exp\Big(-\frac{c_1^{\gamma_1}}{2^{2+\gamma_1}}(A2^{-\ell})^{\gamma_1}\Big) + c_4 A^{\gamma_1/\gamma} \exp\Big(-\frac{1}{4}(A2^{-\ell})^{\gamma_1}\Big)\,.$$

Consequently, since $A2^{-\ell} \geq \frac{1}{2}A^{1-\gamma}(\log A)^{-\gamma/\gamma_1}$, there exists positive constants ν_1 and ν_2 depending only on c, γ and γ_1 such that

$$\sigma^2 \leq A\big(50v^2 + \nu_1 \exp(-\nu_2 A^{\gamma_1(1-\gamma)}(\log A)^{-\gamma})\big) = AV(A)\,. \tag{3.37}$$

Starting from the decomposition (3.29) and the bounds (3.32), (3.33) and (3.34), we aggregate the contributions of the terms by using Lemma 3 given in the appendix. Then, by taking into account the bounds (3.36) and (3.37), Proposition 2 follows. $\diamond$

3.2 Proof of Theorem 1

For any positive M and any positive integer i, we set

$$\overline{X}_M(i) = \varphi_M(X_i) - \mathbb{E}\varphi_M(X_i)\,.$$

• If $\lambda \geq n^{\gamma_1/\gamma}$, setting $M = \lambda/n$, we have:

$$\sum_{i=1}^n |\overline{X}_M(i)| \leq 2\lambda,$$

which ensures that

$$\mathbb{P}\Big(\sup_{j\leq n}|S_j| \geq 3\lambda\Big) \leq \mathbb{P}\Big(\sum_{i=1}^n |X_i - \overline{X}_M(i)| \geq \lambda\Big).$$

Now

$$\mathbb{P}\Big(\sum_{i=1}^n |X_i - \overline{X}_M(i)| \geq \lambda\Big) \leq \frac{1}{\lambda}\sum_{i=1}^n \mathbb{E}|X_i - \overline{X}_M(i)| \leq \frac{2n}{\lambda}\int_M^\infty H(x)dx.$$

Now recall that $\log H(x) = 1 - x^{\gamma_2}$. It follows that the function $x \to \log(x^2 H(x))$ is nonincreasing for $x \geq (2/\gamma_2)^{1/\gamma_2}$. Hence, for $M \geq (2/\gamma_2)^{1/\gamma_2}$,

$$\int_M^\infty H(x)dx \leq M^2 H(M)\int_M^\infty \frac{dx}{x^2} = MH(M).$$

Whence

$$\mathbb{P}\Big(\sum_{i=1}^n |X_i - \overline{X}_M(i)| \geq \lambda\Big) \leq 2n\lambda^{-1}MH(M) \ \text{ for any } \ M \geq (2/\gamma_2)^{1/\gamma_2}. \tag{3.38}$$

Consequently our choice of M together with the fact that $(\lambda/n)^{\gamma_2} \geq \lambda^\gamma$ lead to

$$\mathbb{P}\Big(\sup_{j\leq n} |S_j| \geq 3\lambda\Big) \leq 2\exp(-\lambda^\gamma)$$

provided that $\lambda/n \geq (2/\gamma_2)^{1/\gamma_2}$. Now since $\lambda \geq n^{\gamma_1/\gamma}$, this condition holds if $\lambda \geq (2/\gamma_2)^{1/\gamma}$. Consequently for any $\lambda \geq n^{\gamma_1/\gamma}$, we get that

$$\mathbb{P}\Big(\sup_{j\leq n} |S_j| \geq 3\lambda\Big) \leq e\exp(-\lambda^\gamma/C_1)\,,$$

as soon as $C_1 \geq (2/\gamma_2)^{1/\gamma}$.

• Let $\zeta = \mu \vee (2/\gamma_2)^{1/\gamma_1}$ where μ is defined by (3.22). Assume that $(4\zeta)^{\gamma_1/\gamma} \leq \lambda \leq n^{\gamma_1/\gamma}$. Let p be a real in $[1, \frac{n}{2}]$, to be chosen later on. Let

$$A = \Big[\frac{n}{2p}\Big],\ k = \Big[\frac{n}{2A}\Big] \text{ and } M = H^{-1}(\tau(c^{-\frac{1}{\gamma_1}}A))\,.$$

For any set of natural numbers K, denote

$$\bar{S}_M(K) = \sum_{i\in K}\overline{X}_M(i)\,.$$

For i integer in $[1, 2k]$, let $I_i = \{1 + (i-1)A, \dots, iA\}$. Let also $I_{2k+1} = \{1 + 2kA, \dots, n\}$. Set

$$\bar{S}_1(j) = \sum_{i=1}^j \bar{S}_M(I_{2i-1}) \ \text{ and } \bar{S}_2(j) = \sum_{i=1}^j \bar{S}_M(I_{2i}).$$

We then get the following inequality

$$\sup_{j\leq n}|S_j| \leq \sup_{j\leq k+1}|\bar{S}_1(j)| + \sup_{j\leq k}|\bar{S}_2(j)| + 2AM + \sum_{i=1}^n |X_i - \overline{X}_M(i)|\,. \tag{3.39}$$

Using (3.38) together with (2.5) and our selection of M, we get for all positive λ that

$$\mathbb{P}\Big(\sum_{i=1}^{n}|X_i-\overline{X}_M(i)|\geq\lambda\Big)\leq 2n\lambda^{-1}M\exp(-A^{\gamma_1})\quad\text{for}\quad A\geq(2/\gamma_2)^{1/\gamma_1}\,.$$

By using Lemma 5 in Dedecker and Prieur (2004), we get the existence of independent random variables $(\bar{S}^*_M(I_{2i}))_{1\leq i\leq k}$ with the same distribution as the random variables $\bar{S}_M(I_{2i})$ such that

$$\mathbb{E}|\bar{S}_M(I_{2i})-\bar{S}^*_M(I_{2i})|\leq A\tau(A)\leq A\exp\big(-cA^{\gamma_1}\big)\,. \tag{3.40}$$

The same is true for the sequence $(\bar{S}_M(I_{2i-1}))_{1\leq i\leq k+1}$. Hence for any positive λ such that $\lambda\geq 2AM$,

$$\begin{aligned}\mathbb{P}\Big(\sup_{j\leq n}|S_j|\geq 6\lambda\Big) &\leq \lambda^{-1}A(2k+1)\exp\big(-cA^{\gamma_1}\big)+2n\lambda^{-1}M\exp(-A^{\gamma_1})\\ &\quad+\mathbb{P}\Big(\max_{j\leq k+1}\Big|\sum_{i=1}^{j}\bar{S}^*_M(I_{2i-1})\Big|\geq\lambda\Big)+\mathbb{P}\Big(\max_{j\leq k}\Big|\sum_{i=1}^{j}\bar{S}^*_M(I_{2i})\Big|\geq\lambda\Big)\,.\end{aligned}$$

For any positive t, due to the independence and since the variables are centered, $(\exp(t\bar{S}_M(I_{2i})))_i$ is a submartingale. Hence Doob's maximal inequality entails that for any positive t,

$$\mathbb{P}\Big(\max_{j\leq k}\sum_{i=1}^{j}\bar{S}^*_M(I_{2i})\geq\lambda\Big)\leq e^{-\lambda t}\prod_{i=1}^{k}\mathbb{E}\Big(\exp(t\bar{S}_M(I_{2i}))\Big)\,.$$

To bound the Laplace transform of each random variable $\bar{S}_M(I_{2i})$, we apply Proposition 2 to the sequences $(X_{i+s})_{i\in\mathbb{Z}}$ for suitable values of s. Hence we derive that, if $A\geq\mu$ then for any positive t such that $t<\nu A^{\gamma_1(\gamma-1)/\gamma}$ (where ν is defined by (3.23)),

$$\sum_{i=1}^{k}\log\mathbb{E}\Big(\exp(t\bar{S}_M(I_{2i}))\Big)\leq Akt^2\frac{V(A)}{1-t\nu^{-1}A^{\gamma_1(1-\gamma)/\gamma}}\,.$$

Obviously the same inequalities hold true for the sums associated to $(-X_i)_{i\in\mathbb{Z}}$. Now some usual computations (see for instance page 153 in Rio (2000)) lead to

$$\mathbb{P}\Big(\max_{j\leq k}\Big|\sum_{i=1}^{j}\bar{S}^*_M(I_{2i})\Big|\geq\lambda\Big)\leq 2\exp\Big(-\frac{\lambda^2}{4AkV(A)+2\lambda\nu^{-1}A^{\gamma_1(1-\gamma)/\gamma}}\Big)\,.$$

Similarly, we obtain that

$$\mathbb{P}\Big(\max_{j\leq k+1}\Big|\sum_{i=1}^{j}\bar{S}^*_M(I_{2i-1})\Big|\geq\lambda\Big)\leq 2\exp\Big(-\frac{\lambda^2}{4A(k+1)V(A)+2\lambda\nu^{-1}A^{\gamma_1(1-\gamma)/\gamma}}\Big)\,.$$

Let now $p = n\lambda^{-\gamma/\gamma_1}$. It follows that $2A \le \lambda^{\gamma/\gamma_1}$ and, since $M \le (2A)^{\gamma_1/\gamma_2}$, we obtain $2AM \le (2A)^{\gamma_1/\gamma} \le \lambda$. Also, since $\lambda \ge (4\zeta)^{\gamma_1/\gamma}$, we have $n \ge 4p$ implying that $A \ge 4^{-1}\lambda^{\gamma/\gamma_1} \ge \zeta$. The result follows from the previous bounds.

To end the proof, we mention that if $\lambda \le (4\zeta)^{\gamma_1/\gamma}$, then

$$\mathbb{P}\Big(\sup_{j\le n} |S_j| \ge \lambda\Big) \le 1 \le e \exp\Big(-\frac{\lambda^\gamma}{(4\zeta)^{\gamma_1}}\Big),$$

which is less than $n\exp(-\lambda^\gamma/C_1)$ as soon as $n \ge 3$ and $C_1 \ge (4\zeta)^{\gamma_1}$. $\diamond$

3.3 Proof of Remark 3

Setting $W_i = \varphi_M(X_i)$ we first bound $\mathrm{Cov}(W_i, W_{i+k}))$. Applying Inequality (4.2) of Proposition 1 in Dedecker and Doukhan (2003), we derive that, for any positive k,

$$|\mathrm{Cov}(W_i, W_{i+k})| \le 2\int_0^{\gamma(\mathcal{M}_i, W_{i+k})/2} Q_{|W_i|} \circ G_{|W_{i+k}|}(u)du$$

where

$$\gamma(\mathcal{M}_i, W_{i+k}) = \|\mathbb{E}(W_{i+k}|\mathcal{M}_i) - \mathbb{E}(W_{i+k})\|_1 \le \tau(k)\,,$$

since $x \mapsto \varphi_M(x)$ is 1-Lipschitz. Now for any j, $Q_{|W_j|} \le Q_{|X_j|} \le Q$, implying that $G_{|W_j|} \ge G$, where G is the inverse function of $u \to \int_0^u Q(v)dv$. Taking $j = i$ and $j = i + k$, we get that

$$|\mathrm{Cov}(W_i, W_{i+k})| \le 2\int_0^{\tau(k)/2} Q_{|X_i|} \circ G(u)du\,.$$

Making the change-of-variables $u = G(v)$ we also have

$$|\mathrm{Cov}(W_i, W_{i+k})| \le 2\int_0^{G(\tau(k)/2)} Q_{X_i}(u)Q(u)du\,, \tag{3.41}$$

proving the remark.

3.4 Proof of Theorem 2

For any $n \ge 1$, let $T = T_n$ where (T_n) is a sequence of real numbers greater than 1 such that $\lim_{n\to\infty} T_n = \infty$, that will be specified later. We truncate the variables at the level T_n. So we consider

$$X'_i = \varphi_{T_n}(X_i) - \mathbb{E}\varphi_{T_n}(X_i) \text{ and } X''_i = X_i - X'_i\,.$$

Let $S'_n = \sum_{i=1}^n X'_i$ and $S''_n = \sum_{i=1}^n X''_i$. To prove the result, by exponentially equivalence lemma in Dembo and Zeitouni (1998, Theorem 4.2.13. p130), it suffices to prove that for any $\eta > 0$,

$$\limsup_{n\to\infty} a_n \log \mathbb{P}\big(\frac{\sqrt{a_n}}{\sigma_n}|S''_n| \ge \eta\big) = -\infty\,, \tag{3.42}$$

and

$$\{\frac{1}{\sigma_n}S'_n\} \text{ satisfies (1.4) with the good rate function } I(t)=\frac{t^2}{2}\,. \tag{3.43}$$

To prove (3.42), we first notice that $|x-\varphi_T(x)|=(|x|-T)_+$. Consequently, if

$$W'_i = X_i - \varphi_T(X_i)\,,$$

then $Q_{|W'_i|} \le (Q-T)_+$. Hence, denoting by V''_{T_n} the upper bound for the variance of S''_n (corresponding to V for the variance of S_n) we have, by Remark 3 ,

$$V''_{T_n} \le \int_0^1 (Q(u)-T_n)_+^2 du + 4\sum_{k>0}\int_0^{\tau_{W'}(k)/2}(Q(G_{T_n}(v))-T_n)_+ dv\,.$$

where G_T is the inverse function of $x \to \int_0^x (Q(u)-T)_+du$ and the coefficients $\tau_{W'}(k)$ are the τ-mixing coefficients associated to $(W'_i)_i$. Next, since $x\to x-\varphi_T(x)$ is 1-Lipschitz, we have that $\tau_{W'}(k)\le \tau_X(k)=\tau(k)$. Moreover, $G_T\ge G$, because $(Q-T)_+\le Q$. Since Q is nonincreasing, it follows that

$$V''_{T_n} \le \int_0^1 (Q(u)-T_n)_+^2 du + 4\sum_{k>0}\int_0^{\tau(k)/2}(Q(G(v))-T_n)_+ dv.$$

Hence

$$\lim_{n\to+\infty} V''_{T_n} = 0. \tag{3.44}$$

The sequence (X''_i) satisfies (2.5) and we now prove that it satisfies also (2.6) for n large enough. With this aim, we first notice that, since $|\mathbb{E}(X''_i)|=|\mathbb{E}(X'_i)|\le T$, $|X''_i|\le |X_i|$ if $|X_i|\ge T$. Now if $|X_i|<T$ then $X''_i=\mathbb{E}(\varphi_T(X_i))$, and

$$\big|\mathbb{E}(\varphi_{T_n}(X_i))\big| \le \int_{T_n}^{\infty} H(x)dx < 1 \text{ for } n \text{ large enough.}$$

Then for $t\ge 1$,

$$\sup_{i\in[1,n]} \mathbb{P}(|X''_i|\ge t)\le H(t)\,,$$

proving that the sequence (X''_i) satisfies (2.6) for n large enough. Consequently, for n large enough, we can apply Theorem 1 to the sequence (X''_i), and we get that, for any $\eta>0$,

$$\mathbb{P}\Big(\sqrt{\frac{a_n}{\sigma_n^2}}|S''_n|\ge\eta\Big)\le n\exp\Big(-\frac{\eta^\gamma\sigma_n^\gamma}{C_1 a_n^{\frac{\gamma}{2}}}\Big)+\exp\Big(-\frac{\eta^2\sigma_n^2}{C_2a_n(1+nV''_{T_n})}\Big)+\exp\Big(-\frac{\eta^2\sigma_n^2}{C_3na_n}\exp\Big(\frac{\eta^\delta\sigma_n^\delta}{C_4a_n^{\frac{\delta}{2}}}\Big)\Big)\,,$$

where $\delta=\gamma(1-\gamma)/2$. This proves (3.42), since $a_n\to 0$, $a_nn^{\gamma/(2-\gamma)}\to\infty$, $\lim_{n\to\infty}V''_{T_n}=0$ and $\liminf_{n\to\infty}\sigma_n^2/n>0$.

We turn now to the proof of (3.43). Let $p_n = [n^{1/(2-\gamma)}]$ and $q_n = \delta_n p_n$ where δ_n is a sequence of integers tending to zero and such that

$$\delta_n^{\gamma_1} n^{\gamma_1/(2-\gamma)} / \log n \to \infty \text{ and } \delta_n^{\gamma_1} a_n n^{\gamma_1/(2-\gamma)} \to \infty$$

(this is always possible since $\gamma_1 \geq \gamma$ and by assumption $a_n n^{\gamma/(2-\gamma)} \to \infty$). Let now $m_n = [n/(p_n + q_n)]$. We divide the variables $\{X_i'\}$ in big blocks of size p_n and small blocks of size q_n, in the following way: Let us set for all $1 \leq j \leq m_n$,

$$Y_{j,n} = \sum_{i=(j-1)(p_n+q_n)+1}^{(j-1)(p_n+q_n)+p_n} X_i' \quad \text{and} \quad Z_{j,n} = \sum_{i=(j-1)(p_n+q_n)+p_n+1}^{j\,(p_n+q_n)} X_i' \,.$$

Then we have the following decomposition:

$$S_n' = \sum_{j=1}^{m_n} Y_{j,n} + \sum_{j=1}^{m_n} Z_{j,n} + \sum_{i=m_n(p_n+q_n)+1}^{n} X_i' \,. \tag{3.45}$$

For any $j = 1, \ldots, m_n$, let now

$$I(n,j) = \{(j-1)(p_n+q_n)+1, \ldots, (j-1)(p_n+q_n)+p_n\} \,.$$

These intervals are of cardinal p_n. Let

$$\ell_n = \inf\{k \in \mathbb{N}^*, 2^k \geq \varepsilon_n^{-1} p_n^{\gamma/2} a_n^{-1/2}\} \,,$$

where ε_n a sequence of positive numbers tending to zero and satisfying

$$\varepsilon_n^2 a_n n^{\gamma/(2-\gamma)} \to \infty. \tag{3.46}$$

For each $j \in \{1, \ldots, m_n\}$, we construct discrete Cantor sets, $K_{I(n,j)}^{(\ell_n)}$, as described in the proof of Proposition 1 with $A = p_n$, $\ell = \ell_n$, and the following selection of c_0,

$$c_0 = \frac{\varepsilon_n}{1+\varepsilon_n} \frac{2^{(1-\gamma)/\gamma} - 1}{2^{1/\gamma} - 1} \,.$$

Notice that clearly with the selections of p_n and ℓ_n, $p_n 2^{-\ell_n} \to \infty$. In addition with the selection of c_0 we get that for any $1 \leq j \leq m_n$,

$$\mathrm{Card}(K_{I(n,j)}^{(\ell_n)})^c \leq \frac{\varepsilon_n p_n}{1+\varepsilon_n}$$

and

$$K_{I(n,j)}^{(\ell_n)} = \bigcup_{i=1}^{2^{\ell_n}} I_{\ell_n,i}(p_n, j) \,,$$

where the $I_{\ell_n,i}(p_n, j)$ are disjoint sets of consecutive integers, each of same cardinal such that

$$\frac{p_n}{2^{\ell_n}(1+\varepsilon_n)} \le \mathrm{Card} I_{\ell_n,i}(p_n, j) \le \frac{p_n}{2^{\ell_n}} . \tag{3.47}$$

With this notation, we derive that

$$\sum_{j=1}^{m_n} Y_{j,n} = \sum_{j=1}^{m_n} S'\big(K^{(\ell_n)}_{I(n,j)}\big) + \sum_{j=1}^{m_n} S'\big((K^{(\ell_n)}_{I(n,j)})^c\big) . \tag{3.48}$$

Combining (3.45) with (3.48), we can rewrite S'_n as follows

$$S'_n = \sum_{j=1}^{m_n} S'\big(K^{(\ell_n)}_{I(n,j)}\big) + \sum_{k=1}^{r_n} \tilde{X}_i , \tag{3.49}$$

where $r_n = n - m_n \mathrm{Card} K^{(\ell_n)}_{I(n,1)}$ and the $\tilde{X}_i$ are obtained from the X'_i and satisfied (2.5) and (3.5) with the same constants. Since $r_n = o(n)$, applying Theorem 1 and using the fact that $\liminf_{n\to\infty} \sigma_n^2/n > 0$, we get that for any $\eta > 0$,

$$\limsup_{n\to\infty} a_n \log \mathbb{P}\Big(\frac{\sqrt{a_n}}{\sigma_n} \sum_{k=1}^{r_n} \tilde{X}_i \ge \eta\Big) = -\infty . \tag{3.50}$$

Hence to prove (3.43), it suffices to prove that

$$\{\sigma_n^{-1} \sum_{j=1}^{m_n} S'\big(K^{(\ell_n)}_{I(n,j)}\big)\} \text{ satisfies (1.4) with the good rate function } I(t) = t^2/2. \tag{3.51}$$

With this aim, we choose now $T_n = \varepsilon_n^{-1/2}$ where ε_n is defined by (3.46).

By using Lemma 5 in Dedecker and Prieur (2004), we get the existence of independent random variables $\big(S^*(K^{(\ell_n)}_{I(n,j)})\big)_{1\le j\le m_n}$ with the same distribution as the random variables $S'(K^{(\ell_n)}_{I(n,j)})$ such that

$$\sum_{j=1}^{m_n} \mathbb{E}|S'(K^{(\ell_n)}_{I(n,j)}) - S^*(K^{(\ell_n)}_{I(n,j)})| \le \tau(q_n) \sum_{j=1}^{m_n} \mathrm{Card} K^{(\ell_n)}_{I(n,j)} .$$

Consequently, since $\sum_{j=1}^{m_n} \mathrm{Card} K^{(\ell_n)}_{I(n,j)} \le n$, we derive that for any $\eta > 0$,

$$a_n \log \mathbb{P}\Big(\frac{\sqrt{a_n}}{\sigma_n} \Big|\sum_{j=1}^{m_n} (S'(K^{(\ell_n)}_{I(n,j)}) - S^*(K^{(\ell_n)}_{I(n,j)}))\Big| \ge \eta\Big) \le a_n \log\Big(\eta^{-1}\sigma_n^{-1} n \sqrt{a_n} \exp\big(-c\delta_n^{\gamma_1} n^{\gamma_1/(2-\gamma)}\big)\Big) ,$$

which tends to $-\infty$ by the fact that $\liminf_n \sigma_n^2/n > 0$ and the selection of δ_n. Hence the proof of the MDP for $\{\sigma_n^{-1} \sum_{j=1}^{m_n} S'\big(K^{(\ell_n)}_{I(n,j)}\big)\}$ is reduced to proving the MDP for $\{\sigma_n^{-1} \sum_{j=1}^{m_n} S^*\big(K^{(\ell_n)}_{I(n,j)}\big)\}$. By Ellis Theorem, to prove (3.51) it remains then to show that, for any real t,

$$a_n \sum_{j=1}^{m_n} \log \mathbb{E} \exp\Big(t S'\big(K^{(\ell_n)}_{I(n,j)}\big)/\sqrt{a_n \sigma_n^2}\Big) \to \frac{t^2}{2} \text{ as } n \to \infty . \tag{3.52}$$

As in the proof of Proposition 1, we decorrelate step by step. Using Lemma 2 and taking into account the fact that the variables are centered together with the inequality (3.12), we obtain, proceeding as in the proof of Proposition 1, that for any real t,

$$\Big|\sum_{j=1}^{m_n}\log\mathbb{E}\exp\Big(tS'\big(K^{(\ell_n)}_{I(n,j)}\big)/\sqrt{a_n\sigma_n^2}\Big)-\sum_{j=1}^{m_n}\sum_{i=1}^{2^{\ell_n}}\log\mathbb{E}\exp\Big(tS'\big(I_{\ell_n,i}(p_n,j)\big)/\sqrt{a_n\sigma_n^2}\Big)\Big|$$
$$\le\frac{|t|m_np_n}{\sqrt{a_n\sigma_n^2}}\Big(\exp\Big(-c\frac{c_0^{\gamma_1}}{4}\frac{p_n^{\gamma_1}}{2^{\ell_n\gamma_1}}+2\frac{|t|}{\sqrt{\varepsilon_na_n\sigma_n^2}}\frac{p_n}{2^{\gamma\ell_n}}\Big)+\sum_{j=0}^{k_{\ell_n}}\exp\Big(-c\frac{c_0^{\gamma_1}}{4}\frac{p_n^{\gamma_1}}{2^{j\gamma_1/\gamma}}+2\frac{|t|}{\sqrt{\varepsilon_na_n\sigma_n^2}}\frac{p_n}{2^j}\Big)\Big)\,,$$

where $k_{\ell_n}=\sup\{j\in\mathbb{N}\,,\,j/\gamma<\ell_n\}$. By the selection of p_n and ℓ_n, and since $\liminf_{n\to\infty}\sigma_n^2/n>0$ and $\varepsilon_n^2a_nn^{\gamma/(2-\gamma)}\to\infty$, we derive that for n large enough, there exists positive constants K_1 and K_2 depending on c, γ and γ_1 such that

$$a_n\Big|\sum_{j=1}^{m_n}\log\mathbb{E}\exp\Big(tS'\big(K^{(\ell_n)}_{I(n,j)}\big)/\sqrt{a_n\sigma_n^2}\Big)-\sum_{j=1}^{m_n}\sum_{i=1}^{2^{\ell_n}}\log\mathbb{E}\exp\Big(tS'\big(I_{\ell_n,i}(p_n,j)\big)/\sqrt{a_n\sigma_n^2}\Big)\Big|$$
$$\le\ K_1|t|\sqrt{a_nn}\log(n)\exp\Big(-K_2\big(\varepsilon_n^2a_nn^{\gamma/(2-\gamma)}\big)^{\gamma/2}n^{\gamma(1-\gamma)/(2-\gamma)}\Big)\,,\tag{3.53}$$

which converges to zero by the selection of ε_n.

Hence (3.52) holds if we prove that for any real t

$$a_n\sum_{j=1}^{m_n}\sum_{k=1}^{2^{\ell_n}}\log\mathbb{E}\exp\Big(tS'\big(I_{\ell_n,i}(p_n,j)\big)/\sqrt{a_n\sigma_n^2}\Big)\to\frac{t^2}{2}\text{ as }n\to\infty\,.\tag{3.54}$$

With this aim, we first notice that, by the selection of ℓ_n and the fact that $\varepsilon_n\to0$,

$$\|S'\big(I_{\ell_n,i}(p_n,j)\|_\infty\le 2T_n2^{-\ell_n}p_n=o(\sqrt{na_n})=o(\sqrt{\sigma_n^2a_n})\,.\tag{3.55}$$

In addition, since $\lim_n V''_{T_n}=0$ and the fact that $\liminf_n\sigma_n^2/n>0$, we have $\lim_n\sigma_n^{-2}\mathrm{Var}S'_n=1$. Notice that by (3.49) and the fact that $r_n=o(n)$,

$$\mathrm{Var}S'_n=\mathbb{E}\Big(\sum_{j=1}^{m_n}\sum_{i=1}^{2^{\ell_n}}S'(I_{\ell_n,i}(p_n,j))\Big)^2+o(n)\text{ as }n\to\infty\,.$$

Also, a straightforward computation as in Remark 3 shows that under (2.5) and (2.6) we have

$$\mathbb{E}\Big(\sum_{j=1}^{m_n}\sum_{i=1}^{2^{\ell_n}}\big(\ S'\big(I_{\ell_n,i}(p_n,j)\big)\Big)^2=\sum_{j=1}^{m_n}\sum_{i=1}^{2^{\ell_n}}\mathbb{E}\big(S'^2\big(I_{\ell_n,i}(p_n,j)\big)\big)+o(n)\text{ as }n\to\infty\,.$$

Hence

$$\lim_{n\to\infty}(\sigma_n)^{-2}\sum_{j=1}^{m_n}\sum_{i=1}^{2^{\ell_n}}\mathbb{E}\big(S'^2\big(I_{\ell_n,i}(p_n,j)\big)\big)=1\,.\tag{3.56}$$

Consequently (3.54) holds by taking into account (3.55) and (3.56) and by using Lemma 2.3 in Arcones (2003). ⋄

3.5 Proof of Corollary 1

We have to show that

$$\lim_{n\to\infty} \mathrm{Var}(S_n)/n = \sigma^2 > 0\,. \tag{3.57}$$

Proceeding as in the proof of Remark 3, we get that for any positive k,

$$|\mathrm{Cov}(X_0, X_k)| \le 2\int_0^{G(\tau(k)/2)} Q^2(u)du\,,$$

which combined with (2.5) and (2.6) imply that $\sum_{k>0} k|\mathrm{Cov}(X_0, X_k)| < \infty$. This condition together with the fact that $\mathrm{Var}(S_n) \to \infty$ entails (3.57) (see Lemma 1 in Bradley (1997)).

4 Appendix

We first give the following decoupling inequality.

Lemma 2. *Let Y_1, ..., Y_p be real-valued random variables each a.s. bounded by M. For every $i \in [1, p]$, let $\mathcal{M}_i = \sigma(Y_1, ..., Y_i)$ and for $i \ge 2$, let Y_i^* be a random variable independent of $\mathcal{M}_{i-1}$ and distributed as Y_i. Then for any real t,*

$$|\mathbb{E}\exp\Big(t\sum_{i=1}^p Y_i\Big) - \prod_{i=1}^p \mathbb{E}\exp(tY_i)| \le |t|\exp(|t|Mp)\sum_{i=2}^p \mathbb{E}|Y_i - Y_i^*|\,.$$

In particular, we have for any real t,

$$|\mathbb{E}\exp\Big(t\sum_{i=1}^p Y_i\Big) - \prod_{i=1}^p \mathbb{E}\exp(tY_i)| \le |t|\exp(|t|Mp)\sum_{i=2}^p \tau(\sigma(Y_1, \dots, Y_{i-1}), Y_i)\,,$$

where τ is defined by (2.2).

Proof of Lemma 2. Set $U_k = Y_1 + Y_2 + \cdots + Y_k$. We first notice that

$$\mathbb{E}\big(e^{tU_p}\big) - \prod_{i=1}^p \mathbb{E}\big(e^{tY_i}\big) = \sum_{k=2}^p \Big(\mathbb{E}\big(e^{tU_k}\big) - \mathbb{E}\big(e^{tU_{k-1}}\big)\mathbb{E}\big(e^{tY_k}\big)\Big)\prod_{i=k+1}^p \mathbb{E}\big(e^{tY_i}\big) \tag{4.1}$$

with the convention that the product from $p+1$ to p has value 1. Now

$$|\mathbb{E}\exp(tU_k) - \mathbb{E}\exp(tU_{k-1})\mathbb{E}\exp(tY_k)| \le \|\exp(tU_{k-1})\|_\infty \|\mathbb{E}\big(e^{tY_k} - e^{tY_k^*}|\mathcal{M}_{k-1}\big)\|_1\,.$$

Using (3.13) we then derive that

$$|\mathbb{E}\exp(tU_k) - \mathbb{E}\exp(tU_{k-1})\mathbb{E}\exp(tY_k)| \le |t|\exp(|t|kM)\|Y_k - Y_k^*\|_1\,. \tag{4.2}$$

Since the variables are bounded by M, starting from (4.1) and using (4.2), the result follows.$\diamond$

One of the tools we use repeatedly is the technical lemma below, which provides bounds for the log-Laplace transform of any sum of real-valued random variables.

Lemma 3. *Let $Z_0, Z_1, \dots$ be a sequence of real valued random variables. Assume that there exist positive constants $\sigma_0, \sigma_1, \dots$ and $c_0, c_1, \dots$ such that, for any positive i and any t in $[0, 1/c_i[$,*

$$\log \mathbb{E} \exp(tZ_i) \leq (\sigma_i t)^2/(1 - c_i t)\,.$$

Then, for any positive n and any t in $[0, 1/(c_0 + c_1 + \cdots + c_n)[$,

$$\log \mathbb{E} \exp(t(Z_0 + Z_1 + \cdots + Z_n)) \leq (\sigma t)^2/(1 - Ct),$$

where $\sigma = \sigma_0 + \sigma_1 + \cdots + \sigma_n$ and $C = c_0 + c_1 + \cdots + c_n$.

Proof of Lemma 3. Lemma 3 follows from the case $n = 1$ by induction on n. Let L be the log-Laplace of $Z_0 + Z_1$. Define the functions γ_i by

$$\gamma_i(t) = (\sigma_i t)^2/(1 - c_i t) \quad \text{for } t \in [0, 1/c_i[\quad \text{and } \gamma_i(t) = +\infty \quad \text{for } t \geq 1/c_i.$$

For u in $]0, 1[$, let $\gamma_u(t) = u\gamma_1(t/u) + (1 - u)\gamma_0(t/(1 - u))$. From the Hölder inequality applied with $p = 1/u$ and $q = 1/(1 - u)$, we get that $L(t) \leq \gamma_u(t)$ for any nonnegative t. Now, for t in $[0, 1/C[$, choose $u = (\sigma_1/\sigma)(1 - Ct) + c_1 t$ (here $C = c_0 + c_1$ and $\sigma = \sigma_0 + \sigma_1$). With this choice $1 - u = (\sigma_0/\sigma)(1 - Ct) + c_0 t$, so that u belongs to $]0, 1[$ and

$$L(t) \leq \gamma_u(t) = (\sigma t)^2/(1 - Ct),$$

which completes the proof of Lemma 3. $\diamond$

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